Applications of the Neutrosophic Poisson Distribution for Bi-Univalent Functions Involving the Modified Caputo’s Derivative Operator

S. Santhiya and K. Thilagavathi *

School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India
* Correspondence: kthilagavathi@vit.ac.in

Abstract: This paper establishes the upper bounds for the second and third coefficients of holomorphic and bi-univalent functions in a family which involves Bazilevic functions and $\mu$-pseudo-starlike functions under a new operator, joining the neutrosophic Poisson distribution with the modified Caputo’s derivative operator. We also discuss Fekete–Szegö’s function problem in this family. Examples are given to support our case for the neutrosophic Poisson distribution. The fields of physics, mechanics, engineering, and biology all make extensive use of fractional derivatives.

Keywords: Bi-univalent functions; neutrosophic Poisson distribution; modified Caputo’s derivative operator; Bazilevic functions; $\mu$-pseudo-starlike functions; $(G, H)$ Lucas polynomial

MSC: 30C45; 30C50; 30C20; 30C80

1. Introduction

Let $A$ denote the class of functions $f$ of the form
\[ f(\xi) = \xi + d_2 \xi^2 + \ldots. \] (1)
which contains all univalent functions of the form in Equation (1). Biberbach [1] first presented the familiar coefficient conjecture for the function $f \in A$ of the form in Equation (1) and was supported by de Branges [2] in 1985. Between 1916 and 1985, this idea was the subject of numerous study attempts.

Let $S$ be the subcollection of $A$ that contains functions that have univalent values in $U$. Every function $f \in S$ follows the Koebe one-quarter theorem (see [3]) and has an inverse $f^{-1}$ such that $f^{-1}(f(\xi)) = \xi, \xi \in U$ and $f(f^{-1}(\omega)) = \omega, |\omega| < r_0(f), r_0(f) \geq \frac{1}{4}$. If $f$ is of the form (1.1), then
\[ f^{-1}(\omega) = \omega - d_2(\omega)^2 - (2d_2^2 - d_3)(\omega)^3 \ldots \quad |\omega| < r_0(f) \] (2)

If both $f$ and $f^{-1}$ are univalent in $U$, then a function $f \in A$ is said to be bi-univalent in $U$. Regarding the set of bi-univalent functions in $U$, Srivastava et al. [4] reportedly revived the study of holomorphic and bi-univalent functions in recent years, which we denote with $\Sigma$. This was followed by pieces by authors such as Frasin and Aouf [5], Goyal and Goswami [6], Srivastava and Bansal [7] and others (see, for example, [8–12]).

For the polynomials $G(x)$ and $H(x)$ with real coefficients, the $(G, H)$ Lucas polynomials $L_{G, H, k}(x)$ are defined by the following recurrence relation (see [13,14]):
\[ L_{G, H, k}(x) = G(x)L_{G, H, k-1}(x) + H(x)L_{G, H, k-2}(x) \quad k \geq 2. \]
In a variety of fields in the mathematical, statistical, physical, and engineering sciences, the Lucas polynomials are crucial (see, for instance, [15–18]). The generating function of the \((G, H)\) Lucas polynomial \(L_{G,H,k}(x)\) (see [16]) is given by

\[
M_{G(x),H(x)}(\zeta) = \sum_{k=2}^{\infty} L_{G,H,k}(x) \zeta^k = \frac{2 - G(x)\zeta}{1 - G(x)\zeta - H(x)\zeta^2}.
\] (3)

If the holomorphic functions of \(f\) and \(g\) are in \(U\), then \(f\) is subordinate to \(g\), which implies \(f(\zeta) \prec g(\zeta)\). If there are Schwarz functions \(|w(\zeta)| < 1\) present for all \(\zeta \in U\) and \(w(0) = 0\), then we have the following condition (see also [19]):

\[
f \prec g \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).
\]

A function \(f \in A\) is called a Bazilevic function of the order \(\lambda\), where \(\lambda \geq 0\), if (see [20])

\[
Re \left( \frac{f'(\zeta)}{f(\zeta)^{1-\lambda}} \right), \quad \zeta \in U.
\]

A function \(f \in A\) is called a \(\delta\)-pseudo-starlike function of the order \(\mu\), where \(\mu \geq 1\), if (see [21])

\[
Re \left( \frac{f'(\zeta)^\mu}{f(\zeta)} \right), \quad \zeta \in U.
\]

In recent years, the distributions of random variables have generated a great deal of interest. Their probability density functions have played an important role in statistics and probability theory. This brand new type of thought in fuzzy logic offers a fresh framework for addressing problems with ambiguous data (see [22] for neutrosophic numbers and the references therein). The application of neutrosophic crisp set theory to the classical probability distributions, particularly the Poisson, exponential, and Uniform distributions, creates a new path for addressing problems that adhere to the classical distributions while also containing inaccurately characterised data.

We will now study the following issues by assuming that \(\xi N(\zeta)\) represents the neutrosophic Poisson distribution series. The neutrosophic probability distribution is deeply concerned with certaining more broad and obvious values, whereas the classical probability distributions only deal with specific data and set parameter values. In actuality, a classical Poisson distribution of \(x\) with an uncertain parameter value is the neutrosophic Poisson distribution of a discrete variable \(\kappa\). A variable is said to have the neutrosophic Poisson distribution if its probability with the value \(n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}\) is

\[
NP(\kappa = n) = \frac{(\theta_N)^n}{n!} e^{-\theta_N}.
\]

Hence, for the neutrosophic statistical number \(N = d + i\), the distribution parameter \(\theta_N\) is the expected value and the variance, or \(NE(x) = NV(x) = \theta_N\) (see [23,24] and the sources referenced). We create a power series by using the probabilities for the neutrosophic Poisson distribution as its coefficients:

\[
\Psi(\theta_N, \zeta) = \zeta + \sum_{k=2}^{\infty} \frac{(\theta_N)^{k-1}}{(k-1)!} e^{-\theta_N} \zeta^k.
\]

For \(f \in A\), we use the convolution operator \(*\) to introduce the linear operator

\[
\mathcal{A}_{\theta_N} f(\zeta) = \Psi(\theta_N, \zeta) * f(\zeta) = \zeta + \sum_{k=2}^{\infty} \frac{(\theta_N)^{k-1}}{(k-1)!} e^{-\theta_N} a_k \zeta^k.
\] (4)
Definition 1. The fractional integral of the order $\delta$ is defined for a function $f \in A$ by
\[
D^\delta f(\zeta) = \frac{1}{\Gamma(1-\delta)} \frac{d}{d\zeta} \int_a^b \frac{f(\tau)}{(\zeta - \tau)^{1-\delta}} d\tau \quad 0 \leq \delta < 0
\]
where the multiplicity of $(\zeta - \tau)^{1-\delta}$ is removed by requiring $\log(\zeta - \tau)$ to be real when $\zeta - \tau > 0$.

Definition 2. The fractional integral of the order $\delta$ is defined for a function $f \in A$ by
\[
D^\delta f(\zeta) = \frac{1}{\Gamma(1-\delta)} \frac{d}{d\zeta} \int_a^b \frac{f(\tau)}{(\zeta - \tau)^{1-\delta}} d\tau \quad 0 \leq \delta < 0
\]
where the multiplicity of $(\zeta - \tau)^{1-\delta}$ is removed by requiring $\log(\zeta - \tau)$ to be real when $\zeta - \tau > 0$.

Definition 3 ({[25]}). Caputo’s definition of a fractional-order derivative is given by
\[
D^\gamma f(\tau) = \frac{1}{\Gamma(n-\gamma)} \int_0^\tau f^{(n)}(t) (\tau-t)^{n-\gamma-1} dt
\]
where $n-1 \leq \text{Re}(\gamma) \leq n, n \in \mathbb{N}$, and $\gamma$ is allowed to be a real or complex number and is the initial value of the function $f$.

Definition 4 ({[26]}). The modified Caputo’s derivative operator is given by
\[
I^\eta_0 f(\zeta) = \frac{\Gamma(2+\eta-\delta)}{\eta-\delta} \zeta^{\eta-\delta} \int_0^\zeta \frac{\Psi^\eta f(\tau)}{(\zeta - \tau)^{\delta+1-\eta}} d\tau
\]
where $\eta$ is a real number, $-1 < \delta \leq \eta < 2$, and $\Psi^\eta f(\tau) = \Gamma(2-\eta)\tau^\eta D^\eta f(\tau)$. Now, if $\zeta + \sum_{k=2}^\infty a_k \zeta^k$ is an analytic function in $A$, then
\[
I^\eta_0 f(\zeta) = \zeta + \sum_{k=2}^\infty (\frac{\Gamma(k+1)^2}{\Gamma(2k+\eta-\delta+1)(2k+\eta-\delta+1)}) a_k \zeta^k, \quad \zeta \in U.
\]

In this paper, for $f \in A$, we introduce a new linear operator $M^\eta_{\delta,\theta_N} : A \to A$:
\[
M^\eta_{\delta,\theta_N} f(\zeta) = I^\eta_0 \ast A_{\theta_N}.
\]

It is easy to obtain from Equation (9) that
\[
M^\eta_{\delta,\theta_N} f(\zeta) = \zeta + \sum_{k=2}^\infty \frac{(\theta_N)^{k-1}2\Gamma(2+\eta-\delta)\Gamma(2-\eta)}{(k+\eta-\delta+1)(2+\eta-\delta+1)} a_k \zeta^k, \quad \zeta \in U,
\]
where $-1 < \delta \leq \eta < 2$.

Abiodun Tinuoye Olad [27] focused on the use of bounds for the neutrosophic Poisson distribution, and Wanas and Sokol [28] obtained a Poisson distribution with a Ruscheweyh derivative operator. In their results, $\theta_N$ was not precisely defined. Classical probability distributions only deal with specified data, and their parameters are always given with a specified value, while the neutrosophic probability distribution gives a more general clarity of the study issues when $\theta_N$ is an interval. We obtained the second and third inequality as well as the Fekete–Szegö inequality for the function $f \in Y(\alpha, \lambda, \mu, \eta, \delta, \theta_N; e)$. Additionally, examples were given to support our case for the neutrosophic Poisson distribution.

2. Main Results

We begin this section by defining the family $Y(\alpha, \lambda, \mu, \eta, \delta, \theta_N; e)$ as follows:
Definition 5. Assume that $\lambda \geq 0, \mu \geq 1, 0 \leq \alpha \leq 1, \eta - 1 < \delta \leq \eta < 2, \theta_N \in [1, \infty)$, and $e$ is analytic in $U, e(0) = 1$. The function $f \in \Sigma$ is in the family $Y_\Sigma(\alpha, \lambda, \mu, \eta, \delta, \theta_N; e)$ if it fulfills the subordinations

$$
(1 - \alpha) \frac{\xi^{(1-\lambda)} \left( M_\delta^{\eta} \tilde{f}(\zeta) \right)'}{\left( M_\delta^{\eta} \tilde{f}(\zeta) \right)^{(1-\lambda)}} + \alpha \frac{\zeta \left( \left( M_\delta^{\eta} \tilde{f}(\zeta) \right) \right)'^\mu}{M_\delta^{\eta} \tilde{f}(\zeta)} < e(\zeta)
$$

and

$$
(1 - \alpha) \frac{\xi^{(1-\lambda)} \left( M_\delta^{\eta} f^{-1}(\omega) \right)'}{\left( M_\delta^{\eta} f^{-1}(\omega) \right)^{(1-\lambda)}} + \alpha \frac{\zeta \left( \left( M_\delta^{\eta} f^{-1}(\omega) \right) \right)'^\mu}{M_\delta^{\eta} f^{-1}(\omega)} < 1 + h_1(\omega) + h_2(\omega)^2 + \ldots.
$$

where $f^{-1}$ is given by Equation (2).

In particular, if we choose $\alpha = 1$ in Definition 5, then the family $Y_\Sigma(\alpha, \lambda, \mu, \eta, \delta, \theta_N; e)$ reduces to the family $N_\Sigma(\mu, \eta, \delta, \theta_N; e)$ of $\mu$-pseudo-bi-starlike functions which satisfy the following subordinations:

$$
\frac{\zeta \left( \left( M_\delta^{\eta} \tilde{f}(\zeta) \right) \right)'^\mu}{M_\delta^{\eta} \tilde{f}(\zeta)} < 1 + h_1(\zeta) + h_2(\zeta)^2 + \ldots.
$$

and

$$
\omega \left( \left( M_\delta^{\eta} f^{-1}(\omega) \right) \right)'^\mu < 1 + h_1(\omega) + h_2(\omega)^2 + \ldots.
$$

If we choose $\alpha = 0$ in Definition 5, then the family $Y_\Sigma(\alpha, \lambda, \mu, \eta, \delta, \theta_N; e)$ reduces to the family $C_\Sigma(\lambda, \eta, \delta, \theta_N; e)$ of bi-Bazilevic univalent functions which satisfy the following subordinations:

$$
\xi^{(1-\lambda)} \left( M_\delta^{\eta} \tilde{f}(\zeta) \right)'</div>

and

$$
\omega^{(1-\lambda)} \left( M_\delta^{\eta} f^{-1}(\omega) \right)'</div>

If we choose $\alpha = \mu = 1$ in Definition 5, then the family $Y_\Sigma(\alpha, \lambda, \mu, \eta, \delta, \theta_N; e)$ reduces to the family $D_\Sigma(\eta, \delta, \theta_N; e)$ of bi-starlike functions which satisfy the following subordinations:

$$
\frac{\xi \left( \left( M_\delta^{\eta} \tilde{f}(\zeta) \right) \right)'}{M_\delta^{\eta} \tilde{f}(\zeta)} < 1 + h_1(\zeta) + h_2(\zeta)^2 + \ldots.
$$
and
\[ \frac{M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega)}{M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega)}' < 1 + h_1(\omega) + h_2(\omega)^2 + \ldots . \]

In the following sections, we obtain the Fekete–Szegö inequality results for the function 
\( f \in \mathcal{Y}(a, \lambda, \mu, \eta, \delta, \theta_N; e) \):

**Theorem 1.** Assume that \( \lambda \geq 0, \mu \geq 1, 0 \leq a \leq 1, \eta - 1 < \delta \leq \eta < 2, \) and \( \theta_N \in [1, \infty) \). If 
\( f \in \Sigma \) of the form in Equation (1) is in the class \( \mathcal{Y}(a, \lambda, \mu, \eta, \delta, \theta_N; e) \) with \( \epsilon(\xi) = 1 + h_1(\xi)^+ \)
\[ |h_1| = \frac{|h_1|}{4E} \] 
and
\[ |d_3| \leq \min \left\{ \max \left\{ \left| \frac{h_1}{F} \right|, \left| \frac{h_2}{F} \right| - \frac{Jh_1^2}{16E^2F} \right\}, \max \left\{ \left| \frac{h_1}{F} \right|, \left| \frac{h_2}{F} \right| - \frac{(2F + J)h_1^2}{16E^2F} \right\} \right\}, \] 
where
\[ E = (1 + \eta - \delta)(1 - \eta)(1 - a)(\lambda + 1) + 2\delta(\mu - 1)|\theta_N e^{-\theta_N}|, \]
\[ F = (2 + \eta - \delta)(2 - \eta)(18(1 - a)(\lambda + 1) + 54\delta(\mu - 1)|\theta_N|^2) e^{-2\theta_N}, \] 
\[ J = (1 + \eta - \delta)(1 - \eta)^2 \left( 1 - a \right)^2 \left( \lambda^2 + 31\lambda - 32 \right) + 32\delta(\mu^2 - 3\mu + 2) \left( \theta_N^2 \right)|\theta_N| e^{-2\theta_N}. \]

**Proof.** Suppose that \( f \in \mathcal{Y}(a, \lambda, \mu, \eta, \delta, \theta_N; e) \). Then, there exist two holomorphic functions \( \chi, \varphi : U \to U \) given by
\[ \chi(\xi) = b_1 \xi + b_2 \xi^2 + \ldots, \quad \xi \in U \] 
\[ \varphi(\omega) = l_1 \omega + l_2 \omega^2 + \ldots, \quad \omega \in U \]
with \( \chi(0) = \varphi(0) = 0, |\chi(\xi)| < 1, |\varphi(\omega)| < 1, \) and \( \chi, \omega \in U \) such that

\[ (1 - a) \frac{\xi^{(1 - \lambda)} \left( M_{\lambda,\delta,\chi}^\eta f(\xi) \right)'}{M_{\lambda,\delta,\chi}^\eta f(\xi)} + a \frac{\xi \left( M_{\lambda,\delta,\chi}^\eta f(\xi) \right)'}{M_{\lambda,\delta,\chi}^\eta f(\xi)} = 1 + h_1 \chi(\xi) + h_2 \chi^2(\xi) + \ldots \] 
\[ \frac{(1 - a) \omega^{(1 - \lambda)} \left( M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega) \right)'}{M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega)} + a \frac{\xi \left( M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega) \right)'}{M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega)} = 1 + h_1 \varphi(\omega) + h_2 \varphi^2(\omega) + \ldots \]

Combining Equations (14) and (13) as well as Equations (15) and (16) yields

\[ (1 - a) \frac{\xi^{(1 - \lambda)} \left( M_{\lambda,\delta,\chi}^\eta f(\xi) \right)'}{M_{\lambda,\delta,\chi}^\eta f(\xi)} = 1 + h_1 b_1 + \left[ h_1 b_2 + h_2 b_2^2 \right] (\xi)^2 + \ldots \] 
\[ (1 - a) \frac{\omega^{(1 - \lambda)} \left( M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega) \right)'}{M_{\lambda,\delta,\chi}^\eta f^{-1}(\omega)} = 1 + h_1 l_1 + \left[ h_1 l_2 + h_2 l_2^2 \right] (\omega)^2 + \ldots \]

It is quite well known that if \( |\chi(\xi) < 1|, |\varphi(\omega) < 1|, \) and \( \xi, \omega \in U \), then we obtain
\[ |b_n| \leq 1, |l_n| < 1, \quad n \in \mathbb{N}. \]
In light of Equations (18) and (19), after simplifying, we find that

\[
4(1 + \eta - \delta)(1 - \eta)[(1 - \alpha)(\lambda + 1) + 2\delta(\mu - 1)]\theta_N e^{-\delta N}d_2 = h_1 b_1, \\
(2 + \eta - \delta)(2 - \eta)[18(1 - \alpha)(\lambda + 1) + 54\delta(\mu - 1)](\theta_N)^2 e^{-\delta N}d_3 \\
+ (1 + \eta - \delta)^2(1 - \eta)^2\left[(1 - \alpha)(\lambda^2 + 31\lambda - 32) + 32\delta(\mu^2 - 3\mu + 2)\right](\theta_N)^2 e^{-2\delta N}d_2^2
\]

\[
= h_1 b_2 + h_2 b_1^2,
\]

\[
-4(1 + \eta - \delta)(1 - \eta)[(1 - \alpha)(\lambda + 1) + 2\delta(\mu - 1)]\theta_N e^{-\delta N}d_2 = h_1 l_1, \\
(2 + \eta - \delta)(2 - \eta)[18(1 - \alpha)(\lambda + 1) + 54\delta(\mu - 1)](\theta_N)^2 e^{-\delta N}(2d_2^2 - d_3) \\
+ (1 + \eta - \delta)^2(1 - \eta)^2\left[(1 - \alpha)(\lambda^2 + 31\lambda - 32) + 32\delta(\mu^2 - 3\mu + 2)\right](\theta_N)^2 e^{-2\delta N}d_2^2
\]

\[
= h_1 l_2 + h_2 l_1^2.
\]

The inequality in Equation (11) follows from Equation (21). If we apply the notation Equation (13), then Equation (21) becomes

\[
4Ed_2 = h_1 b_1, \quad Fd_3 + Jd_2^2 = h_1 b_2 + h_2 b_1^2
\]

This yields

\[
Fd_3 = h_1 b_2 + \left(\frac{h_2}{16E^2} - \frac{Jh_1^2}{16E^2}\right),
\]

In addition, upon using the known sharp result ([3], p. 10), we obtain

\[
|b_2 - \gamma b_1^2| \leq \max\{1, |\gamma|\},
\]

where for all \(\gamma \in C\), we obtain

\[
|d_3| \leq \max\left\{\left|\frac{h_1}{F}\right|, \left|\frac{h_2}{F} - \frac{Jh_1^2}{16E^2}\right|\right\}.
\]

In the same way, Equations (23) and (24) become

\[
-4Ed_2 = h_1 l_1, \quad F(2d_2^2 - d_3) + Jd_2^2 = h_1 l_2 + h_2 l_1^2.
\]

This yields

\[
-\frac{F}{h_1}d_3 = l_2 + \left(\frac{h_2}{h_1} - \left(\frac{2F + J}{16E^2}\right)\right)l_1^2,
\]

where, upon using the known sharp result ([3], p. 10), we obtain

\[
|l_2 - \gamma l_1^2| \leq \max\{1, |\gamma|\},
\]

such that for all \(\gamma \in C\), we obtain

\[
\left|\frac{F}{h_1}\right| |d_3| \leq \max\left\{1, \left|\frac{h_2}{h_1} - \left(\frac{2F + J}{16E^2}\right)\right|\right\}.
\]

The inequality in Equation (12) follows from Equations (28) and (32). Hence, the proof is complete. \(\square\)

If we take the generating function in Equation (3) of the \((G, H)\) Lucas polynomial \(L_{G,H,k}(x)\) as \(e(z) + 1\), then from Equation (2), we have \(h_1 = G(x)\) and \(h_2 = G^2(x) + 2H(x)\), and Theorem 1 becomes the following corollary:
Corollary 1. If $f \in \Sigma$ of the form in Equation (1) is in the class $\Psi_{\Sigma}(\alpha, \lambda, \mu, \eta, \delta, \theta_N; M_{G(x), H(x)} - 1)$, then
\[
|d_2| \leq \frac{|G(x)|}{4(1+\eta-\delta)(1-\eta)(1-\alpha)(\lambda+1) + 2\delta(\mu-1)\theta_N e^{-\theta_N}}.
\]
\[
|d_3| \leq \min \left\{ \max \left\{ \frac{|G(x)|}{F}, \frac{|G^2(x) + 2H(x)|}{|JG^2(x)|} \right\}, \max \left\{ \frac{|F(x)|}{F}, \frac{|G^2(x) + 2H(x)|}{|JG^2(x)|} \right\} \right\}
\]
(33)

for all $\alpha, \lambda, \mu, \eta, \delta, \theta_N$, and $x$ such that $\lambda \geq 0, \mu \geq 1, 0 \leq \alpha \leq 1, \eta - 1 < \delta \leq \eta < 2$, $\theta_N \in [1, \infty]$, and $x \in R$, where $E, F$, and $J$ are given by Equation (13) and $M_{G(x), H(x)}$ is given by Equation (3).

In the next theorem, we discuss a bound for $|d_3 - \beta d_2^2|$, called Fekete–Szegö’s problem:

Theorem 2. If $f \in \Sigma$ of the form in Equation (1) is in the class $\Psi_{\Sigma}(\alpha, \lambda, \mu, \eta, \delta, \theta_N; e)$, then
\[
|d_3 - \beta d_2^2| \leq \frac{|h_1|}{F} \max \left\{ 1, \frac{|b_2 - (f - \beta)h_1|}{|h_1 - (J - \beta F)h_1|} \right\}
\]
(34)

for all $\alpha, \lambda, \mu, \eta, \delta, \theta_N$, and $x$ such that $\lambda \geq 0, \mu \geq 1, 0 \leq \alpha \leq 1, \eta - 1 < \delta \leq \eta < 2$, $\theta_N \in [1, \infty]$, and $\beta \in C$, where $E, F$, and $J$ are given by Equation (13).

Proof. We apply the notations from the proof of Theorem 1. From Equations (25) and (26), we have
\[
d_3 - \beta d_2^2 = \frac{h_1}{F} \left( b_2 + \left( \frac{h_2}{h_1} - \frac{Jh_1}{16E^2} \right) b_1^2 \right) - \beta \frac{h_2^2b_1^2}{16E^2},
\]

(35)

where upon using the known sharp result ([3], p. 10), we obtain
\[
|b_2 - \gamma b_1^2| \leq \max \{ 1, |\gamma| \},
\]

(36)

where for all $\gamma \in C$, we obtain
\[
|d_3 - \beta d_2^2| \leq \frac{|h_1|}{F} \max \left\{ 1, \frac{|b_2 - (J - \beta F)h_1|}{|h_1 - (J - \beta F)h_1|} \right\}.
\]

(37)

From Equations (29) and (30), we have
\[
d_3 - \beta d_2^2 = \frac{h_1}{F} \left( b_2 + \left( \frac{h_2}{h_1} - \frac{(J + \beta F)h_1}{16E^2} \right) b_1^2 \right),
\]

(38)

where upon using the known sharp result ([3], p. 10), we obtain
\[
|b_2 - \gamma b_1^2| \leq \max \{ 1, |\gamma| \},
\]

(39)

where for all $\gamma \in C$, we obtain
\[
|d_3 - \beta d_2^2| \leq \frac{|h_1|}{F} \max \left\{ 1, \frac{|b_2 - (J + \beta F)h_1|}{|h_1 - (J + \beta F)h_1|} \right\}.
\]

(40)

The inequality in Equation (34) follows from Equations (37) and (40). \qed
Corollary 2. If $f \in \Sigma$ of the form in Equation (1) is in the class $\mathcal{Y}_\Sigma(\alpha, \lambda, \mu, \eta, \delta, \vartheta_N; M_{G(x),H(x)} - 1)$, then
\[
|d_3 - \beta d^2_2| \leq \frac{|G(x)|}{F} \min \left\{ \max \left\{ 1, \left| \frac{G^2(x) + 2H(x)}{G(x)} - \left( \frac{(J - \beta F)G(x)}{16E^2} \right) \right\} \right\},
\]
\[
, \max \left\{ 1, \left| \frac{G^2(x) + 2H(x)}{G(x)} - \left( \frac{(2F + J - \beta F)G(x)}{16E^2} \right) \right\} \right\}
\]
for all $\alpha, \lambda, \mu, \eta, \delta, \vartheta_N$, and $x$ such that $\lambda \geq 0, \mu \geq 1, \eta - 1 \leq \delta \leq \eta < 2$, $\vartheta_N \in [1, \infty]$, and $x \in \mathbb{R}$, where $E, F$, and $J$ are given by Equation (13) and $M_{G(x),H(x)}$ is given by Equation (3).

3. Instance of a Case Study

When a phone employee at a corporation receives calls at a rate of $[1, 3]$ calls per minute, we will determine the likelihood that the employee will not receive any calls within a minute. The given corollary follows from Theorems 1 and 2:

Corollary 3. If $f \in \Sigma$ of the form in Equation (1) is in the class $\mathcal{Y}_\Sigma(0, \lambda, \mu, \eta, \delta, \vartheta_N; e)$, then
\[
|d_2| \leq \frac{|h_1|}{4(1 + \eta - \delta)(1 - \eta)(\lambda + 1) + 2\delta(\mu - 1)](1, 3)e^{-[1, 3]} = \frac{|h_1|}{4E},
\]
and
\[
|d_3| \leq \min \left\{ \max \left\{ \frac{|h_1|}{F}, \left| \frac{h_2}{F} - \frac{h_1^2}{16E^2F} \right| \right\}, \max \left\{ 1, \left| \frac{h_2}{h_1} - \left( \frac{(2F + J)h_1}{16E^2F} \right) \right| \right\} \right\},
\]
\[
|d_3 - \beta d^2_2| \leq \left\{ \frac{|h_1|}{F} \max \left\{ 1, \left| \frac{h_2}{h_1} - \left( \frac{(J - \beta F)h_1}{16E^2} \right) \right| \right\}, \max \left\{ 1, \left| \frac{h_2}{h_1} - \left( \frac{(2F + J - \beta F)h_1}{16E^2} \right) \right| \right\} \right\},
\]
for all $\alpha, \lambda, \mu, \eta, \delta, \vartheta_N$, and $x$ such that $\lambda \geq 0, \mu \geq 1, \alpha = 0, \eta - 1 < \delta \leq \eta < 2$, $\vartheta_N \in [1, 3]$, and $\beta \in C$, where
\[
E = (1 + \eta - \delta)(1 - \eta)(\lambda + 1) + 2\delta(\mu - 1)](1, 3)e^{-[1, 3]},
\]
\[
F = (2 + \eta - \delta)(2 - \eta)(\lambda + 1) + 54\delta(\mu - 1)](1, 3)]^2 e^{-[1, 3]},
\]
\[
J = (1 + \eta - \delta)^2(1 - \eta)^2[\left( \lambda^2 + 31\lambda - 32 \right) + 32\delta(\mu^2 - 3\mu + 2)](1, 3)^2 e^{-2[1, 3]}.
\]

4. The Importance of the Fekete–Szegö Inequality Results

Every aspect of human endeavours depends on probability and statistics, which are particularly important quantitative tools in the fields of economics and finance. Any set of actions will produce varied results, depending on some circumstances. Therefore, there are many fascinating decisions that may be made with different selections of values for the parameter $\vartheta_N$. Additionally, from an economic perspective, our findings will be beneficial in decision-making processes.

5. Conclusions

This paper deals with the applications of the neutrosophic Poisson distribution for bi-univalent functions involving the modified Caputo’s derivative operator. In addition, we found the Fekete–Szegö inequality results to be in the subclass of holomorphic functions. Furthermore, the Hankel determinant may be investigated for this distribution in the future. We anticipate that Caputo’s derivative operator will be important in several fields related to mathematics, science, and technology.
Author Contributions: Conceptualization, S.S. and K.T.; methodology, S.S. and K.T.; validation, S.S. and K.T.; formal analysis, S.S. and K.T.; investigation, S.S. and K.T.; resources, S.S. and K.T.; data curation, S.S. and K.T.; writing—original draft preparation, S.S.; writing—review and editing, S.S. and K.T.; visualization, S.S. and K.T.; supervision, K.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References


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