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Article

A New Contraction with an Application for Integral Equations

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Abstract: In this study, we introduce a new definition called δ-contraction. However, we prove some theorems for mappings satisfying the δ-contraction and touch upon fixed points. In the obtained theorems, we also show the existence and uniqueness of fixed points. In order to prove the validity of the results of our main theorems, we give a few examples as well as an application that reveals the solution of some integral equations.

Keywords: δ-contraction; fractional integral equation; fixed point

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1. Introduction and Preliminaries

Banach's work, which is significant in many fields of science, mentions the existence of a fixed point of a mapping on a defined metric space [1]. This important principle has been used for the solution to some equations encountered in many different fields of mathematics. In addition, fixed point theorem has been studied by some authors, and important results have been obtained in various spaces [2,3].

On the other hand, fuzzy subjects gained importance when Zadeh first defined the fuzzy set [4]. The metric definition in a fuzzy set was then given [5]. Then, the fuzzy metric space was defined [6], and later, a different version of the fuzzy metric space was established by considering the condition of G-completeness [7,8]. On the other hand, the subject of fixed point theory, which has been dealt with by most scientists, has also had an important place in fuzzy metric spaces. This subject has been studied on two different versions of fuzzy metric spaces: M-complete and G-complete [9]. In this process, some of the most important theorems and results were obtained [10]. Some authors have made important contributions to the subject of fuzzy metric spaces [3,11–16]. After these important studies, fixed point theory has since become popular in some branches of fuzzy mathematics. Many authors have presented many important and diverse studies on fixed point theory [6,17].

In this work, we introduce the definition of a δ-contraction bounded $[0,1)$ semi-open interval to give new theorems related to fixed points and the application of the results of these theorems. First, we set a nondecreasing condition on this δ-contraction. Second, we want the image under the δ-contraction to approach the value 1 as the limit of a sequence in which the elements are defined in the semi-open interval $[0,1)$ approaches the value 1. We prove the existence and uniqueness of the fixed point for a δ-contraction with the G-completeness condition of non-Archimedean fuzzy metric spaces. We present an example for the results obtained. Additionally, we set up a few applications to show that solutions to integral equations can be found using our main results.

Throughout this study, short versions of some terms will be given. NAFMS will be written instead of the expression of non-Archimedean fuzzy metric spaces, and FMS will be written instead of the expression of fuzzy metric spaces.

Definition 1 ([12])**.** *Let $\nabla : [0,1] \times [0,1] \to [0,1]$ be a commutative, associative, and continuous binary operation. Then, ∇ is a continuous t-norm, first, if the $\nabla(\varsigma, 1) = \varsigma$ condition is provided for every $\varsigma \in [0,1]$ and, second, the $\nabla(\varsigma, \varrho) \leq \nabla(\sigma, \rho)$ condition is provided whenever $\varsigma \leq \sigma$, $\varrho \leq \rho$ and $\varsigma, \varrho, \sigma, \rho \in [0,1]$.*

Definition 2 ([7]). *Let Θ be a nonempty set, ∇ be a continuous t-norm, and F_{ms} be a fuzzy set on $\Theta^2 \times (0,\infty)$ for the triple (Θ, F_{ms}, ∇). For all $\varsigma, \varrho, \sigma \in \Theta$, $\acute{r}, \check{t} > 0$, if the following conditions are provided, then (Θ, F_{ms}, ∇) is called FMS:*

1. $F_{ms}(\varsigma, \varrho, \acute{r}) > 0$;
2. $F_{ms}(\varsigma, \varrho, \acute{r}) = 1$ *iff* $\varsigma = \varrho$;
3. $F_{ms}(\varsigma, \varrho, \acute{r}) = F_{ms}(\varrho, \varsigma, \acute{r})$;
4. $F_{ms}(\varsigma, \varrho, \acute{r}) \nabla F_{ms}(\varrho, \sigma, \check{t}) \leq F_{ms}(\varsigma, \sigma, \acute{r} + \check{t})$;
5. $F_{ms}(\varsigma, \varrho, \cdot) : (0,\infty) \to (0,1]$ *is continuous.*
   If the condition (4) replaced by

4^o. $F_{ms}(\varsigma, \sigma, max\{\acute{r}, \check{t}\}) \geq F_{ms}(\varsigma, \varrho, \acute{r}) \bigtriangledown F_{ms}(\varrho, \sigma, \acute{r})$ *for all* $\varsigma, \varrho, \sigma \in \Theta$ *and* $\acute{r}, \check{t} > 0$,
   or equivalently,
   $F_{ms}(\varsigma, \sigma, \acute{r}) \geq F_{ms}(\varsigma, \varrho, \acute{r}) \bigtriangledown F_{ms}(\varrho, \sigma, \acute{r})$,
   then (Θ, F_{ms}, ∇) *is called NAFMS [13].*

Definition 3. *Let (Θ, F_{ms}, ∇) be an NAFMS. Then, it will be necessary to mention some definitions of any sequence $\{\varsigma_\flat\}$ in Θ below:*

(1) Convergence: If the limit of $F_{ms}(\varsigma_\flat, \varsigma, \acute{r})$ becomes 1 as n approaches ∞ for all $\acute{r} > 0$, i.e., for each $\lambda \in (0,1)$ and $\acute{r} > 0$, there exists a $\flat_0 \in \mathbb{N}$ such that $F_{ms}(\varsigma_\flat, \varsigma, \check{t}) > 1 - \lambda$ for all $\flat \geq \flat_0$. Then, it is said to be $\{\varsigma_\flat\}$ convergent toward ς in Θ and is denoted by $\varsigma_\flat \to \varsigma$ [6,13].

(2) Being G-Cauchy: If the limit of $F_{ms}(\varsigma_\flat, \varsigma_{\flat+p}, \acute{r})$ becomes 1 as $\flat$ approaches ∞ for any $p > 0$ and $\acute{r} > 0$, then $\{\varsigma_\flat\}$ is said to be G-Cauchy [9,13,15].

(3) Completeness: If every G-Cauchy sequence converges, then (Θ, F_{ms}, ∇) is said to be complete [10,15].

Definition 4 ([18]). *Let £ be the family of all ξ functions such that $\xi : [0,1) \to \mathbb{R}$ is a continuous and strictly increasing mapping and for each sequence $\{\varsigma_\flat\} \subset [0,1)$ of positive numbers $\lim_{\flat\to\infty} \varsigma_\flat = 1$ if and only if $\lim_{\flat\to\infty} \xi(\varsigma_\flat) = +\infty$.*

The following definitions are defined for all $\varsigma, \varrho \in \Theta$, $\xi \in$ £ and for a real number $\theta \in (0,1)$.

(1) The mapping $\Re : \Theta \to \Theta$ is called a ξ-contraction such that

$$F_{ms}(\Re\varsigma, \Re\varrho, \acute{r}) < 1 \Rightarrow \xi(F_{ms}(\Re\varsigma, \varrho, \acute{r})) \geq \xi(F_{ms}(\varsigma, \varrho, \acute{r})) + \theta$$

is satisfied.

(2) The mapping $\Re : \Theta \to \Theta$ is called a ξ-weak contraction such that

$$\begin{aligned} F_{ms}(\Re\varsigma, \Re\varrho, \acute{r}) &< 1 \\ &\Rightarrow \xi(F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})) \\ &\geq \xi(\min\{F_{ms}(\varsigma, \varrho, \acute{r}), F_{ms}(\varsigma, \Re\varsigma, \acute{r}), F_{ms}(\varrho, \Re\varrho, \acute{r})\}) + \theta \end{aligned}$$

is satisfied.

In this work, we use the special version of the above definition below:

Let $\delta : [0,1) \to [0,1]$ be a nondecreasing mapping such that for every sequence $\{\varsigma_\flat\} \subset [0,1)$,

$$\lim_{\flat\to\infty} \delta(\varsigma_\flat) = 1 \text{ iff} \lim_{\flat\to\infty} \varsigma_\flat = 1 \tag{1}$$

is provided.

Throughout this work, we consider the set of δ as $\Im$.

2. Main Results

Definition 5. *The mapping $\Re : \Theta \to \Theta$ is called a δ-contraction such that there exists a $\kappa \in (0,1)$, when*

$$\delta(F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})) \geq [\delta(\min\{F_{ms}(\varsigma, \varrho, \acute{r}), F_{ms}(\varsigma, \Re\varsigma, \acute{r}), F_{ms}(\varrho, \Re\varrho, \acute{r})\})]^{\kappa} \tag{2}$$

is satisfied for all $\varsigma, \varrho \in \Theta$, $\delta \in \Im$.

Theorem 1. *Let the triple (Θ, F_{ms}, ∇) be a complete NAFMS and $\Re : \Theta \to \Theta$ be a δ-contraction. If $\Re$ or δ is continuous, then $\Re$ has a unique fixed point.*

Proof. Let $\varsigma_0 \in \Theta$ and assign the sequence $\{\varsigma_\flat\}$ by $\varsigma_{\flat+1} = \Re(\varsigma_\flat)$ for all $\flat \in \mathbb{N}$. If $\varsigma_{\flat+1} = \varsigma_\flat$, then $\varsigma_{\flat+1}$ is a fixed point of $\Re$. Let $\varsigma_{\flat+1} \neq \varsigma_\flat$ for all natural numbers $\flat$ with the point 0. Therefore, from (2), we have

$$\begin{aligned}
\delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})) &= \delta(F_{ms}(\Re\varsigma_{\flat-1}, \Re\varsigma_\flat, \acute{r})) \\
&\geq [\delta(\min\{F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}), F_{ms}(\varsigma_{\flat-1}, \Re\varsigma_{\flat-1}, \acute{r}), F_{ms}(\varsigma_\flat, \Re\varsigma_\flat, \acute{r})\})]^{\kappa} \\
&= [\delta(\min\{F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}), F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}), F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})\})]^{\kappa} \\
&= [\delta(\min\{F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}), F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})\})]^{\kappa}.
\end{aligned} \tag{3}$$

Assume that $\min\{F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}), F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})\} = F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})$. Then, from the inequality (3), we have

$$\delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})) \geq [\delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r}))]^{\kappa} > \delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r}))$$

which is a contradiction. Therefore, we have $\min\{F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}), F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})\} = F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r})$. Then, from the inequality (3), we have

$$\delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})) \geq [\delta(F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}))]^{\kappa}.$$

By repeating the process, then, we obtain

$$\delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})) \geq [\delta(F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}))]^{\kappa} \geq ... \geq [\delta(F_{ms}(\varsigma_0, \varsigma_1, \acute{r}))]^{\kappa}. \tag{4}$$

As the limit $\flat$ goes to ∞ in (4), we have

$$\lim_{\flat\to\infty} \delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})) = 1.$$

Then, from the property of δ, we have

$$\lim_{\flat\to\infty} F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r}) = 1. \tag{5}$$

The sequence $\{\varsigma_\flat\}$ being G-Cauchy: In order to show that, if the inequality (4) is used, then, we have

$$\begin{aligned}
\delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+p}, \acute{r})) &\geq [\delta(F_{ms}(\varsigma_{\flat-1}, \varsigma_{\flat+p-1}, \acute{r}))]^{\kappa} \\
&\geq [\delta(F_{ms}(\varsigma_{\flat-2}, \varsigma_{\flat+p-2}, \acute{r}))]^{\kappa} \\
&\quad ... \\
&\geq [\delta(F_{ms}(\varsigma_0, \varsigma_p, \acute{r}))]^{\kappa}.
\end{aligned} \tag{6}$$

Since for some fixed p, $\delta(F_{ms}(\flat_0, \flat_p, s))$ is fixed and as the limit as $\flat$ goes to ∞ in the inequality (6), we have

$$\lim_{n\to\infty} \delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+p}, \acute{r})) = 1.$$

Then, from the property of δ, we have

$$\lim_{n\to\infty} F_{ms}(\varsigma_\flat, \varsigma_{\flat+p}, \acute{r}) = 1.$$

Therefore, $\{\varsigma_\flat\}$ is a G-Cauchy sequence. From the completeness of Θ , there exists $\varsigma \in \Theta$ such that $\lim\limits_{\flat\to\infty} \varsigma_\flat = \varsigma$. If G is continuous, from (5), we have

$$F_{ms}(\varsigma, \Re\varsigma, \acute{r}) = \lim_{\flat\to\infty} F_{ms}(\varsigma_\flat, \Re\varsigma_\flat, \acute{r}) = \lim_{\flat\to\infty} F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r}) = 1.$$

This proves that ς is a fixed point of $\Re$. Let δ be continuous; then, we claim that $\Re\varsigma = \varsigma$. On the contrary, let $\Re\varsigma \neq \varsigma$. In that case, there is an $\flat_0 \in \mathbb{N}$ such that $\{\varsigma_{\flat_k}\}$ is a subsequence of $\{\varsigma_\flat\}$ for which $F_{ms}(\varsigma_{\flat_k}, \Re\varsigma, \acute{r}) < 1$ for all $\flat_k \geq \flat_0$. If the inequality (2) is used, then, we have

$$\begin{aligned} & \delta(F_{ms}(\varsigma_{\flat_{k+1}}, \Re\varsigma, \acute{r})) \\ = \; & \delta(F_{ms}(\Re\varsigma_{\flat_k}, \Re\varsigma, \acute{r})) \\ \geq \; & [\delta(\min\{F_{ms}(\varsigma_{\flat_k}, \varsigma, \acute{r}), F_{ms}(\varsigma_{\flat_k}, \Re\varsigma_{\flat_k}, \acute{r}), F_{ms}(\varsigma, \Re\varsigma, \acute{r})\})]^{\kappa} \\ = \; & [\delta(\min\{F_{ms}(\varsigma_{\flat_k}, \varsigma, \acute{r}), F_{ms}(\varsigma_{\flat_k}, \varsigma_{\flat_{k+1}}, \acute{r}), F_{ms}(\varsigma, \Re\varsigma, \acute{r})\})]^{\kappa}. \end{aligned} \tag{7}$$

As $\flat$ approaches ∞ in (7), we obtain

$$\delta(F_{ms}(\varsigma, \Re\varsigma, \acute{r})) \geq [\delta(F_{ms}(\varsigma, \Re\varsigma, \acute{r}))]^{\kappa} > \delta(F_{ms}(\varsigma, \Re\varsigma, \acute{r}))$$

a contradiction. Hence, ς is the fixed point of $\Re$.

Now, to show the uniqueness of the fixed point $\Re$, we assume that $\varsigma, \varrho \in \Theta$ are two fixed points of $\Re$. If $\varsigma \neq \varrho$, then there exists $\acute{r} > 0$ such that $0 < F_{ms}(\varsigma, \varrho, \acute{r}) < 1$, and hence,

$$\begin{aligned} \delta(F_{ms}(\varsigma, \varrho, \acute{r})) &= \delta((\Re\varsigma, \Re\varrho, \acute{r})) \\ &\geq [\delta(\min\{F_{ms}(\varsigma, \varrho, \acute{r}), F_{ms}(\varsigma, \Re\varsigma, \acute{r}), F_{ms}(\varrho, \Re\varrho, \acute{r})\})]^{\kappa} \\ &= [\delta(F_{ms}(\varsigma, \varrho, \acute{r}))]^{\kappa} \\ &> \delta(F_{ms}(\varsigma, \varrho, \acute{r})) \end{aligned}$$

a contradiction. Hence, the fixed point of $\Re$ is unique. □

Remark 1. *Let (Θ, F_{ms}, ∇) be a complete NAFMS and $\Re : \Theta \to \Theta$ satisfying the following condition:*

$$F_{ms}(\Re\varsigma, \Re\varrho, \acute{r}) < 1 \Rightarrow \delta(F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})) \geq [\delta(F_{ms}(\varsigma, \varrho, \acute{r}))]^{\kappa} \tag{8}$$

for all $\varsigma, b \in \Theta$, $\delta \in \Im$ and $\kappa \in (0,1)$. Then, $\Re$ is a continuous mapping.

Proof. From continuity of δ and (8), it is seen that $\Re$ is a contractive mapping, i.e.,

$$F_{ms}(\Re\varsigma, \Re\varrho, \acute{r}) > F_{ms}(\varsigma, \varrho, \acute{r})$$

for all $\varsigma, \varrho \in \Theta$, $\Re\varsigma \neq \Re\varrho$. Thus, $\Re$ is a continuous mapping. □

Corollary 1. *Let (Θ, F_{ms}, ∇) be a G-complete NAFMS and $\Re : \Theta \to \Theta$ be a mapping. Assume that there exist $\delta \in \Im$ and $\kappa \in (0,1)$ such that*

$$F_{ms}(\Re\varsigma, \Re\varrho, \acute{r}) < 1 \Rightarrow \delta(F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})) \geq [\delta(F_{ms}(\varsigma, \varrho, \acute{r}))]^{\kappa} \tag{9}$$

for all $\varsigma, \varrho \in \Theta$. Then, $\Re$ has a unique fixed point.

Proof. Let the sequence $\{\varsigma_\flat\}$ be defined as in Theorem 1 and $\varsigma_{\flat+1} \neq \varsigma_\flat$ for all $\flat \in \mathbb{N} \cup \{0\}$. Therefore, by the inequality (9), we have

$$\begin{aligned} \delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})) &= \delta(F_{ms}(\Re\varsigma_{\flat-1}, \Re\varsigma_\flat, \acute{r})) \\ &\geq [\delta(F_{ms}(\varsigma_{\flat-1}, \varsigma_\flat, \acute{r}))]^\kappa \\ &\geq [\delta(F_{ms}(\varsigma_{\flat-2}, \varsigma_{\flat-1}, \acute{r}))]^{\kappa^2} \\ &\ldots \\ &= [\delta(F_{ms}(\varsigma_0, \varsigma_1, \acute{r}))]^{\kappa^\flat}. \end{aligned} \tag{10}$$

As the limit goes to ∞ in (10), we have

$$\lim_{\flat\to\infty} \delta(F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r})) = 1.$$

Then, from the property of δ, we have

$$\lim_{\flat\to\infty} F_{ms}(\varsigma_\flat, \varsigma_{\flat+1}, \acute{r}) = 1.$$

Similar to Theorem 1, it can be shown that the sequence $\{\varsigma_\flat\}$ is a G-Cauchy sequence. Since Θ is complete, there exists $\varsigma \in \Theta$ such that $\lim_{n\to\infty} \varsigma_\flat = \varsigma$. On the other hand, note that $\Re$ is continuous. Then, we obtain $\Re\varsigma = \varsigma$.
Now, to show the uniqueness of the fixed point $\Re$, we assume that $\varsigma, \varrho \in \Theta$ are two fixed points of $\Re$. If $\varsigma \neq \varrho$, then there exists $\acute{r} > 0$ such that $0 < F_{ms}(\varsigma, \varrho, \acute{r}) < 1$, and hence,

$$\delta(F_{ms}(\varsigma, \varrho, \acute{r})) = \delta(F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})) \geq [\delta(F_{ms}(\varsigma, \varrho, \acute{r}))]^\kappa > \delta(F_{ms}(\varsigma, \varrho, \acute{r}))$$

which is a contradiction. Hence, the fixed point of $\Re$ is unique. □

Example 1. *Let $\Theta = [1,4]$ and $F_{ms}(\varsigma, \varrho, \acute{r}) = \frac{\min\{\varsigma,\varrho\}}{\max\{\varsigma,\varrho\}}$ for all $\varsigma, \varrho \in \Theta$ and $\acute{r} > 0$ with $t_1 \nabla t_2 = t_1 t_2$. It is clear that (Θ, F_{ms}, ∇) is complete NAFMS also from the work of Romaguera et al. [19]. Let $\delta : [0,1) \to [0,1]$ be such that $\delta\varpi = e^{-\sqrt{1-\varpi}}$ for all $\varpi \in [0,1)$. It is clear that all the properties of the δ are satisfied. Define $\Re : \Theta \to \Theta$ by $\Re\upsilon = \sqrt{\upsilon}$ for all $\upsilon \in \Theta$. We want to show that $\Re$ satisfies the inequality (9). Let $\varsigma < \varrho$ for all $\varsigma, \varrho \in \Theta$. Then,*

$$\begin{aligned} F_{ms}(\Re\varsigma, \Re\varrho, \acute{r}) &< 1 \\ &\Rightarrow \delta(F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})) \\ &= e^{-\sqrt{1-F_{ms}(\Re\varsigma,\Re\varrho,\acute{r})}} \\ &\geq e^{-\kappa\sqrt{1-F_{ms}(\varsigma,\varrho,\acute{r})}} \\ &= [\delta(F_{ms}(\varsigma, \varrho, \acute{r}))]^\kappa \end{aligned} \tag{11}$$

for some $\kappa \in (0,1)$. The inequality (11) is equivalent to

$$\sqrt{1 - F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})} \leq \kappa\sqrt{1 - F_{ms}(\varsigma, \varrho, \acute{r})}.$$

Then, we have

$$\frac{\sqrt{1 - F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})}}{\sqrt{1 - F_{ms}(\varsigma, \varrho, \acute{r})}} \leq \kappa \Rightarrow \frac{1 - F_{ms}(\Re\varsigma, \Re\varrho, \acute{r})}{1 - F_{ms}(\varsigma, \varrho, \acute{r})} \leq \kappa^2 \tag{12}$$

for some $\kappa \in (0,1)$. Thus, we have

$$\frac{1 - \frac{\sqrt{\varsigma}}{\sqrt{\varrho}}}{1 - \frac{\varsigma}{\varrho}} = \frac{\frac{\sqrt{\varrho}-\sqrt{\varsigma}}{\sqrt{\varrho}}}{\frac{\varrho-\varsigma}{\varrho}} = \frac{\sqrt{\varrho}}{\sqrt{\varrho} + \sqrt{\varsigma}} \leq \frac{2}{3}.$$

Hence, the inequaity (12) is satisfied with $\kappa = \sqrt{\frac{2}{3}}$. *Therefore, all the conditions of Corollary 1 are satisfied and* $v = 1$ *is the unique fixed point of* $\Re$.

3. Application to Integral Equations

Let $Y = C([\varsigma, \varrho], \mathbb{R})$ be the set of real continuous functions for $\varsigma \geq 0$. Define $F_{ms} : Y \times Y \times (0, \infty) \to [0, 1]$ as

$$F_{ms}(\eta, \mu, \acute{r}) = e^{-\frac{d(\eta,\mu)}{\acute{r}}}$$

for all $\eta, \mu \in Y$ and $\acute{r} > 0$, where $d : Y \times Y \to [0, \infty)$ is defined by

$$d(\eta, \mu) = \sup_{s \in [\varsigma, \varrho]} |\eta(s) - \mu(s)|$$

with the continuous t-norm ∇ such that $t_1 \nabla t_2 = \min\{t_1, t_2\}$. Since (Y, d) is a complete metric space, then (Y, F_{ms}, ∇) is complete NAFMS. Consider the Fredholm-type integral equation as follows:

$$\eta(r) = u(r) + \int_{\varsigma}^{\varrho} G(r, s)\phi(s, \eta(s))ds \tag{13}$$

and consider the mapping $\Re : Y \to Y$ by

$$\Re\eta(r) = u(r) + \int_{\varsigma}^{\varrho} G(r, s)\phi(s, \eta(s))ds$$

for all $r, s \in [\varsigma, \varrho]$ where

(Â-1) $\phi : [\varsigma, \varrho] \times \mathbb{R} \to \mathbb{R}$ is continuous;
(Â-2) $u : [\varsigma, \varrho] \to \mathbb{R}$ is continuous;
(Â-3) $G : [\varsigma, \varrho] \times [\varsigma, \varrho] \to \mathbb{R}$ is continuous;
(Â-4) If $F_{ms}(\Re\eta, \Re\mu, \acute{r}) < 1$, then

$$|\phi(s, \eta(s)) - \phi(s, \mu(s))| \leq |\eta(s) - \mu(s)|$$

for all $\eta, \mu \in Y$ and $s \in [\varsigma, \varrho]$;
(Â-5) The following inequality holds:

$$\int_{\varsigma}^{\varrho} G(r, s)ds \leq \frac{1}{2}$$

for all $r, s \in [\varsigma, \varrho]$.

Theorem 2. *Under the assumptions (Â-1)–(Â-5), the Fredholm-type integral Equation (14) has a solution in* Y.

Proof. Here, we show that $\Re$ satisfies all the conditions of Corollary 1. For any $\eta, \mu \in Y$, we have

$$\begin{aligned} |\Re\eta(r) - \Re\mu(r)| &= \left| \int_{\varsigma}^{\varrho} G(r, s)[\phi(s, \eta(s)) - \phi(s, \mu(s))]ds \right| \\ &\leq \int_{\varsigma}^{\varrho} G(r, s)|\phi(s, \eta(s)) - \phi(s, \mu(s))|\, ds \end{aligned}$$

$$\begin{aligned} &\leq \int_{\varsigma}^{\varrho} G(r,s)|\eta(s)-\mu(s)|\,ds \\ &\leq d(\eta,\mu)\int_{\varsigma}^{\varrho} G(r,s)ds \\ &\leq \frac{1}{2}d(\eta,\mu). \end{aligned}$$

Therefore, we have

$$d(\Re\eta,\Re\mu) \leq \frac{1}{2}d(\eta,\mu)$$

and so we obtain

$$e^{-\frac{d(\Re\eta,\Re\mu)}{\acute{r}}} \geq e^{-\frac{1}{2}\frac{d(\eta,\mu)}{\acute{r}}}.$$

Define $\delta : [0,1) \to [0,1]$ by $\delta(\varpi) = \varpi$ for all $\varpi \in [0,1)$. Using (9) and the property of δ, we have

$$\delta(F_{ms}(\Re\eta,\Re\mu,\acute{r})) \geq [\delta(F_{ms}(\eta,\mu,\acute{r}))]^{\frac{1}{2}}.$$

If we choose $\kappa = \frac{1}{2} \in (0,1)$ such that

$$\delta(F_{ms}(\Re\eta,\Re\mu,\acute{r})) \geq [\delta(F_{ms}(\eta,\mu,\acute{r}))]^{\kappa}.$$

Therefore, $\Re$ has a unique fixed point, that is, the integral Equation (11) has a unique solution in $\Upsilon = C([\varsigma,\varrho],\mathbb{R})$. □

Fixed point theory has many applications in some branches of mathematics. In particular, fixed point theory solutions are used to find the solution of some kinds of equations. For example, Abdeljawad et al. [20] recently presented an application for fractional differential equations. In the following section, we show the result of our main theorem and the existence of solutions of fractional differential equations.

4. Application to Fractional Differential Equations

In this section, we show that a nonlinear fractional differential equations has a solution in the sense of the Caputo derivative. Recall that the Caputo fractional derivative of $\gamma(s)$ order $\vartheta > 0$ is denoted by ${}^{C}D^{\vartheta}\gamma(s)$, and it is defined as follows:

$${}^{C}D^{\vartheta}\gamma(s) = \frac{1}{\Gamma(\lambda-\vartheta)}\int_{0}^{s}(s-\rho)^{\lambda-\vartheta-1}\gamma^{\lambda}(\rho)d\rho$$

with $\lambda = [|\vartheta|]+1 \in \mathbb{N}$, where $\vartheta \in [\lambda-1,\lambda)$ and $[|\vartheta|]$ denotes the greatest integer of ϑ and $\gamma : [0,\infty) \to \mathbb{R}$ is continuous. Let $X = C([0,1]\mathbb{R})$ be the set of all continuous functions from $[0,1]$ into $\mathbb{R}$. Define $F_{ms} : X \times X \times (0,\infty) \to [0,1]$ as

$$F_{ms}(\eta,\mu,\acute{r}) = e^{-\frac{d(\eta,\mu)}{\acute{r}}}$$

for all $\eta,\mu \in X$ and $\acute{r} > 0$, where $d : X \times X \to [0,\infty)$ is defined by

$$d(\eta,\mu) = \sup_{s\in[0,1]} |\eta(s)-\mu(s)|$$

with the continuous t-norm ∇ such that $t_1 \nabla t_2 = \min\{t_1, t_2\}$. Since (X,d) is complete metric space, then (X, F_{ms}, ∇) is complete NAFMS. Consider the fractional integral equation as follows:

$$\eta(s) = \frac{1}{\Gamma(\vartheta)} \int_0^s (s-\rho)^{\vartheta-1} \phi(\rho, \eta(\rho)) d\rho - \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^1 (1-\rho)^{\vartheta-1} \phi(\rho, \eta(\rho)) d\rho$$

$$+ \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^t \left(\int_0^\rho (\rho - v)^{\vartheta-1} \phi(v, \eta(v)) dv \right) d\rho \tag{14}$$

and consider the mapping $T : X \to X$ by

$$T\eta(s) = \frac{1}{\Gamma(\vartheta)} \int_0^s (s-\rho)^{\vartheta-1} \phi(\rho, \eta(\rho)) d\rho - \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^1 (1-\rho)^{\vartheta-1} \phi(\rho, \eta(\rho)) d\rho + \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^t \left(\int_0^\rho (\rho - v)^{\vartheta-1} \phi(v, \eta(v)) dv \right) d\rho$$

where

($\hat{\text{H}}$-1) $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and ${}^C D^\vartheta(\eta(s)) = \phi(s, \eta(s))$ with the boundary conditions

$$\eta(0) = 0, \eta(1) = \int_0^t \eta(\rho) d\rho$$

for all $\eta \in X$, $s, t \in (0,1)$, $\vartheta \in (1,2]$.

($\hat{\text{H}}$-2) If $F_{ms}(T\eta, T\mu, \acute{r}) < 1$, then

$$|\phi(b, \eta) - \phi(b, \mu)| \leq \Gamma(\vartheta + 1)\kappa|\eta - \mu|$$

for all $\eta, \mu \in X$, $\kappa \in (0,1)$ and $b \in [0,1]$.

Theorem 3. *Under the assumptions ($\hat{\text{H}}$-1) and ($\hat{\text{H}}$-2), the nonlinear fractional differential equation has a solution in X.*

Proof. Here, we show that T satisfies all the conditions of Corollary 1. For any $\eta, \mu \in X$, we have

$$\begin{aligned}
& |T\eta(s)) - T\mu(s)| \\
= & \Bigg| \frac{1}{\Gamma(\vartheta)} \int_0^s (s-\rho)^{\vartheta-1} \phi(\rho, \eta(\rho)) d\rho - \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^1 (1-\rho)^{\vartheta-1} \phi(\rho, \eta(\rho)) d\rho \\
& + \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^t \left(\int_0^\rho (\rho - v)^{\vartheta-1} \phi(v, \eta(v)) dv \right) d\rho \\
& - \frac{1}{\Gamma(\vartheta)} \int_0^s (s-\rho)^{\vartheta-1} \phi(\rho, \mu(\rho)) d\rho + \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^1 (1-\rho)^{\vartheta-1} \phi(\rho, \mu(\rho)) d\rho \\
& - \frac{2s}{(2-t^2)\Gamma(\vartheta)} \int_0^t \left(\int_0^\rho (\rho - v)^{\vartheta-1} \phi(v, \mu(v)) dv \right) d\rho \Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\vartheta)}\int_0^s (s-\rho)^{\vartheta-1}|\phi(\rho,\eta(\rho))-\phi(\rho,\mu(\rho))|d\rho \\
&\quad+\frac{2s}{(2-t^2)\Gamma(\vartheta)}\int_0^1 (1-\rho)^{\vartheta-1}|\phi(\rho,\eta(\rho))-\phi(\rho,\mu(\rho))|d\rho \\
&\quad+\frac{2s}{(2-t^2)\Gamma(\vartheta)}\int_0^t \left|\int_0^\rho (\rho-v)^{\vartheta-1}[\phi(v,\eta(v))-\phi(v,\mu(v))]dv\right|d\rho \\
&\leq \frac{1}{\Gamma(\vartheta)}\int_0^s (s-\rho)^{\vartheta-1}\Gamma(\vartheta+1)\kappa|\eta(\rho)-\mu(\rho)|d\rho \\
&\quad+\frac{2s}{(2-t^2)\Gamma(\vartheta)}\int_0^1 (1-\rho)^{\vartheta-1}\Gamma(\vartheta+1)\kappa|\eta(\rho)-\mu(\rho)|d\rho \\
&\quad+\frac{2s}{(2-t^2)\Gamma(\vartheta)}\int_0^t\int_0^\rho (\rho-v)^{\vartheta-1}\Gamma(\vartheta+1)\kappa|\eta(\rho)-\mu(\rho)|dvd\rho \\
&\leq \Gamma(\vartheta+1)\kappa|\eta(\rho)-\mu(\rho)|\frac{1}{\Gamma(\vartheta)}\left(\int_0^s (s-\rho)^{\vartheta-1}d\rho+\frac{2s}{(2-t^2)}\int_0^1 (1-\rho)^{\vartheta-1}d\rho\right. \\
&\quad\left.+\frac{2s}{(2-t^2)}\int_0^t\int_0^\rho (\rho-v)^{\vartheta-1}dvd\rho\right) \\
&\leq \Gamma(\vartheta+1)\kappa d(\eta,\mu)\frac{1}{\Gamma(\vartheta)}\left(\frac{s^\vartheta}{\vartheta}+\frac{2s}{(2-t^2)}\frac{1}{\vartheta}+\frac{2s}{(2-t^2)}\frac{s^{\vartheta+1}}{\vartheta(\vartheta+1)}\right) \\
&\leq \Gamma(\vartheta+1)\kappa d(\eta,\mu)\frac{1}{\Gamma(\vartheta+1)}\sup_{s\in[0,1]}\left(s^\vartheta+\frac{2s}{(2-t^2)}+\frac{2s}{(2-t^2)}\frac{s^{\vartheta+1}}{(\vartheta+1)}\right) \\
&\leq \kappa d(\eta,\mu).
\end{aligned}
$$

Thus, we obtain

$$|T\eta(s))-T\mu(s)|\leq\kappa d(\eta,\mu).$$

Therefore,

$$
\begin{aligned}
\sup_{s\in[0,1]}|T\eta(s))-T\mu(s)| &\leq \kappa d(\eta,\mu) \\
&\Rightarrow d(T\eta,T\mu)\leq\kappa d(\eta,\mu),
\end{aligned}
$$

and so, we have

$$e^{-\frac{d(T\eta,T\mu)}{\acute{r}}}\geq e^{-\kappa\frac{d(\eta,\mu)}{\acute{r}}}.$$

Define $\delta:[0,1)\to[0,1]$ by $\delta(\omega)=\omega$ for all $\omega\in[0,1)$, using (9) and the property of δ, we have

$$\delta(F_{ms}(T\eta,T\mu,\acute{r}))\geq[\delta(F_{ms}(\eta,\mu,\acute{r}))]^{\kappa}.$$

Therefore, T has a unique fixed point, that is, the nonlinear fractional differential Equation (14) has a unique solution in $X=C([0,1],\mathbb{R})$. $\square$

5. Conclusions

Jleli and Samet [2] principle and proved an important result, showing the existence of fixed points for such contractions in metric spaces. By taking the concept of θ-contraction from work in [2], of δ-contraction, which is a different and new contraction obtained by changing some conditions in the ξ-contraction in the previous study. We have proved the existence and uniqueness of the fixed point of functions by using this δ-contraction condition in complete NAFMS. We have proved some fixed point results by using the δ-contraction. We have provided an example, and we have presented an application to show that solutions to integral equations can be found.

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