Compound Adaptive Fuzzy Synchronization Controller Design for Uncertain Fractional-Order Chaotic Systems

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Abstract: In this paper, the synchronization of two fractional-order chaotic systems with uncertainties and external disturbances is considered. A fuzzy logic system is utilized to estimate uncertain nonlinearity, and its estimation accuracy is improved by constructing a series-parallel model. A disturbance observer is implemented to estimate bounded disturbance. To solve the “explosion of complexity” problem in the backstepping scheme, fractional-order command filters are employed to estimate virtual control inputs and their derivatives, and error compensation signals are devised to reduce filtering errors. Based on the fractional-order Lyapunov criterion, the proposed compound adaptive fuzzy backstepping control strategy can guarantee that the synchronization error converges to a small neighborhood of the origin. At last, the validity of the proposed control strategy is verified via a numerical simulation.

Keywords: fractional-order chaotic system; adaptive backstepping control; command filter; series-parallel model; disturbance observer

1. Introduction

Fractional calculus has been developed for over three centuries, which can be seen as an extension of ordinary calculus. It plays an important role in dealing with the control problem of non-integer order systems, which has received extensive attention due to its fascinating properties and potential application values. Over the past few decades, it has been discovered that in some real-world systems related to time series, compared with integer-order systems, fractional-order systems have better modeling accuracy due to the memory and inheritance of fractional calculus, such as financial systems [1], viscoelastic systems [2], dielectric polarization [3], and electrode–electrolyte polarization [4].

Nowadays, some studies have indicated that a lot of fractional-order systems exhibit chaotic phenomena, which is nonlinearity with complexity, randomness, unpredictability and extreme sensitivity to initial values [5–8], and has great application value in secure communication [9], signal processing, mathematical biology and machinery [5], etc. So far, scholars have proposed various control strategies to achieve the synchronization of fractional-order chaotic systems (FOCSs), e.g., sliding mode control [10], adaptive control [11], backstepping control [12], etc. Among them, backstepping control is an effective and commonly utilized method to realize the synchronization of FOCSs with the form of strict feedback. On the other hand, in actual modeling, due to the measurement errors, parameter uncertainties, external disturbances and other factors, the uncertain items often appear in the FOCSs which will affect the synchronization performance. Therefore, an ingenious idea is to combine backstepping control with function estimation techniques to handle its uncertainties so as to achieve the control objective. In [13], Ha et al. devised an adaptive fuzzy backstepping control (AFBC) method to realize the synchronization of FOCSs. Simultaneously, the fractional derivative of the virtual control function was regarded as a part of the uncertainties and was estimated by the fuzzy logic system (FLS)
so as to enhance the performance of synchronization and reduce the amount of calculation. In [14], a backstepping sliding mode control scheme was devised for a typical fractional-order quadrotor unmanned aerial vehicle, which can reduce the chattering phenomenon and improve the response and convergence speeds. In [15], a hybrid control method combining AFBC and sliding mode control was adopted to achieve fast and high-precision control of nonlinear fully actuated and underactuated mechanical systems in the presence of uncertainties. It is worth emphasizing that the backstepping control schemes utilized in the above literature can effectively achieve the control objectives, but the disadvantage of these schemes is that they will lead to an increase in computational complexity due to repeated differentiation of virtual control inputs. Therefore, how to devise a suitable method to reduce the computational complexity is meaningful and necessary work.

In recent years, some scholars have introduced the fractional-order command filter to estimate the derivatives of virtual control inputs to settle the “explosion of complexity” problem, and some interesting research results have been realized. In [16], Ha et al. devised an AFBC method to achieve the control objective, and adopted a fractional-order command filter to approximate the derivative of the virtual input. Simultaneously, an error compensation mechanism was utilized to handle errors generated in the filtering process. In [17], Han et al. devised an AFBC scheme with a fractional-order command filter for fractional-order systems. Among them, the fractional-order command filter was employed to eliminate the differential expansion of the backstepping controller, and the sliding mode controller was utilized to enhance the robustness of the system. In [18], a method combining AFBC and finite-time sliding mode control was employed for fractional-order systems with actuator faults, which not only exhibits superior robustness, but also helps to complete all the required tracking control targets in some finite time. It is noted that the filter used in the above literature has a strong approximation effect on the virtual control input and its derivative, and finally achieved a good control effect, but the discussion on the estimation accuracy of parameters is rare. However, in many practical systems, such as navigation systems and industrial lockstitch machine systems, the requirements for parameter accuracy are very high. Therefore, how to improve the estimation accuracy of parameters deserves further consideration.

In order to solve the above problem, a natural idea is to increase number of rules or neural nodes when estimating uncertain terms by FLSs or neural networks, but this method will increase the computational burden. Therefore, to avoid this situation, a series-parallel model is introduced, whose main idea is to construct a prediction error as an additional feedback term for the adaptive law. In [19], Liu et al. exploited a compound learning sliding mode control strategy for triangular fractional-order systems, in which a prediction error is defined and combined with a tracking error to construct a compound learning law to improve the estimation accuracy of the parameters. In [20], Zhou et al. devised a T-S fuzzy-based compound learning control strategy to synchronize two FOCSs with both uncertain parameters and time-varying delays. Furthermore, the proposed interval excitation condition can ensure a fast convergence of the parameters. In [21], Han et al. utilized online recorded data and instantaneous data to define a prediction error and combined a synchronization error to build a series-parallel model, and the proposed composite learning sliding mode control scheme ensured that the synchronization error asymptotically tends to zero. However, note that the disturbance is considered in the above literature and handled by an adaptive law, but this method will produce a disturbance error and have a negative impact on the control performance.

As far as we know, an effective solution to deal with the effects of disturbances is to devise a disturbance observer (DOB). In [22], Mofid et al. combined a DOB with an adaptive sliding mode control scheme to achieve a rapid response for the synchronization of three-dimensional FOCSs. In [23], Li et al. adopted a saturation control scheme to control a double-pendulum offshore crane subjected to matching disturbance, and constructed a fractional-order DOB to eliminate the effect of the matching disturbance. In [24], Guha et al. developed a DOB to estimate disturbance in an endogenous or exogenous system to
accelerate dynamic response with minimal flutter, and utilized the Mittag–Leffler stability theorem to ensure the finite time convergence of disturbance error. The above literature provides a lot of outstanding solutions for dealing with disturbances. How to combine the DOB with the backstepping control technology to handle the synchronous control of FOCSs deserves further study.

Inspired by the above discussion, this paper studies the synchronization of FOCSs with uncertainties and external disturbances based on compound AFBC technology. First, FLSs are utilized to estimate uncertain terms, and series-parallel models are constructed to enhance their estimation accuracy, while DOBs are devised to estimate disturbances. For two different chaotic systems, the main objective of this paper is to design a compound adaptive fuzzy controller to ensure that synchronization errors converge to a small neighborhood of the origin and all signals are bounded. Simultaneously, the effectiveness of the proposed method is demonstrated by the Lyapunov stability theory. Finally, the contributions of this article are shown below. (1) It is difficult to achieve a very ideal effect by using traditional fuzzy methods to approximate uncertain nonlinear terms or parameters in the system. However, in many practical systems, the requirement of parameter accuracy is very high. The series parallel model proposed in this paper can effectively improve the approximation accuracy without increasing the number of fuzzy rules. (2) Compared with the traditional DOB, the proposed DOB in this paper adds a prediction error as a feedback term to further improve the observation accuracy.

Here is the remaining part of this study. In Section 2, the notation and preliminaries are displayed. In Section 3, the master–slave systems are introduced, and DOBs and adaptive fuzzy controllers are designed, while stability analysis is performed by using the Lyapunov method. In Section 4, a simulation example is employed to check the feasibility and validity of this means. Furthermore, the conclusion is given in Section 5.

2. Preliminaries
2.1. Definitions and Lemmas

In this part, several fundamental definitions and lemmas with respect to fractional-order integrals and derivatives will be presented, which are useful for stability analysis.

**Definition 1** ([25]). The fractional integral with order $\theta \in (0, 1)$ is defined as

$$I^{\theta}f(t) = \frac{1}{\Gamma(\theta)} \int_{t_0}^{t} \frac{f(\varrho)}{(t-\varrho)^{1-\theta}} d\varrho,$$

where $t \geq t_0$, $\Gamma(\theta) = \int_{0}^{\infty} e^{-t\tau} \tau^{\theta-1} d\tau$.

In this paper, we mainly consider the Caputo fractional derivative, owing to the fact that its initial conditions for fractional differential equations possess the identical physical meaning with the integer-order one.

**Definition 2** ([26]). The Caputo fractional derivative with order $\theta \in (0, 1)$ is described as

$$^{c}D^{\theta}f(t) = \frac{1}{\Gamma(1-\theta)} \int_{t_0}^{t} \frac{f'(\varrho)}{(t-\varrho)^{\theta}} d\varrho,$$

where $f(t) \in C^n([t_0, +\infty), \mathbb{R})$. Hereafter, $D^{\theta}$ represents $^{c}D^{\theta}$ for convenience.

**Definition 3** ([27]). The Mittag–Leffler function is

$$E_{\nu,\kappa}(\phi) = \sum_{k=0}^{\infty} \frac{\phi^k}{\Gamma(\kappa v)},$$
where $\iota, \nu \in \mathbb{R}^+$, and $\phi \in \mathbb{C}$, and its Laplace transformation is
\[
\mathcal{L}\{t^{\nu-1}E_{\iota,\nu}(-ct^\iota)\} = \frac{s^{\nu-\nu}}{s^\iota + c}.
\] (4)

Lemma 1 ([28]). Let $\omega(t) \in \mathbb{R}$ be a real-valued continuous differentiable vector function, then we have
\[
\frac{1}{2} \mathcal{D}^\theta \omega^T \omega \leq \omega^T \mathcal{D}^\theta \omega \quad \forall \theta \in (0, 1], \quad t \geq 0.
\] (5)

Lemma 2 ([29]). (Yong’s Inequality) For $\forall a, b \geq 0$, the following inequality holds:
\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\] (6)

Lemma 3 ([30]). For $\nu \in \mathbb{C}$, $0 < \iota < 1$ and $\frac{\pi}{2} < \phi < \min\{\pi, \pi \nu\}$, we can obtain
\[
E_{\iota,\nu}(\phi) = -\sum_{j=1}^n \frac{1}{\Gamma(\nu - ij)} \phi^j + o\left(\frac{1}{|\phi|^\iota + 1}\right),
\] (7)

when $n \geq 1$, $|\phi| \to \infty$, $\phi \leq |\arg(\phi)| \leq \pi$.

Lemma 4 ([31]). Supposing $\sigma$ is a normal number, the following holds:
\[
|\eta| - \eta \tanh \left(\frac{\eta}{\sigma}\right) \leq 0.2785\sigma = \sigma'.
\] (8)

Lemma 5 ([31]). Suppose that $0 < \nu < 2$ and $\xi \in \mathbb{R}$. If there exists $\xi' \in \mathbb{R}^+$, such that $(\frac{\pi \nu}{2}) < \xi' \leq \min\{\pi, \pi \nu\}$, then we have
\[
|E_{\iota,\nu}(\phi)| \leq \frac{p}{1 + |\phi|},
\] (9)

where $p \in \mathbb{R}^+$, $\xi \leq |\arg(\phi)| \leq \pi$.

2.2. The Fuzzy Logic System
In the next part, in order to estimate the uncertainty, an FLS is required and characterized as below.

The $i$th rule is portrayed as below: $\mathbb{R}^{(i)}$: if $\mu_1 \in Y_1^i, \ldots, \mu_n \in Y_n^i$, then $\Phi = \Xi_i$, where $Y_1^i, \ldots, Y_i^i$ are fuzzy sets and $\Phi$ is the output. Through taking advantage of the defuzzification, the FLS output can be manifested as shown below:
\[
\Phi(\mu) = \frac{\sum_{j=1}^n \rho_j (\prod_{j=1}^n \delta_{Y_j^i}(\mu_j))}{\sum_{j=1}^n (\prod_{j=1}^n \delta_{Y_j^i}(\mu_j))} = \rho^T \xi(\mu),
\] (10)

where $\Phi(\mu)$ is the approximation of the $\Phi(\mu)$, $\delta_{Y_j^i}(\mu_j)$ is the degree of membership of $\mu_j$ to $Y_j^i$, $q$ is the amount of fuzzy rules, $\rho = [\rho_1, \ldots, \rho_q]^T$ is the adjustable parameter vector, and $\xi = [\xi_1, \ldots, \xi_q]^T$, where
\[
\xi_i(\mu) = \frac{\prod_{j=1}^n \delta_{Y_j^i}(\mu_j)}{\sum_{j=1}^n (\prod_{j=1}^n \delta_{Y_j^i}(\mu_j))}.
is the fuzzy basis function (FBF), assuming that the picked FBFs have at least one active rule count, i.e., \( \sum_{i=1}^{q} (\prod_{j=1}^{n} \delta_{i}^{j}(\mu_{j})) > 0. \)

Later, the unknown function \( \Phi_{i}(\mu) \) can be estimated by employing (10):

\[
\Phi_{i}(\mu_{i}) = \rho_{i}^{T} \xi_{i}^{*}(\mu_{i}) + \epsilon_{i}(\mu_{i}),
\]

where \( \epsilon_{i}(\mu_{i}) \) is the optimal estimation error, and the optimal estimation of \( \rho_{i} \) can be described as

\[
\rho_{i}^{*} = \arg \min_{\rho_{i}} \left[ \sup_{\mu \in \Omega_{\rho_{i}}} |\Phi_{i}(\mu_{i}) - \Phi_{i}(\mu_{i}, \rho_{i})| \right],
\]

and note that the vector \( \rho_{i}^{*} \) is only utilized for stability analysis. It is not necessary to calculate its specific values in the implementation process.

Therefore, from (10) and (11), we have

\[
\hat{\Phi}_{i}(\mu_{i}, \rho_{i}) - \Phi_{i}(\mu_{i}) = \hat{\Phi}_{i}(\mu_{i}, \rho_{i}) - \hat{\Phi}_{i}(\mu_{i}, \rho_{i}^{*}) + \hat{\Phi}_{i}(\mu_{i}, \rho_{i}^{*}) - \Phi_{i}(\mu_{i})
\]

\[
= \rho_{i}^{* T} \xi_{i}^{*}(\mu_{i}) - \rho_{i}^{T} \xi_{i}^{*}(\mu_{i}) + \epsilon_{i}(\mu_{i})
\]

\[
= \hat{\rho}_{i}^{T} \xi_{i}^{*}(\mu_{i}) + \epsilon_{i}(\mu_{i}),
\]

where \( \hat{\rho}_{i} = \rho_{i} - \rho_{i}^{*} \), and \( \epsilon_{i}(\mu_{i}) = \Phi_{i}(\mu_{i}, \rho_{i}^{*}) - \Phi_{i}(\mu_{i}). \)

**Remark 1.** Obviously, the FLS (10) is greatly universally employed in control applications. According to the general approximation outcomes, the FLS (10) can approximate any nonlinear smooth function \( \Phi \) with arbitrary precision in a compact operation space, presuming that the construction of the FLS (10), i.e., membership function parameters and number of rules, is correctly predesigned by the designer and the parameter \( \rho \) is defined by the learning algorithm.

### 3. Main Results

#### 3.1. System Description

**Consider a class of uncertain FOCSSs portrayed as below.**

The master system is

\[
\begin{align*}
D^{\delta} x_{i}(t) &= x_{i+1}(t) + f_{i}(x_{i}(t)), \quad (i = 1, \ldots, n - 1), \\
D^{\delta} x_{n}(t) &= f_{n}(x_{n}(t)),
\end{align*}
\]

where \( x_{i} \in \mathbb{R} \) is the measurable state vector, \( x_{i} = [x_{1}, x_{2}, \ldots, x_{i}]^{T} \in \mathbb{R}^{i} \), and \( f_{i}(x_{i}) \in \mathbb{R} \) is an unknown smooth function.

The slave system is

\[
\begin{align*}
D^{\delta} y_{i}(t) &= y_{i+1}(t) + g_{i}(y_{i}(t)) + \gamma_{i}(t), \quad (i = 1, \ldots, n - 1), \\
D^{\delta} y_{n}(t) &= g_{n}(y_{n}(t)) + u(t) + \gamma_{n}(t),
\end{align*}
\]

where \( y_{i} \in \mathbb{R} \) is the measurable state vector, \( y_{i} = [y_{1}, y_{2}, \ldots, y_{i}]^{T} \in \mathbb{R}^{i} \), and \( g_{i}(y_{i}) \in \mathbb{R} \) is an unknown smooth function, \( \gamma(t) = [\gamma_{1}, \ldots, \gamma_{n}]^{T} \) is an unknown disturbance vector, and \( u = [u_{1}, \ldots, u_{n}]^{T} \in \mathbb{R}^{n} \) represents the control input vector.

From (14) and (15), the error system can be characterized as follows:

\[
\begin{align*}
D^{\delta} e_{i}(t) &= e_{i+1}(t) + \Phi_{i}(x_{i}, y_{i}) + \gamma_{i}(t), \\
D^{\delta} e_{n}(t) &= \Phi_{n}(x_{n}, y_{n}) + u(t) + \gamma_{n}(t),
\end{align*}
\]

where \( e_{i+1}(t) = y_{i+1} - x_{i+1} \), and \( \Phi_{i}(x_{i}, y_{i}) = g_{i}(y_{i}(t)) - f_{i}(x_{i}(t)) \). Hereafter, we will replace \( \Phi_{i}(x_{i}, y_{i}) \) with \( \Phi_{i}(\mu_{i}) \) for convenience.
Note that the unknown function $\Phi_i(\mu_i)$ can be directly estimated by the FLS (10), but it is difficult to obtain satisfactory estimation accuracy without increasing the number of fuzzy rules. Thus, to settle this problem, a series-parallel model is presented below:

$$
\begin{align*}
D^\alpha \hat{e}_i &= \hat{e}_{i+1}(t) + \Phi_i(\mu_i, \rho_i) + k_i \check{e}_i + \gamma_i, \\
D^\alpha \hat{e}_i &= \hat{e}_{i+1}(t) + \Phi_i(\mu_i, \rho_i) + k_i \check{e}_i + \frac{\beta_i}{\beta_{i-1}} \hat{e}_{i-1} + \gamma_i, \\
D^\alpha \hat{e}_n &= \Phi_n(\mu_n, \rho_n) + k_n \hat{e}_n + \frac{\beta_n}{\beta_{n-1}} \hat{e}_{n-1} + \gamma_n,
\end{align*}
$$

(17)

where $k_i \in \mathbb{R}^+$, and the prediction error is $\hat{e}_i = e_i - \hat{e}_i$.

### 3.2. Design of the Disturbance Observer

As we all know, unknown external disturbances often appear in real-world systems, which will affect the performance of the system. Therefore, immediately after, in order to estimate unknown disturbances of the slave system, DOBs will be devised.

To facilitate the design of DOBs, auxiliary variables are brought by

$$
\begin{align*}
Q_i(t) &= \delta_i e_i(t) - \gamma_i(t), \\
Q_n(t) &= \delta_n e_n(t) - \gamma_n(t).
\end{align*}
$$

(18)

From (16) and (18), the fractional derivative of $Q_i(t)$ can be obtained as

$$
D^\alpha Q_i = \delta_i D^\alpha e_i - D^\alpha \gamma_i
= \delta_i(e_{i+1} + \Phi_i(\mu_i) + \gamma_i) - D^\alpha \gamma_i
= \delta_i e_{i+1} + \delta_i \Phi_i(\mu_i) - \delta_i Q_i + \delta_i \check{e}_i - D^\alpha \gamma_i.
$$

(19)

Similar to the $Q_i(t)$, the $Q_n(t)$ is calculated as

$$
D^\alpha Q_n = \delta_n \Phi_n(\mu_n) + \delta_n u - \delta_n Q_n + \delta_n \check{e}_n - D^\alpha \gamma_n,
$$

(20)

and in order to enhance the estimation performance of the DOB, the estimation of the $Q$ is introduced as

$$
\begin{align*}
D^\alpha \hat{Q}_i &= \delta_i e_{i+1} + \delta_i \Phi_i(\mu_i) - \delta_i \hat{Q}_i + \delta_i \check{e}_i - \frac{1}{\beta_i} \hat{e}_i - \eta_i, \\
D^\alpha \hat{Q}_n &= \delta_n \Phi_n(\mu_n) + \delta_n u - \delta_n \hat{Q}_n + \delta_n \check{e}_n - \frac{1}{\beta_n} \hat{e}_n - \eta_n.
\end{align*}
$$

(21)

Next, the estimation of disturbance can be described as

$$
\hat{\gamma}_i = \delta_i e_i - \hat{Q}_i,
$$

(22)

whose disturbance estimation error is

$$
\bar{\gamma}_i = \gamma_i - \hat{\gamma}_i.
$$

(23)

Furthermore, on the basis of (18), (22) and (23), we have

$$
\hat{Q}_i = Q_i - \bar{Q}_i = \gamma_i - \gamma_i = \hat{\gamma}_i.
$$

(24)

Now, by using (13), (19), (20) and (21), the fractional derivative of $\hat{Q}_i$ can be obtained as

$$
D^\alpha \check{Q}_i = \delta_i \check{Q}_i(\mu_i) - \delta_i e_i(\mu_i) - \delta_i \hat{Q}_i - D^\alpha \gamma_i + \frac{1}{\beta_i} \check{e}_i + \eta_i, (i = 1, \ldots, n)
$$

(25)
Remark 2. Different from the handling of the disturbance term in [13,25], the DOB proposed above has better approximation accuracy, which can achieve a more satisfactory control effect.

3.3. Controller Design

In the following part of this section, we will promote proper synchronization between the systems (14) and (15) by devising a fuzzy adaptive control law, while guaranteeing that all signals are bounded and the synchronization error asymptotically converges to zero. However, in the traditional backstepping control method, its computational complexity often increases sharply with the increase in the order, i.e., the explosion of complexity problem. Thus, to solve this problem, fractional-order command filters are adopted as

\[
\begin{align*}
D^\alpha e_i^c &= -\lambda_i (e_i^c - \alpha_i), \\
D^\alpha e_n^c &= -\lambda_n (e_n^c - u),
\end{align*}
\]

where \( \lambda_i \in \mathbb{R}^+ \), \( \alpha_i \) and \( u \) are the virtual control input and actual control input, respectively, \( e_i^c \) and \( e_n^c \) are the output, and the initial conditions are \( e_i^c(0) = \alpha_i(0) \) and \( e_n^c(0) = u(0) \), respectively.

Further, the tracking error is defined as

\[
\begin{align*}
\omega_1 &= e_1, \\
\omega_i &= e_i - e_{i-1}.
\end{align*}
\]

However, fractional-order command filters will bring filtering errors, which will affect the performance of synchronization. Thus, error compensation signals are employed to handle this problem, which are characterized as

\[
\begin{align*}
D^\alpha \tau_1 &= -s_1 \tau_1 + \tau_2 + e_1^c - \alpha_1, \\
D^\alpha \tau_i &= -s_i \tau_i + \tau_{i-1} + \tau_{i+1} + e_i^c - \alpha_i, \\
D^\alpha \tau_n &= -s_n \tau_n + \tau_{n-1},
\end{align*}
\]

where \( s_i \in \mathbb{R}^+ \), \( \tau_1(0) = 0 \), and \( \tau_i = e_i^c - \alpha_i \).

The compensated tracking error signal is characterized as

\[
\eta_i = \omega_i - \tau_i.
\]

The virtual control input \( \alpha_i(t) \) and actual control input \( u(t) \) are devised as

\[
\begin{align*}
\alpha_1 &= -s_1 \omega_1 - \rho_1^T \xi_1 - \text{tanh} \left( \frac{\eta_1}{\sigma_1} \right) - \hat{\gamma}_1, \\
\alpha_i &= -s_i \omega_i - \omega_{i-1} - \rho_i^T \xi_i - \text{tanh} \left( \frac{\eta_i}{\sigma_i} \right) + D^\alpha e_{i-1}^c - \hat{\gamma}_i, \\
\alpha_n &= -s_n \omega_n - \omega_{n-1} - \rho_n^T \xi_n - \text{tanh} \left( \frac{\eta_n}{\sigma_n} \right) + D^\alpha e_{n-1}^c - \hat{\gamma}_n.
\end{align*}
\]

Remark 3. It can be noticed that there exist some control methods that suffer from chattering problems, such as Refs. [13,25]. To handle this problem, here, we take advantage of a smooth hyperbolic tangent function instead of the sign function.

Next, the adaptive law is devised as

\[
D^\alpha \theta_i = \theta_i (|\eta_i| + \frac{\xi_i}{\beta_i}) - m_i \theta_i,
\]

where \( \theta_i \) and \( m_i \) are normal numbers.
In light of the above discussion, our key results can be described below.

**Theorem 1.** Premeditating systems (14) and (15), suppose that the series-parallel model is devised as (17), the fractional-order command filter is chosen as (26), and the error compensation signal is portrayed as (28), while presuming the DOB, the controller, and the fuzzy adaptive law are characterized as (22), (30) and (31), respectively. Then the system can ensure that the properties below hold:

- All signals remain bounded in a closed-loop system.
- The synchronization error eventually converges to a small neighborhood of zero.

First, to accomplish the proof of the Theorem 1, an assumption is reasonable to be given.

**Assumption 1.** The \( \gamma_i(t) \) and its fractional-order derivative are bounded, i.e., \( |\gamma_i(t)| \leq \bar{\gamma}_i(t) \) and \( D^\delta \gamma_i(t) \leq \bar{h}_i \), where \( \bar{\gamma}_i(t) \) and \( \bar{h}_i \) are the unknown positive function and positive constant, respectively.

**Proof.** Step 1: Select the Lyapunov function as
\[
V_1 = V_{11} + V_{12} + V_{13} + V_{14}
\]
\[
= \frac{1}{2} \eta_1^2 + \frac{1}{2\beta_1} \bar{\gamma}_1^2 + \frac{1}{2\theta_1} \rho_1^2 + \frac{1}{2} \gamma_1^2. \tag{32}
\]

From (13), (16), (23), (27), (28), (29), (30), (31), Lemma 1 and Lemma 4, the fractional-order derivative of \( V_{11} \) is calculated as
\[
D^\delta V_{11} \leq \eta_1 (e_2 + \Phi_1 + \gamma_1 - D^\delta \tau_1)
= \eta_1 (e_2 + \Phi_1 - \Phi_1 + \Phi_1 + \gamma_1 - D^\delta \tau_1)
= \eta_1 (e_2 - \rho_1^T \hat{\tau}_1 - \epsilon_1 + \rho_1^T \hat{\tau}_1 + \gamma_1 + s_1 \tau_1 - \tau_2 - \epsilon_1 + \alpha_1)
= \eta_1 (e_2 - \rho_1^T \hat{\tau}_1 - \epsilon_1 + \rho_1^T \hat{\tau}_1 + \gamma_1 + s_1 \tau_1 + \alpha_1) \tag{33}
\]
\[
\leq \eta_1 \eta_2 - \eta_1 \rho_1^T \hat{\tau}_1 + |\eta_1| - \eta_1 \tanh \left( \frac{\eta_1}{\bar{\epsilon}_1} \right) - s_1 \eta_1^2 - \eta_1 \bar{\gamma}_1
\]
\[
\leq \eta_1 \eta_2 - \eta_1 \rho_1^T \hat{\tau}_1 + \sigma_1' - s_1 \eta_1^2 - \eta_1 \bar{\gamma}_1.
\]

Next, by using (16), (17) and Lemma 1, we have
\[
D^\delta V_{12} \leq \frac{1}{\beta_1} \bar{\epsilon}_1 (D^\delta e_1 - D^\delta \bar{\epsilon}_1)
= \frac{1}{\beta_1} \bar{\epsilon}_1 (e_2 + \Phi_1 + \gamma_1 - \bar{\epsilon}_2 - \Phi_1 - k_1 \bar{\epsilon}_1 - \bar{\gamma}_1)
= \frac{1}{\beta_1} \bar{\epsilon}_1 (e_2 - \rho_1^T \hat{\tau}_1 - \epsilon_1 - k_1 \bar{\epsilon}_1 - \bar{\gamma}_1) \tag{34}
\]
\[
\leq \frac{1}{\beta_1} \bar{\epsilon}_1 \bar{\epsilon}_2 - \frac{1}{\beta_1} \bar{\epsilon}_1 \rho_1^T \hat{\tau}_1 + \frac{1}{\beta_1} |\bar{\epsilon}_1| - \frac{1}{\beta_1} k_1 \bar{\epsilon}_1^2 - \frac{1}{\beta_1} \bar{\epsilon}_1 \bar{\gamma}_1.
\]

Simultaneously, according to Lemma 1, we obtain
\[
D^\delta V_{13} \leq \frac{1}{\beta_1} \rho_1^T D^\delta \bar{\rho}_1. \tag{35}
\]

In addition, it follows from (24), (25) and Lemma 1 that
\[
D^\delta V_{14} \leq \bar{\gamma}_1 (\delta_1 \rho_1^T \hat{\tau}_1 (\mu_1) - \delta_1 \bar{\epsilon}_1 (\mu_1) - \delta_1 \bar{Q}_1 - D^\delta \bar{\gamma}_1 + \frac{1}{\beta_1} \bar{\epsilon}_1 + \eta_1). \tag{36}
\]
Then, from (31), (33), (34), (35) and (36), we can obtain
\[ D^\delta V_1 \leq \eta_1 \eta_2 - \eta_1 \rho_1^T \dot{\xi}_1 + \sigma'_1 - s_1 \eta_1^2 - \eta_1 \dot{\gamma}_1 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + \frac{1}{\beta_1} |\dot{\varepsilon}_1| \\
- \frac{1}{\beta_1} \dot{\varepsilon}_1 \rho_1^T \xi_1 - \frac{1}{\beta_1} k_1 \dot{\varepsilon}_1^2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \gamma_1 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \gamma_1 + \frac{1}{\beta_1} \gamma_1 \dot{\varepsilon}_1 - \frac{m_1}{\theta_1} \dot{\rho}_1^T \rho_1 \\
\leq \eta_1 \eta_2 + \sigma'_1 - s_1 \eta_1^2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + \frac{1}{\beta_1} |\dot{\varepsilon}_1| - \frac{1}{\beta_1} k_1 \dot{\varepsilon}_1^2 + \frac{m_1}{\theta_1} \dot{\rho}_1^T \rho_1 \\
+ \delta_1 \rho_1^T \dot{\xi}_1 + \delta_1 |\dot{\gamma}_1| - \delta_1 \dot{\gamma}_1^2 - \gamma_1 D^\delta \gamma_1 \\
\leq \eta_1 \eta_2 + \sigma'_1 - s_1 \eta_1^2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + \frac{1}{\beta_1} |\dot{\varepsilon}_1| - \frac{1}{\beta_1} k_1 \dot{\varepsilon}_1^2 - \frac{m_1}{\theta_1} \dot{\rho}_1^T \rho_1 \\
- \frac{m_1}{\theta_1} \dot{\rho}_1^T \rho_1 + \delta_1 \rho_1^T \dot{\xi}_1 + \delta_1 |\dot{\gamma}_1| - \delta_1 \dot{\gamma}_1^2 - \gamma_1 D^\delta \gamma_1. \quad (37) \]

Noting that \( \bar{c}_1^2 \leq 1 \), and on the basis of Lemma 2, we have
\[
\begin{aligned}
\begin{cases}
\frac{1}{\beta_1} |\dot{\varepsilon}_1| &\leq \frac{1}{2\beta_1} \dot{\varepsilon}_1^2 + \frac{1}{2\beta_1}, \\
- \dot{\rho}_1^T \rho_1 &\leq \frac{1}{2} \dot{\rho}_1^T \rho_1 + \frac{1}{2} \dot{\rho}_1^T \rho_1, \\
\delta_1 \rho_1^T \dot{\xi}_1 &\leq \frac{1}{4} \delta_1 \dot{\varepsilon}_1^2 + \rho_1^T \rho_1, \\
\delta_1 |\dot{\gamma}_1| &\leq \frac{1}{4} \dot{\gamma}_1 \dot{\gamma}_1 + 1, \\
\gamma_1 D^\delta \gamma_1 &\leq \frac{1}{4} \dot{\gamma}_1^2 + (D^\delta \gamma_1)^2.
\end{cases}
\end{aligned} \quad (38) \]
Substituting (38) into (37) yields
\[
D^\delta V_1 \leq \eta_1 \eta_2 + \sigma'_1 - s_1 \eta_1^2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 \\
- \frac{k_1}{\beta_1} \dot{\varepsilon}_1^2 = \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 \\
+ \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 \\
\leq -s_1 \eta_1^2 + \frac{2k_1}{\beta_1} \dot{\varepsilon}_1 - \frac{m_1 - 2\theta_1}{\beta_1} \dot{\rho}_1^T \rho_1 - \frac{1}{\beta_1} (\dot{\varepsilon}_1^2 - \delta_1 \dot{\varepsilon}_1^2 + \frac{1}{2} \dot{\gamma}_1^2 + \eta_1 \eta_2) \\
+ \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 = \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 \\
\leq -b_1 V_1 + \eta_1 \eta_2 + \frac{1}{\beta_1} \dot{\varepsilon}_1 \dot{\varepsilon}_2 + z_1,
\end{aligned} \quad (39) \]

where \( b_1 = \min \{2s_1, 2k_1 - 1, m_1 - 2\theta_1, \dot{\varepsilon}_1^2 - 2\delta_1 + \frac{1}{2} \} \in \mathbb{R}^+ \) and \( z_1 = \frac{m_1}{\beta_1} \dot{\rho}_1^T \rho_1 + \sigma'_1 + \frac{1}{\beta_1} + h_1^2 + 1 \in \mathbb{R}^+ \).

Step i: Choosing the Lyapunov function as
\[
V_i = V_{i-1} + V_{i2} + V_{i3} + V_{i4} \\
= V_{i-1} + \frac{1}{2} \eta_i^2 + \frac{1}{2\beta_1} \dot{\varepsilon}_1^2 + \frac{1}{2\beta_1} \dot{\rho}_1^T \rho_1 + \frac{1}{2} \gamma_1^2. \quad (40) \]
Similar to the Step 1, the fractional-order derivative of $V_1$ is calculated as

$$\mathcal{D}^\alpha V_1 \leq \eta_i(\epsilon_{i+1} + \Phi_i + \gamma_i - \mathcal{D}^\alpha e_{i-1} - \mathcal{D}^\alpha \tau_i)$$

$$= \eta_i(e_{i+1} + \Phi_i - \Phi_i + \gamma_i - \mathcal{D}^\alpha e_{i-1} - \mathcal{D}^\alpha \tau_i)$$

$$= \eta_i(e_{i+1} + \bar{\beta}_i e_i - e_i + \bar{\rho}^T_i \tilde{\xi}_i + \gamma_i - \mathcal{D}^\alpha e_{i-1} + s_i \tau_i + \tau_{i-1} - \tau_{i+1} - e_i + \alpha_i)$$

$$= \eta_i(\eta_{i+1} - \bar{\beta}_i e_i - e_i + \bar{\rho}^T_i \tilde{\xi}_i + \gamma_i - \mathcal{D}^\alpha e_{i-1} + s_i \tau_i + \tau_{i-1} + \tau_{i+1} + \alpha_i)$$

$$\leq \eta_i e_i + \eta_i \gamma_i + \eta_i \gamma_i + |\eta_i| - \eta_i \tanh\left(\frac{\eta_i}{\alpha_i}\right) - s_i \eta_i^2 - \eta_i^{-1} \eta_i$$

Next, by using (16), (17) and Lemma 1, we have

$$\mathcal{D}^\alpha V_2 \leq \frac{1}{\bar{\beta}_i} \tilde{e}_i(e_{i+1} + \Phi_i + \gamma_i - \tilde{e}_{i+1} - \tilde{e}_{i-1} - \tilde{\gamma}_i)$$

$$= \frac{1}{\bar{\beta}_i} \tilde{e}_i(e_{i+1} - \bar{\beta}_i \eta_i - e_i - \gamma_i - k_i e_i - \frac{\bar{\beta}_i}{\bar{\beta}_{i-1}} \tilde{e}_{i-1})$$

$$\leq \frac{1}{\bar{\beta}_i} \tilde{e}_i e_i + \frac{1}{\bar{\beta}_i} \tilde{e}_i \bar{\rho}_i^T \tilde{\xi}_i + \frac{1}{\bar{\beta}_i} |\tilde{e}_i| - \frac{1}{\bar{\beta}_i} \tilde{e}_i \gamma_i - \frac{1}{\bar{\beta}_i} k_i e_i^2 - \frac{1}{\bar{\beta}_{i-1}} \tilde{e}_{i-1} e_i.$$  \hspace{1cm} (42)

Simultaneously, in line with Lemma 1, we obtain

$$\mathcal{D}^\alpha V_3 \leq \frac{1}{\bar{\beta}_i} \bar{\beta}_i^T \mathcal{D}^\alpha \bar{\beta}_i.$$  \hspace{1cm} (43)

In addition, it follows from (24), (25) and Lemma 1 that

$$\mathcal{D}^\alpha V_4 \leq \bar{\gamma}_i(\delta_i \bar{\rho}_i^T \tilde{\xi}_i(\mu_i) - \delta_i e_i(\mu_i) - \delta_i \tilde{Q}_i) - \mathcal{D}^\alpha \gamma_i + \frac{1}{\bar{\beta}_i} \tilde{e}_i + \eta_i).$$  \hspace{1cm} (44)

Then, from (31), (41), (42), (43) and (44), we can obtain

$$\mathcal{D}^\alpha V_i \leq -b_{i-1} V_{i-1} + \eta_i e_i + \eta_i \eta_i e_i + \eta_i \eta_i e_i + s_i \eta_i^2 + \frac{1}{\bar{\beta}_i} |e_i|$$

$$- \frac{1}{\bar{\beta}_i} \tilde{e}_i \bar{\beta}_i^T \tilde{\xi}_i - \frac{1}{\bar{\beta}_i} k_i e_i^2 - \frac{1}{\bar{\beta}_{i-1}} \tilde{e}_{i-1} e_i + \frac{1}{\bar{\beta}_i} \tilde{e}_i |\tilde{e}_i|$$

$$\leq -b_{i-1} V_{i-1} + s_i \eta_i^2 + \frac{1}{\bar{\beta}_i} |e_i|$$

$$- \frac{1}{\bar{\beta}_i} k_i e_i^2 - \frac{m_i}{\bar{\beta}_i} \bar{\rho}_i + \delta_i \bar{\rho}_i^T \tilde{\xi}_i + \delta_i |\tilde{\gamma}_i| - \delta_i \tilde{e}_i^2 - \tilde{e}_i \mathcal{D}^\alpha \gamma_i$$

$$\leq -b_{i-1} V_{i-1} + s_i \eta_i^2 + \frac{1}{\bar{\beta}_i} |e_i|$$

$$- \frac{1}{\bar{\beta}_i} k_i e_i^2 - \frac{m_i}{\bar{\beta}_i} \bar{\rho}_i + \delta_i \bar{\rho}_i^T \tilde{\xi}_i + \delta_i |\tilde{\gamma}_i| - \delta_i \tilde{e}_i^2 - \tilde{e}_i \mathcal{D}^\alpha \gamma_i.$$  \hspace{1cm} (45)
Similar to (38), we have

\[
\begin{aligned}
\frac{1}{\beta_i}|v_i| & \leq \frac{1}{2\beta_i}e_i^2 + \frac{1}{2\beta_i'}, \\
-\tilde{\rho}_i^T\tilde{\rho}_i^* & \leq \frac{1}{2}\tilde{\rho}_i^T\tilde{\rho}_i + \frac{1}{2}\tilde{\rho}_i'^T\tilde{\rho}_i', \\
\delta_i\tilde{\rho}_i^T\tilde{\rho}_i & \leq \frac{1}{4}\delta_i^2\tilde{\gamma}_i^2 + \rho_i^T\rho_i, \\
\delta_i|\tilde{\gamma}_i| & \leq \frac{1}{4}\delta_i^2\tilde{\gamma}_i^2 + 1, \\
\gamma_iD^\theta\gamma_i & \leq \frac{1}{4}\tilde{\gamma}_i^2 + (D^\theta\gamma_i)^2.
\end{aligned}
\]  

(46)

Substituting (46) into (45) yields

\[
\begin{aligned}
D^\theta V_i & \leq -b_iV_{i-1} + z_i + \eta_i\eta_{i-1} + \sigma_i - s_i\eta_i^2 + \frac{1}{\beta_i}\tilde{e}_i\tilde{e}_{i-1}' + \frac{1}{2\beta_i}e_i^2 + \frac{1}{2\beta_i} - k_i\xi_i^2 \\
& \quad - \frac{m_i}{2\beta_i}\tilde{\rho}_i^T\tilde{\rho}_i + \frac{m_i}{2\beta_i}\rho_i^T\rho_i' + \frac{1}{4}\delta_i^2\tilde{\gamma}_i^2 + \rho_i^T\rho_i + \frac{1}{4}\delta_i^2\gamma_i^2 + 1 + \frac{1}{4}\tilde{\gamma}_i^2 + (D^\theta\gamma_i)^2 \\
& \quad = -b_iV_{i-1} - s_i\eta_i^2 - \frac{2k_i - 1}{2\beta_i}e_i^2 - \frac{m_i - 2\theta_i}{2\beta_i}\tilde{\rho}_i^T\tilde{\rho}_i - \frac{1}{2}(\delta_i^2 - 2\delta_i + 1)\tilde{\gamma}_i^2 \\
& \quad + \eta_i\eta_{i+1} + \frac{1}{\beta_i}\tilde{e}_i\tilde{e}_{i+1} + s_i + \frac{m_i}{2\beta_i}\rho_i^T\rho_i' + \sigma_i + \frac{1}{2\beta_i} + h_i^2 + 1 \\
& \leq -b_iV_i + \eta_i\eta_{i+1} + \frac{1}{\beta_i}\tilde{e}_i\tilde{e}_{i+1} + z_i,
\end{aligned}
\]  

(47)

where \(b_i = \min\{b_i, 2s_i, 2k_i, 1, m_i - 2\theta_i, \delta_i^2 - 2\delta_i + 1\} \in \mathbb{R}^+\) and \(z_i = z_{i-1} + \frac{m_i}{2\beta_i}\rho_i^T\rho_i' + \sigma_i + \frac{1}{2\beta_i} + h_i^2 + 1 \in \mathbb{R}^+\).

Step n: Next, we will complete the controller design and stability analysis. Construct the Lyapunov function as

\[
V_n = V_{n-1} + V_{n1} + V_{n2} + V_{n3} + V_{n4}
\]

\[
= V_{n-1} + \frac{1}{2}\eta_n^2 + \frac{1}{2\beta_n}e_n^2 + \frac{1}{2\beta_n}\rho_n^T\rho_n + \frac{1}{2}\gamma_n^2.
\]  

(48)

Similar to the Step i, we can obtain

\[
D^\theta V_{n1} \leq \eta_n(\Phi_n + \gamma + u - D^\theta\xi_{n-1} - D^\theta\tau_n)
\]

\[
= \eta_n(\Phi_n - \Phi_n + \Phi_n + \gamma + u - D^\theta\xi_{n-1} - D^\theta\tau_n)
\]

\[
= \eta_n(-\tilde{\rho}_n^T\tilde{\rho}_n - \eta_n + \rho_n^T\rho_n + \gamma + u - D^\theta\xi_{n-1} + s_n\tau_n + \tau_n)
\]  

\[
\leq -\eta_n\tilde{\rho}_n^T\tilde{\rho}_n + |\eta_n| + \eta_n\tilde{\gamma} - \eta_n\tanh\left(\frac{\eta_n}{\sigma_n}\right) - s_n\eta_n^2 - \eta_n - 1\eta_n
\]

\[
\leq -\eta_n\tilde{\rho}_n^T\tilde{\rho}_n + \sigma_n' + \eta_n\tilde{\gamma} - s_n\eta_n^2 - \eta_n - 1\eta_n.
\]  

(49)

\[
D^\theta V_{n2} \leq \frac{1}{\beta_n}\tilde{e}_n(\Phi_n + \gamma + u - \Phi_n - k_n\tilde{e}_n - u - \frac{\beta_n}{\beta_n-1}\tilde{e}_{n-1} - \tilde{\gamma})
\]

\[
= \frac{1}{\beta_n}\tilde{e}_n(\rho_n^T\rho_n - \epsilon_n + \rho_n^T\rho_n + \gamma + u - \frac{\beta_n}{\beta_n-1}\tilde{e}_{n-1})
\]

\[
\leq -\frac{1}{\beta_n}\tilde{e}_n\tilde{\rho}_n^T\tilde{\rho}_n + \frac{1}{\beta_n}e_n^2 + \frac{1}{\beta_n}\tilde{e}_n\tilde{\gamma} - \frac{1}{\beta_n}k_n\tilde{e}_n^2 - \frac{1}{\beta_n-1}\tilde{e}_{n-1}e_n.
\]  

(50)

\[
D^\theta V_{n3} = \frac{1}{2\beta_n}D^\theta\tilde{\rho}_n\tilde{\rho}_n \leq \frac{1}{\beta_n}\tilde{\rho}_n^T\tilde{\rho}_n D^\theta\rho_n = \frac{1}{\beta_n}\tilde{\rho}_n^T D^\theta\rho_n.
\]  

(51)
\[ D^\alpha V_n \leq \hat{\gamma}_n (\delta_n \hat{p}_n^T \hat{v}_n(\mu_n) - \delta_n \varepsilon_n(\mu_n) - \delta_n \hat{Q}_n - D^\alpha \gamma_n + \frac{1}{\hat{p}_n} \varepsilon_n + \eta_n). \]  

(52)

Then, from (31), (49), (50), (51) and (52), we can obtain

\[
D^\alpha V_n \leq D^\alpha V_{n-1} + \eta_n D^\alpha \eta_n + \frac{1}{\hat{p}_n} \varepsilon_n D^\alpha \eta_n + \frac{1}{\hat{p}_n} \varepsilon_n D^\alpha \rho_n + \frac{1}{\hat{p}_n} \hat{p}_n^T D^\alpha \rho_n + \gamma_n D^\alpha \gamma_n
\]

\[
\leq -b_{n-1} V_{n-1} + z_{n-1} + \sigma'_n - s_n \eta_n^2 + \frac{1}{\hat{p}_n} |\varepsilon_n| - \frac{1}{\hat{p}_n} k_n^2 + \frac{m_n}{\theta_n} \hat{p}_n^T \hat{p}_n + \delta_n \hat{r}_n^T \hat{v}_n + \delta_n |\gamma_n| - \delta_n \gamma_n^2 - \gamma_n D^\alpha \gamma_n
\]

\[
\leq -b_{n-1} V_{n-1} + z_{n-1} + \sigma'_n - s_n \eta_n^2 + \frac{1}{\hat{p}_n} |\varepsilon_n| - \frac{1}{\hat{p}_n} k_n^2 + \frac{m_n}{\theta_n} \hat{p}_n^T \hat{p}_n + \delta_n \hat{r}_n^T \hat{v}_n + \delta_n |\gamma_n| - \delta_n \gamma_n^2 - \gamma_n D^\alpha \gamma_n.
\]

(53)

According to Lemma 2, we obtain that

\[
\begin{align*}
\frac{1}{\hat{p}_n} |\varepsilon_n| & \leq \frac{1}{2 \hat{p}_n} \delta_n^2 + \frac{1}{2 \hat{p}_n}, \\
\hat{p}_n^T \hat{p}_n^* & \leq \frac{1}{2} \hat{p}_n^T \hat{p}_n + \frac{1}{2} \hat{p}_n^T \hat{p}_n^*, \\
\delta_n \hat{r}_n^T \hat{v}_n & \leq \frac{1}{4} \delta_n \gamma_n^2 + \frac{1}{4} \delta_n \gamma_n^2 + 1, \\
\delta_n |\gamma_n| & \leq \frac{1}{4} \delta_n \gamma_n^2 + 1, \\
\gamma_n D^\alpha \gamma_n & \leq \frac{1}{4} \gamma_n^2 + (D^\alpha \gamma_n)^2.
\end{align*}
\]

(54)

Substituting (54) into (53) yields

\[
D^\alpha V_n \leq -b_{n-1} V_{n-1} + z_{n-1} + \sigma'_n - s_n \eta_n^2 + \frac{1}{\hat{p}_n} \varepsilon_n^2 + \frac{1}{\hat{p}_n} |\varepsilon_n| - \frac{1}{\hat{p}_n} k_n^2 - m_n \hat{p}_n^T \hat{p}_n
\]

\[
+ \frac{m_n}{2 \hat{p}_n} \hat{p}_n^T \hat{p}_n^* + \frac{1}{4} \delta_n \gamma_n^2 + \frac{1}{4} \delta_n \gamma_n^2 + 1 + \frac{1}{4} \gamma_n^2 + (D^\alpha \gamma_n)^2
\]

\[
= -b_{n-1} V_{n-1} - s_n \eta_n^2 - \frac{2}{\hat{p}_n} \varepsilon_n^2 - m_n - \frac{1}{2 \hat{p}_n} k_n \eta_n^2 - \frac{1}{2} \hat{p}_n^T \hat{p}_n
\]

\[
+ \frac{m_n}{2 \hat{p}_n} \hat{p}_n^T \hat{p}_n^* + \frac{1}{2 \hat{p}_n} \hat{h}_n^2 + 1
\]

\[
\leq -b_n V_n + z_n,
\]

where \( b_n = \min \{b_{n-1}, 2s_n, 2k_n - 1, m_n - 2\theta_n, \delta_n^2 - 2\delta_n + \frac{1}{2} \} \in \mathbb{R}^+ \), and \( z_n = z_{n-1} + \sigma'_n + \frac{m_n}{2 \hat{p}_n} \hat{p}_n^T \hat{p}_n^* + \frac{1}{2 \hat{p}_n} \hat{h}_n^2 + 1 \in \mathbb{R}^+ \).

In accordance with (55), there exists a non-negative function \( b(t) \) such that the equation holds below:

\[ D^\alpha V_n + a(t) = -b_n V_n + z_n, \]

(56)

and its Laplace transformation can be obtained by

\[ s^\alpha V_n(s) - s^{\alpha-1} V_n(0) + A(s) = -b_n V_n(s) + \frac{z_n}{s}. \]

(57)

Further, we can obtain

\[ V_n(s) = \frac{s^{\alpha-1}}{s^\alpha + b_n} V_n(0) + \frac{z_n}{s(s^\alpha + b_n)} - \frac{A(s)}{s^\alpha + b_n}. \]

(58)
In line with (4), we can obtain
\[ |V_n(t)| \leq |V_n(0)|E_{\vartheta,1}(-b_n t^\vartheta) + z_n t^\vartheta E_{\vartheta,\vartheta+1}(-b_n t^\vartheta). \]  
(59)

In light of Lemmas 3 and 5, \( \forall \epsilon > 0 \), there exists \( t > t_1 > 0 \), then the following inequalities can hold:
\[ z_n t^\vartheta E_{\vartheta,\vartheta+1}(-b_n t^\vartheta) \leq \frac{z_n}{b_n} + \frac{\epsilon}{3}, \]  
(60)
\[ |V_n(0)|E_{\vartheta,1}(-b_n t^\vartheta) \leq \frac{\epsilon}{3}, \]  
(61)
and the value of the parameter can be adjusted as \( \frac{z_n}{b_n} \leq \frac{\epsilon}{3} \).

It follows from (59), (60) and (61) that
\[ |V_n(t)| \leq \epsilon. \]  
(62)

Equation (62) implies that the control objective has been achieved, and then this completes the proof process.

**Remark 4.** It is worth noting that parameters designed above play a vital role in the control and synchronization performance of the system. Therefore, based on the past experience and the effect of simulation, the following reference opinions for parameter selection are given. (1) The control gain coefficient \( s_i \) should be chosen to be larger in order to obtain satisfactory control performance, but if it is too large, it will result in more energy loss. (2) To obtain a more accurate system model, the coefficient \( \theta_i \) should be designed to be larger, but it is necessary to avoid parameter drift caused by designing it to be too large. Simultaneously, to ensure the boundedness of \( \rho_i \), parameter \( m_i \) should also be selected to be large, but it should be noted that if it is too large, it may cause \( \rho_i \) to converge to zero quickly, which will cause the FLS to not work. (3) To obtain a better disturbance observation result, the parameter \( \delta_i \) should be designed to be smaller. (4) To make the constructed series-parallel model play a good role, parameter \( k_i \) should be chosen to be as large as possible, and parameter \( \beta_i \) should be chosen to be as small as possible.

**Remark 5.** From (31), it can be found that different from the usual control scheme, the prediction error \( \tilde{e}_i \) is included in the adaptive law (31), and it is utilized as an additional feedback term of \( \rho_i \) to improve the estimation accuracy of the FLS. Simultaneously, to improve the observation effect of the DOB, the prediction error \( \tilde{e}_i \) is also included in Equation (21).

### 4. Simulation Results

So as to certify the feasibility of the put forward controller for uncertain FOCSs, we premeditate the following systems.

The master system [32] is

\[
\begin{aligned}
D^\vartheta x_1 &= x_2 + \frac{10}{7} (x_1 - x_1^3), \\
D^\vartheta x_2 &= x_3 + 10 x_1 - x_2, \\
D^\vartheta x_3 &= -\frac{100}{7} x_2.
\end{aligned}
\]  
(63)

The slave system [32] is

\[
\begin{aligned}
D^\vartheta y_1 &= y_2 + \frac{10}{7} (y_1 - y_1^3) + 0.15 \sin x_1 + 0.1 \sin(x_1 x_2), \\
D^\vartheta y_2 &= y_3 + 10 y_1 - y_2 + 0.5 \sin(x_1^2 + x_2^2) + \frac{1}{10 + x_1 x_2 \cos^2 t}, \\
D^\vartheta y_3 &= -\frac{100}{7} y_2 + u(t) + 1.5 \sin x_2 + \frac{0.3 \sin(x_1 x_2)}{x_1^2 + x_2^2 + x_3^2}. 
\end{aligned}
\]  
(64)
The initial conditions are given as $x(0) = [x_1(0), x_2(0), x_3(0)]^T = [0.5, 1.5, 1.5]^T$, $y(0) = [y_1(0), y_2(0), y_3(0)]^T = [0.8, 0.5, 0.5]^T$, $e_1(0) = e_2(0) = e_3(0) = 0$, $a_1(0) = a_2(0) = u(0) = 0$, $\hat{e}_1 = \hat{e}_2 = \hat{e}_3 = 0$, $\hat{\gamma} = 0$, $\hat{Q} = 0$, $d = 0$. The opted parameters are revealed as $\theta = 0.98$, $\lambda_1 = \lambda_2 = \lambda_3 = 28$, $\theta_1 = \theta_2 = \theta_3 = 2$, $k_1 = k_2 = 20$, $k_3 = 80$, $s_1 = s_2 = s_3 = 10$, $c_1 = c_2 = c_3 = 12$, $\beta_1 = \beta_2 = \beta_3 = 0.001$, $m_1 = m_2 = m_3 = 9$, $\delta_1 = \delta_2 = \delta_3 = 9.5$.

There are three FLSs adopted in the devised controller. The first FLS takes $x_1$ and $y_1$ as its input and defines nine Gaussian membership functions evenly distributed on $[-2, 2]$ for every input. The initial condition is picked as $\rho_1(0) = [1, \ldots, 1]^T \in \mathbb{R}^{81}$. The second FLS takes $x_2, y_1$ and $y_2$ as its input and defines four Gaussian membership functions evenly distributed on $[-3, 3]$ for every input. The initial condition is picked as $\rho_2(0) = [1, \ldots, 1]^T \in \mathbb{R}^{256}$. The third FLS takes $x_3$ and $y_3$ as its input and defines 10 Gaussian membership functions evenly distributed on $[-25, 20]$ for every input. The initial condition is picked as $\rho_3(0) = [1, \ldots, 1]^T \in \mathbb{R}^{100}$.

Ultimately, the simulation outcomes are exhibited in Figures 1–5. Figure 1 exhibits that the master–slave systems are practically synchronized and their synchronization errors converge to a small neighborhood of the origin, which shows that the proposed method has excellent control performance. Figure 2 displays the approximation effect of $\Phi_i$, and the simulation results show that the proposed compound AFBC method has obvious improvement effect on FLS. Figure 3 exhibits the approximation effect of the virtual input $\alpha_i$ and the actual input $u$, from which we can find that the filter adopted has a strong approximation effect. The tracking errors by using the compound AFBC and the tracking error compensation signals are exhibited in Figure 4. Figure 5 displays the approximation effect of $\gamma_i$, and the simulation results show that the proposed DOB can effectively improve the approximation effect by adding a series parallel model compared with the existing DOB. The above results show that the synchronization error has a fast convergence and the controller is able to work well with external disturbances and model uncertainties. Further, we can also observe that all signals can remain bounded, so the control objective is achieved.

**Figure 1.** Synchronization performance: (a) synchronization performance among $x_1$ and $y_1$; (b) synchronization performance among $x_2$ and $y_2$; (c) synchronization performance among $x_3$ and $y_3$; (d) synchronization errors.
Figure 2. Comparison results: (a) the approximation effect of $\Phi_1$ by utilizing the compound AFBC; (b) the approximation effect of $\Phi_1$ by utilizing the AFBC; (c) the approximation effect of $\Phi_2$ by utilizing the compound AFBC; (d) the approximation effect of $\Phi_2$ by utilizing the AFBC; (e) the approximation effect of $\Phi_3$ by utilizing the compound AFBC; (f) the approximation effect of $\Phi_3$ by utilizing the AFBC.
Figure 3. Time response of filters: (a) virtual input $a_1$ and filter output $e_1'$; (b) virtual input $a_2$ and filter output $e_2'$; (c) actual input $u$ and filter output $e_3'$; (d) filter errors.

Figure 4. Auxiliary variables: (a) tracking errors by using the compound AFBC; (b) tracking error compensation signals.
Figure 5. Comparison results: (a) the approximation effect of $\gamma_1$ by utilizing the compound AFBC; (b) the approximation effect of $\gamma_1$ by utilizing the AFBC; (c) the approximation effect of $\gamma_2$ by utilizing the compound AFBC; (d) the approximation effect of $\gamma_2$ by utilizing the AFBC; (e) the approximation effect of $\gamma_3$ by utilizing the compound AFBC; (f) the approximation effect of $\gamma_3$ by utilizing the AFBC.
5. Conclusions

In this paper, a compound AFBC scheme is proposed for two FOCSs with strict feedback. At each step, a FLS is utilized to approximate a unknown function, and a command filter is taken to approximate the derivative of the virtual control input. Simultaneously, a DOB is adopted to approximate a unknown disturbance. All signals are shown to be bounded and the synchronization errors asymptotically approached to zero by the advanced adaptive fuzzy backstepping controller. Finally, from the simulation results, it can also be intuitively seen that the proposed scheme achieves satisfactory synchronization. How to devise a compound AFBC scheme for FOCSs with input delay is one of our next research directions.

Author Contributions: Project administration, X.Z.; Writing—original draft, F.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the Natural Science Research Project of Guangxi Minzu University (2020KJYB002), and the Key Natural Science Projects of Anhui Universities (KJ2020A0643).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data that support the findings of this study is in the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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