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A Three-Field Variational Formulation for a Frictional Contact Problem with Prescribed Normal Stress

Andaluzia Cristina Matei

Department of Mathematics, University of Craiova, A.I. Cuza 13, 200585 Craiova, Romania; andaluzia.matei@edu.ucv.ro; Tel.: +40-251-413-728

Abstract: In the present work, we address a nonlinear boundary value problem that models frictional contact with prescribed normal stress between a deformable body and a foundation. The body is nonlinearly elastic, the constitutive law being a subdifferential inclusion. We deliver a three-field variational formulation by means of a new variational approach governed by the theory of bipotentials combined with a Lagrange-multipliers technique. In this new approach, the unknown of the mechanical model is a triple consisting of the displacement field, a Lagrange multiplier related to the friction force and the Cauchy stress tensor. We obtain existence, uniqueness, boundedness and convergence results.

Keywords: boundary value problem; multi-valued elastic operator; frictional contact with prescribed normal stress; fractional Sobolev spaces; separable bipotential; Lagrange multipliers; variational inequalities; minimization; weak solution; approximation

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1. Introduction

Everywhere around us we can see deformable bodies in interaction. However, even though very common, the contact phenomenon is not a trivial one; in addition, in engineering, handling the interactions between deformable bodies and obstacles is very important and requires advanced applied mathematics. The contact phenomenon can be mathematically modeled by means of boundary value problems governed by partial differential equations. Actually, the topic is very complex, involving continuum mechanics, differential equations, function spaces, calculus of variations, nonlinear analysis, control theory and numerical analysis. The importance and the abundance of the applications of contact problems in the real world has motivated a large number of scientists to investigate this kind of model. It is worth underlining that, due to their complexity, contact models do not have classical solutions. Thus, the variational methods play a crucial role in the qualitative and quantitative analysis.

In the present paper, we focus on the well-posedness and approximation results addressing a stationary frictional contact problem with prescribed normal stress, for materials governed by a multi-valued elastic operator. Using the bipotential theory, we deliver a variational formulation of the mechanical model in a form of a variational system consisting of three inequalities. Placing us in an appropriate functional setting governed by Lebesgue and Sobolev spaces for vector functions including fractional spaces on the boundary, we apply the saddle point theory and a minimization technique in order to prove the existence of at least one solution. We also pay attention to the uniqueness of the solution. Firstly, we draw attention to a partial uniqueness result related to the uniqueness in the first component. Subsequently, we discuss a global uniqueness result, not for the original problem, but for a perturbed version of it. After we investigate the boundedness of the solution of the perturbed problem, we prove a convergence result allowing an approximation of a weak solution of the contact model under consideration. The present study can be seen as a continuation of [1]; there, a two-field variational formulation for the same model was delivered; with the well-posedness of the model being studied under a more restrictive hypothesis for the prescribed normal stress. In [1], the weak solution is a pair consisting of the displacement vector and the Cauchy stress tensor. In the present study, the weak solution is a triple by considering a Lagrange multiplier related to the friction force as a component of the weak solution, in addition to the displacement vector and the Cauchy stress tensor. From the mathematical point of view, the new study is more complex. Besides the theory of bipotentials and the minimization techniques, the present study requires a saddle point technique, fractional Lebesgue and Sobolev spaces, weak topologies and a convergence of the Mosco type. It is worth emphasizing that the variational approach we propose leads to a new class of variational problems governed by Lagrange multipliers. From the mechanical point of view, the new variational approach we propose is important because it makes possible an estimation of the friction force, even a numerical computation, after passing from the qualitative study to the quantitative analysis in a future investigation.

Let us specify some helpful references for background knowledge: for the mechanics of deformable solids/contact mechanics see, e.g., [2–9]; for bipotential theory, we refer to, e.g., [10–15]; for the saddle point theory see, e.g., [16–20]; and for the functional spaces the reader can consult, e.g., [21–27]. For recent results related to the topic of the present work, we refer, for instance, to [28–30]. However, in order to increase the clarity of the exposure, we specify herein some mathematical tools that will play a crucial role.

Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ be a Hilbert space.

Definition 1. *A bipotential is a function $B : X \times X \to (-\infty, \infty]$ with the following three properties:*

(i) *B is convex and lower semicontinuous in each argument;*

(ii) *For each $x,\, y \in X$, we have $B(x,y) \geq (x,y)_X$;*

(iii) *For each $x,\, y \in X$, $y \in \partial B(\cdot,y)(x) \Leftrightarrow x \in \partial B(x,\cdot)(y) \Leftrightarrow B(x,y) = (x,y)_X$.*

Recall that the Fenchel conjugate of a functional $\phi : X \to (-\infty, \infty]$ is the functional

$$\phi^* : X \to (-\infty, \infty], \quad \phi^*(x^*) = \sup_{x \in X}\{(x^*, x) - \phi(x)\}.$$

Always, the Fenchel conjugate is a convex and lower semicontinuous functional, see, e.g., [22].

Theorem 1. *Let $\phi : X \to (-\infty, \infty]$ be a proper, convex, lower semicontinuous functional. Then:*

(i) *For each $x,\, y \in X$, we have $\phi(x) + \phi^*(y) \geq (x,y)_X$;*

(ii) *For each $x,\, y \in X$, $y \in \partial\phi(x) \Leftrightarrow x \in \partial\phi^*(y) \Leftrightarrow \phi(x) + \phi^*(y) = (x,y)_X$.*

Notice that if ϕ is a proper, convex, lower semicontinuous functional, then its Fenchel conjugate has all these properties too, see, e.g., [22]. For some details and additional elements in the convex analysis, we refer to, e.g., [18,31–34].

In addition, we shall need the following theorem.

Theorem 2 (See, e.g., [35])**.** *Let $(X, \|\cdot\|_X)$ be a reflexive Banach space and let $K \subset X$ be a nonempty, convex, closed, unbounded subset of X. Suppose $\varphi : K \to \mathbb{R}$ is coercive, convex and lower semicontinuous. Then, φ is bounded from below on K and attains its infimum in K. If φ is strictly convex, then φ has a unique minimizer.*

We recall that $\varphi : K \to \mathbb{R}$ is coercive if, for all $u \in K$, we have $\varphi(u) \to \infty$ as $\|u\|_X \to \infty$.

Theorem 3. *Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two Hilbert spaces and let $A \subseteq X$, $B \subseteq Y$ be nonempty, closed, convex subsets. Assume that a bifunctional $\mathcal{L} : A \times B \to \mathbb{R}$ satisfies the following conditions*

$$v \to \mathcal{L}(v,\mu) \text{ is convex and lower semi-continuous } \text{ for all } \mu \in B,$$

$$\mu \to \mathcal{L}(v,\mu) \text{ is concave and upper semi-continuous } \text{ for all } v \in A.$$

Moreover, we assume that

$$A \text{ is bounded or } \lim_{\|v\|_X \to \infty,\, v \in A} \mathcal{L}(v,\mu_0) = \infty \text{ for some } \mu_0 \in B$$

and

$$B \text{ is bounded or } \lim_{\|\mu\|_Y \to \infty,\, \mu \in B} \inf_{v \in A} \mathcal{L}(v,\mu) = -\infty.$$

Then:

(a) *The bifunctional $\mathcal{L}(\cdot,\cdot)$ has at least one saddle point;*
(b) *The set $A_0 \times B_0$ of the saddle points of $\mathcal{L}$ is convex, where $A_0 \subset A$ and $B_0 \subset B$;*
(c) *If $v \to \mathcal{L}(v,\mu)$ is strictly convex for all $\mu \in B$, then A_0 contains at most one point;*
(d) *If $\mu \to \mathcal{L}(v,\mu)$ is strictly concave for all $v \in A$, then B_0 contains at most one point.*

For a proof of (a), see [18] (p. 176). For a proof of (b), (c), (d), see [18] (p. 169).

The rest of the paper has the following structure. In Section 2 we provide the functional setting we use. In Section 3 we describe the mechanical model and deliver its three-field variational formulation. In Section 4 we obtain existence, uniqueness, boundedness and convergence results. The last section provides some conclusions and final comments.

2. Functional Setting

Let $\Omega \subset \mathbb{R}^N$ $(N > 1)$ be a bounded domain with a regular enough boundary denoted by Γ. We use standard notation for $L^p(\Omega)$, $W^{1,p}(\Omega)$, $L^p(\Gamma)$, $H^1(\Omega)$, see, e.g., [21–27].

Let $2 \le p < \infty$. Recall that, according to the trace theorem—see, e.g., [6] (p. 34)—there exists a unique linear continuous operator $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ such that

(a) $\gamma\, u = u|_\Gamma$ if $u \in C^1(\overline{\Omega})$;
(b) $\|\gamma\, u\|_{L^p(\Gamma)} \le c_{tr} \|u\|_{W^{1,p}(\Omega)}$ with $c_{tr} = c_{tr}(p,\Omega) > 0$;
(c) If $1 < p < \infty$, then $\gamma(W^{1,p}(\Omega)) = W^{1-1/p,p}(\Gamma)$;
(d) If $1 < p < N$, then $\gamma : W^{1,p}(\Omega) \to L^r(\Gamma)$ is compact for any r such that $1 \le r < \frac{Np-p}{N-p}$.

The function $\gamma\, u$ is called the trace of the scalar function u on Γ and the operator $\gamma : W^{1,p}(\Omega) \to L^p(\Gamma)$ is called the trace operator. The trace operator is neither an injection, nor a surjection from $W^{1,p}(\Omega)$ to $L^p(\Gamma)$. However, according to, e.g., Theorem 1.5.1.2 in [23], there exists a unique linear continuous and surjective operator from $W^{1,p}(\Omega)$ to $\gamma(W^{1,p}(\Omega)) = W^{1-1/p,p}(\Gamma)$. Therefore, there are some $\bar{d} = \bar{d}(p,\Omega)$, such that

$$\|\gamma u\|_{W^{1-\frac{1}{p},p}(\Gamma)} \le \bar{d} \|u\|_{W^{1,p}(\Omega)} \quad \text{for all } u \in W^{1,p}(\Omega).$$

Furthermore, there exists a map called the right inverse of γ, denoted here by ℓ,

$$\ell : W^{1-1/p,p}(\Gamma) \to W^{1,p}(\Omega), \qquad \gamma(\ell(\zeta)) = \zeta \quad \text{for all} \quad \zeta \in W^{1-1/p,p}(\Gamma);$$

see, e.g., [36]. Notice that the right inverse operator is a linear and continuous map.

The space

$$W^{1-1/p,p}(\Gamma) = \{u \in L^p(\Gamma) \,|\, \exists\, v \in W^{1,p}(\Omega) \text{ such that } u = \gamma v \text{ a.e. on } \Gamma\}$$

is a normed space endowed with the following norm

$$\|u\|_{W^{1-1/p,p}(\Gamma)} = \left(\|u\|^p_{L^p(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|u(\boldsymbol{x}) - u(\boldsymbol{y})|^p}{\|\boldsymbol{x} - \boldsymbol{y}\|^{N+p-2}} ds(\boldsymbol{x}) ds(\boldsymbol{y}) \right)^{1/p},$$

see, e.g., [26], (p. 43). Actually, the space $(W^{1-1/p,p}(\Gamma), \|\cdot\|_{W^{1-1/p,p}(\Gamma)})$ is a Banach space, see, e.g., [24], (p. 332).

If $p = 2$, $\gamma(H^1(\Omega)) = H^{1/2}(\Gamma)$. The space $H^{1/2}(\Gamma)$ is a Hilbert space endowed with the inner product

$$(v, w)_{H^{1/2}(\Gamma)} = (v, w)_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{(v(\boldsymbol{x}) - v(\boldsymbol{y}))(w(\boldsymbol{x}) - w(\boldsymbol{y}))}{\|\boldsymbol{x} - \boldsymbol{y}\|^N} d\Gamma(\boldsymbol{x}) d\Gamma(\boldsymbol{y})$$

and the corresponding Sobolev–Slobodeckij norm

$$\|v\|_{H^{1/2}(\Gamma)} = \left(\|v\|^2_{L^2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{(v(\boldsymbol{x}) - v(\boldsymbol{y}))^2}{\|\boldsymbol{x} - \boldsymbol{y}\|^N} d\Gamma(\boldsymbol{x}) d\Gamma(\boldsymbol{y}) \right)^{1/2}.$$

Notice that

$$\|u\|_{L^2(\Gamma)} \le \|u\|_{H^{1/2}(\Gamma)} \text{ for all } u \in H^{1/2}(\Gamma),$$

so $H^{1/2}(\Gamma)$ is continuously embedded in $L^2(\Gamma)$. The space $H^{1/2}(\Gamma)$ is called the image of $H^1(\Omega)$ by the trace operator γ.

To proceed, we consider $N = 3$. The fields in $\mathbb{R}^3$ will be typeset in boldface. The inner product and the Euclidean norm on $\mathbb{R}^3$ will be denoted by $\cdot$ and $\|\cdot\|$, respectively.

Let us introduce the vector spaces

$$\begin{aligned} L^2(\Omega; \mathbb{R}^3) &= \{\boldsymbol{w} = (w_i) \mid w_i \in L^2(\Omega),\ 1 \le i \le 3\}; \\ H^1(\Omega; \mathbb{R}^3) &= \{\boldsymbol{w} = (w_i) \mid w_i \in H^1(\Omega),\ 1 \le i \le 3\}; \\ H^{1/2}(\Gamma; \mathbb{R}^3) &= \boldsymbol{\gamma}(H^1(\Omega; \mathbb{R}^3)) = \{\boldsymbol{w} = (w_i) \mid w_i \in H^{1/2}(\Gamma),\ 1 \le i \le 3\}. \end{aligned}$$

The spaces $L^2(\Omega; \mathbb{R}^3)$ and $H^1(\Omega; \mathbb{R}^3)$ are Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^3)} = \sum_{i=1}^3 \int_\Omega u_i v_i \, dx, \qquad (\boldsymbol{u}, \boldsymbol{v})_{H^1(\Omega;\mathbb{R}^3)} = \sum_{i=1}^3 (u_i, v_i)_{H^1(\Omega)},$$

and the associated norms $\|\cdot\|_{L^2(\Omega;\mathbb{R}^3)} = \sqrt{(\cdot,\cdot)_{L^2(\Omega;\mathbb{R}^3)}}$ and $\|\cdot\|_{H^1(\Omega;\mathbb{R}^3)} = \sqrt{(\cdot,\cdot)_{H^1(\Omega;\mathbb{R}^3)}}$, respectively. The space $H^{1/2}(\Gamma; \mathbb{R}^3)$ is also a Hilbert space endowed with the inner product

$$\begin{aligned} (\boldsymbol{v}, \boldsymbol{w})_{H^{1/2}(\Gamma;\mathbb{R}^3)} &= \sum_{i=1}^3 (v_i, w_i)_{H^{1/2}(\Gamma)} \\ &= \sum_{i=1}^3 (v_i, w_i)_{L^2(\Gamma)} + \sum_{i=1}^3 \int_\Gamma \int_\Gamma \frac{(v_i(\boldsymbol{x}) - v_i(\boldsymbol{y}))(w_i(\boldsymbol{x}) - w_i(\boldsymbol{y}))}{\|\boldsymbol{x} - \boldsymbol{y}\|^3} d\Gamma(\boldsymbol{x}) d\Gamma(\boldsymbol{y}) \end{aligned}$$

and the corresponding Sobolev–Slobodeckij norm

$$\|\boldsymbol{w}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)} = \left(\|\boldsymbol{w}\|^2_{L^2(\Gamma;\mathbb{R}^3)} + \int_\Gamma \int_\Gamma \frac{\|\boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{w}(\boldsymbol{y})\|^2}{\|\boldsymbol{x} - \boldsymbol{y}\|^3} d\Gamma(\boldsymbol{x}) d\Gamma(\boldsymbol{y}) \right)^{1/2}.$$

We easily observe that

$$\|\boldsymbol{u}\|_{L^2(\Gamma;\mathbb{R}^3)} \le \|\boldsymbol{u}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)} \text{ for all } \boldsymbol{u} \in H^{1/2}(\Gamma; \mathbb{R}^3), \tag{1}$$

so $H^{1/2}(\Gamma; \mathbb{R}^3)$ is continuously embedded in $L^2(\Gamma; \mathbb{R}^3)$.

The trace operator for vector functions $\boldsymbol{\gamma} : H^1(\Omega; \mathbb{R}^3) \to L^2(\Gamma; \mathbb{R}^3)$ is a linear, continuous and compact operator, but it is neither an injection nor a surjection; see, e.g., [37].

$$\|\boldsymbol{\gamma}\, \boldsymbol{u}\|_{L^2(\Gamma;\mathbb{R}^3)} \le \bar{c}_{tr} \|\boldsymbol{u}\|_{H^1(\Omega;\mathbb{R}^3)} \quad \text{for all } \boldsymbol{u} \in H^1(\Omega; \mathbb{R}^3) \quad (\bar{c}_{tr} > 0).$$

Notice that

$$\gamma : H^1(\Omega;\mathbb{R}^3) \to H^{1/2}(\Gamma;\mathbb{R}^3)$$

is a linear, continuous operator but it is not a compact operator. Let us point out that there are some $\widetilde{c} > 0$ such that

$$\|\gamma \boldsymbol{u}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)} \leq \widetilde{c}\|\boldsymbol{u}\|_{H^1(\Omega;\mathbb{R}^3)} \quad \text{for all } \boldsymbol{u} \in H^1(\Omega;\mathbb{R}^3). \tag{2}$$

Furthermore, there exists a linear, continuous operator $\boldsymbol{\ell} : H^{1/2}(\Gamma;\mathbb{R}^3) \to H^1(\Omega;\mathbb{R}^3)$ such that

$$\gamma(\boldsymbol{\ell}(\boldsymbol{\xi})) = \boldsymbol{\xi} \quad \text{for all } \boldsymbol{\xi} \in H^{1/2}(\Gamma;\mathbb{R}^3).$$

The operator $\boldsymbol{\ell}$ is called the *right inverse of the trace operator* γ.

Let $\mathbb{S}^3$ be the space of second-order symmetric tensors on $\mathbb{R}^3$. Every field in $\mathbb{S}^3$ is typeset in boldface. By $:$ and $\|\cdot\|_{\mathbb{S}^3}$ we denote the inner product and the Euclidean norm on $\mathbb{S}^3$.

We introduce now two tensor Lebesgue spaces, as follows.

$$\begin{aligned} L^2(\Omega;\mathbb{S}^3) &= \{\boldsymbol{\mu} = (\mu_{ij}) \,:\, \mu_{ij} \in L^2(\Omega) \text{ for all } i,j \in \{1,2,3\}\}; \\ L^2_s(\Omega;\mathbb{S}^3) &= \{\boldsymbol{\mu} = (\mu_{ij}) \,:\, \mu_{ij} \in L^2(\Omega),\, \mu_{ij} = \mu_{ji} \text{ for all } i,j \in \{1,2,3\}\}. \end{aligned}$$

The space $L^2(\Omega;\mathbb{S}^3)$ is a Hilbert space endowed with the inner product

$$(\boldsymbol{\mu},\boldsymbol{\tau})_{L^2(\Omega;\mathbb{S}^3)} = \int_\Omega \sum_{i,j=1}^{3} \mu_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\,dx;$$

the space $L^2_s(\Omega;\mathbb{S}^3)$ is a Hilbert space endowed with the inner product

$$(\boldsymbol{\mu},\boldsymbol{\tau})_{L^2_s(\Omega;\mathbb{S}^3)} = (\boldsymbol{\mu},\boldsymbol{\tau})_{L^2(\Omega;\mathbb{S}^3)}.$$

Next, we introduce the following space,

$$H_1 = \{\boldsymbol{u} \in L^2(\Omega;\mathbb{R}^3) \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in L^2_s(\Omega;\mathbb{S}^3)\},$$

where

$$\boldsymbol{\varepsilon} : H^1(\Omega;\mathbb{R}^3) \to L^2(\Omega;\mathbb{S}^3)$$

is the linear operator

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

the index that follows a comma indicates a weak partial derivative with respect to the corresponding component of the independent variable.

The space H_1 is a real Hilbert space endowed with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega;\mathbb{R}^3)} + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)}.$$

The associated norm on the space H_1 is denoted by $\|\cdot\|_{H_1}$. According to, e.g., [38], $H_1 = H^1(\Omega;\mathbb{R}^3)$ algebraically and the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H^1(\Omega;\mathbb{R}^3)}$ are equivalent.

Notice that $\boldsymbol{\varepsilon}$ is a linear and continuous operator from $H^1(\Omega;\mathbb{R}^3)$ to $L^2_s(\Omega;\mathbb{S}^3)$.

Let Γ_1 be a measurable part of Γ with positive surface measure.

We consider the space

$$V = \{\, \boldsymbol{v} \in H_1 \mid \gamma\boldsymbol{v} = \boldsymbol{0} \text{ almost everywhere on } \Gamma_1\}. \tag{3}$$

This is a closed subspace of $H^1(\Omega;\mathbb{R}^3)$, so $(V, (\cdot,\cdot)_{H^1(\Omega;\mathbb{R}^3)}, \|\cdot\|_{H^1(\Omega;\mathbb{R}^3)})$ is a Hilbert space.

Let us recall Korn's inequality: there exists $c_K = c_K(\Omega, \Gamma_1) > 0$ such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2_s(\Omega;\mathbb{S}^3)} \geq c_K \|\boldsymbol{v}\|_{H_1} \quad \text{for all } \boldsymbol{v} \in V;$$

see, e.g., [38,39]. Using this inequality, it can be proved that the space V is a Hilbert space endowed with the following inner product,

$$(\cdot,\cdot)_V : V \times V \to \mathbb{R}; \quad (\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{4}$$

and the corresponding norm

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L^2_s(\Omega;\mathbb{S}^3)} \quad \text{for all } \boldsymbol{v} \in V.$$

Notice that, since the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H^1(\Omega;\mathbb{R}^3)}$ are equivalent, then there exists $\widetilde{c}_K = \widetilde{c}_K(\Omega, \Gamma_1) > 0$ such that

$$\|\boldsymbol{v}\|_V \geq \widetilde{c}_K \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^3)} \quad \text{for all } \boldsymbol{v} \in V. \tag{5}$$

We proceed by introducing a closed subspace of $H^{1/2}(\Gamma;\mathbb{R}^3)$ as follows:

$$\boldsymbol{\gamma}(V) = \{\widetilde{\boldsymbol{v}} \in H^{1/2}(\Gamma;\mathbb{R}^3) \,|\, \widetilde{\boldsymbol{v}} = \boldsymbol{\gamma}\boldsymbol{v} \text{ for some } \boldsymbol{v} \in V\}. \tag{6}$$

According to [40], the space $\boldsymbol{\gamma}(V)$ is a closed subspace of $H_\Gamma = H^{1/2}(\Gamma;\mathbb{R}^3)$. Thus, $\boldsymbol{\gamma}(V)$ is a Hilbert space endowed with the inner product

$$(\cdot,\cdot)_{\gamma(V)} : \boldsymbol{\gamma}(V) \times \boldsymbol{\gamma}(V) \to R, \quad (\boldsymbol{\zeta}, \boldsymbol{\phi})_{\gamma(V)} = (\boldsymbol{\zeta}, \boldsymbol{\phi})_{H^{1/2}(\Gamma;\mathbb{R}^3)} \quad \text{for all } \boldsymbol{\zeta},\, \boldsymbol{\phi} \in \boldsymbol{\gamma}(V).$$

We note that

$$\boldsymbol{\gamma}(\boldsymbol{\ell}(\boldsymbol{\gamma}\boldsymbol{v})) = \boldsymbol{\gamma}\boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in V.$$

As

$$\boldsymbol{\ell}(\boldsymbol{\zeta}) \in V \quad \text{for all } \boldsymbol{\zeta} \in \boldsymbol{\gamma}(V),$$

we can introduce an operator as follows

$$R: \ \boldsymbol{\gamma}(V) \to V, \quad R(\boldsymbol{\zeta}) = \boldsymbol{\ell}(\boldsymbol{\zeta}).$$

The operator R is a linear and continuous operator. Hence, there are some $\bar{c} > 0$ such that

$$\|R\widetilde{\boldsymbol{v}}\|_V \leq \bar{c} \|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)} \quad \text{for all } \widetilde{\boldsymbol{v}} \in \boldsymbol{\gamma}(V). \tag{7}$$

3. The Model and Its Three-Field Variational Formulation

The physical setting is as follows: a deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth enough boundary Γ, partitioned in three measurable parts Γ_1, Γ_2 and Γ_3 with positive surface measures. The body is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω, surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 while on Γ_3 the body is in frictional contact with a foundation.

According to this physical setting, we state the following boundary value problem.

Problem 1. *Find* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and* $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$*, such that*

$$\begin{aligned}
\operatorname{Div} \boldsymbol{\sigma}(\boldsymbol{x}) + \boldsymbol{f}_0(\boldsymbol{x}) &= \mathbf{0} \ in\ \Omega, && (8)\\
\boldsymbol{\sigma}(\boldsymbol{x}) &\in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}))\ in\ \Omega, && (9)\\
\boldsymbol{u}(\boldsymbol{x}) &= \mathbf{0}\ on\ \Gamma_1, && (10)\\
\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) &= \boldsymbol{f}_2(\boldsymbol{x})\ on\ \Gamma_2, && (11)\\
-\sigma_\nu(\boldsymbol{x}) &= F(\boldsymbol{x})\ on\ \Gamma_3, && (12)\\
\|\boldsymbol{\sigma}_\tau(\boldsymbol{x})\| \le k(\boldsymbol{x})\,|\sigma_\nu(\boldsymbol{x})|, \quad \boldsymbol{\sigma}_\tau(\boldsymbol{x}) &= -\,k(\boldsymbol{x})\,|\sigma_\nu(\boldsymbol{x})|\,\frac{\boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|} \quad \text{if} \quad \boldsymbol{u}_\tau(\boldsymbol{x}) \neq \mathbf{0} \text{ on } \Gamma_3. && (13)
\end{aligned}$$

As usual, $\boldsymbol{u} = (u_i)$ denotes the displacement field, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ denotes the infinitesimal strain tensor, $\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the Cauchy stress tensor, ω is a constitutive map, $\boldsymbol{\nu}$ stands for the unit outward normal to Γ, the normal and the tangential components of the displacement vector on the boundary are defined by the formulas $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$ and the normal and the tangential components of the Cauchy vector on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$.

Due to the condition

$$-\sigma_\nu = F,$$

according to the engineering literature, the frictional contact model we treat is a bilateral frictional contact problem. In this context, we have to mention that the bilateral frictional contact phenomenon can be found in many components of mechanical equipment. Thus, many real-world examples can be envisaged. The mathematical and the engineering literature contains relevant applications of bilateral frictional contact models; see, e.g., [41,42]. Referring to the behavior of the materials, for significant examples of nonlinearly elastic constitutive laws described by means of subdifferential inclusions for various constitutive maps ω, see, e.g., [38] and the references therein. For the convenience of the reader, we indicate here an example of such a constitutive map:

$$\omega : \mathbb{S}^3 \to \mathbb{R}, \quad \omega(\boldsymbol{\tau}) = \frac{1}{2}\mathcal{A}\boldsymbol{\tau} : \boldsymbol{\tau} + \frac{k}{2}\|\boldsymbol{\tau} - \mathcal{P}_K\boldsymbol{\tau}\|^2_{\mathbb{S}^3}, \tag{14}$$

where $\mathcal{A} : \mathbb{S}^3 \to \mathbb{S}^3$, $\mathcal{A} = (\mathcal{A}_{ijkl})$, $\mathcal{A}_{ijkl} = \lambda_0\delta_{ij}\delta_{kl} + \mu_0(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, $1 \le i,j,k,l \le 3$, with λ_0, μ_0 and k small enough positive material coefficients, and $\mathcal{P}_K : \mathbb{S}^3 \to K$ denotes the projection operator on the closed and convex set $K \subset \mathbb{S}^3$ which contains $0_{\mathbb{S}^3}$.

In order to study Problem 1, we make the following assumptions.

Assumption 1. *$\omega : \mathbb{S}^3 \to \mathbb{R}$ is a convex and lower semicontinuous functional. In addition, there exist α, β such that $1 > \beta \ge \alpha > 0$ and $\beta\|\boldsymbol{\varepsilon}\|^2_{\mathbb{S}^3} \ge \omega(\boldsymbol{\varepsilon}) \ge \alpha\|\boldsymbol{\varepsilon}\|^2_{\mathbb{S}^3}$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^3$.*

Assumption 2.

$$\boldsymbol{f}_0 \in L^2(\Omega;\mathbb{R}^3) \ and\ \boldsymbol{f}_2 \in L^2(\Gamma_2;\mathbb{R}^3).$$

Assumption 3. *$F \in L^{\frac{2-\delta}{1-\delta}}(\Gamma_3)$ $(0 < \delta < 1)$ and $F(\boldsymbol{x}) \ge 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.*

Assumption 4. *The coefficient of friction satisfies $k \in L^\infty(\Gamma_3)$ and $k(\boldsymbol{x}) \ge 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.*

Notice that the example in (14) fulfills Assumption 1.

Let $(\boldsymbol{u}, \boldsymbol{\sigma})$ be a pair of smooth-enough functions that verify Problem 1. Using Green's formula

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^3)} + (\operatorname{Div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^3)} = \int_\Gamma \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})\cdot\gamma\boldsymbol{v}(\boldsymbol{x})\,d\Gamma \quad \text{for all } \boldsymbol{v} \in H^1(\Omega;\mathbb{R}^3),$$

(see, e.g., [3], (p. 145)), by taking into account (8), (10) and (11) we obtain, for all $\boldsymbol{v} \in V$

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^3)} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \gamma\boldsymbol{v}(\boldsymbol{x})\, d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \cdot \gamma\boldsymbol{v}(\boldsymbol{x})\, d\Gamma, \quad (15)$$

where V is the space defined in (3).

As

$$\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \cdot \gamma\boldsymbol{v}(\boldsymbol{x}) = \sigma_\nu(\boldsymbol{x})v_\nu(\boldsymbol{x}) + \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x}) \text{ a.e. on } \Gamma,$$

then by (15) we infer that

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^3)} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, d\Gamma + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x})\, d\Gamma - \int_{\Gamma_3} F(\boldsymbol{x})v_\nu(\boldsymbol{x})\, d\Gamma.$$

Herein and everywhere below, $v_\nu(\boldsymbol{x}) = \gamma\boldsymbol{v}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x})$ and $\boldsymbol{v}_\tau(\boldsymbol{x}) = \gamma\boldsymbol{v}(\boldsymbol{x}) - v_\nu(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x})$ a.e. $\boldsymbol{x} \in \Gamma$. By taking into consideration H3, H4, using the trace theorem for $N = 3$ and $p = 2$ and the Hölder's inequality, it follows that $F(\cdot)v_\nu(\cdot) \in L^1(\Gamma_3)$ for all $\boldsymbol{v} \in V$.

Since

$$V \ni \boldsymbol{v} \to (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \gamma\boldsymbol{v}(\boldsymbol{x})\, d\Gamma - \int_{\Gamma_3} F(\boldsymbol{x})v_\nu(\boldsymbol{x}) d\Gamma \in \mathbb{R}$$

is a linear and continuous map, according to Riesz's representation theorem, there exists a unique element $\boldsymbol{f} \in V$ such that

$$(\boldsymbol{f}, \boldsymbol{v})_V = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \gamma\boldsymbol{v}(\boldsymbol{x}) d\Gamma - \int_{\Gamma_3} F(\boldsymbol{x})v_\nu(\boldsymbol{x}) d\Gamma \quad \text{for all } \boldsymbol{v} \in V.$$

Therefore,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^3)} = (\boldsymbol{f}, \boldsymbol{v})_V + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x})\, d\Gamma. \quad (16)$$

Let D be the dual of the space $\gamma(V)$ defined in (6).

We define $\boldsymbol{\lambda} \in D$ such that

$$\langle \boldsymbol{\lambda}, \widetilde{\boldsymbol{v}} \rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \widetilde{\boldsymbol{v}}_\tau(\boldsymbol{x})\, d\Gamma, \qquad \text{for all } \widetilde{\boldsymbol{v}} \in \gamma(V), \quad (17)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D and $\gamma(V)$ and

$$\widetilde{\boldsymbol{v}}_\tau(\boldsymbol{x}) = \widetilde{\boldsymbol{v}}(\boldsymbol{x}) - \widetilde{v}_\nu(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}), \qquad \widetilde{v}_\nu(\boldsymbol{x}) = \widetilde{\boldsymbol{v}}(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \quad (18)$$

Furthermore, we define a form $c(\cdot, \cdot)$ as follows,

$$c : V \times D \to \mathbb{R}, \quad c(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \gamma\boldsymbol{v} \rangle, \quad \text{for all } \boldsymbol{v} \in V, \boldsymbol{\mu} \in D. \quad (19)$$

Important properties of the form $c(\cdot, \cdot)$ are given by the following lemma.

Lemma 1. *The form $c(\cdot, \cdot)$ is bilinear. In addition,*

- *$c(\cdot, \cdot)$ is continuous of rank $M_c > 0$ i.e.,*

$$|c(\boldsymbol{v}, \boldsymbol{\zeta})| \le M_c \|\boldsymbol{v}\|_V \|\boldsymbol{\zeta}\|_D \textit{ for all } \boldsymbol{v} \in V, \boldsymbol{\zeta} \in D,$$

- *$c(\cdot, \cdot)$ verifies the inf-sup property:*

$$\inf_{\boldsymbol{\zeta} \in D, \boldsymbol{\zeta} \neq 0_D} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq 0_V} \frac{c(\boldsymbol{v}, \boldsymbol{\zeta})}{\|\boldsymbol{v}\|_V \|\boldsymbol{\zeta}\|_D} \ge \alpha_c \textit{ for some } \alpha_c > 0.$$

Proof. The bilinearity of $c(\cdot, \cdot)$ is obvious keeping in mind the linearity of the trace operator γ.

Let $\boldsymbol{v} \in V$ and $\boldsymbol{\zeta} \in D$ be arbitrarily given.

$$
\begin{aligned}
|c(\boldsymbol{v}, \boldsymbol{\zeta})| &\leq \|\boldsymbol{\zeta}\|_D \|\gamma \boldsymbol{v}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)} \\
&\leq \widetilde{c}\|\boldsymbol{\zeta}\|_D \|\boldsymbol{v}\|_{H^1(\Omega;\mathbb{R}^3)} \\
&\leq \frac{\widetilde{c}}{\widetilde{c}_K}\|\boldsymbol{\zeta}\|_D \|\boldsymbol{v}\|_V.
\end{aligned}
$$

We can consider $M_c = \frac{\widetilde{c}}{\widetilde{c}_K}$ where $\widetilde{c} > 0$ appears in (2) and $\widetilde{c}_K > 0$ appears in (5).

Next, we prove that $c(\cdot, \cdot)$ verifies the inf-sup property, or equivalently, there are some α_c>0 such that $\|\boldsymbol{\zeta}\|_D \leq \frac{1}{\alpha_c} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq 0_V} \frac{c(\boldsymbol{v},\boldsymbol{\zeta})}{\|\boldsymbol{v}\|_V}$ for all $\boldsymbol{\zeta} \in D, \boldsymbol{\zeta} \neq 0_D$.

Indeed, let $\boldsymbol{\zeta} \in D, \boldsymbol{\zeta} \neq 0_D$.

$$
\|\boldsymbol{\zeta}\|_D = \sup_{\widetilde{\boldsymbol{w}} \in \gamma(V), \widetilde{\boldsymbol{w}} \neq \mathbf{0}_{\gamma(V)}} \frac{< \boldsymbol{\zeta}, \widetilde{\boldsymbol{w}} >}{\|\widetilde{\boldsymbol{w}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}}.
$$

As $\langle \boldsymbol{\zeta}, \widetilde{\boldsymbol{w}} \rangle = \langle \boldsymbol{\zeta}, \gamma(\boldsymbol{\ell}\widetilde{\boldsymbol{w}}) \rangle = \langle \boldsymbol{\zeta}, \gamma(R\widetilde{\boldsymbol{w}})) \rangle$, then

$$
\|\boldsymbol{\zeta}\|_D = \sup_{\widetilde{\boldsymbol{w}} \in \gamma(V), \widetilde{\boldsymbol{w}} \neq \mathbf{0}_{\gamma(V)}} \frac{\langle \boldsymbol{\zeta}, \gamma(R\widetilde{\boldsymbol{w}})) \rangle}{\|\widetilde{\boldsymbol{w}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} = \sup_{\widetilde{\boldsymbol{w}} \in \gamma(V), \widetilde{\boldsymbol{w}} \neq \mathbf{0}_{\gamma(V)}} \frac{c(R\widetilde{\boldsymbol{w}}, \boldsymbol{\zeta})}{\|\widetilde{\boldsymbol{w}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} = \sup_{\widetilde{\boldsymbol{w}} \in \gamma(V), R\widetilde{\boldsymbol{w}} \neq 0_V} \frac{c(R\widetilde{\boldsymbol{w}}, \boldsymbol{\zeta})}{\|\widetilde{\boldsymbol{w}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}}.
$$

Using (7), we can write

$$
\|\boldsymbol{\zeta}\|_D \leq \bar{c} \sup_{\widetilde{\boldsymbol{w}} \in \gamma(V), R\widetilde{\boldsymbol{w}} \neq 0_V} \frac{c(R\widetilde{\boldsymbol{w}}, \boldsymbol{\zeta})}{\|R\widetilde{\boldsymbol{w}}\|_V} \leq \bar{c} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq 0_V} \frac{c(\boldsymbol{v}, \boldsymbol{\zeta})}{\|\boldsymbol{v}\|_V}.
$$

Thus, we can take $\alpha_c = \dfrac{1}{\bar{c}}$. □

Using the hypotheses H3, H4, as $\gamma \boldsymbol{v} \in L^{2-\delta}(\Gamma;\mathbb{R}^3)$ and $\|\boldsymbol{v}_\tau(\boldsymbol{x})\| \leq \|\gamma \boldsymbol{v}(\boldsymbol{x})\|$ a.e. $\boldsymbol{x} \in \Gamma$, we deduce that $k(\cdot)\, F(\cdot)\, \|\boldsymbol{v}_\tau(\cdot)\| \in L^1(\Gamma_3)$ for all $\boldsymbol{v} \in V$ using the generalized Hölder's inequality.

Let us introduce the following subset of D,

$$
\Lambda = \left\{ \boldsymbol{\zeta} \in D \,:\, \langle \boldsymbol{\zeta},\, \widetilde{\boldsymbol{v}} \rangle \leq \int_{\Gamma_3} k(\boldsymbol{x})\, F(\boldsymbol{x})\, \|\widetilde{\boldsymbol{v}}_\tau(\boldsymbol{x})\|\, d\Gamma \quad \text{for all } \widetilde{\boldsymbol{v}} \in \gamma(V) \right\}; \tag{20}
$$

see (18) for the definition of $\widetilde{\boldsymbol{v}}_\tau$.

It is easy to observe that $\boldsymbol{\lambda} \in \Lambda$ by using (17) and (20).

Lemma 2. *The set Λ is a closed convex bounded subset of D that contains 0_D.*

Proof. It is easy to observe that $0_D \in \Lambda$. In addition, the convexity can be easily obtained by means of the definition of the convex sets.

Let $(\boldsymbol{\zeta}_n) \subset \Lambda$ be a convergent sequence,

$$
\boldsymbol{\zeta}_n \to \boldsymbol{\zeta} \quad \text{in } D \text{ as } n \to \infty.
$$

We have to prove that $\boldsymbol{\zeta} \in \Lambda$. As the sequence $(\boldsymbol{\zeta}_n)$ converges strongly to $\boldsymbol{\zeta}$, then $\boldsymbol{\zeta}_n \rightharpoonup \boldsymbol{\zeta}$ in D as $n \to \infty$ and then

$$
\boldsymbol{\zeta}_n \rightharpoonup^* \boldsymbol{\zeta} \text{ in } D \text{ as } n \to \infty.
$$

Using the definition of the weak* convergence, we infer that

$$
\langle \boldsymbol{\zeta}_n,\, \widetilde{\boldsymbol{v}} \rangle \to \langle \boldsymbol{\zeta},\, \widetilde{\boldsymbol{v}} \rangle \text{ as } n \to \infty \quad \text{for all } \widetilde{\boldsymbol{v}} \in \gamma(V).
$$

Since $(\zeta_n) \subset \Lambda$, then

$$\langle \zeta_n, \widetilde{\boldsymbol{v}} \rangle \leq \int_{\Gamma_3} k(\boldsymbol{x})\, F(\boldsymbol{x})\, \|\widetilde{\boldsymbol{v}}_\tau(\boldsymbol{x})\|\, d\Gamma \quad \text{for all } \widetilde{\boldsymbol{v}} \in \gamma(V).$$

Passing to the limit $n \to \infty$ in the previous inequality, we conclude that $\zeta \in \Lambda$. As a result, Λ is a closed set.

Let $\zeta \in \Lambda$ be arbitrarily fixed.

$$\|\zeta\|_D = \sup_{\widetilde{\boldsymbol{v}} \in \gamma(V), \widetilde{\boldsymbol{v}} \neq 0_{\gamma(V)}} \frac{\langle \zeta, \widetilde{\boldsymbol{v}} \rangle}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}}.$$

Let $\widetilde{\boldsymbol{v}} \in \gamma(V)$, $\widetilde{\boldsymbol{v}} \neq 0_{\gamma(V)}$.

$$\begin{aligned}
\frac{\langle \zeta, \widetilde{\boldsymbol{v}} \rangle}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} &\leq \frac{\int_{\Gamma_3} k(\boldsymbol{x}) F(\boldsymbol{x}) \|\widetilde{\boldsymbol{v}}_\tau(\boldsymbol{x})\|\, d\Gamma}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} \\
&\leq \frac{\int_{\Gamma_3} k(\boldsymbol{x}) F(\boldsymbol{x}) \|\widetilde{\boldsymbol{v}}(\boldsymbol{x})\|\, d\Gamma}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} \\
&\leq \frac{\|k\|_{L^\infty(\Gamma_3)} \|F\|_{L^{\frac{2-\delta}{1-\delta}}(\Gamma_3)} \|\widetilde{\boldsymbol{v}}\|_{L^{2-\delta}(\Gamma;\mathbb{R}^3)}}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} \\
&\leq \frac{\|k\|_{L^\infty(\Gamma_3)} \|F\|_{L^{\frac{2-\delta}{1-\delta}}(\Gamma_3)} c(meas(\Gamma), \delta) \|\widetilde{\boldsymbol{v}}\|_{L^{2}(\Gamma;\mathbb{R}^3)}}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}}.
\end{aligned}$$

As

$$\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)} \geq \|\widetilde{\boldsymbol{v}}\|_{L^{2}(\Gamma;\mathbb{R}^3)},$$

see (1), then

$$\frac{\langle \zeta, \widetilde{\boldsymbol{v}} \rangle}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} \leq \|k\|_{L^\infty(\Gamma_3)} \|F\|_{L^{\frac{2-\delta}{1-\delta}}(\Gamma_3)} c(meas(\Gamma), \delta).$$

As a result,

$$\|\zeta\|_D = \sup_{\widetilde{\boldsymbol{v}} \in \gamma(V), \widetilde{\boldsymbol{v}} \neq 0_{\gamma(V)}} \frac{\langle \zeta, \widetilde{\boldsymbol{v}} \rangle}{\|\widetilde{\boldsymbol{v}}\|_{H^{1/2}(\Gamma;\mathbb{R}^3)}} \leq \|k\|_{L^\infty(\Gamma_3)} \|F\|_{L^{\frac{2-\delta}{1-\delta}}(\Gamma_3)} c(meas(\Gamma), \delta).$$

Therefore, Λ is a bounded set. □

Next, we claim that

$$c(\boldsymbol{u}, \boldsymbol{\lambda}) = \int_{\Gamma_3} k(\boldsymbol{x})\, F(\boldsymbol{x})\, \|\boldsymbol{u}_\tau(\boldsymbol{x})\| d\Gamma. \tag{21}$$

Indeed, since $c(\boldsymbol{u}, \boldsymbol{\lambda}) = \langle \boldsymbol{\lambda}, \gamma \boldsymbol{u} \rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{u}_\tau(\boldsymbol{x})\, d\Gamma$, in order to justify (21), we have to prove that

$$-\boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{u}_\tau(\boldsymbol{x}) = k(\boldsymbol{x}) F(\boldsymbol{x}) \|\boldsymbol{u}_\tau(\boldsymbol{x})\| \text{ a.e. on } \Gamma_3. \tag{22}$$

To prove (22), we consider $\boldsymbol{x} \in \Gamma_3$ arbitrarily fixed.

- If $\boldsymbol{u}_\tau(\boldsymbol{x}) = \boldsymbol{0}$ then $\boldsymbol{0} = \boldsymbol{0}$ and, thus, (22) holds true.
- If $\boldsymbol{u}_\tau(\boldsymbol{x}) \neq \boldsymbol{0}$ then

$$-\boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{u}_\tau(\boldsymbol{x}) = k(\boldsymbol{x}) F(\boldsymbol{x}) \frac{\boldsymbol{u}_\tau(\boldsymbol{x}) \cdot \boldsymbol{u}_\tau(\boldsymbol{x})}{\|\boldsymbol{u}_\tau(\boldsymbol{x})\|} = k(\boldsymbol{x}) F(\boldsymbol{x}) \|\boldsymbol{u}_\tau(\boldsymbol{x})\|.$$

Hence, (22) holds true in this situation too.

As a result, (21) is fulfilled.

On the other hand, by (20),

$$c(\boldsymbol{u},\boldsymbol{\zeta}) \leq \int_{\Gamma_3} k(\boldsymbol{x})F(\boldsymbol{x})\|\boldsymbol{u}_\tau(\boldsymbol{x})\|\, d\Gamma \qquad \text{for all } \boldsymbol{\zeta} \in \Lambda. \tag{23}$$

Therefore, by (21) and (23), we obtain

$$c(\boldsymbol{u},\boldsymbol{\zeta}-\boldsymbol{\lambda}) \leq 0 \text{ for all } \boldsymbol{\zeta} \in \Lambda. \tag{24}$$

Let ω^* be the Fenchel conjugate of the constitutive function ω,

$$\omega^* : \mathbb{S}^3 \to (-\infty,\infty], \quad \omega^*(\boldsymbol{\tau}) = \sup_{\boldsymbol{\xi}\in\mathbb{S}^3}\{\boldsymbol{\tau} : \boldsymbol{\xi} - \omega(\boldsymbol{\xi})\}.$$

This is a proper, convex and lower semicontinuous functional.

Notice that

$$(1-\beta)\|\boldsymbol{\tau}\|^2 \leq \omega^*(\boldsymbol{\tau}) \leq \frac{1}{4\alpha}\|\boldsymbol{\tau}\|^2 \quad \text{for all } \boldsymbol{\tau} \in \mathbb{S}^3, \tag{25}$$

where α, β are the constants in the hypothesis H1; see [43] for a proof of (25).

Due to (25),

$$\omega^*(\boldsymbol{\tau}(\cdot)) \in L^1(\Omega) \quad \text{for all } \boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}^3), \tag{26}$$

and, due to the hypothesis H1,

$$\omega(\boldsymbol{\tau}(\cdot)) \in L^1(\Omega) \quad \text{for all } \boldsymbol{\tau} \in L^2(\Omega;\mathbb{S}^3).$$

Thus,

$$\omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot))) \in L^1(\Omega) \quad \text{for all } \boldsymbol{v} \in V. \tag{27}$$

Let us introduce the bifunctional $B : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{R}$,

$$B(\boldsymbol{\tau},\boldsymbol{\mu}) = \omega(\boldsymbol{\tau}) + \omega^*(\boldsymbol{\mu}) \quad \text{for all } \boldsymbol{\tau},\boldsymbol{\mu} \in \mathbb{S}^3. \tag{28}$$

Using Theorem 1 for $X = \mathbb{S}^3$ and $\phi = \omega$, we deduce that $B(\cdot,\cdot)$ is a bipotential in the sense of Definition 1.

By (27) and (26) we are lead to

$$B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\cdot)),\boldsymbol{\tau}(\cdot)) \in L^1(\Omega) \quad \text{for all } \boldsymbol{v} \in V,\ \boldsymbol{\tau} \in L^2_s(\Omega;\mathbb{S}^3).$$

This regularity allows us to define a form $b(\cdot,\cdot)$ as follows,

$$b : V \times L^2_s(\Omega;\mathbb{S}^3) \to \mathbb{R} \quad b(\boldsymbol{v},\boldsymbol{\mu}) = \int_\Omega B(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})),\boldsymbol{\mu}(\boldsymbol{x}))\, dx. \tag{29}$$

Since $\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}))$ in Ω and $B(\cdot,\cdot)$ defined in (28) is a bipotential, then a.e. $\boldsymbol{x} \in \Omega$,

$$\begin{aligned} B(\boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}),\boldsymbol{\sigma}(\boldsymbol{x})) &= \boldsymbol{\sigma}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{u})(\boldsymbol{x}), \\ B(\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}),\boldsymbol{\mu}(\boldsymbol{x})) &\geq \boldsymbol{\mu}(\boldsymbol{x}) : \boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}). \end{aligned}$$

After integration over Ω, we obtain

$$b(\boldsymbol{u},\boldsymbol{\sigma}) = (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2_s(\Omega;\mathbb{S}^3)}, \tag{30}$$

$$b(\boldsymbol{v},\boldsymbol{\mu}) \geq (\boldsymbol{\mu},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)} \quad \text{for all } \boldsymbol{v} \in V, \boldsymbol{\mu} \in L^2_s(\Omega;\mathbb{S}^3). \tag{31}$$

In particular, setting in (31) $\boldsymbol{v} = \boldsymbol{u}$ and $\boldsymbol{\mu} = \boldsymbol{\sigma}$, respectively, the following inequalities can be written:

$$
\begin{aligned}
b(\boldsymbol{u},\boldsymbol{\mu}) &\geq (\boldsymbol{\mu},\boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2_s(\Omega;\mathbb{S}^3)} \quad \text{for all } \boldsymbol{\mu}\in L^2_s(\Omega;\mathbb{S}^3); &(32)\\
b(\boldsymbol{v},\boldsymbol{\sigma}) &\geq (\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)} \quad \text{for all } \boldsymbol{v}\in V. &(33)
\end{aligned}
$$

Hence, by (32) and (30), we arrive at

$$b(\boldsymbol{u},\boldsymbol{\mu})-b(\boldsymbol{u},\boldsymbol{\sigma})\geq(\boldsymbol{\mu}-\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2_s(\Omega;\mathbb{S}^3)}. \tag{34}$$

Consider now a variable subset of $L^2_s(\Omega;\mathbb{S}^3)$ defined as follows: given $\boldsymbol{\zeta}\in\Lambda$,

$$\Sigma(\boldsymbol{\zeta})=\{\boldsymbol{\mu}\in L^2_s(\Omega;\mathbb{S}^3)\ :\ (\boldsymbol{\mu},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)}+c(\boldsymbol{v},\boldsymbol{\zeta})=(\boldsymbol{f},\boldsymbol{v})_V \quad \text{for all } \boldsymbol{v}\in V\}.$$

Lemma 3. *Let $\boldsymbol{\zeta}\in\Lambda$. The set $\Sigma(\boldsymbol{\zeta})$ is a nonempty, convex, closed, unbounded subset of $L^2_s(\Omega;\mathbb{S}^3)$.*

Proof. Let $\boldsymbol{\zeta}\in\Lambda$ and let us define

$$\varphi_\zeta : V\to\mathbb{R}, \quad \varphi_\zeta(\boldsymbol{v})=c(\boldsymbol{v},\boldsymbol{\zeta}).$$

As $c(\cdot,\cdot)$ is a bilinear form, then the linearity of φ_ζ is obvious. Let us prove its continuity. In order to do this, we have to prove that there exists $K>0$ such that

$$|\varphi_\zeta(\boldsymbol{v})|\leq K\|\boldsymbol{v}\|_V \text{ for all } \boldsymbol{v}\in V.$$

Indeed, let $\boldsymbol{v}\in V$.

$$|\varphi_\zeta(\boldsymbol{v})|=|c(\boldsymbol{v},\boldsymbol{\zeta}|\leq\frac{\widetilde{c}}{\widetilde{c}_K}\|\boldsymbol{\zeta}\|_D\|\boldsymbol{v}\|_V.$$

We can set $K=\frac{\widetilde{c}}{\widetilde{c}_K}$, where $\widetilde{c}$ is the constant in (2) and $\widetilde{c}_K$ is the constant in (5). As a result, φ_ζ is a linear and continuous map. Then, due to Riesz's representation theorem, there exists a unique $\boldsymbol{G}_\zeta\in V$ such that

$$c(\boldsymbol{v},\boldsymbol{\zeta})=(\boldsymbol{G}_\zeta,\boldsymbol{v})_V \quad \text{for all } \boldsymbol{v}\in V.$$

Hence,

$$(\boldsymbol{f},\boldsymbol{v})_V-c(\boldsymbol{\zeta},\boldsymbol{v})=(\boldsymbol{f},\boldsymbol{v})_V-(\boldsymbol{G}_\zeta,\boldsymbol{v})_V=(\boldsymbol{f}-\boldsymbol{G}_\zeta,\boldsymbol{v})_V,$$

and keeping in mind the definition of the inner product on V, (4), actually we can write

$$(\boldsymbol{f},\boldsymbol{v})_V-c(\boldsymbol{\zeta},\boldsymbol{v})=(\boldsymbol{\varepsilon}(\boldsymbol{f}-\boldsymbol{G}_\zeta),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)}.$$

Let us take $\boldsymbol{\mu}=\boldsymbol{\varepsilon}(\boldsymbol{f}-\boldsymbol{G}_\zeta)$ to conclude that $\Sigma(\boldsymbol{\zeta})$ is a nonempty subset of $L^2_s(\Omega;\mathbb{S}^3)$. The convexity can be easily proved by using the definition of the convex sets.

In order to prove that $\Sigma(\boldsymbol{\zeta})$ is a closed subset of $L^2_s(\Omega;\mathbb{S}^3)$, we consider $(\boldsymbol{\mu}_n)\subset\Sigma(\boldsymbol{\zeta})$ such that $\boldsymbol{\mu}_n\to\boldsymbol{\mu}$ in $L^2_s(\Omega;\mathbb{S}^3)$ as $n\to\infty$ and we prove that $\boldsymbol{\mu}\in\Sigma(\boldsymbol{\zeta})$. As

$$(\boldsymbol{\mu}_n,\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)}+c(\boldsymbol{v},\boldsymbol{\zeta})=(\boldsymbol{f},\boldsymbol{v})_V \text{ for all } \boldsymbol{v}\in V \tag{35}$$

and

$$(\boldsymbol{\mu}_n,\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)}\to(\boldsymbol{\mu},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)} \text{ as } n\to\infty \text{ for each } \boldsymbol{v}\in V$$

then, passing to the limit $n\to\infty$ in (35), we obtain

$$(\boldsymbol{\mu},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)}+c(\boldsymbol{v},\boldsymbol{\zeta})=(\boldsymbol{f},\boldsymbol{v})_V \text{ for all } \boldsymbol{v}\in V.$$

Hence, $\boldsymbol{\mu}\in\Sigma(\boldsymbol{\zeta})$ and, thus, $\Sigma(\boldsymbol{\zeta})$ is a closed set.

Finally, let us prove that $\Sigma(\boldsymbol{\zeta})$ is an unbounded subset. Indeed, there exist at least one sequence $(\boldsymbol{\mu}_n)_n \subset \Lambda$ such that $\|\boldsymbol{\mu}_n\|_{L^2_s(\Omega;\mathbb{S}^3)} \to \infty$ as $n \to \infty$. We can construct such a sequence as follows: we take $\boldsymbol{\tau} \in \boldsymbol{\varepsilon}(V)^\perp$ and for each positive integer n we define

$$\boldsymbol{\mu}_n = \boldsymbol{\varepsilon}(\boldsymbol{f} - \boldsymbol{G}_\zeta) + n\boldsymbol{\tau}, \tag{36}$$

where $\boldsymbol{\varepsilon}(V)^\perp = \{\boldsymbol{\tau} \in L^2_s(\Omega;\mathbb{S}^3) \,|\, (\boldsymbol{\tau},\boldsymbol{\sigma})_{L^2_s(\Omega;\mathbb{S}^3)} = 0 \text{ for all } \boldsymbol{\sigma} \in \boldsymbol{\varepsilon}(V)\}$, with $\boldsymbol{\varepsilon}(V) = \{\boldsymbol{\varepsilon}(\boldsymbol{v}) \,|\, \boldsymbol{v} \in V\}$. Recall that $\boldsymbol{\varepsilon}(V)$ and $\boldsymbol{\varepsilon}(V)^\perp$ are closed subspaces of the space $L^2_s(\Omega;\mathbb{S}^3)$ and $L^2_s(\Omega;\mathbb{S}^3) = \boldsymbol{\varepsilon}(V) \oplus \boldsymbol{\varepsilon}(V)^\perp$, see, e.g., Theorem 1.16 in [8]. As $\boldsymbol{f} - \boldsymbol{G}_\zeta \in V$ and $\boldsymbol{\tau} \in \boldsymbol{\varepsilon}(V)^\perp$, by using (36) it follows that $\boldsymbol{\mu}_n \in L^2_s(\Omega;\mathbb{S}^3)$ for all positive integer n. Moreover,

$$(\boldsymbol{\mu}_n, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)} + \langle \boldsymbol{\zeta}, \gamma\boldsymbol{v}\rangle = (\boldsymbol{\varepsilon}(\boldsymbol{f}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)}.$$

Taking into consideration (4) we can write

$$(\boldsymbol{\mu}_n, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2_s(\Omega;\mathbb{S}^3)} + \langle \boldsymbol{\zeta}, \gamma\boldsymbol{v}\rangle = (\boldsymbol{f}, \boldsymbol{v})_V.$$

Thus, $(\boldsymbol{\mu}_n)_n \subset \Sigma(\boldsymbol{\zeta})$.
Furthermore, $\|\boldsymbol{\mu}_n\|_{L^2_s(\Omega;\mathbb{S}^3)} \to \infty$ as $n \to \infty$. Indeed,

$$\|\boldsymbol{\mu}_n\|_{L^2_s(\Omega;\mathbb{S}^3)} = n\left\|\frac{\boldsymbol{\varepsilon}(\boldsymbol{f}) - \boldsymbol{\varepsilon}(\boldsymbol{G}_\zeta)}{n} + \boldsymbol{\tau}\right\|_{L^2_s(\Omega;\mathbb{S}^3)}$$

and

$$\left(\frac{\boldsymbol{\varepsilon}(\boldsymbol{f}) - \boldsymbol{\varepsilon}(\boldsymbol{G}_\zeta)}{n} + \boldsymbol{\tau}\right)$$

is a bounded sequence. Therefore, the subset $\Sigma(\boldsymbol{\zeta})$ is unbounded. $\square$

We observe that $\boldsymbol{\sigma} \in \Sigma(\boldsymbol{\lambda})$.
Let $\boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda})$. Then, $(\boldsymbol{\mu}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2_s(\Omega;\mathbb{S}^3)} = (\boldsymbol{\varepsilon}(\boldsymbol{f}) - \boldsymbol{\varepsilon}(\boldsymbol{G}_\lambda), \boldsymbol{u})_V = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2_s(\Omega;\mathbb{S}^3)}$. Hence,

$$(\boldsymbol{\mu} - \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2_s(\Omega;\mathbb{S}^3)} = 0.$$

By (34), we obtain

$$b(\boldsymbol{u},\boldsymbol{\mu}) - b(\boldsymbol{u},\boldsymbol{\sigma}) \geq 0 \quad \text{for all } \boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda}). \tag{37}$$

On the other hand, by (33) and (30) combined with (16), (17) and (19), we can write

$$\begin{aligned} b(\boldsymbol{v},\boldsymbol{\sigma}) - b(\boldsymbol{u},\boldsymbol{\sigma}) &\geq (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2_s(\Omega;\mathbb{S}^3)} \\ &= (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_V - \langle \boldsymbol{\lambda}, \gamma\boldsymbol{v} - \gamma\boldsymbol{u}\rangle \\ &= (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_V - c(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\lambda}). \end{aligned}$$

As a consequence,

$$b(\boldsymbol{v},\boldsymbol{\sigma}) - b(\boldsymbol{u},\boldsymbol{\sigma}) + c(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\lambda}) \geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_V \quad \text{for all } \boldsymbol{v} \in V. \tag{38}$$

Keeping in mind (38), (24) and (37), we are led to the following variational problem.

Problem 2. *Find $\boldsymbol{u} \in V$, $\boldsymbol{\lambda} \in \Lambda$, $\boldsymbol{\sigma} \in \Sigma(\boldsymbol{\lambda})$ such that*

$$\begin{aligned} b(\boldsymbol{v},\boldsymbol{\sigma}) - b(\boldsymbol{u},\boldsymbol{\sigma}) + c(\boldsymbol{v}-\boldsymbol{u},\boldsymbol{\lambda}) &\geq (\boldsymbol{f}, \boldsymbol{v}-\boldsymbol{u})_V \quad \textit{for all } \boldsymbol{v} \in V \\ c(\boldsymbol{u},\boldsymbol{\zeta}-\boldsymbol{\lambda}) &\leq 0 \quad \textit{for all } \boldsymbol{\zeta} \in \Lambda \\ b(\boldsymbol{u},\boldsymbol{\mu}) - b(\boldsymbol{u},\boldsymbol{\sigma}) &\geq 0 \quad \textit{for all } \boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda}). \end{aligned}$$

Each solution $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\sigma}) \in V \times \Lambda \times \Sigma(\boldsymbol{\lambda})$ of Problem 2 is called a weak solution of Problem 1.

Keeping in mind (29) and (28), the bifunctional $b(\cdot, \cdot)$ can be written by means of two functionals $J(\cdot)$ and $J^*(\cdot)$ as follows,

$$b(\boldsymbol{v}, \boldsymbol{\mu}) = J(\boldsymbol{v}) + J^*(\boldsymbol{\mu}),$$

where

$$J : V \to \mathbb{R}, \quad J(\boldsymbol{v}) = \int_\Omega \omega(\boldsymbol{\varepsilon}(\boldsymbol{v})(\boldsymbol{x}))\, dx$$

and

$$J^* : L^2_s(\Omega; \mathbb{S}^3) \to \mathbb{R}, \quad J^*(\boldsymbol{\mu}) = \int_\Omega \omega^*(\boldsymbol{\mu}(\boldsymbol{x}))\, dx.$$

Then, we observe that

$$\begin{aligned} b(\boldsymbol{v}, \boldsymbol{\sigma}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) &= J(\boldsymbol{v}) - J(\boldsymbol{u}), \\ b(\boldsymbol{u}, \boldsymbol{\mu}) - b(\boldsymbol{u}, \boldsymbol{\sigma}) &= J^*(\boldsymbol{\mu}) - J^*(\boldsymbol{\sigma}). \end{aligned}$$

In consequence, Problem 2 can be equivalently written as follows.

Problem 3. *Find $\boldsymbol{u} \in V$, $\boldsymbol{\lambda} \in \Lambda$, $\boldsymbol{\sigma} \in \Sigma(\boldsymbol{\lambda})$ such that*

$$\begin{aligned} J(\boldsymbol{v}) - J(\boldsymbol{u}) + c(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{\lambda}) &\geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \textit{for all } \boldsymbol{v} \in V && (39) \\ c(\boldsymbol{u}, \boldsymbol{\zeta} - \boldsymbol{\lambda}) &\leq 0 \quad \textit{for all } \boldsymbol{\zeta} \in \Lambda && (40) \\ J^*(\boldsymbol{\mu}) - J^*(\boldsymbol{\sigma}) &\geq 0 \quad \textit{for all } \boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda}). \end{aligned}$$

The weak solvability of Problem 1 will be studied by means of the variational formulation stated in Problem 3.

4. Well-Posedness and a Convergence Result

This section is devoted to the solvability of Problem 3.

Theorem 4. *Assume that the hypotheses H1–H4 hold true. Then, Problem 3 has at least one solution $(\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\sigma}) \in V \times \Lambda \times \Sigma(\boldsymbol{\lambda})$. If ω is, in addition, strictly convex, there is uniqueness in the first component.*

Proof. Let us introduce the following bifunctional:

$$\mathcal{L} : V \times \Lambda \to R\,, \quad \mathcal{L}(\boldsymbol{v}, \boldsymbol{\zeta}) = J(\boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_V + c(\boldsymbol{v}, \boldsymbol{\zeta}).$$

A pair $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$ verifies (39) and (40) if and only if it is a saddle point of the bifunctional $\mathcal{L}(\cdot, \cdot)$, i.e.,

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{\zeta}) \leq \mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda}) \leq \mathcal{L}(\boldsymbol{v}, \boldsymbol{\lambda}), \quad \text{for all } \boldsymbol{v} \in V, \text{ for all } \boldsymbol{\zeta} \in \Lambda. \tag{41}$$

Indeed, (40) is equivalent with the first inequality in the chain above. On the other hand, using the definition of $\mathcal{L}(\cdot, \cdot)$, by a similar technique with that used in [44], we deduce that (39) is equivalent with the second inequality in (41).

Keeping in mind H1, we immediately conclude that $J(\cdot)$ is a convex lower semicontinuous functional such that

$$J(\boldsymbol{v}) \geq \alpha \|\boldsymbol{v}\|_V^2 \quad \text{for all } \boldsymbol{v} \in V.$$

Furthermore, according to Lemma 1, $c(\cdot, \cdot)$ is a bilinear continuous form. Therefore, $\mathcal{L}(\cdot, \cdot)$ fulfills the conditions in Theorem 3. Moreover, according to Lemma 2, Λ is a closed convex bounded set which contains 0_D. In addition, we note that

$$\mathcal{L}(\boldsymbol{v}, 0_D) = J(\boldsymbol{v}) + c(\boldsymbol{v}, 0_D) - (\boldsymbol{f}, \boldsymbol{v})_X \geq \alpha \|\boldsymbol{v}\|_V^2 - \|\boldsymbol{f}\|_V \|\boldsymbol{v}\|_V,$$

which allows us to write

$$\lim_{\|\boldsymbol{v}\|_V \to \infty} \mathcal{L}(\boldsymbol{v}, 0_D) = \infty.$$

Therefore, applying Theorem 3, we conclude that the functional $\mathcal{L}(\cdot,\cdot)$ has at least one saddle point $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*) \in V \times \Lambda$.

Let us introduce the set $\Sigma(\boldsymbol{\lambda}^*)$. According to Lemma 3, this is a nonempty closed convex unbounded subset of $L^2_s(\Omega;\mathbb{S}^3)$. Moreover, due to (25), we infer that $J^*(\cdot)$ is convex, lower semicontinuous and

$$J^*(\boldsymbol{\mu}) \geq (1-\beta)\|\boldsymbol{\mu}\|^2_{L^2_s(\Omega;\mathbb{S}^3)}.$$

Using Theorem 2, we conclude that $J^*(\cdot)$ has a unique minimizer $\boldsymbol{\sigma}^*$ on $\Sigma(\boldsymbol{\lambda}^*)$. Consequently, the triple $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*) \in V \times \Lambda \times \Sigma(\boldsymbol{\lambda}^*)$ is a solution of Problem 3.

Finally, if ω is, in addition, strictly convex, using Theorem 3 we immediately conclude that Problem 3 has at least one solution which is unique in its first component. □

In order to obtain a global uniqueness result, one option could be to perturb Problem 3 as follows.

Problem 4. *Let $\epsilon > 0$. Find $\boldsymbol{u}_\epsilon \in V$, $\boldsymbol{\lambda}_\epsilon \in \Lambda$, $\boldsymbol{\sigma}_\epsilon \in \Sigma(\boldsymbol{\lambda}_\epsilon)$ such that*

$$J(\boldsymbol{v}) - J(\boldsymbol{u}_\epsilon) + c(\boldsymbol{v} - \boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_\epsilon)_V \quad \text{for all } \boldsymbol{v} \in V \tag{42}$$

$$c(\boldsymbol{u}_\epsilon, \boldsymbol{\zeta} - \boldsymbol{\lambda}_\epsilon) - \epsilon(\|\boldsymbol{\zeta}\|^2_D - \|\boldsymbol{\lambda}_\epsilon\|^2_D) \leq 0 \quad \text{for all } \boldsymbol{\zeta} \in \Lambda \tag{43}$$

$$J^*(\boldsymbol{\mu}) - J^*(\boldsymbol{\sigma}_\epsilon) \geq 0 \quad \text{for all } \boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda}_\epsilon). \tag{44}$$

Let us introduce the following perturbed bifunctional.

$$\mathcal{L}_\epsilon : V \times \Lambda \to R\,, \quad \mathcal{L}(\boldsymbol{v}, \boldsymbol{\zeta}) = J(\boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v})_V + c(\boldsymbol{v}, \boldsymbol{\zeta}) - \epsilon\|\boldsymbol{\zeta}\|^2_D \quad (\epsilon > 0).$$

Using similar techniques with those used in [44], it can be verified that a pair $(\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon) \in V \times \lambda$ verifies (42) and (43) if and only if it is a saddle point of the bifunctional $\mathcal{L}_\epsilon(\cdot,\cdot)$, i.e.,

$$\mathcal{L}_\epsilon(\boldsymbol{u}_\epsilon, \boldsymbol{\zeta}) \leq \mathcal{L}_\epsilon(\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon) \leq \mathcal{L}_\epsilon(\boldsymbol{v}, \boldsymbol{\lambda}_\epsilon), \qquad \text{for all } \boldsymbol{v} \in V,\ \boldsymbol{\zeta} \in \Lambda.$$

We observe that $\mathcal{L}_\epsilon(\cdot,\cdot)$ is strictly concave in the second argument.

Theorem 5. *Let $\epsilon > 0$. Assume H1–H4 hold true. In addition, we assume that ω and ω^* are strictly convex functionals. Then, Problem 4 has a unique solution $(\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon, \boldsymbol{\sigma}_\epsilon) \in V \times \Lambda \times \Sigma(\boldsymbol{\lambda}_\epsilon)$. Moreover, if ω is Lipschitz continuous of rank L then*

$$\|\boldsymbol{u}_\epsilon\|_V \leq \frac{\|\boldsymbol{f}\|_V}{\alpha}; \tag{45}$$

$$\|\boldsymbol{\lambda}_\epsilon\|_D \leq \frac{\|\boldsymbol{f}\|_V + L}{\alpha_c}; \tag{46}$$

$$\|\boldsymbol{\sigma}_\epsilon\|_{L^2_s(\Omega;\mathbb{S}^3)} \leq \frac{1}{\sqrt{4\alpha(1-\beta)}}\left(\|\boldsymbol{f}\|_V + \frac{M_c(\|\boldsymbol{f}\|_V + L)}{\alpha_c}\right). \tag{47}$$

Proof. Let $\epsilon > 0$. We observe that $\mathcal{L}_\epsilon(\cdot,\cdot)$ is strictly convex in the first argument and strictly concave in the second argument. We apply Theorem 3 in order to conclude that there exists a unique $(\boldsymbol{u}^*_\epsilon, \boldsymbol{\lambda}^*_\epsilon) \in V \times \Lambda$ such that (42) and (43) are fulfilled. Let $\Sigma(\boldsymbol{\lambda}^*_\epsilon)$. As $J^*(\cdot)$ is strictly convex, lower semicontinuous and coercive we deduce that it has a unique minimizer $\boldsymbol{\sigma}^*_\epsilon \in \Sigma(\boldsymbol{\lambda}^*_\epsilon)$. The triple $(\boldsymbol{u}^*_\epsilon, \boldsymbol{\lambda}^*_\epsilon, \boldsymbol{\sigma}^*_\epsilon) \in V \times \Lambda \times \Sigma(\boldsymbol{\lambda}^*_\epsilon)$ is the unique solution of Problem 4.

In order to prove the boundedness of the solution, let us set $\boldsymbol{v} = 0_V$ in (42) and $\boldsymbol{\zeta} = 0_D$ in (43). With this choice, (42) and (43) lead us to

$$J(\boldsymbol{u}_\epsilon) \leq (\boldsymbol{f}, \boldsymbol{u}_\epsilon)_V,$$

since, due to H1, $J(0_V) = 0$. Additionally, from this,

$$\alpha \|\boldsymbol{u}_\epsilon\|_V^2 \leq \|\boldsymbol{f}\|_V \|\boldsymbol{u}_\epsilon\|_V.$$

Hence, (45) holds true.

By (42), setting $\boldsymbol{v} = \boldsymbol{u}_\epsilon - \boldsymbol{w}$ with $\boldsymbol{w} \in V$ arbitrarily chosen, we immediately obtain

$$c(\boldsymbol{w}, \boldsymbol{\lambda}_\epsilon) \leq (\boldsymbol{f}, \boldsymbol{w})_V + J(\boldsymbol{u}_\epsilon - \boldsymbol{w}) - J(\boldsymbol{u}_\epsilon).$$

As ω is Lipschitz continuous of rank L, J is Lipschitz continuous of the same rank L. Therefore,

$$c(\boldsymbol{w}, \boldsymbol{\lambda}_\epsilon) \leq (\|\boldsymbol{f}\|_V + L)\|\boldsymbol{w}\|_V \text{ for all } \boldsymbol{w} \in V.$$

Let $\boldsymbol{w} \neq 0_V$. Then

$$\sup_{\boldsymbol{w} \in V, \boldsymbol{w} \neq 0_V} \frac{c(\boldsymbol{w}, \boldsymbol{\lambda}_\epsilon)}{\|\boldsymbol{w}\|_V} \leq \|\boldsymbol{f}\|_V + L.$$

By using the inf-sup property of the form $c(\cdot, \cdot)$, we can write

$$\alpha_c \|\boldsymbol{\lambda}_\epsilon\|_D \leq \|\boldsymbol{f}\|_V + L.$$

As a result, we obtain (46).

According to (44) and keeping in mind (25),

$$(1-\beta)\|\boldsymbol{\sigma}_\epsilon\|^2_{L^2_s(\Omega;\mathbb{S}^3)} \leq J^*(\boldsymbol{\sigma}_\epsilon) \leq J^*(\boldsymbol{\mu}) \leq \frac{1}{4\alpha}\|\boldsymbol{\mu}\|^2_{L^2_s(\Omega;\mathbb{S}^3)} \text{ for all } \boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda}_\epsilon). \tag{48}$$

Let $\boldsymbol{G}_{\lambda_\epsilon}$ be the unique element of V such that

$$c(\boldsymbol{v}, \boldsymbol{\lambda}_\epsilon) = (\boldsymbol{G}_{\lambda_\epsilon}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V.$$

Using $\boldsymbol{G}_{\lambda_\epsilon}$ we define

$$\boldsymbol{\mu} = \boldsymbol{\varepsilon}(\boldsymbol{f}) - \boldsymbol{\varepsilon}(\boldsymbol{G}_{\lambda_\epsilon}).$$

Then,

$$\begin{aligned} \|\boldsymbol{\mu}\|_{L^2_s(\Omega;\mathbb{S}^3)} &\leq \|\boldsymbol{\varepsilon}(\boldsymbol{f})\|_{L^2_s(\Omega;\mathbb{S}^3)} + \|\boldsymbol{\varepsilon}(\boldsymbol{G}_{\lambda_\epsilon})\|_{L^2_s(\Omega;\mathbb{S}^3)} \\ &= \|\boldsymbol{f}\|_V + \|\boldsymbol{G}_{\lambda_\epsilon}\|_V. \end{aligned}$$

As

$$\|\boldsymbol{G}_{\lambda_\epsilon}\|_V = \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq 0_V} \frac{(\boldsymbol{G}_{\lambda_\epsilon}, \boldsymbol{v})_V}{\|\boldsymbol{v}\|_V} = \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \neq 0_V} \frac{c(\boldsymbol{v}, \boldsymbol{\lambda}_\epsilon)}{\|\boldsymbol{v}\|_V} \leq M_c \|\boldsymbol{\lambda}_\epsilon\|_D,$$

then

$$\|\boldsymbol{\mu}\|_{L^2_s(\Omega;\mathbb{S}^3)} \leq \|\boldsymbol{f}\|_V + M_c \|\boldsymbol{\lambda}_\epsilon\|_D.$$

and from this, keeping in mind (46), we deduce that

$$\|\boldsymbol{\mu}\|_{L^2_s(\Omega;\mathbb{S}^3)} \leq \|\boldsymbol{f}\|_V + \frac{M_c(\|\boldsymbol{f}\|_V + L)}{\alpha_c}.$$

Using now (48), we can write

$$\|\boldsymbol{\sigma}_\epsilon\|_{L^2_s(\Omega;\mathbb{S}^3)} \leq \frac{1}{\sqrt{4\alpha(1-\beta)}}\|\boldsymbol{\mu}\|_{L^2_s(\Omega;\mathbb{S}^3)}.$$

Combining this last inequality with the previous one, we immediately obtain (47). □

With these preliminaries, an approximation result can be obtained. Everywhere below, we keep the assumptions H1, H2, H3, H4. In addition, we assume that ω and ω^* are strictly convex. In addition, we assume that ω is Lipschitz continuous of rank L and ω^* is upper semicontinuous.

Theorem 6 (A convergence result)**.** *Let $\epsilon > 0$ and let $(\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon, \boldsymbol{\sigma}_\epsilon)$ be the unique solution of Problem 4. Then, passing eventually to a subsequence $\Big((\boldsymbol{u}_{\epsilon'}, \boldsymbol{\lambda}_{\epsilon'}, \boldsymbol{\sigma}_{\epsilon'})\Big)$ of the sequence $\Big((\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon, \boldsymbol{\sigma}_\epsilon)\Big)$, the subsequence $\Big((\boldsymbol{u}_{\epsilon'}, \boldsymbol{\lambda}_{\epsilon'}, \boldsymbol{\sigma}_{\epsilon'})\Big)$ is weakly convergent to a solution of Problem 3 as $\epsilon' \to 0$.*

Proof. Let $\epsilon > 0$ and let $(\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon, \boldsymbol{\sigma}_\epsilon)$ be the unique solution of Problem 4. Let $\boldsymbol{G}_{\lambda_\epsilon}$ be the unique element of V such that $c(\boldsymbol{v}, \boldsymbol{\lambda}_\epsilon) = (\boldsymbol{G}_{\lambda_\epsilon}, \boldsymbol{v})$ for all $\boldsymbol{v} \in V$. As

$$\|\boldsymbol{G}_{\lambda_\epsilon}\|_V = \sup_{\boldsymbol{v}\in V, \boldsymbol{v}\neq 0_V} \frac{(\boldsymbol{G}_{\lambda_\epsilon}, \boldsymbol{v})_V}{\|\boldsymbol{v}\|_V} = \sup_{\boldsymbol{v}\in V, \boldsymbol{v}\neq 0_V} \frac{c(\boldsymbol{v}, \boldsymbol{\lambda}_\epsilon)}{\|\boldsymbol{v}\|_V} \leq M_c\|\boldsymbol{\lambda}_\epsilon\|_D \leq \frac{M_c(\|\boldsymbol{f}\|_V + L)}{\alpha_c},$$

then, $(\boldsymbol{G}_{\lambda_\epsilon})$ is a bounded sequence. This conclusion together with the boundedness results (45)–(47) allow us to assert that $\Big((\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon, \boldsymbol{\sigma}_\epsilon, \boldsymbol{G}_\epsilon)\Big)$ is a bounded sequence. Thus, passing eventually to a subsequence $\Big((\boldsymbol{u}_{\epsilon'}, \boldsymbol{\lambda}_{\epsilon'}, \boldsymbol{\sigma}_{\epsilon'}, \boldsymbol{G}_{\lambda_{\epsilon'}})\Big)$ of the sequence $\Big((\boldsymbol{u}_\epsilon, \boldsymbol{\lambda}_\epsilon, \boldsymbol{\sigma}_\epsilon, \boldsymbol{G}_{\lambda_\epsilon})\Big)$, we can write

$$\begin{aligned} \boldsymbol{u}_{\epsilon'} &\rightharpoonup \boldsymbol{u}^* \text{ in } V \text{ as } \epsilon' \to 0; \\ \boldsymbol{\lambda}_{\epsilon'} &\rightharpoonup \boldsymbol{\lambda}^* \text{ in } D \text{ as } \epsilon' \to 0; \\ \boldsymbol{\sigma}_{\epsilon'} &\rightharpoonup \boldsymbol{\sigma}^* \text{ in } L^2_s(\Omega;\mathbb{S}^3) \text{ as } \epsilon' \to 0; \\ \boldsymbol{G}_{\lambda_{\epsilon'}} &\rightharpoonup \boldsymbol{G}^* \text{ in } V \text{ as } \epsilon' \to 0. \end{aligned} \tag{49}$$

To proceed, we prove that $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*, \boldsymbol{\sigma}^*)$ is a solution of Problem 3.

Passing to the limit $\epsilon' \to 0$ in (42) and (43), we observe that $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*) \in V \times D$ verifies (39) and the inequality in (40). Moreover, Λ being a convex and closed set, it is weakly closed too. Therefore, $\boldsymbol{\lambda}^* \in \Lambda$. Thus, (40) is verified.

Next, we claim that

$$\boldsymbol{G}_{\lambda_{\epsilon'}} \rightharpoonup \boldsymbol{G}_{\lambda^*} \text{ in } V \text{ as } \epsilon' \to 0, \tag{50}$$

where $\boldsymbol{G}_{\lambda^*}$ is the unique element of V such that $c(\boldsymbol{v}, \boldsymbol{\lambda}^*) = (\boldsymbol{G}_{\lambda^*}, \boldsymbol{v})$ for all $\boldsymbol{v} \in V$. Indeed, (49) implies that $\boldsymbol{\lambda}_{\epsilon'} \rightharpoonup *\boldsymbol{\lambda}^*$. Therefore,

$$\langle \boldsymbol{\lambda}_{\epsilon'}, \widetilde{\boldsymbol{v}} \rangle \to \langle \boldsymbol{\lambda}^*, \widetilde{\boldsymbol{v}} \rangle \text{ as } \epsilon' \to 0 \text{ for all } \widetilde{\boldsymbol{v}} \in \gamma(V).$$

Let $\boldsymbol{v} \in V$ be arbitrarily fixed. As $\langle \boldsymbol{\lambda}_{\epsilon'}, \gamma\boldsymbol{v} \rangle \to \langle \boldsymbol{\lambda}^*, \gamma\boldsymbol{v} \rangle$ as $\epsilon' \to \infty$ for all $\boldsymbol{v} \in V$, then

$$c(\boldsymbol{v}, \boldsymbol{\lambda}_{\epsilon'}) \to c(\boldsymbol{v}, \boldsymbol{\lambda}^*) \text{ as } \epsilon' \to 0 \text{ for all } \boldsymbol{v} \in V.$$

and from this,

$$(\boldsymbol{G}_{\lambda_{\epsilon'}}, \boldsymbol{v})_V \to (\boldsymbol{G}_{\lambda^*}, \boldsymbol{v})_V \text{ as } \epsilon' \to 0 \text{ for all } \boldsymbol{v} \in V.$$

Hence, (50) holds true.

Next, by (50) we can prove that

$$\varepsilon(G_{\lambda_{\epsilon'}}) \to \varepsilon(G_{\lambda^*}) \text{ in } L^2_s(\Omega;\mathbb{S}^3) \text{ as } \epsilon' \to 0. \tag{51}$$

Indeed, ε being a continuous operator, it is also a lower semicontinuous and upper semicontinuous operator. On the other hand, ε is a linear operator so it is also a convex operator. Therefore, ε is a weakly lower semicontinuous operator and a weakly upper semicontinuous operator as well. Hence, keeping in mind (50), we can write

$$\limsup\ \varepsilon(G_{\lambda_{\epsilon'}}) \le \varepsilon(G_{\lambda^*}) \le \liminf\ \varepsilon(G_{\lambda_{\epsilon'}}) \text{ as } \epsilon' \to 0.$$

As a result, (51) is true.

Let $\boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda}^*)$. Then, there exists $\boldsymbol{\mu}_0 \in \varepsilon(V)^{\perp}$ such that

$$\boldsymbol{\mu} = \varepsilon(\boldsymbol{f}) - \varepsilon(G_{\lambda^*}) + \boldsymbol{\mu}_0.$$

We claim that there exists $(\boldsymbol{\mu}_{\epsilon'}) \subset L^2_s(\Omega;\mathbb{S}^3)$ such that

$$\boldsymbol{\mu}_{\epsilon'} \in \Sigma(\boldsymbol{\lambda}_{\epsilon'}), \quad \boldsymbol{\mu}_{\epsilon'} \to \boldsymbol{\mu} \text{ in } L^2_s(\Omega;\mathbb{S}^3) \text{ as } \epsilon' \to 0. \tag{52}$$

Indeed, let us define

$$\boldsymbol{\mu}_{\epsilon'} = \varepsilon(\boldsymbol{f}) - \varepsilon(G_{\lambda_{\epsilon'}}) + \boldsymbol{\mu}_0.$$

Clearly, $\boldsymbol{\mu}_{\epsilon'} \in \Sigma(\boldsymbol{\lambda}_{\epsilon'})$.

Due to (51), we immediately observe that (52) holds true.

Passing to the superior limit $\epsilon' \to 0$ in

$$J^*(\boldsymbol{\mu}_{\epsilon'}) - J^*(\boldsymbol{\sigma}_{\epsilon'}) \ge 0,$$

we obtain

$$J^*(\boldsymbol{\mu}) - J^*(\boldsymbol{\sigma}^*) \ge 0.$$

As $\boldsymbol{\mu}$ was arbitrarily chosen in $\Sigma(\boldsymbol{\lambda}^*)$, we conclude that

$$J^*(\boldsymbol{\mu}) - J^*(\boldsymbol{\sigma}^*) \ge 0 \quad \text{for all } \boldsymbol{\mu} \in \Sigma(\boldsymbol{\lambda}^*).$$

It remains to justify that

$$\boldsymbol{\sigma}^* \in \Sigma(\boldsymbol{\lambda}^*).$$

Indeed, as

$$\boldsymbol{\sigma}_{\epsilon'} \rightharpoonup \boldsymbol{\sigma}^* \text{ as } \epsilon' \to 0,$$

$$\boldsymbol{\lambda}_{\epsilon'} \rightharpoonup \boldsymbol{\lambda}^* \text{ as } \epsilon' \to 0,$$

keeping in mind the definition of $\Sigma(\boldsymbol{\lambda}_{\epsilon'})$ and $\Sigma(\boldsymbol{\lambda}^*)$, we obtain the conclusion passing to the limit $\epsilon' \to 0$ in

$$(\boldsymbol{\sigma}_{\epsilon'}, \varepsilon(\boldsymbol{v}))_{L^2(\Omega;\mathbb{S}^3)} + c(\boldsymbol{v}, \boldsymbol{\lambda}_{\epsilon'}) = (\boldsymbol{f}, \boldsymbol{v})_V.$$

□

5. Conclusions and Final Comments

In the present paper, we address a frictional contact model with prescribed normal stress. We deliver a new weak formulation that is a three-field variational formulation governed by a bipotential related to the constitutive function ω and a Lagrange multiplier related to the friction force σ_τ. We establish existence, uniqueness, boundedness and convergence results. Theorem 6 indicates us that the unique solution of the perturbed Problem 4 helps us to approximate a weak solution of Problem 1. Delivering uniqueness results by omitting a perturbation technique is left open.

The advantage of the approach we propose is twofold. On the one hand, the new approach allows the inclusion of the friction force in the unknown in addition to the displacement field and the Cauchy stress tensor. On the other hand, the qualitative analysis we perform allows moving on to the quantitative analysis in order to efficiently approximate the triple weak solutions.

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