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Unique Solvability of the Initial-Value Problem for Fractional Functional Differential Equations—Pantograph-Type Model

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Abstract: Contrary to the initial-value problem for ordinary differential equations, where the classical theory of establishing the exact unique solvability conditions exists, the situation with the initial-value problem for linear functional differential equations of the fractional order is usually non-trivial. Here we establish the unique solvability conditions for the initial-value problem for linear functional differential equations of the fractional order. The advantage is the lack of the calculation of fractional derivatives, which is a complicated task. The unique solution is represented by the Neumann series. In addition, as examples, the model with a discrete memory effect and a pantograph-type model from electrodynamics are studied.

Keywords: fractional order functional differential equations; unique solvability; Caputo derivative; the model with a discrete memory effect; the pantograph-type model from electrodynamics

MSC: 26A33; 34A08; 34K08; 34K37; 47H07; 74D05

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1. Introduction

Using fractional calculus, one can more accurately explain natural phenomena by mathematical models. Therefore, fractional functional differential equations (FFDEs) have been undergoing intensive development recently [1–13].

The variety of studies in the FFDEs better describes the diverse specifics of the theory of boundary value problems, see [5–9,12].

The primary interest of our investigation is to find precise requirements sufficient for the existence of the unique solution of the initial-value problem for FFDEs, in contrast with the more general conditions described in [2,5,8]. The main result is represented by Theorem 3. The novelty is given by the fact that it is not necessary to calculate fractional derivatives here. Therefore, such a condition is much more practical to use in variable FFDEs. The main focus of the investigations is on obtaining the new method practiced with exact results for linear FFDEs that could be useful for various applications. We expect that the proposed procedure may have significance above average because of its simplicity. The discrete memory effect model (see, for example, [7,11]) and the model of the pantograph type from electrodynamics (see, for example, [1,3,13]) are studied.

We consider the fractional functional differential problem

$$D_a^q x(t) = (lx)(t) + r(t), \quad t \in [a,b] \tag{1}$$

$$x(a) = c, \tag{2}$$

where D_a^q is the Caputo derivative of a fractional order q, $0 < q < 1$, with the lower limit zero, function $r \in C([a,b], \mathbb{R}^n)$, $c \in \mathbb{R}^n$, and operator l maps space of absolutely continuous functions $AC([a,b], \mathbb{R}^n)$ to the space of continuous functions $C([a,b], \mathbb{R}^n)$.

Here are used spaces:

- $C([a,b],\mathbb{R}^n)$ is the Banach space of continuous functions $[a,b] \to \mathbb{R}^n$:

$$\|x\|_{C([a,b],\mathbb{R}^n)} := \max_{t\in[a,b]} |x(t)|_\infty = \max_{t\in[a,b]} \text{ess sup}\, |x(t)|;$$

- $AC([a,b],\mathbb{R}^n)$ is the Banach space of absolutely continuous functions $[a,b] \to \mathbb{R}^n$:

$$\|x\|_{AC([a,b],\mathbb{R}^n)} := \int_a^b \|x'(t)\|dt + \|x(0)\|;$$

- $AC_a([a,b],\mathbb{R}^n)$ is the Banach space of $x \in AC([a,b],\mathbb{R}^n)$ that satisfies condition (2).

Definition 1 ([14])**.** *A solution of the linear Cauchy problem* (1), (2) *is understood as an absolutely continuous vector-function* $x : [a,b] \to \mathbb{R}^n$ *with the property* (2) *and satisfying the Equation* (1) *for almost all* $t \in [a,b]$.

Definition 2 ([4])**.** *For an absolutely continuous function* $x \in [a,b]$*, the Caputo fractional derivative of order* q *is determined by*

$$D_a^q x(t) = \frac{1}{\Gamma(1-q)}\left(\frac{d}{dt}\right)\int_a^t (t-\xi)^{-q}\big(x(\xi)-x(a)\big)d\xi,\ 0<q<1,$$

where $\Gamma(q) : [0,\infty) \to \mathbb{R}$ *is Gamma-function:*

$$\Gamma(q) := \int_0^\infty \xi^{q-1}e^{-\xi}d\xi. \tag{3}$$

Definition 3. *For every* $\sigma_i \in \{-1,1\}$, $i = 1,2,\ldots,n$, *and*

$$\varsigma := \begin{pmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n \end{pmatrix}, \tag{4}$$

an operator $l : AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ *is* ς*-positive operator if the relation*

$$\varsigma x(t) \geq 0, \quad t \in [a,b] \tag{5}$$

implies

$$\varsigma(lx)(t) \geq 0, \quad \textit{for a.e.} \quad t \in [a,b]. \tag{6}$$

Lemma 1 (Fredholm alternative)**.** *The nonhomogeneous problem* (1), (2) *is uniquely solvable if the corresponding homogeneous problem*

$$x(a) = 0 \tag{7}$$

for linear FFDE

$$D_a^q x(t) = (lx)(t), \quad t \in [a,b], \tag{8}$$

only has a trivial solution.

Lemma 2 (Theorem 3.7 and Theorem 3.8 [4])**.** *Assume that* $0 < q < 1$ *and* $x(t) \in C([a,b],\mathbb{R}^n)$, *then*

$$D_a^q I_a^q x(t) = x(t) \quad \textit{almost everywhere on} \quad [a,b].$$

If $x(t) \in AC([a,b],\mathbb{R}^n)$, *then*

$$I_a^q D_a^q x(t) = x(t) - x(a) \quad \textit{almost everywhere on} \quad [a,b],$$

where

$$I_a^q x(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} x(\xi) d\xi,$$

and Γ*-function is defined by* (3).

Taking into account Definition 1, Lemma 2 and Formula (3), the following Lemma is fulfilled.

Lemma 3. *The Cauchy problem* (1), (2) *on* $[a,b]$ *is equivalent to the equation*

$$x(t) = x(a) + \frac{1}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} (lx)(\xi) d\xi + \frac{1}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} r(\xi) d\xi.$$

2. Main Results

Let us fix $z_0 \in C_a([a,b], \mathbb{R}^n)$ and consider the sequence of functions (see, for example, [6,15]) for $t \in [a,b]$ and $k = 1, 2, \ldots$

$$z_k(t) := \frac{1}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} (l z_{k-1})(\xi) d\xi. \tag{9}$$

Suppose that for ς defined by (4) the next conditions

$$\varsigma z_0(t) > 0, \quad t \in (a,b], \quad z_0(a) = 0 \tag{10}$$

are true. The following Lemma is necessary to set the main result.

Lemma 4. *Let a certain function* $z_0 \in AC([a,b], \mathbb{R}^n)$ *satisfy the condition*

$$\varsigma z_0(t) \geq 0, \quad t \in [a,b], \tag{11}$$

and the linear operator $l : AC([a,b], \mathbb{R}^n) \to C([a,b], \mathbb{R}^n)$ *be a* ς*-positive operator, then,*

$$\varsigma z_k(t) \geq 0, \quad t \in [a,b], \quad k = 1, 2, \ldots, \tag{12}$$

where z_k *is defined by* (9).

Proof. Let us construct z_1 by the Formula (9):

$$z_1(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} (l z_0)(s) ds, \quad t \in [a,b]. \tag{13}$$

By ς-positivity of the operator l and property (11), we have that

$$\varsigma (l z_0)(t) \geq 0, \quad t \in [a,b],$$

which immediately implies that $\varsigma z_1(t) \geq 0$. It is easy to show by induction that condition (12) is fulfilled for all $k \geq 1$. The Lemma 4 is proved. □

Theorem 1. *Assume that the linear operator* l *in FFDEs* (1) *and* (8) *is* ς*-positive. Suppose also that there exists vector function* $z_0 \in AC_a([a,b], \mathbb{R}^n)$ *with properties* (10) *and a natural number* m*, a real number* $\rho \in (1, +\infty)$*, and* $\{v_i\}_{i=0}^m$*,* $v_0 > 0$*, such that the functions* $z_1, z_2, \ldots, z_m$*, defined by* (9)*, satisfy the inequality*

$$\varsigma \Big(v_0 D_a^q z_0(t) + \rho \sum_{i=0}^{m-1} (v_{i+1} - v_i) \rho^i (l z_i)(s) - \rho^{m+1} v_m (l z_m(t)) \Big) \geq 0 \quad \text{for a.e.} \quad t \in [a,b]. \tag{14}$$

Then, the homogeneous initial-value problem (7), (8) only has the trivial solution and the non-homogeneous initial value problem (1), (2) has a unique solution $u(\cdot)$ for arbitrary $r \in C([a,b],\mathbb{R}^n)$. Moreover, this solution is representable in the form of the uniformly convergent Neumann series

$$x(t) = r_c(t) + \frac{1}{\Gamma(q)}\int_a^t (t-\xi)^{q-1}(lr_c)(\xi)d\xi + \\ + \frac{1}{\Gamma(q)}\int_a^t (t-\xi)^{q-1} l\Big(\frac{1}{\Gamma(q)}\int_a^{\xi}(t-\eta)^{q-1}(lr_c)(\eta)d\eta\Big)(\xi)d\xi + \dots, \quad (15)$$

where

$$r_c(t) = c + \frac{1}{\Gamma(q)}\int_a^t (t-\xi)^{q-1} r(\xi)d\xi.$$

Moreover, if c and r satisfy the condition

$$\varsigma\Big(c + \frac{1}{\Gamma(q)}\int_a^t (t-\xi)^{q-1} r(\xi)d\xi\Big) \geq 0, \quad \textit{for a.e.} t \in [a,b], \quad (16)$$

then the unique solution (15) of problem (1), (2) is nonnegative in the sense (5).

Proof. To prove Theorem 1, we use Theorem 4 from [6] on the unique solvability of the initial-value problem (1), (2), and (8), (7).

Theorem 2 (Theorem 4 [6])**.** *Assume that the linear operator $l = (l_k)_{k=1}^n$ in FFDEs (1) and (8) is ς-positive. Let there exist such a number $\beta > 1$ a function $z_0 \in AC_a([a,b],\mathbb{R}^n)$ with properties (10). Additionally, the following fractional functional differential inequality*

$$\varsigma\Big(D_a^q z_0(t) - \beta(lz_0)(t)\Big) \geq 0 \quad \textit{for a.e.} \quad t \in [a,b] \quad (17)$$

is fulfilled.

Then, the assertion of Theorem 1 is true for the inhomogeneous (1), (2), and homogeneous Cauchy problems (7), (8).

Let us consider the function

$$z(t) = v_0 z_0(t) + v_1\rho z_1(t) + \dots + v_m \rho^m z_m(t), \quad t \in [a,b], \quad (18)$$

where the vector-functions $z_1, z_2, \dots$ are defined by (9) and the function z_0 has properties (10).

It is obvious from (10) and Lemma 4 that Formula (18) for the function z can be rewritten in the form

$$z(t) = v_0 z_0(t) + \zeta(t), \quad t \in [a,b], \quad (19)$$

where ζ is nonnegative in the sense

$$\varsigma\zeta(t) \geq 0, \quad t \in [a,b].$$

Since function z_0 has properties (10) it follows from Formula (19) that relations (10) are fulfilled for the function z.

By assumption, function z_0 is absolutely continuous, then from recurrence Formula (9), it is obvious that the function z constructed by (18) is also an absolutely continuous function.

Let us consider the continuous function

$$\omega(t) := D_a^q z(t) - \rho(lz)(t). \quad (20)$$

By (9) and (18), we have

$$\omega(t) = v_0 D_a^q z_0(t) + v_1 \rho D_a^q z_1(t) + \cdots + v_m \rho^m D_a^q z_m(t) - \\ - \rho l \big(v_0 z_0(t) + v_1 \rho z_1(t) + \cdots + v_m \rho^m z_m(t)\big). \quad (21)$$

Taking into account (9) and Lemma 2, we have that

$$D_a^q z_k(t) = (l z_{k-1})(s), \quad k \geq 1. \quad (22)$$

So,

$$\omega(t) = v_0 D_a^q z_0(t) + \\ + v_1 \rho (l z_0)(t) + v_2 \rho^2 (l z_1)(t) + \cdots + v_m \rho^m (l z_{m-1})(s) - \\ - v_0 \rho (l z_0)(t) - v_1 \rho^2 (l z_1)(t) - \cdots - v_{m-1} \rho^m (l z_{m-1})(s) ds - \\ - v_m \rho^{m+1} (l z_m)(s) ds.$$

Next, the last relation is equivalent to the formula

$$\omega(t) = v_0 D_a^q z_0(t) + \rho\big(v_1 - v_0\big)(l z_0)(t) + \cdots + \rho\big(v_i - v_{i-1}\big)\rho^i (l z_i)(t) + \dots \\ + \rho\big(v_m - v_{m-1}\big)\rho^{m-1}(l z_{m-1})(t) - \rho^{m+1} v_m (l z_m)(t).$$

Taking into account (14), we get that

$$\varsigma \omega(t) \geq 0, \quad t \in [a, b]. \quad (23)$$

In view of (20), the inequality (23) means that the function z defined by (19) is a solution of the fractional differential inequality (17). It is easy to see that the absolutely continuous function z constructed by (18) fulfills the conditions of Theorem 4 from [6] (see Theorem 2) with $\beta = \frac{\rho}{v_0}$. So, one can apply the mentioned theorem to the initial value problems (8), (7), and (1), (2) and get the assertions required. Theorem 1 is proved. □

Theorem 3. *Assume that operator l in FFDEs (1) and (8) is ς-positive, and suppose that there exist a real number $0 < \gamma < 1$ and integer numbers $k \geq 0$ and $j \geq 1$ for which the following inequality*

$$\varsigma\big(\gamma z_k(t) - z_{k+j}(t)\big) \geq 0, \quad \textit{for a.e.} \quad t \in [a, b] \quad (24)$$

is satisfied, where $z_0 \in C_a([a, b], \mathbb{R}^n)$ is a certain increasing function with properties (10). Then, the assertion of Theorem 1 is true for the inhomogeneous (1), (2), and homogeneous Cauchy problems (7), (8).

Proof. Let us consider the function z from (18) in view:

$$z(t) = (1 - \alpha) \sum_{i=0}^{k} \rho^i z_i(t) + \sum_{i=k+1}^{k+j} \rho^i z_i(t), \quad 0 < \alpha < 1, \quad \rho > 1, \quad (25)$$

where $t \in [a, b]$, $\alpha \in (0, 1)$, $v_i = 1 - \alpha$ for $i = 0, 1, \dots, k$ and $v_i = 1$ for $i = k + 1, \dots, k + j$, functions $z_1, z_2, \dots$ are defined by (9) and function z_0 has properties (10).

Taking into account (25), we can rewrite (20) by the next way:

$$\omega(t) = (1 - \alpha) D_a^q z_0(t) + \sum_{i=1}^{k} (1 - \alpha)\rho^i (l z_{i-1})(t) + \sum_{i=k+1}^{k+j} \rho^i (l z_{i-1})(t) - \\ - \rho l \Big(\sum_{i=0}^{k} (1 - \alpha)\rho^i z_i(t) + \sum_{i=k+1}^{k+j} \rho^i z_i(t)\Big) \quad (26)$$

or in view of (22) relation (26) can be rewritten by

$$\omega(t) = (1-\alpha)D_a^q z_0(t) + \rho^{k+1} l\Big(\alpha z_k(t) - \rho^j z_{k+j}(t)\Big), \quad j \in \mathbb{N}.$$

Let us put $\gamma = \frac{\alpha}{\rho^j}$. For numbers $0 < \alpha < 1$ and $\rho > 1$, by the ς-positivity of the operator l relation (24) yields

$$\varsigma l\Big(\alpha z_k(t) - \rho^j z_{k+r}(t)\Big) \geq 0, \quad \text{for } a.e. \quad t \in [a,b].$$

Thus, in view of properties (9) and (10) for increasing function z_0, property (24) for ς-positive operator l we have, that $\varsigma\omega(t) \geq 0$, where $\omega(t)$ is constructed by (26) for $t \in [a,b]$. So, we get that the continuous function $\omega(t)$ from (26) implies the condition (14) from Theorem 1. The application of that theorem to the initial-value problem (1), (2), and corresponding homogeneous problem (7), (8) gives the assertions required. Theorem 3 is proved. □

Corollary 1. *Suppose that linear operator $l : AC([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ in FFDEs (1) and (8) is ς-positive, and assume that there exists a real number $0 < \gamma < 1$, the following differential inequality for a.e. $t \in [a,b]$ is satisfied*

$$\frac{\varsigma}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} (lz_0)(\xi) d\xi \leq \varsigma\gamma z_0(t), \tag{27}$$

where $z_0 \in C([a,b],\mathbb{R}^n)$ is a certain increasing function with properties (10).

Then, the assertion of Theorem 1 is true for homogeneous Cauchy problems (7), (8), and the inhomogeneous (1), (2).

Proof. Let us put $k = 0$, and $j = 1$, and (13) in Theorem 3, then we immediately get our Corollary 1. □

3. Application

Let us consider an FFDE

$$D_a^q x(t) = \sum_{i=1}^{t_i \leq t} F_i(t) x(\tau_i(t)) + r(t), \quad t \in [a,b], \tag{28}$$

where $a < t_1 < t_2 < \cdots < t_m < b$ are given, F_i are matrix-functions with continuous components and $\tau_i \in C([a,b],[a,b])$, $i = 1,\ldots,m$. Setting $\tau_i(t) = t_i$, $i = 1,\ldots,m$, in (28), we get

$$D_a^q x(t) = \sum_{i=1}^{t_i \leq t} F_i(t) x(t_i) + r(t), \quad t \in [a,b]. \tag{29}$$

Obviously, (29) is a discrete memory effect model, see, for example, [7,11].

3.1. The Model with a Discrete Memory Effect

Let us consider Cauchy problem (2) for FFDE (29) and fix a function $z_0 \in AC_a([a,b],\mathbb{R}^n)$ defined by

$$z_0(t) = \zeta(t-a), \quad t \in [a,b], \tag{30}$$

where $\zeta > 0$ is a certain number, and introduce the sequence of functions for $t \in [a,b]$ and $k = 1,2,\ldots$

$$z_k(t) := \frac{1}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} \sum_{j=1}^{m} \chi_j(\xi) F_j(\xi) z_{k-1}(t_j) d\xi.$$

where $\chi_j(t)$ is the characteristic function of the interval $[t_j, b]$:

$$\chi_j(t) = 1, \quad \text{if} \quad t \in [t_j, b] \quad \textit{and} \quad \chi_j(t) = 0, \quad \text{if} \quad t \notin [t_j, b]. \tag{31}$$

Theorem 4. *Suppose that*

$$\varsigma F_j(t)\varsigma \geq 0 \quad \textit{for a. a.} \quad t \in [a, b], \quad 1 \leq j \leq m, \tag{32}$$

is fulfilled, where F_j, $j = 1, \dots, m$, are defined by

$$F_j(t) := \begin{pmatrix} f^j_{11}(t) & f^j_{12}(t) & \dots & f^j_{1m}(t) \\ f^j_{21}(t) & f^j_{22}(t) & \dots & f^j_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ f^j_{m1}(t) & f^j_{m2}(t) & \dots & f^j_{mm}(t) \end{pmatrix}, \tag{33}$$

ς is defined by (4), and assume that there exists a real number $0 < \gamma < 1$, such that the inequality

$$\frac{\varsigma}{\Gamma(q)} \int_a^t (t-\xi)^{q-1} \sum_{j=1}^{m} \chi_j(s) F_j(\xi)(t_j - a) d\xi \leq \varsigma\gamma(t-a) \tag{34}$$

is satisfied for almost all t from $[a, b]$.

Then, the nonhomogeneous initial-value problem (29), (2) has a unique solution $x(\cdot)$ for arbitrary $r \in C([a, b], \mathbb{R}^n)$. Moreover, this solution is representable in the form of the uniformly convergent series

$$x(t) = \sum_{k=0}^{n} r^{[k]}, \quad t \in [a, b], \tag{35}$$

where

$$r^{[k]}(t) = \frac{1}{\Gamma(q)} \sum_{j=1}^{m} \int_a^t (t-\xi)^{q-1} \chi_j(\xi) F_j(\xi) r^{[k-1]}(t_j) d\xi, \quad t \in [a, b],$$
$$r^{[0]}(t) = c + \frac{1}{\Gamma(q)} \int_a^t (t-\xi)^{1-q} r(\xi) d\xi.$$

Moreover, if c and r satisfy condition (16), then the unique solution (35) of the problem (29), (2) is nonnegative in the sense (5).

The corresponding homogeneous initial value problem only has the trivial solution.

Proof. To prove Theorem 4, we need the following Lemma.

Lemma 5. *If each of the continuous functions $F_i : [a, b] \times [a, b]$, $i = 1, 2, \dots, m$, are defined by (33), satisfies inequality (32), where ς is defined by (4), then the linear operator*

$$AC([a, b], \mathbb{R}^n) \ni u \mapsto (lx)(t) := \sum_{i=1}^{t_i \leq t} F_i(t) x(t_i) \tag{36}$$

is ς-positive.

Proof. Let vector function $x \in AC_a([a, b], \mathbb{R}^n)$ satisfy condition (5). Taking into account (6), with operator l defined by (36), we have

$$\varsigma l(x)(t) = \varsigma \sum_{i=1}^{t_i \leq t} F_i(t) x(t_i) = \varsigma \sum_{i=1}^{t_i \leq t} F_i(t) \varsigma\varsigma x(t_i)), \tag{37}$$

for $t \in [a, b]$ and ς defined by (4). From (5), (32), and (37) we conclude that

$$\varsigma(lx)(t) = \varsigma \sum_{i=1}^{t_i \le t} F_i(t)x(t_i) \ge 0 \quad \text{for } a.\ a. \quad t \in [a, b].$$

Then operator l given by formula (36) is ς-positive.
Thus, Lemma 5 is proved. □

Obviously, Theorem 4 is the partial case of Corollary 1 with the ς-positive operator l, defined by (36) (see Lemma 5) and absolutely continuous function z_0 defined by (30). □

3.2. Pantograph-Type Model

Now let us consider Equation (28) in view

$$D_a^q x(t) = \sum_{i=1}^{m} \chi_i(t)F_i(t)x(\tau_i(t)) + r(t), \quad t \in [a, b],$$

where $\chi_i(t)$ is defined by (31). Obviously, we can study a general case

$$D_a^q x(t) = \sum_{i=1}^{m} P_i(t)x(\tau_i(t)) + r(t), \quad t \in [a, b], \tag{38}$$

where P_i, $i = 1, \dots, m$, are defined by

$$P_i(t) := \begin{pmatrix} p_{11}^i(t) & p_{12}^i(t) & \dots & p_{1m}^i(t) \\ p_{21}^i(t) & p_{22}^i(t) & \dots & p_{2m}^i(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1}^i(t) & p_{m2}^i(t) & \dots & p_{mm}^i(t) \end{pmatrix}. \tag{39}$$

On the other hand, if P_i are matrix-functions with continuous components, $a = 0$ and $\tau_i(t) = \lambda_i t$ for $\lambda_i \in (0, 1)$ in (38), we get

$$D_0^q x(t) = \sum_{i=1}^{m} P_i(t)x(\lambda_i t) + r(t), \quad t \in [0, 1], \tag{40}$$

which is a pantograph-type model. Equation (40) is a pantograph equation used in electrodynamics [1,3,13]. The pantograph is a device used in electric trains to collect electric currents from the overload lines.

Now, our goal is to establish the precise conditions sufficient for the unique solvability of the Cauchy problem

$$x(0) = c \tag{41}$$

for FFDE (40), where constants λ_i, $i = 1, 2, \dots, m$, belong to the interval $(0, 1)$, function $r \in C([0, 1], \mathbb{R}^n)$ $c \in \mathbb{R}^n$, P_i, $i = 0, 1, \dots, m$, are defined by (39).

We need the next Lemma.

Lemma 6 (Lemma 9 [6])**.** *If each of the matrix-functions $P_i \in C([0, 1], \mathbb{R}^n)$ are defined by (39), it satisfies inequality*

$$\varsigma P_i(t)\varsigma \ge 0 \quad \text{for almost all} \quad t \in [0, 1], \quad i = 1, 2, \dots, m, \tag{42}$$

where ς is defined by (4), then, the linear operator

$$AC([0, 1], \mathbb{R}^n) \ni u \mapsto (lx)(t) := \sum_{i=1}^{m} P_i(t)x(\lambda_i t) \tag{43}$$

is ς-positive, where $\lambda_i \in (0,1)$.

Let us fix increasing function $z_0 \in AC_0([0,1],\mathbb{R}^n)$ defined by

$$z_0(t) = \zeta t, \quad t \in [0,1], \tag{44}$$

where $\zeta \in (0,1)$ is a certain number and introduces the sequence of functions for $t \in [0,1]$ and $k = 1,2,\dots$

$$z_k(t) := \frac{1}{\Gamma(q)} \int_0^t \frac{1}{(t-\xi)^{1-q}} \sum_{j=1}^{m} P_j(\xi) z_{k-1}(\lambda_i \xi) d\xi.$$

Obviously, (10), (12) are true for increasing function y_0.

Theorem 5. *Suppose that (42) is fulfilled, and assume that there exists a real number $0 < \gamma < 1$, such that the fractional functional differential inequality*

$$\frac{\varsigma}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} \sum_{j=1}^{m} P_j(\xi)(\lambda_i \xi) d\xi \le \varsigma t \tag{45}$$

is satisfied for almost all t from $[0,1]$.

Then, the nonhomogeneous initial value problem (40), (41) has a unique solution $x(\cdot)$ for arbitrary $r \in C([0,1],\mathbb{R}^n)$. Moreover, this solution is representable in the form of the uniformly convergent series (35), where

$$r^{[k]}(t) = \frac{1}{\Gamma(q)} \sum_{i=1}^{n} \int_0^t (t-\xi)^{q-1} P_i(\xi) r^{[k-1]}(\lambda_i \xi) d\xi, \quad t \in [0,1],$$
$$r^{[0]}(t) = c + \frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} r(\xi) d\xi.$$

Moreover, if c and r satisfy condition

$$\varsigma\Big(c + \frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} r(\xi) d\xi\Big) \ge 0, \quad \textit{for a.e.} \quad t \in [0,1],$$

then the unique solution (35) of the problem (40), (41) is nonnegative in the sense (5). The corresponding homogeneous initial value problem only has the trivial solution.

Proof. Obviously, Theorem 5 is the partial case of Corollary 1 with the ς-positive operator l, defined by (43) (see Lemma 6) and absolutely continuous function z_0 defined by (44). □

4. Discussion

Here we established new exact solvability conditions on the Cauchy problem (2) for FFDEs (1). Despite the fact that the hypotheses of Theorem 1 are more complicated than the hypotheses of Theorem 2 from [6], and that Theorem 1 is only a consequence of Theorem 2, but Theorem 3 (a consequence of Theorem 1) shows that it is easier to use Theorem 3 then Theorem 2 from [6]. For comparison, please see Theorem 5 from the actual paper, and Theorem 5 from [6].

5. Conclusions

We deal with a functional-analytical approach to the treatment of certain ς-positive systems of linear FFDEs. The received results are much simpler in application than the results from [6] and help establish precise conditions sufficient for the existence of the unique solution of the Cauchy problem for the system of linear FFDEs determined by isotone operators without calculation of the fractional derivatives.

We expect reasonable future interest in applying similar methods to the boundary-value problem for FFDEs.

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