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Abstract Impulsive Volterra Integro-Differential Inclusions

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Abstract: In this work, we provide several applications of (a,k)-regularized C-resolvent families to the abstract impulsive Volterra integro-differential inclusions. The resolvent operator families under our consideration are subgenerated by multivalued linear operators, which can degenerate in the time variable. The use of regularizing operator C seems to be completely new within the theory of the abstract impulsive Volterra integro-differential equations.

Keywords: multivalued linear operators; (a,k)-Regularized C-resolvent families; abstract impulsive fractional integro-differential inclusions; abstract impulsive higher-order Cauchy problems; abstract impulsive Volterra integro-differential inclusions

MSC: 34A37; 34C25; 34C27

1. Introduction

The fractional calculus and the fractional differential equations play a significant role in modeling of complex systems in various disciplines in science and engineering; see, e.g., [1–5] and references cited therein for more details about the subject. On the other hand, the theory of impulsive differential equations has experienced a significant growth in popularity because of its huge potential of applicability in various fields of pure and applied science. For example, the impulsive differential equations are used for modeling processes exhibiting changes at certain moments, negligible compared with the duration of the whole process; these types of processes cannot be described using the classical theory of integer or fractional differential equations. For further information concerning the theory of impulsive differential equations, we refer the reader to [6–17] and references cited therein.

As stated in the abstract, the main purpose of this research article is to provide certain applications of (a,k)-regularized C-resolvent families to the abstract impulsive Volterra integro-differential inclusions in Banach spaces ((a,k)-regularized C-resolvent families in sequentially complete locally convex spaces can be also considered but we will skip all details regarding this topic here). In the currently existing theory for the abstract impulsive Volterra integro-differential equations, it has been commonly used that the linear operator A under consideration is single-valued and generates a strongly continuous semigroup, cosine operator function or fractional resolvent operator family (for some results concerning applications of the almost sectorial operators, one can refer to [18,19]). This is probably the first research paper to consider using the C-regularized solution operator families (even global non-degenerate C-regularized semigroups) or multivalued linear operators in the theory of abstract impulsive Volterra integro-differential equations (some applications of once-integrated semigroups on weakly compactly generated Banach spaces have recently been provided by I. Benedetti, V. Obukhovskii and V. Taddei in [20], where the authors have investigated the solvability of the impulsive Cauchy problem for integro-differential inclusions with non-densely defined linear operators). Concerning the abstract impulsive

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degenerate differential equations with Caputo fractional derivatives, we would like to mention that T. D. Ke and C. T. Kinh have recently analyzed, in [21], the existence and stability of solutions for a class of degenerate impulsive fractional differential equations using the subordination principles and degenerate semigroups of operators (cf. also [22] and Definition 8 below with $a(t) \equiv k(t) \equiv 1$ and $C = I$).

The organization of this paper can be briefly described as follows. In Section 2.1, we recollect the basic definitions and results about the multivalued linear operators and solution operator families subgenerated by them. Section 3 investigates the C-wellposedness of the abstract impulsive differential inclusions of integer order (see Theorem 1 and Corollary 1 for some results obtained in this direction). In Section 3.1, we investigate the C-wellposedness of the abstract degenerate impulsive higher-order Cauchy problem $(ACP)_n$. In the existing literature, we have not been able to locate any research article concerning the use of (ultra-)distribution semigroups ((ultra-)distribution cosine functions) to the abstract impulsive differential equations of first order (second order) or the C-well-posedness of the abstract impulsive higher-order Cauchy problem $(ACP)_n$, even if this problem is non-degenerate, i.e., solvable with the respect to the highest derivative. Motivated by this fact, we present several illustrative applications in Example 1. In Example 2, we consider the abstract impulsive differential equations of first order (second order) with the multivalued linear operators $\mathcal{A}$ satisfying the condition (P), introduced by A. Favini and A. Yagi in the important research monograph [23]; the main importance of Example 3 is to present some applications of entire (g_n, C)-regularized resolvent families in the study of the abstract impulsive Cauchy problem $(ACP)_{n;1}$ clarified below as well as to initiate the research of the abstract Volterra integro-differential equations with impulsive effects in the complex plane (the angular domains of the complex plane).

Before proceeding further with the organization of paper, we would like to state the serious fact that a large number of structural results about the existence and uniqueness of the (abstract) impulsive fractional differential equations have been incorrectly stated in the existing literature because the authors have used the completely wrong formulae for the forms of solutions (see e.g., the discussions carried out in the research articles [15,16,24–27]). In Section 4, we will say just a few words about the abstract impulsive fractional differential inclusions with Caputo derivatives or Riemann–Liouville derivatives. The main aim of this section is a very simple result, Theorem 2, which indicates the incorrectness of many structural results published so far in the currently existing literature. As a simple consequence of the second result of Section 4, Theorem 3, we have that it is very difficult to study the existence and uniqueness of the piecewise continuous solutions of the abstract fractional differential inclusions with the Riemann–Liouville derivatives of order $\alpha \in (0, 1)$.

Section 5 investigates the abstract Volterra integro-differential inclusions with impulsive effects. The main result of this section is Theorem 4, which particularly shows that it is much better to analyze the well-posedness of the abstract Volterra integro-differential inclusions with the kernel $a(t) = g_\alpha(t)$ than the well-posedness of the abstract impulsive fractional differential inclusions with Caputo derivatives (Riemann–Liouville derivatives). In Theorem 5, we provide certain applications of a special class of the exponentially bounded (a, k)-regularized C-resolvent families to the abstract impulsive degenerate Volterra Equation (14). In Section 6, we make final comments and remarks on the abstract impulsive Volterra integro-differential equations considered in this paper. To better demonstrate the main goals and ideas of this article, we will study here the very simple forms of impulsive effects.

2. Preliminaries

We use the standard symbols and notation throughout the paper. By $(E, \| \cdot \|_E)$ and $(X, \| \cdot \|)$ we denote two complex Banach spaces (since no confusion seems likely, the norm $\| \cdot \|_E$ will be also denoted by $\| \cdot \|$ in the sequel). The abbreviation $C(K : X)$, where K is a non-empty compact subset of $\mathbb{R}$, stands for the space of all continuous functions from K into X; $C(K) \equiv C(K : \mathbb{C})$, $\mathbb{N}_n := \{1, \cdot\cdot\cdot, n\}$ and $\mathbb{N}_n^0 := \{0, 1, \cdot\cdot\cdot, n\}$, where $n \in \mathbb{N}$. Let $0 < \tau \leq \infty$

and $a \in L^1_{loc}([0,\tau))$. Then we say that the function $a(t)$ is a kernel on $[0,\tau)$ if and only if for each $f \in C([0,\tau))$ the assumption $\int_0^t a(t-s)f(s)\,ds = 0$, $t \in [0,\tau)$ implies $f(t) = 0$, $t \in [0,\tau)$. Set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, where $\Gamma(\cdot)$ denotes the Euler gamma function, and $g_0(t) := \delta(t)$, the Dirac delta distribution ($\alpha > 0$). We need the following conditions:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta \in \mathbb{R}$ such that $\tilde{k}(\lambda) := (\mathcal{L}k)(\lambda) := \lim_{b\to\infty}\int_0^b e^{-\lambda t}k(t)dt := \int_0^\infty e^{-\lambda t}k(t)dt$ exists for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \beta$. Put $\mathrm{abs}(k)$:=inf$\{\Re\lambda : \tilde{k}(\lambda)$ exists$\}$.

(P2): $k(t)$ satisfies (P1) and $\tilde{k}(\lambda) \neq 0$, $\Re\lambda > \beta$ for some $\beta \geq \mathrm{abs}(k)$.

We say that a function $h(\cdot)$ belongs to the class $LT-E$ if there exist an exponentially bounded function $f \in C([0,\infty) : E)$ and a real number $a > 0$ such that $h(\lambda) = (\mathcal{L}f)(\lambda)$, $\lambda > a$.

Let $T > 0$. Then the space of X-valued piecewise continuous functions on $[0,T]$ is defined by

$$\mathcal{P}C([0,T] : X) \equiv \big\{u : [0,T] \to X : u \in C\big((t_i,t_{i+1}] : X\big),\\ u(t_i-) = u(t_i) \text{ and } u(t_i+) \text{ exist for any } i \in \mathbb{N}_l^0\big\},$$

where $0 \equiv t_0 < t_1 < t_2 < ... < t_l < T \equiv t_{l+1}$ and the symbols $u(t_i-)$ and $u(t_i+)$ denote the left and the right limits of the function $u(t)$ at the point $t = t_i$, $i \in \mathbb{N}_{l-1}^0$, respectively. Let us recall that $\mathcal{P}C([0,T] : X)$ is a Banach space endowed with the norm $\|u\| := \max\{\sup_{t\in[0,T)}\|u(t+)\|, \sup_{t\in(0,T]}\|u(t-)\|\}$. The space of X-valued piecewise continuous functions on $[0,\infty)$, denoted by $\mathcal{P}C([0,\infty) : X)$, if defined as the union of those functions $f : [0,\infty) \to X$ such that the discontinuites of $f(\cdot)$ form a discrete set and that for each $T > 0$ we have $f_{|[0,T]}(\cdot) \in \mathcal{P}C([0,T] : X)$.

2.1. Solution Operator Families Subgenerated by MLOs

Recall that a multivalued mapping $\mathcal{A} : X \to P(X)$ is said to be a multivalued linear operator (MLO in X, or simply, MLO) if and if the following hold:

(1) $D(\mathcal{A}) := \{u \in X : \mathcal{A}u \neq \emptyset\}$ is a linear submanifold of X;
(2) $\mathcal{A}u + \mathcal{A}v \subseteq \mathcal{A}(u+v)$ for $u,\ v \in D(\mathcal{A})$ and $\lambda\mathcal{A}u \subseteq \mathcal{A}(\lambda u)$ for $\lambda \in \mathbb{C}$ and $u \in D(\mathcal{A})$.

For more details about multivalued linear operators and (degenerate) (a,k)-regularized C-resolvent families subgenerated by them, we refer the reader to the research monographs [23] by A. Favini, A. Yagi and [28] by M. Kostić.

It is well known that, for every $u,\ v \in D(\mathcal{A})$ and for every $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}u + \eta\mathcal{A}v = \mathcal{A}(\lambda u + \eta v)$. Furthermore, $\mathcal{A}0$ is a linear manifold in X and $\mathcal{A}u = f + \mathcal{A}0$ for any $u \in D(\mathcal{A})$ and $f \in \mathcal{A}u$. Put $R(\mathcal{A}) := \{\mathcal{A}u : u \in D(\mathcal{A})\}$. The inverse $\mathcal{A}^{-1}$ is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}v := \{u \in D(\mathcal{A}) : v \in \mathcal{A}u\}$. We know that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. Suppose now that $\mathcal{A},\ \mathcal{B}$ are two MLOs in X. Thus its sum $\mathcal{A}+\mathcal{B}$ is defined by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})u := \mathcal{A}u + \mathcal{B}u$ for $u \in D(\mathcal{A}+\mathcal{B})$. Clearly, $\mathcal{A}+\mathcal{B}$ is an MLO in X. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. We have that $\mathcal{B}\mathcal{A}$ is an MLO in X and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The inclusion $\mathcal{A} \subseteq \mathcal{B}$ means that $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$. The integer powers $\mathcal{A}^n$ are defined inductively in the usual way; we set $D_\infty(\mathcal{A}) := \bigcap_{n\in\mathbb{N}} D(\mathcal{A}^n)$.

We say that a multivalued linear operator$\mathcal{A}$ is closed if and only if for any nets (u_τ) in $D(\mathcal{A})$ and (v_τ) in X such that $v_\tau \in \mathcal{A}u_\tau$ for all $\tau \in I$ the assumptions $\lim_{\tau\to\infty} u_\tau = u$ and $\lim_{\tau\to\infty} v_\tau = v$ imply $u \in D(\mathcal{A})$ and $v \in \mathcal{A}u$.

The following lemma (see [28]) is important for our proofs.

Lemma 1. *Assume that $\mathcal{A}$ is a closed* MLO *in X, Ω is a locally compact and separable metric space, as well as that μ is a locally finite Borel measure defined on Ω. If $g : \Omega \to X$ and $h : \Omega \to X$ are μ-integrable, and $h(u) \in \mathcal{A}g(u)$ for $u \in \Omega$, then $\int_\Omega h\,d\mu \in \mathcal{A}\int_\Omega g\,d\mu$ and $\int_\Omega g\,d\mu \in D(\mathcal{A})$.*

Let $\mathcal{A}$ be an MLO in X, $C \in L(X)$ be injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\gamma \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\gamma - \mathcal{A})$;
(ii) $(\gamma - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

For $\gamma \in \rho_C(\mathcal{A})$, the operator $\gamma \mapsto (\gamma - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$; the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_{\mathcal{I}}(\mathcal{A})$, where $\mathcal{I}$ denotes the identity operator on X, and $R(\gamma : \mathcal{A}) \equiv (\gamma - \mathcal{A})^{-1}$ for $\gamma \in \rho(\mathcal{A})$.

Consider now the following abstract degenerate Volterra inclusion:

$$\mathcal{B}u(t) \subseteq \mathcal{A} \int_0^t a(t-s)u(s)ds + \mathcal{F}(t), \quad t \in [0,T], \tag{1}$$

where $\mathcal{A}\colon X \to P(E)$ and $\mathcal{B}\colon X \to P(E)$ are two given mappings (possibly non-linear), $T \in (0,\infty]$, $a \in L^1_{loc}([0,T])$, $a \neq 0$ and $\mathcal{F}\colon [0,T] \to P(E)$. The notion of a pre-solution of (1) and the notion of a (strong) solution of (1) have recently been introduced in ([28], Definition 3.1.1(i)):

Definition 1. *(i) A function $u \in C([0,T] : X)$ is said to be a pre-solution of (1) if and only if $(a * u)(t) \in D(\mathcal{A})$ and $u(t) \in D(\mathcal{B})$ for $t \in [0,T]$, as well as (1) holds.*

(ii) A solution of (1) is any pre-solution $u(\cdot)$ of (1) satisfying additionally that there exist functions $u_{\mathcal{B}} \in C([0,T] : E)$ and $u_{a,\mathcal{A}} \in C([0,T] : E)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$ and $u_{a,\mathcal{A}}(t) \in \mathcal{A}\int_0^t a(t-s)u(s)ds$ for $t \in [0,T]$, as well as

$$u_{\mathcal{B}}(t) \in u_{a,\mathcal{A}}(t) + \mathcal{F}(t), \quad t \in [0,T].$$

(iii) A strong solution of (1) is any function $u \in C([0,T] : X)$ satisfying that there exist two continuous functions $u_{\mathcal{B}} \in C([0,T] : E)$ and $u_{\mathcal{A}} \in C([0,T] : E)$ such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$, $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for all $t \in [0,T]$, and

$$u_{\mathcal{B}}(t) \in (a * u_{\mathcal{A}})(t) + \mathcal{F}(t), \quad t \in [0,T].$$

In the following, unless otherwise specified, we always assume that $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : E \to P(E)$ is an MLO, $C_1 \in L(X,E)$, $C_2 \in L(E)$ is injective, $C \in L(E)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. We will now analyze multivalued linear operators as subgenerators of (a,k)-regularized (C_1,C_2)-existence and uniqueness families and (a,k)-regularized C-resolvent families.

Definition 2. *(see ([28], Definition 3.2.1, Definition 3.2.2))*

(i) $\mathcal{A}$ is said to be a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(X,E) \times L(E)$ if the mappings $t \mapsto R_1(t)y$, $t \geq 0$ and $t \mapsto R_2(t)x$, $t \in [0,\tau)$ are continuous for every fixed $x \in E$ and $y \in X$, as well as the following conditions hold:

$$\left(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1y \right) \in \mathcal{A},\ t \in [0,\tau),\ y \in X \text{ and} \tag{2}$$

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2x, \text{ whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}. \tag{3}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(X,E)$ be strongly continuous. We say that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ if and only if (2) holds.

(iii) *Let* $(R_2(t))_{t\in[0,\tau)} \subseteq L(E)$ *be strongly continuous.* $\mathcal{A}$ *is said a subgenerator of a (local, if* $\tau < \infty$*) mild* (a,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\in[0,\tau)}$ *if and only if (3) holds.*

Definition 3. (i) *Assume that* $\tau \in (0,\infty]$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $k \in C([0,\tau))$, $k \neq 0$, $\mathcal{A} : E \to P(E)$ *is an MLO,* $C \in L(E)$ *is injective and* $C\mathcal{A} \subseteq \mathcal{A}C$*. We say that a strongly continuous operator family* $(R(t))_{t\in[0,\tau)} \subseteq L(E)$ *is an* (a,k)*-regularized* C*-resolvent family with a subgenerator* $\mathcal{A}$ *if and only if* $(R(t))_{t\in[0,\tau)}$ *is a mild* (a,k)*-regularized* C*-uniqueness family having* $\mathcal{A}$ *as subgenerator,* $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ *and* $R(t)C = CR(t)$ *for all* $t \in [0,\tau)$.

(ii) *If* $\tau = \infty$*, then we say that* $(R(t))_{t\geq 0}$ *is exponentially bounded (bounded) if and only if there exists* $\omega \in \mathbb{R}$ *(*$\omega = 0$*) such that the family* $\{e^{-\omega t}R(t) : t \geq 0\}$ *is bounded.*

The integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$) is defined by

$$\mathcal{A}_{int} := \left\{(x,y) \in X \times X : R_2(t)x - k(t)C_2x = \int_0^t a(t-s)R_2(s)y\,ds,\ t \in [0,\tau)\right\}.$$

In this work, we will primarily consider (a,k)-regularized C-resolvent families; for simplicity, we will assume that any (a,k)-regularized C-resolvent family considered below is likewise a mild (a,k)-regularized C-existence family (subgenerated by $\mathcal{A}$). We define the integral generator of an (a,k)-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$ in the same way as above. The notion of a C-regularized semigroup (C-regularized cosine function) with subgenerator (integral generator) $\mathcal{A}$ is obtained by substituting $k(t) \equiv 1$ and $a(t) \equiv 1$ ($a(t) \equiv t$).

Unless stated otherwise, we will always assume henceforth that the operator $C \in L(X)$ is injective.

3. Abstract Impulsive Differential Inclusions of Integer Order

In this section, we will focus on analyze the abstract impulsive differential inclusions of integer order. The application of C-regularized solution operator families is crucial in the case that the order of equation is greater or equal than three.

For the beginning, let us consider the following abstract impulsive higher-order Cauchy inclusion

$$(ACP)_{n;1} : \begin{cases} u^{(n)}(t) \in \mathcal{A}u(t) + f(t), \quad t \in [0,T] \setminus \{t_1,...,t_l\}, \\ \left(\Delta u^{(j)}\right)(t_k) = u^{(j)}(t_k+) - u^{(j)}(t_k-) = Cy_j^k, \quad k \in \mathbb{N}_l,\ j \in \mathbb{N}^0_{n-1}, \\ u^{(j)}(0) = Cu_j, \quad j \in \mathbb{N}^0_{n-1}, \end{cases}$$

where $\mathcal{A}$ is an MLO in X. We will use the following concepts of solutions:

Definition 4. (i) *By a pre-solution of* $(ACP)_{n;1}$ *on* $[0,T]$ *we mean any function* $u(\cdot)$ *which is n-times continuously differentiable on the intervals* $[0,t_1)$, (t_1,t_2), (t_2,t_3), ..., $(t_l,T]$*, the right derivatives* $\lim_{t\to t_i+} u^{(j)}(t)$ *exist for* $0 \leq j \leq n$ *and* $1 \leq i \leq l$*, the left derivatives* $\lim_{t\to t_i-} u^{(j)}(t)$ *exist for* $0 \leq j \leq n$ *and* $1 \leq i \leq l+1$*, and the requirements of* $(ACP)_{n;1}$ *hold.*

A solution of $(ACP)_{n;1}$ *on* $[0,T]$ *is any pre-solution* $u(t)$ *of* $(ACP)_{n;1}$ *on* $[0,T]$ *which additionally satisfies that there exists a function* $u_{\mathcal{A}} : [0,T] \to X$ *such that* $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ *for* $t \in [0,T] \setminus \{t_1,t_2,...,t_l\}$, $u^{(n)}(t) = u_{\mathcal{A}}(t) + f(t)$ *for* $t \in [0,T] \setminus \{t_1,t_2,...,t_l\}$*, the right limits* $\lim_{t\to t_i+} u_{\mathcal{A}}(t)$ *exist for* $1 \leq i \leq l$ *and the left limits* $\lim_{t\to t_i-} u_{\mathcal{A}}(t)$ *exist for* $1 \leq i \leq l+1$.

(ii) *Suppose that* $0 \equiv t_0 < t_1 < ... < t_l < t_{l+1} < ... < +\infty$ *and the sequence* $(t_l)_l$ *has no accumulation point. By a (pre-)solution of* $(ACP)_{n;1}$ *on* $[0,\infty)$ *we mean any function*

$u(\cdot)$ which satisfies that, for every $l \in \mathbb{N}$ and $T \in (t_l, t_{l+1})$, the function $u_{|[0,T]}(\cdot)$ is a (pre-)solution of $(ACP)_{n;1}$ on $[0,T]$.

The main result about the well-posedness of the problem $(ACP)_{n;1}$ reads as follows:

Theorem 1. *Suppose that $\mathcal{A}$ is a closed subgenerator of a local (g_n, C)-regularized resolvent family $(R(t))_{t\in[0,\tau)}$, where $\tau > T$ and $n \in \mathbb{N}$. Suppose that the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ are continuous on the set $[0,T] \setminus \{t_1, ..., t_l\}$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$ for all $t \in [0,T] \setminus \{t_1, ..., t_l\}$, as well as the right limits and the left limits of the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ exist at any point of the set $\{t_1, ..., t_l\}$. Define*

$$\begin{aligned} u(t) := R(t)u_0 + \sum_{j=1}^{n-1} \int_0^t g_j(t-s)R(s)u_j\, ds \\ + \int_0^t \int_0^{t-s} g_{n-1}(t-s-r)R(r)\big(C^{-1}f\big)(s)\, dr\, ds + \omega(t), \quad t \in [0,T], \end{aligned} \tag{4}$$

where

$$\omega(t) := \begin{cases} 0, \quad t \in [0,t_1], \\ \sum_{p=1}^{k} R\big(t-t_p\big)y_0^p + \sum_{p=1}^{k}\sum_{j=1}^{n-1}\int_0^{t-t_p} g_j(t-t_p-s)R(s)y_j^p\, ds, \\ \text{if } t \in \big(t_k, t_{k+1}\big] \text{ for some } k \in \mathbb{N}_{l-1}^0. \end{cases} \tag{5}$$

Then the function $u(t)$ is a unique solution of the problem $(ACP)_{n;1}$, provided that $u_0, ..., u_l \in D(\mathcal{A})$ and $y_k^j \in D(\mathcal{A})$ for all $k \in \mathbb{N}_l$ and $j \in \mathbb{N}_{n-1}^0$.

Proof. The uniqueness of solutions can be simply proved with the help of ([28], Proposition 3.2.8(ii)). Next, we will show that the function

$$u_h(t) := R(t)u_0 + \sum_{j=1}^{n-1} \int_0^t g_j(t-s)R(s)u_j\, ds + \int_0^t \int_0^{t-s} g_{n-1}(t-s-r)R(r)\big(C^{-1}f\big)(s)\, dr\, ds$$

is n-times continuously differentiable on $[0,T]$, $u_h^{(n)}(t) \in \mathcal{A}u_h(t) + f(t)$, $t \in [0,T]$ and $u_h^{(j)}(0) = Cu_j$, $j \in \mathbb{N}_{n-1}^0$. If $(x,y) \in \mathcal{A}$, then we have $R(t)x - Cx = \int_0^t g_n(t-s)R(s)y\, ds$, $t \in [0,T]$ and therefore

$$R^{(n)}(t)x = R(t)y, \quad t \in [0,T]; \; R^{(j)}(t)x = \int_0^t g_{n-j}(t-s)R(s)y\, ds, \; t \in [0,T], \; j \in \mathbb{N}_0^{n-1}. \tag{6}$$

Let $(u_j, v_j) \in \mathcal{A}$, $j \in \mathbb{N}_{n-1}^0$. Since

$$\left(\frac{d^j}{dt^j} \int_0^t \int_0^{t-s} g_{n-1}(t-s-r)R(r)\big(C^{-1}f\big)(s)\, dr\, ds\right)_{t=0} = 0, \quad j \in \mathbb{N}_{n-1}^0,$$

the above equalities simply imply $u_h^{(j)}(0) = Cu_j$, $j \in \mathbb{N}_{n-1}^0$ and

$$\begin{aligned} u^{(n)}(t) = R(t)\big(v_0\big) + \sum_{j=1}^{n-1} \int_0^t g_{n-j}(t-s)R(s)v_j\, ds \\ + \lim_{h\to 0} \int_0^t \frac{R(t+h-s) - R(t-s)}{h}\big(C^{-1}f\big)(s)\, ds \\ + \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} R(t+h-s)\big(C^{-1}f\big)(s)\, ds \end{aligned}$$

$$
\begin{aligned}
&= R(t)(v_0) + \sum_{j=1}^{n-1} \int_0^t g_{n-j}(t-s)R(s)v_j\,ds \\
&+ \lim_{h\to 0} \int_0^t \frac{R(t+h-s) - R(t-s)}{h}\left(C^{-1}f\right)(s)\,ds \\
&+ f(t), \quad t \in [0,T] \setminus \{t_1, ..., t_l\}.
\end{aligned} \tag{7}
$$

Since

$$
R(t-s)\left(C^{-1}f\right)(s) - f(s) = \int_0^{t-s} g_n(t-s-r)R(r)f_{\mathcal{A}}(s)\,dr, \quad 0 \le s \le t,
$$

the dominated convergence theorem shows that

$$
\begin{aligned}
&\lim_{h\to 0} \int_0^t \frac{R(t+h-s) - R(t-s)}{h}\left(C^{-1}f\right)(s)\,ds \\
&\quad = \int_0^t R(t-s)f_{\mathcal{A}}(s)\,ds, \quad t \in [0,T] \setminus \{t_1, ..., t_l\}.
\end{aligned}
$$

Keeping in mind that $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ for $t \in [0,T]$, Lemma 1, the last equality and (7) together imply that the function $u_h(\cdot)$ is n-times continuously differentiable on $[0,T]$ and $u_h^{(n)}(t) \in \mathcal{A}u_h(t) + f(t)$, $t \in [0,T]$. Therefore, it suffices to show that the function $\omega(\cdot)$ is a solution of the problem

$$
\begin{cases}
\omega^{(n)}(t) \in \mathcal{A}\omega(t), \quad t \in [0,T] \setminus \{t_1, ..., t_l\}, \\
\left(\Delta\omega^{(j)}\right)(t_k) = Cy_j^k, \quad k \in \mathbb{N}_l,\ j \in \mathbb{N}_{n-1}^0, \\
\omega^{(j)}(0) = 0, \quad j \in \mathbb{N}_{n-1}^0.
\end{cases}
$$

The third equality is obvious since $\omega(t) = 0$ for $t \in [0, t_1]$. Let $(y_k, z_k) \in \mathcal{A}$ for all $k \in \mathbb{N}_l$ as well as $(y_j^k, z_j^k) \in \mathcal{A}$ for all $k \in \mathbb{N}_l$ and $j \in \mathbb{N}_{n-1}^0$; the second equality simply follows from (6). To verify the first equality, we observe that

$$
\omega^{(n)}(t) = \sum_{p=1}^{k} R(t-t_p)z_0^p + \sum_{p=1}^{k}\sum_{j=1}^{n-1} \int_0^{t-t_p} g_j(t-t_p-s)R(s)z_j^k\,ds
$$

for all $t \in (t_k, t_{k+1}]$ ($k \in \mathbb{N}_{l-1}^0$). Since $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ for $t \in [0,T]$, Lemma 1 yields that $\omega^{(n)}(t) \in \mathcal{A}\omega(t)$ for all $t \in [0,T] \setminus \{t_1, ..., t_l\}$. This completes the proof of the theorem. □

Corollary 1. *Suppose that $\mathcal{A}$ is a closed subgenerator of a global (g_n, C)-regularized resolvent family $(R(t))_{t\ge 0}$, where $n \in \mathbb{N}$. Suppose, further, that $0 < t_1 < ... < t_l < ... < +\infty$, the sequence $(t_l)_l$ has no accumulation point, the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ are continuous on the set $[0,T] \setminus \{t_1, ..., t_l, ...\}$, $f_{\mathcal{A}}(t) \in \mathcal{A}C^{-1}f(t)$ for all $t \in [0,T] \setminus \{t_1, ..., t_l, ...\}$, as well as the right limits and the left limits of the functions $C^{-1}f(\cdot)$ and $f_{\mathcal{A}}(\cdot)$ exist at any point of the set $\{t_1, ..., t_l, ...\}$. Define the functions $u(t)$ and $\omega(t)$ for $t \in [0,T]$ by (4) and (5), respectively. Then the function $u(t)$ is a unique solution of the problem $(ACP)_{n;1}$ for $t \in [0,T] \setminus \{t_1, ..., t_l, ...\}$, provided that $u_0, ..., u_l, ... \in D(\mathcal{A})$ and $y_j^k \in D(\mathcal{A})$ for all $k \in \mathbb{N}$ and $j \in \mathbb{N}_{n-1}^0$.*

Now we will provide the following illustrative applications of Theorem 1 and Corollary 1:

Example 1. *(i)* *Let $\mathcal{A} = A$ be a closed single-valued linear operator and $\lambda \in \rho(A)$. Then it is well known that A is the integral generator of a distribution semigroup (distribution cosine function) if and only if for each $\tau > 0$ there exists $n \in \mathbb{N}$ such that A is the integral generator of a local $(\lambda - A)^{-n}$-regularized semigroup ($(\lambda - A)^{-n}$-regularized cosine function) on $[0, \tau)$; furthermore, there exists an injective operator $C \in L(X)$ such that A is the integral* *generator of a global C-regularized semigroup (C-regularized cosine function); cf. [29] for the notion and more details. Therefore, Theorem 1 and Corollary 1 can be successfully applied in the case that $n = 1$ ($n = 2$).*

(ii) *Suppose that the sequence (M_p) of positive real numbers satisfies $M_0 = 1$, (M.1), (M.2) and (M.3) as well as that a closed linear operator A generates a regular ultradistribution semigroup (regular ultradistribution cosine function) of (M_p)-class; then there exists an injective operator $C \in L(X)$ such that A is the integral generator of a global C-regularized semigroup (C-regularized cosine function); cf. ([29], Sections 3.5 and 3.6) for the notion and more details. Consequently, Theorem 1 and Corollary 1 can be successfully applied in the case that $n = 1$ ($n = 2$). For some important examples of (differential) operators generating ultradistribution semigroups (ultradistribution cosine functions), we refer the reader to ([29], Example 3.5.18, Example 3.5.23, Example 3.5.30(ii), Example 3.5.39).*

(iii) *Suppose that $k \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, $0 \le |\alpha| \le k$, $a_\alpha \ne 0$ for some α with $|\alpha| = k$, $P(x) = \sum_{|\alpha| \le k} a_\alpha i^{|\alpha|} x^\alpha$, $x \in \mathbb{R}^n$, $\omega := \sup_{x \in \mathbb{R}^n} \Re(P(x)) < +\infty$ (condition ([4], (W), p. 68) holds), and X is one of the spaces $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$. Define*

$$P(D) := \sum_{|\alpha| \le k} a_\alpha f^{(\alpha)} \text{ and } D(P(D)) := \{f \in E : P(D)f \in E \text{ distributionally}\}.$$

Then it is well known that the operator $P(D)$ generates an exponentially bounded C-regularized semigroup (C-regularized cosine function) with an appropriately chosen regularizing operator $C \in L(X)$, so that Corollary 1 can be successfully applied in the case that $n = 1$ ($n = 2$).

Example 2. *(i) We can analyze the well-posedness of the abstract impulsive inclusion $(ACP)_{1;1}$ for the multivalued linear operators $\mathcal{A}$ satisfying the following condition:*

(P) There exist finite constants $c,\ M > 0$ and $\beta \in (0, 1]$ such that

$$\Psi := \Psi_c := \Big\{\lambda \in \mathbb{C} : \Re\lambda \ge -c\big(|\Im\lambda| + 1\big)\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda : \mathcal{A})\| \le M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

Then the degenerate semigroup $(T(t))_{t>0}$ generated by $\mathcal{A}$ has an integrable singularity at zero but we can still apply the method obeyed in the proof of Theorem 1 if the function $f(t)$ satisfies the requirements of ([23], Theorem 3.7) and there exist vectors $z_1, ..., z_k, ...$ from the continuity set of the semigroup $(T(t))_{t>0}$ such that $z_k \in \mathcal{A}y_k$, $k = 1, ..., l, ...$ The established conclusion can be simply applied in the analysis of the following abstract impulsive Poisson heat equation in the space $X = L^p(\Omega)$:

$$\begin{cases} \frac{d}{dt}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), \quad t \ge 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(t_k+, x) - m(x)v(t_k-, x) = f_k(x),\ k \in \mathbb{N}, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

under certain logical assumptions; keeping in mind the consideration carried out in ([30], Example 3.10.4), we can also provide certain applications of the almost sectorial operators to the abstract impulsive differential equations of first order in Hölder spaces.

(ii) *Suppose that A, B and C are closed linear operators in X, $D(B) \subseteq D(A) \cap D(C)$, $B^{-1} \in L(X)$ and the conditions ([23], (6.4)-(6.5)) are satisfied with some numbers $c > 0$ and $0 < \beta \le \alpha = 1$; cf. also ([30], Example 3.10.10). In ([23], Chapter VI), the following second order differential equation without impulsive conditions*

$$\frac{d}{dt}\big(Cu'(t)\big) + Bu'(t) + Au(t) = f(t),\ t > 0; \quad u(0) = u_0,\ Cu'(0) = Cu_1$$

has been analyzed by the usual converting into the first order matricial system

$$\frac{d}{dt}Mz(t) = Lz(t) + F(t),\ t > 0; \quad Mz(0) = Mz_0,$$

where

$$M = \begin{bmatrix} I & O \\ O & C \end{bmatrix},\ L = \begin{bmatrix} O & I \\ -A & -B \end{bmatrix},\ z_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \text{ and } F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}\ (t > 0).$$

The argumentation contained in the proof of ([23], Theorem 6.1) shows that the multivalued linear operator $(L_{[D(B)]\times X} - \omega M_{[D(B)]\times X})(M_{[D(B)]\times X})^{-1}$ *satisfies the condition (P) for a sufficiently large number* $\omega > 0$, *in the pivot space* $[D(B)] \times X$. *Hence, this MLO generates a degenerate semigroup* $(T(t))_{t>0}$ *in* $[D(B)] \times X$, *having an integrable singularity at zero and exponentially decaying growth rate at infinity. Then we can apply ([23], Theorem 3.8, Theorem 3.9) in the analysis of existence and uniqueness of solutions of the abstract degenerate Cauchy problem without impulsive conditions:*

$$\frac{d}{dt}Mz(t) = (L - \omega M)z(t) + F(t),\ t > 0; \quad Mz(0) = Mz_0,$$

Furthermore, we can apply Corollary 1 with $n = 1$, $u(t) = Mz(t)$, $t \geq 0$ *and* $\mathcal{A} = (L - \omega M)M^{-1}$ *in the analysis of the existence and uniqueness of the piecewise continuously differentiable solutions of the following second-order impulsive differential equation:*

$$\begin{cases} \frac{d}{dt}\big(Cu'(t)\big) + (2\omega C + B)u'(t) + \big(A + \omega B + \omega^2 C\big)u(t) = f(t),\ t > 0; \\ u(t_k+) - u(t_k-) = y_k,\ C\big[u'(t_k+) + \omega u(t_k+)\big] - C\big[u'(t_k-) + \omega u(t_k-)\big] = z_k,\ k \in \mathbb{N}, \\ u(0) = u_0,\ C\big[u'(0) + \omega u_0\big] = Cu_1. \end{cases}$$

As is well known, we can simply incorporate this result in the analysis of existence and uniqueness of piecewise continuously differentiable solutions of the following damped Poisson-wave type equation in the spaces $X := H^{-1}(\Omega)$ *or* $X := L^p(\Omega)$:

$$\begin{cases} \frac{\partial}{\partial t}\big(m(x)\frac{\partial u}{\partial t}\big) + \big(2\omega m(x) - \Delta\big)\frac{\partial u}{\partial t} + \big(A(x;D) - \omega\Delta + \omega^2 m(x)\big)u(x,t) = f(x,t), \\ t \geq 0,\ x \in \Omega\ ;\ u = \partial u/\partial t = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ u(x,t_k+) - u(x,t_k-) = y_k(x),\ k \in \mathbb{N}, \\ C\big[(\partial u/\partial t)(x,t_k+) + \omega u(x,t_k+)\big] - C\big[(\partial u/\partial t)(x,t_k-) + \omega u(x,t_k-)\big] = z_k(x),\ k \in \mathbb{N}, \\ u(0,x) = u_0(x),\ m(x)\big[(\partial u/\partial t)(x,0) + \omega u_0\big] = m(x)u_1(x), \quad x \in \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ *is a bounded open domain with smooth boundary,* $1 < p < \infty$ *and some extra assumptions are satisfied; see ([23], Example 6.1) for further information.*

We continue this section by providing an illustrative application of Corollary 1 with $n \geq 3$:

Example 3. *Let* $(E, \|\cdot\|)$ *be a complex Banach space,* $s \in \mathbb{N}$ *and let* iA_j, $1 \leq j \leq n$ *be commuting generators of bounded* C_0*-groups on* E. *Further on, let* $P(x) = \sum_{|\eta|\leq d} P_\eta x^\eta$ $(P_\eta \in M_m,\ x \in \mathbb{R}^s)$ *be a polynomial matrix; cf. [4] for the notion and the notation. Then, due to ([4], Theorem 2.3.3), we know that there exists an injective operator* $C \in L(E)$ *with dense range such that the operator* $\overline{P(A)}$, *defined in the usual way, is the integral generator of a global* (g_n, C_m)*-regularized resolvent family* $(W_n(t))_{t\geq 0}$ *on* E^m, *where* $C_m = CI_{m,m}$ *and* $I_{m,m}$ *denotes the identity matrix of format* $m \times m$. *Furthermore, the mapping* $t \mapsto W_n(t)$, $t \geq 0$ *can be extended to the whole complex plane and the following holds:*

(i) $R(W_n(z)) \subseteq D_\infty(P(A))$, $z \in \mathbb{C}$ *and*

$$\overline{P(A)} \int_0^z g_n(z-s)W_n(s)\vec{x}\,ds = W_n(z)\vec{x} - C_m\vec{x},\ z \in \mathbb{C},\ \vec{x} \in E^m.$$

(iii) *The mapping* $z \mapsto W_n(z)$, $z \in \mathbb{C}$ *is entire.*

Suppose now that $m = 1$, $p_{11}(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$, $x \in \mathbb{R}^s$ $(a_\alpha \in \mathbb{C})$ *and* E *is a function space on which translations are uniformly bounded and strongly continuous. In this case,* $\overline{P(A)}$ *is just the operator* $\sum_{|\alpha| \leq d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha$ *with its maximal distributional domain; for example, let* $E = L^p(\mathbb{R}^s)$, *where* $1 \leq p < +\infty$. *Let us also assume that* $0 < t_1 < \ldots < t_l < \ldots < +\infty$ *and the sequence* $(t_l)_l$ *has no accumulation point.*

1. Using Corollary 1 and the above result, we obtain that there exists a dense subset $E_{0,n}$ *of* $L^p(\mathbb{R}^s)$ *such that the following abstract impulsive Cauchy problem:*

$$\begin{cases} \frac{d^n}{dt^n} u(t,x) = \sum_{|\alpha| \leq d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha u(t,x),\ t \geq 0,\ x \in \mathbb{R}^s, \\ \frac{d^j}{dt^j} u\big(t_k+,x\big) - \frac{d^j}{dt^j} u\big(t_k-,x\big) = f_j^k(x),\ j \in \mathbb{N}_{n-1}^0,\ k \in \mathbb{N},\ x \in \mathbb{R}^s, \\ \frac{\partial^l}{\partial t^l} u(t,x)_{|t=0} = f_l(x),\ \ x \in \mathbb{R}^s,\ l = 0,1,\cdot\cdot\cdot,n-1, \end{cases}$$

has a unique solution provided $f_l(\cdot) \in E_{0,n}$, $l \in \mathbb{N}_{n-1}^0$ *and* $f_j^k(\cdot) \in E_{0,n}$, $j \in \mathbb{N}_{n-1}^0$, $k \in \mathbb{N}$.

2. Let $z_k = t_k + is_k$, *where* $s_k \in \mathbb{R}$ $(k \in \mathbb{N})$. *Define*

$$u_h(z) := W_n(z)f_0 + \sum_{j=1}^{n-1} \int_0^z g_j(z-s)W_n(s)f_j\,ds + \omega(z), \quad z \in \mathbb{C},$$

where

$$\omega(z) := \begin{cases} 0, \quad \text{if } \Re z \leq t_1, \\ \sum_{k=1}^l W_n\big(z-t_k\big)f_j^k + \sum_{j=1}^{n-1} \int_0^z g_j(z-s)W_n\big(s-t_k\big)f_j^k\,ds, \\ \ \text{if } \Re z \in \big(t_k, t_{k+1}\big] \text{ for some } k \in \mathbb{N}_{l-1}^0. \end{cases}$$

Then the function $u(t) := u_h(z) + \omega(z)$, $z \in \mathbb{C}$ *is a unique solution of the following abstract impulsive Cauchy problem in the complex plane:*

$$\begin{cases} \frac{d^n}{dz^n} u(z,x) = \sum_{|\alpha| \leq d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha u(z,x),\ z \in \mathbb{C},\ x \in \mathbb{R}^s, \\ \lim_{z \to z_k,\ \Re z > t_k} \frac{d^j}{dz^j} u\big(z,x\big) - \lim_{z \to z_k,\ \Re z < t_k} \frac{d^j}{dz^j} u\big(z,x\big) = f_j^k(x), \\ \text{for any } j \in \mathbb{N}_{n-1}^0,\ k \in \mathbb{N},\ x \in \mathbb{R}^s, \\ \frac{\partial^l}{\partial z^l} u(z,x)_{|z=0} = f_l(x),\ \ x \in \mathbb{R}^s,\ l = 0,1,\cdot\cdot\cdot,n-1, \end{cases}$$

provided $f_l(\cdot) \in E_{0,n}$, $l \in \mathbb{N}_{n-1}^0$ *and* $f_j^k(\cdot) \in E_{0,n}$, $j \in \mathbb{N}_{n-1}^0$, $k \in \mathbb{N}$.

Without going into further details, we will only emphasize here that we can similarly consider the C-wel-lposedness of the following abstract degenerate impulsive Cauchy problem:

$$\begin{cases} \frac{d^n}{dt^n} \sum_{|\alpha| \leq d'} a'_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha u(t,x) = \sum_{|\alpha| \leq d} a_\alpha i^{|\alpha|}(\partial/\partial x)^\alpha u(t,x),\ t \geq 0,\ x \in \mathbb{R}^s, \\ \frac{d^j}{dt^j} u\big(t_k+,x\big) - \frac{d^j}{dt^j} u\big(t_k-,x\big) = f_j^k(x),\ j \in \mathbb{N}_{n-1}^0,\ k \in \mathbb{N},\ x \in \mathbb{R}^s, \\ \frac{\partial^l}{\partial t^l} u(t,x)_{|t=0} = f_l(x),\ \ x \in \mathbb{R}^s,\ l = 0,1,\cdot\cdot\cdot,n-1; \end{cases}$$

see ([28], Theorem 2.3.20, Remark 2.3.21) for more details about the subject.

3. It is obvious that we can similarly study the well-posedness of the abstract impulsive Cauchy problems in the angular domains of the complex plane, provided that the corresponding C-regularized solution operator family is analytic in a sector around the non-negative real axis.

3.1. The Abstract Impulsive Higher-Order Cauchy Problems

Let us consider now the following abstract impulsive higher-order Cauchy problem

$$(ACP)_n : \begin{cases} Bu^{(n)}(t) + A_{n-1}u^{(n-1)}(t) + ... + A_1u'(t) + A_0u(t) = f(t), \\ t \in [0,\infty) \setminus \{t_1,...,t_l,...\}, \\ \left(\Delta u^{(j)}\right)(t_k) = u^{(j)}(t_k+) - u^{(j)}(t_k-) = Cy_j^k, \quad k \in \mathbb{N},\ j \in \mathbb{N}_{n-1}^0, \\ u^{(j)}(0) = Cu_j, \quad j \in \mathbb{N}_{n-1}^0, \end{cases}$$

where $0 < t_1 < ... < t_l < ... < +\infty$, the sequence $(t_l)_l$ has no accumulation point and $B,\ A_{n-1},...,A_0$ are closed linear operators on X.

1. We will first assume that $B = I$ and the corresponding abstract Cauchy problem $(ACP)_n$ without impulsive effects is strongly C-well-posed in the sense of ([31], Definition 5.1), which means that there exists a strong C-propagation family

$$[(S_0(t))_{t\geq 0},...,S_{n-1}(t))_{t\geq 0}]$$

for this problem. Under certain logical assumptions (see e.g., ([31], p. 50) for the case in which $C = I$), the function

$$u_h = \sum_{j=0}^{n-1} S_j(t)u_k + \int_0^t S_{n-1}(t-s)C^{-1}f(s)\,ds, \quad t \geq 0$$

is a unique strong solution of the problem $(ACP)_n$ without impulsive effects. On the other hand, it is very simple to show that the function $\omega : [0,\infty) \to X$, defined by $\omega(t) := 0$, if $t \in [0,t_1]$, and $\omega(t) := \sum_{j=0}^{n-1}[S_j(t-t_1)y_j^1 + ... + S_j(t-t_k)y_j^k]$ if $t \in (t_k, t_{k+1}]$ for some $k \in \mathbb{N}$, is a unique solution of the following problem:

$$(ACP)_n^i : \begin{cases} \omega^{(n)}(t) + A_{n-1}\omega^{(n-1)}(t) + ... + A_1\omega'(t) + A_0\omega(t) = 0, \\ t \in [0,\infty) \setminus \{t_1,...,t_l,...\}; \\ \left(\Delta\omega^{(j)}\right)(t_k) = Cy_j^k, \quad k \in \mathbb{N},\ j \in \mathbb{N}_{n-1}^0, \\ \omega^{(j)}(0) = 0, \quad j \in \mathbb{N}_{n-1}^0. \end{cases}$$

Summa summarum, the function $u(t) := u_h(t) + \omega(t)$, $t \geq 0$ is a unique solution of the abstract impulsive higher-order Cauchy problem $(ACP)_n$ with $B = I$. Keeping in mind the results established in ([31], Theorem 5.6, Corollary 5.7, Example 5.8) and the above conclusion, one can simply consider the C-well-posedness of the abstract impulsive higher-order Cauchy problems in L^p-spaces; in particular, one can study the C-well-posedness of the damped Klein–Gordon equation with impulsive effects. In the same way as above, we can consider the C-well-posedness of the abstract impulsive higher-order Cauchy problem $(ACP)_{n;1}$, where the single-valued linear operator $\mathcal{A} = A$ satisfies the requirements of ([31], Theorem 6.2).

2. Let us consider now the general abstract Cauchy problem $(ACP)_n$ with $B \neq I$. We assume that there exists a global C_1-propagation family $(E(t))_{t\geq 0} \subseteq L(E,X)$ for this problem, where $C_1 \in L(E,X)$, as well as that $C_1u_i \in \mathbf{D}_i$ for $0 \leq i \leq n-1$ and $C_1A_j \subseteq A_jC_1$ for $0 \leq i,\ j \leq n-1$; concerning the inhomogeneity $f(\cdot)$, we will assume here that there is a function $g \in C^1([0,\infty) : E)$ satisfying $C_1g(t) = f(t)$, $t \geq 0$ (cf. ([28], Definition 2.3.29, Definition 2.3.31) for the notion). Then we know that the function

$$u_h(t) = \sum_{i=0}^{n-1} g_{i+1}(t)Cu_i - \sum_{i=0}^{n-1}\sum_{j\in\mathbb{N}_{n-1}^0\setminus D_i}\left(g_{n-j} * E^{(n-1-i)}\right)(t)A_ju_i$$
$$+ \int_0^t E(t-s)G(s)\,ds, \quad t \geq 0,$$

is a strong solution of problem $(ACP)_n$. Fix now an index $i \in \mathbb{N}^0_{n-1}$ and define

$$S_i(t)y := g_{i+1}(t)Cy - \sum_{j \in \mathbb{N}^0_{n-1} \setminus D_i} \Big(g_{n-j} * E^{(n-1-i)} \Big)(t) A_j y, \quad t \geq 0,$$

for any $y \in E$ such that $C_1 y \in \mathbf{D}_i$. Suppose now that $y^k_j \in E$ and $C_1 y^k_j \in \mathbf{D}_i$ for $0 \leq i,\ j \leq n-1$. Define a function $\omega : [0, \infty) \to X$ by $\omega(t) := 0$, if $t \in [0, t_1]$, and $\omega(t) := \sum_{j=0}^{n-1} [S_j(t-t_1) y^1_j + ... + S_j(t-t_k) y^k_j]$ if $t \in (t_k, t_{k+1}]$ for some $k \in \mathbb{N}$. Then the function $u(t) := u_h(t) + \omega(t)$, $t \geq 0$ is a strong solution of the abstract impulsive Cauchy problem $(ACP)_n$, as easily inspected. An application to the abstract differential impulsive Cauchy problems in L^p-spaces can be given following our consideration from ([28], Example 2.3.44).

3. In ([23], Section 5.7, Section 5.8), the abstract Cauchy problem $(ACP)_n$ without impulsive effects has been considered by using the operator matrices and the equivalent abstract degenerate first-order differential equation on the product space. Considering the solutions $S_i(t)$ of the abstract Cauchy problem ([23], (5.34), p. 163) with $u_j = 0$ for $j \neq i$ $(0 \leq i \leq n-1)$, we can similarly construct the function $\omega(t)$ solving the problem $(ACP)^i_n$ with the term $\omega^{(n)}(t)$ replaced therein with the term $B\omega^{(n)}(t)$. Following this approach, we can also analyze the well-posedness of the abstract Cauchy problem $(ACP)_n$ with impulsive effects.

4. On Abstract Impulsive Fractional Differential Inclusions

The main purpose of this section is to explain that the methods proposed in many published research articles cannot be adequately used in the analysis of the well-posedness of the abstract impulsive fractional differential inclusions with Caputo derivatives, unfortunately.

Suppose first that $I = (0, T)$ for some $T \in (0, \infty]$, $\alpha > 0$ and $m = \lceil \alpha \rceil$. The Caputo fractional derivative $\mathbf{D}^\alpha_t u(t)$ is traditionally defined for those functions $u \in C^{m-1}([0, \infty) : E)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0, \infty) : E)$, by

$$\mathbf{D}^\alpha_t u(t) := \frac{d^m}{dt^m} \left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1} \right) \right]; \tag{8}$$

here and hereafter, $u_k = u^{(k)}(0)$ for $0 \leq k \leq m-1$. In our striving to investigate the abstract impulsive integro-differential equations with Caputo fractional derivatives, we need to slightly weaken the assumption $u \in C^{m-1}([0, \infty) : E)$ in the above definition. We propose the following notion:

Definition 5. *A function $u : [0, T] \to X$ belongs to the space $\mathrm{A}_\alpha([0, T] : X)$ if and only if there exist $l \in \mathbb{N}$ and points $t_0 \equiv 0 < t_1 < t_2 < ... < t_l < T \equiv t_{l+1}$ such that the following conditions hold:*

- *(i) The function $u(\cdot)$ is $(m-1)$-times continuously differentiable on the intervals $[0, t_1)$, (t_1, t_2), (t_2, t_3), ..., $(t_l, T]$;*
- *(ii) The right derivatives $\lim_{t \to t_i+} u^{(j)}(t)$ exist for $0 \leq j \leq m-1$ and $1 \leq i \leq l$; the left derivatives $\lim_{t \to t_i-} u^{(j)}(t)$ exist for $0 \leq j \leq m-1$ and $1 \leq i \leq l+1$.*

It is clear that the assumption $u \in \mathrm{A}_\alpha([0, T] : X)$ implies that the function $u^{(j)}(\cdot)$ is essentially bounded on the segment $[0, T]$ for $0 \leq j \leq m-1$. In the following definition, we will introduce the following generalization of the Caputo fractional derivative $\mathbf{D}^\alpha_t u(t)$; the notion of Sobolev space $W^{m,1}((0, T) : X)$ is taken in the sense of [1]:

Definition 6. *(i) Suppose that $u \in \mathrm{A}_\alpha([0, T] : X)$. Then the Caputo fractional derivative $\mathbf{D}^\alpha_t u(t)$ is defined if and only if $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in W^{m,1}((0, T) : X)$, by (8).*

(ii) *Suppose that $u \in L^1_{loc}([0,\infty):X)$. Then the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined for $t \geq 0$ if and only if for each $T > 0$ we have $u_{|[0,T]} \in \mathrm{A}_\alpha([0,T]:X)$ and the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined on the segment $[0,T]$.*

The assumption $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in W^{m,1}((0,T):X)$ is almost mandatory in any reasonable definition of the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$. Unfortunately, this assumption has several very unpleasant consequences if we want to study the well-posedness of the abstract impulsive fractional differential inclusions with Caputo derivatives following certain methods proposed in the existing literature. More precisely, we have the following result:

Theorem 2. *Suppose that $\mathcal{A}$ is the integral generator of a local (g_α,k)-uniqueness C_2-resolvent family, where $\tau > T$, $\alpha \in (0,\infty) \setminus \mathbb{N}$ and $k(t)$ is a kernel. Suppose that $\omega \in \mathrm{A}_\alpha([0,T]:X)$ and*

$$\begin{cases} \mathbf{D}_t^\alpha \omega(t) \in \mathcal{A}\omega(t),\ t \in [0,T] \setminus \{t_1,t_2,...,t_l\}; \\ \omega^{(j)}(0) = 0, \quad 0 \leq j \leq m-1. \end{cases} \tag{9}$$

Then $\omega(t) = 0$ for all $t \in [0,T]$.

Proof. Since $\omega^{(j)}(0) = 0,\ 0 \leq j \leq m-1$ and $\omega(\cdot)$ is $(m-1)$-times continuously differentiable on $[0,t_1]$, we have that the function $(g_{m-\alpha} * \omega)(\cdot)$ is likewise $(m-1)$-times continuously differentiable on $[0,t_1]$ and $(g_{m-\alpha} * \omega)^{(j)}(0) = 0,\ 0 \leq j \leq m-1$. Applying the partial integration and the above equalities, we obtain:

$$\begin{aligned}\left(g_\alpha * \mathbf{D}_t^\alpha \omega\right)(t) &= \left(g_\alpha * \frac{d^m}{dt^m}\left[g_{m-\alpha} * \omega\right]\right)(t) \\ &= \left(g_{\alpha+1-m} * \frac{d}{dt}\left[g_{m-\alpha} * \omega\right]\right)(t), \quad t \in [0,T].\end{aligned}$$

Since the function $(g_{m-\alpha-m} * \omega)(\cdot)$ is absolutely continuous on $[0,T]$, the last equality implies

$$(g_1 * \omega)(t) = \left(g_{\alpha+1} * \mathbf{D}_t^\alpha \omega\right)(t), \quad t \in [0,T].$$

Hence,

$$(g_1 * \omega)(t) \in \left(g_\alpha * \mathcal{A}\left[g_1 * \omega\right]\right)(t), \quad t \in [0,T].$$

Then an application of ([28], Theorem 3.2.8(ii)) gives $(g_1 * \omega)(t) = 0,\ t \in [0,T]$. Since $\omega \in \mathrm{A}_\alpha([0,T]:X)$, the above yields $\omega(t) = 0$ for all $t \in [0,T]$. □

Remark 1. *Suppose that a piecewise-continuous function $u:[0,T] \to X$ is $(m-1)$-times continuously differentiable on the interval $[0,t_1)$. Then we can define the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ in the same way as in Definition 6. Then we have $\mathbf{D}_t^\alpha u(t) = D_t^\alpha u(t)$ provided that $u^{(j)}(0) = 0, 0 \leq j \leq m-1$; here, $D_t^\alpha u(t)$ denotes the Riemann–Liouville fractional derivative of function $u(t)$ defined by the equations [1, (1.11)-(1.12), p. 10]. Even in this situation, the proof of Theorem 2 shows that $\omega(t) = 0$ for all $t \in [0,T]$.*

Before going any further, we would like to observe that Prof. M. Fečkan, Y. Zhou and J. R. Wang have noticed, in the concluding remark of research article [25], that we must use certain generalizations of the Caputo fractional derivatives (the Riemann–Liouville fractional derivative or some other types of fractional derivatives) in order to study the well-posedness of the abstract impulsive fractional Cauchy problems. It is our strong belief that this is the only correct way for the investigations of the abstract impulsive fractional Cauchy problems and that anything else is completely misleading and wrong.

In a recent series of studies, many authors have established important results concerning the existence and uniqueness of the almost periodic type solutions for various classes of the abstract fractional differential inclusions with the Riemann–Liouville derivatives of order $\alpha \in (0,1)$; see e.g., the references quoted in [30]. We define here the Riemann–Liouville fractional derivative $D^{\alpha}_{t,+}u(t)$ in a very general manner: $D^{\alpha}_{t,+}u(t)$ is defined for those locally integrable functions $u : \mathbb{R} \to X$ such that, for almost every $t \in \mathbb{R}$, there exists $\delta_t > 0$ such that the function $s \mapsto g_{1-\alpha}(t-s)u(s)$, $s \in (t-\delta_t, t+\delta_t)$ belongs to the space $W^{1,1}((t-\delta_t, t+\delta_t) : X)$, by

$$D^{\alpha}_{t,+}u(t) := \frac{d}{dt}\int_{-\infty}^{t} g_{1-\alpha}(t-s)u(s)\,ds.$$

If $u \in L^1(\mathbb{R} : X)$, then the Fourier transform of the functions $u(t)$ and $D^{\alpha}_{t,+}u(t)$ can be defined and we have:

$$\Big(\mathcal{F}D^{\alpha}_{t,+}u\Big)(\lambda) = (i\lambda)^{\alpha}(\mathcal{F}u)(\lambda), \quad \lambda \in \mathbb{R}; \tag{10}$$

see e.g., ([3], Property 2.15, p. 90). Now we are ready to state and prove the following simple result:

Theorem 3. *Suppose that $\mathcal{A}$ is a closed* MLO *which satisfies that, for almost every $\lambda \in \mathbb{R}$, the multivalued linear operator $\mathcal{A}_{\lambda} = (i\lambda)^{\alpha} - \mathcal{A}$ is injective. Suppose that $\omega \in L^1(\mathbb{R} : X)$, $(t_l)_{l \in \mathbb{Z}}$ is a sequence without accumulation points and*

$$D^{\alpha}_{t,+}\omega(t) \in \mathcal{A}\omega(t),\ t \in \mathbb{R} \setminus \{..., t_{-l}, ..., t_0, ..., t_l, ...\}. \tag{11}$$

Then $\omega(t) = 0$ for a.e. $t \in \mathbb{R}$.

Proof. Since $\mathcal{A}$ is closed and $\omega \in L^1(\mathbb{R} : X)$, the use of (10) implies

$$[(i\lambda)^{\alpha} - \mathcal{A}](\mathcal{F}\omega)(\lambda) = 0$$

for all $\lambda \in \mathbb{R}$. Our assumption and the Riemann–Lebesgue lemma together imply that $(\mathcal{F}\omega)(\lambda) = 0$, $\lambda \in \mathbb{R}$, and therefore, $\omega(t) = 0$ for a.e. $t \in \mathbb{R}$. □

It is clear that Theorem 3 indicates that it is very difficult to study the existence and uniqueness of piecewise continuous solutions of the abstract fractional differential inclusion

$$D^{\alpha}_{t,+}u(t) \in \mathcal{A}u(t) + f(t), \quad t \in \mathbb{R} \setminus \{..., t_{-l}, ..., t_0, ..., t_l, ...\}.$$

Almost nothing can be said if $\mathcal{A}$ is a subgenerator of an exponentially bounded (g_{β}, g_{ζ})-regularized C-resovent family for some $\beta \in (\alpha, 1]$ and $\zeta \geq 1$ since, in this case, the multivalued linear operator $\mathcal{A}_{\lambda} = (i\lambda)^{\alpha} - \mathcal{A}$ is injective for every $\lambda \in \mathbb{R}$.

5. Abstract Volterra Integro-Differential Inclusions with Impulsive Effects

In this section, we consider the well-posedness of the following abstract impulsive Volterra integro-differential inclusion:

$$\begin{aligned} \mathcal{B}u(t) \subseteq & \mathcal{A}\int_0^t a(t-s)u(s)ds + \mathcal{F}(t), \quad t \in [0,T] \setminus \{t_1, ..., t_l\}; \\ & (\Delta u)(t_m) = Cy_m, \ \ m = 1, ..., l, \end{aligned} \tag{12}$$

where $0 \equiv t_0 < t_1 < ... < t_l < T \equiv t_{l+1}$, where $0 < T < \infty$, $a \in L^1_{loc}([0,T])$, $a \neq 0$, $\mathcal{F} \colon [0,T] \to P(E)$, and $\mathcal{A} \colon X \to P(E)$, $\mathcal{B} \colon X \to P(E)$ are two given mappings, as well as the well-posedness of the following abstract impulsive Volterra integro-differential inclusion:

$$\begin{aligned} \mathcal{B}u(t) \subseteq & \mathcal{A}\int_0^t a(t-s)u(s)ds + \mathcal{F}(t), \quad t \in [0,\infty) \setminus \{t_1, ..., t_l, ...\}; \\ & (\Delta u)(t_l) = Cy_l, \ \ l \geq 1, \end{aligned} \tag{13}$$

where $0 \equiv t_0 < t_1 < ... < t_l < t_{l+1} < ... < +\infty$, the sequence $(t_l)_l$ has no accumulation point, $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $\mathcal{F}\colon [0,\infty) \to P(E)$, and $\mathcal{A}\colon X \to P(E)$, $\mathcal{B}\colon X \to P(E)$ are two given mappings.

We will use the following notion:

Definition 7. *(i) A function $u \in \mathcal{PC}([0,T] : X)$ is said to be a pre-solution of (12) on $[0,T]$ if and only if $u(\cdot)$ is continuous on the set $[0,T] \setminus \{t_1,...,t_l\}$, $(a * u)(t) \in D(\mathcal{A})$ and $u(t) \in D(\mathcal{B})$ for $t \in [0,T] \setminus \{t_1,...,t_l\}$, as well as (12) holds.*

(ii) A solution of (12) on $[0,T]$ is any pre-solution $u(\cdot)$ of (12) on $[0,T]$ satisfying additionally that there exist functions $u_{\mathcal{B}} \in \mathcal{PC}([0,T] : E)$ and $u_{a,\mathcal{A}} \in \mathcal{PC}([0,T] : E)$, continuous on the set $[0,T] \setminus \{t_1,...,t_l\}$, such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$ and $u_{a,\mathcal{A}}(t) \in \mathcal{A}\int_0^t a(t-s)u(s)ds$ for $t \in [0,T] \setminus \{t_1,...,t_l\}$, as well as

$$u_{\mathcal{B}}(t) \in u_{a,\mathcal{A}}(t) + \mathcal{F}(t), \quad t \in [0,T] \setminus \{t_1,...,t_l\}.$$

(iii) A strong solution of (12) on $[0,T]$ is any function $u \in \mathcal{PC}([0,T] : X)$, continuous on the set $[0,T] \setminus \{t_1,...,t_l\}$, satisfying that there exist two functions $u_{\mathcal{B}} \in \mathcal{PC}([0,T] : E)$ and $u_{\mathcal{A}} \in \mathcal{PC}([0,T] : E)$, continuous on the set $[0,T] \setminus \{t_1,...,t_l\}$, such that $u_{\mathcal{B}}(t) \in \mathcal{B}u(t)$, $u_{\mathcal{A}}(t) \in \mathcal{A}u(t)$ for all $t \in [0,T] \setminus \{t_1,...,t_l\}$, and

$$u_{\mathcal{B}}(t) \in (a * u_{\mathcal{A}})(t) + \mathcal{F}(t), \quad t \in [0,T] \setminus \{t_1,...,t_l\}.$$

(iv) Suppose that $0 \equiv t_0 < t_1 < ... < t_l < t_{l+1} < ... < +\infty$ and the sequence $(t_l)_l$ has no accumulation point. By a (pre-)solution [solution, strong solution] of (13) we mean any function $u(\cdot)$ which satisfies that, for every $l \in \mathbb{N}$ and $T \in (t_l, t_{l+1})$, the function $u_{|[0,T]}(\cdot)$ is a (pre-)solution [solution, strong solution] of (12) on $[0,T]$.

The following important results can be simply deduced:

(i) If $\mathcal{A}$ and $\mathcal{B}$ are two MLOs and $\mathcal{A}$ is closed, then any strong solution of (12) on $[0,T]$ [(13)] is already a solution of (12) on $[0,T]$ [(13)].

(ii) If $\mathcal{B} = I$, $a(t)$ and $k(t)$ are kernels and $\mathcal{A}$ is a closed subgenerator of a mild (a,k)-regularized C_2-uniqueness family, then any pre-solution of (12) on $[0,T]$ [(13)] is unique (see [28], Proposition 3.2.8(ii)).

The following essential result can be simply reformulated in the global setting:

Theorem 4. *(i) Suppose $a(t)$ and $k(t)$ are kernels, $k(0) = 1$, $C_2 \in L(X)$ and $\mathcal{A}$ is a closed subgenerator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$, where $\tau > T$. Define $\mathcal{F}(t) := 0$ for $t \in [0,t_1]$ and $\mathcal{F}(t) := \sum_{s=1}^{m} k(t-t_s)C_2 y_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}_l$. Define also $u(t) := 0$ for $t \in [0,t_1]$ and $u(t) := \sum_{s=1}^{l} R_2(t-t_s)y_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}_l$. If $y_1,...,y_l \in D(\mathcal{A})$, then $u(t)$ is a unique strong solution of problem (12) on $[0,T]$, with the operator C replaced therein with the operator C_2.*

(ii) Suppose that $a(t)$ and $k(t)$ are kernels, $k(0) = 1$, $C_1 \in L(X,E)$ and $\mathcal{A}$ is a closed subgenerator of a mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)} \subseteq L(X,E)$ such that $R_1(0) = C_1$, where $\tau > T$. Define $\mathcal{F}(t)$ and $u(t)$ in the same way as above, with the operator C_2 replaced therein with the operator C_1 and the elements $y_1,...,y_l \in X$. Then $u(t)$ is a solution of problem (12) on $[0,T]$, with the operator C replaced therein with the operator C_1.

Proof. We will prove only (i). The uniqueness of a strong solution of problem (12) on $[0,T]$ follows from the above observations. Let $z_m \in \mathcal{A}y_m$, $m = 1,...,l$; then we have $R_2(t)y_m - k(t)C_2 y_m = \int_0^t a(t-s)R_2(s)z_m\,ds$ for all $t \in [0,\tau)$ and $m = 1,...,l$ so that $R_2(t-t_m)y_m - k(t-t_m)Cy_m = \int_0^{t-t_m} a(t-s-t_m)R_2(s)z_m\,ds$ for all $t \in [0,\tau)$ and $m = 1,...,l$. Adding these equalities for $s = 1,...,m$, we simply obtain that $u(t) = (a * u_{\mathcal{A}})(t) + \mathcal{F}(t)$ on $(t_m, t_{m+1}]$, where $m \in \mathbb{N}_l$ is fixed and $u_{\mathcal{A}}(t)$ is defined in the same way as $u(t)$ with the elements $y_1,...,y_l$ replaced therein with the elements $z_1,...,z_l$ (we only want to notice that it is necessary to divide the segment of the integration $[0, t]$ into the segments $[0, t_1]$, $[t_1, t_2]$, ..., $[t_{m-1}, t_m]$ and $[t_m, t]$). Since $k(0) = 1$, we have $R_2(0)x = C_2 x$ for all $x \in D(\mathcal{A})$, which simply completes the proof of theorem. $\square$

It is clear that Theorem 4 can be applied to a wide class of the abstract impulsive Volterra integro-differential inclusions; see the research monograph [28,29,32] and references cited therein for fairly complete information about the subject. Concerning the use of C-regularized solution operator families (integrated solution operator families and convoluted solution operator families with $k(0) = 0$ cannot be used for providing certain applications of Theorem 4), we may refer e.g., to ([29], Example 2.1.9, Example 2.1.10(ii); Section 2.5) and ([28], Theorem 3.2.21; see also pp. 323–324); for the sake of completeness, we will present the following illustrative applications of Theorem 4, only:

Example 4. *(i)* *Suppose that Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$ and $1 < p < \infty$. Let B be the multiplication in $L^p(\Omega)$ with $m(x)$, and let $A = \Delta - b$ act with the Dirichlet boundary conditions. Then our analysis from ([28], Example 3.2.23) shows that that there exists an operator $C_1 \in L(L^p(\Omega))$ such that the MLO $\mathcal{A} = -AB^{-1}$ is a subgenerator of an entire (g_1, g_1)-regularized C_1-existence family. Consider now the following degenerate Volterra integral equation associated to the abstract backward Poisson heat equation in the space $X = L^p(\Omega)$:*

$$(PR): \begin{cases} m(x)v(t,x) = u_0(x) + \int_0^t (-\Delta + b)v(s,x)\,ds, \\ t \in [0,\infty) \setminus \{t_1, ..., t_l, ...\},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(t_l+,x) - m(x)v(t_l-,x) = C_1\big[f_1(x) + ... + f_l(x)\big], \\ x \in \Omega,\ t \in \big(t_l, t_{l+1}\big] \ \ (l \in \mathbb{N}), \end{cases}$$

where $t_1 < t_2 < ... < t_l < ...$, the sequence (t_l) has no accumulation point and $f_l \in X$ for all $l \geq 1$. Then Theorem 4 and its proof yield that there exists a global solution of the problem (PR) which can be extended to an analytic function defined on the set $\mathbb{C} \setminus \{t_1, ..., t_l, ...\}$; note that we can also apply Theorem 1 here by assuming that $\{t_1, ..., t_l, ...\}$ is a complex sequence obeying certain properties, as well as that it is still not clear how we can consider the well-posedness of the fractional analogue of problem (PR) obtained by replacing the first equation of (PR) with the equation $m(x)v(t,x) = u_0(x) + \int_0^t g_\alpha(t-s)(-\Delta + b)v(s,x)\,ds$ for some number $\alpha \in (0,1)$.

(ii) *Our analysis from ([28], Example 3.10.7) and the second equality in ([1], (1.21)) shows that we can similarly analyze the following degenerate Volterra integral equation closely connected with the inverse generator problem and the abstract backward Poisson heat equation in the space $X = L^p(\Omega)$:*

$$\begin{cases} (\Delta - b)v(t,x) = (\Delta - b)v(0,x) + t\Big(\frac{d}{ds}[(\Delta - b)v(s,x)]\Big)_{s=0} \\ + \int_0^t g_\alpha(t-s)m(s)v(s,x)\,ds, \\ t \in [0,\infty) \setminus \{t_1, ..., t_l, ...\},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ (\Delta - b)v(t_l+,x) - (\Delta - b)v(t_l-,x) = C_1\big[f_1(x) + ... + f_l(x)\big], \\ x \in \Omega,\ t \in \big(t_l, t_{l+1}\big] \ \ (l \in \mathbb{N}), \end{cases}$$

where $t_1 < t_2 < ... < t_l < ...$, the sequence (t_l) has no accumulation point $f_l \in X$ for all $l \geq 1$ and a number $\alpha \in (1,2)$ satisfies an extra assumption.

We continue this section with the observation that the theory of abstract degenerate Volterra integro-differential equations is rather non-trivial as well as that the use of multivalued linear operators is not sufficiently adequate to cover all related problems within this theory. Consider now the following problem:

$$\begin{aligned} Bu(t) &= f(t) + \int_0^t a(t-s)Au(s)ds, \quad t \in [0,T] \setminus \{t_1, ..., t_l\}; \\ B(\Delta u)(t_m) &= CBy_m, \quad m = 1, ..., l, \end{aligned} \tag{14}$$

where $T \in (0,\infty)$ and $t \mapsto f(t)$ for $t \in [0,T]$ is a Lebesgue measurable mapping with values in X, $a \in L^1_{loc}([0,T])$ and A, B are closed linear operators with domain and range contained in X. We refer the reader to ([28], Definition 2.2.1) for the notion of a mild (strong) solution of (14) without impulsive effects; we similarly define the notion of a mild (strong) solution of (14) with impulsive effects.

The notion of an exponentially bounded (a,k)-regularized C-resolvent family for (14) has recently been introduced in ([28], Definition 2.2.2):

Definition 8. *Assume that the functions $a(t)$ and $k(t)$ satisfy (P1), as well as that $R(t) \colon D(B) \to E$ is a linear mapping for $t \geq 0$. Suppose that $C \in L(E)$ is injective and $CA \subseteq AC$. Then the operator family $(R(t))_{t\geq 0}$ is said to be an exponentially bounded (a,k)-regularized C-resolvent family for* (14) *if and only if there exists*

$$\omega \geq \max(0, \mathrm{abs}(a), \mathrm{abs}(k))$$

such that the following holds:

(i) The mapping $t \mapsto R(t)x$, $t \geq 0$ is continuous for every fixed element $x \in D(B)$.
(ii) There exist $M \geq 1$ and $\omega \geq 0$ such that $\|R(t)\| \leq Me^{\omega t}$, $t \geq 0$.
(iii) For every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$, the operator $B - \tilde{a}(\lambda)A$ is injective, $C(R(B)) \subseteq R(B - \tilde{a}(\lambda)A)$ and

$$\tilde{k}(\lambda)(B - \tilde{a}(\lambda)A)^{-1}CBx = \int_0^\infty e^{-\lambda t}R(t)x\,dt, \quad x \in D(B).$$

We will use the following lemma (cf. [28], Theorem 2.2.8):

Lemma 2. *Assume that the functions $|a|(t)$ and $k(t)$ satisfy* (P1)*, and let $(R(t))_{t\geq 0}$ is an exponentially bounded (a,k)-regularized C-resolvent family for* (14)*, satisfying* (ii) *of* Definition 8 *with $\omega \geq \max(0, \mathrm{abs}(|a|), \mathrm{abs}(k))$.*

(i) Suppose that $v_0 \in D(B)$ and the following condition holds:
  (i.1) for every $x \in D(B)$, there exista function $h(\lambda; x) \in LT - E$ and a number $\omega_0 > \omega$ such that $h(\lambda; x) = \tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}CBx$ provided $\tilde{k}(\lambda) \neq 0$ and $\Re\lambda > \omega_0$.

  Then the function $u(t) = R(t)v_0$, $t \geq 0$ is a mild solution of (14) *with $f(t) = k(t)CBv_0$, $t \geq 0$ and without impulsive effects. The uniqueness of mild solutions holds if we suppose additionally that $CB \subseteq BC$ and the function $k(t)$ satisfies* (P2).
(ii) Suppose that $CB \subseteq BC$, $v_0 \in D(A) \cap D(B)$ and the following condition holds:
  (ii.1) for every $x \in E$, there exist a function $h(\lambda; x) \in LT - E$ and a number $\omega_1 > \omega$ such that $h(\lambda; x) = \tilde{k}(\lambda)B(B - \tilde{a}(\lambda)A)^{-1}Cx$ provided $\tilde{k}(\lambda) \neq 0$ and $\Re\lambda > \omega_1$.

  Then the function $u(t) = R(t)v_0$, $t \geq 0$ is a strong solution of (14) *with $f(t) = k(t)CBv_0$, $t \geq 0$ and without impulsive effects. The uniqueness of strong solutions holds if we suppose additionally that the function $k(t)$ satisfies* (P2).

Let us note that the requirements of Lemma 2(i) imply that

$$A(a * R)(t)v_0 = BR(t)v_0 - k(t)CBv_0, \quad t \geq 0,\ v_0 \in D(B). \tag{15}$$

By applying (15), Lemma 2 and the argumentation contained in the proof of Theorem 4, one may simply obtain the following result:

Theorem 5. *(i) Suppose that the requirements of* Lemma 2(i) *hold and $k(0) = 1$. Define $\mathcal{F}(t) := 0$ for $t \in [0, t_1]$ and $\mathcal{F}(t) := \sum_{s=1}^{m} k(t - t_s)CBy_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}_l$. Define also $u(t) := 0$ for $t \in [0, t_1]$ and $u(t) := \sum_{s=1}^{l} R(t - t_s)y_s$ if $t \in (t_m, t_{m+1}]$ for some integer $m \in \mathbb{N}_l$. If $y_1, ..., y_l \in D(B)$, then $u(t)$ is a mild solution of problem* (14) *on $[0, T]$. The uniqueness of mild solutions of problem* (14) *holds if we suppose additionally that $CB \subseteq BC$ and the function $k(t)$ satisfies* (P2).

(ii) Suppose that the requirements of Lemma 2(ii) *hold and $k(0) = 1$. Define $\mathcal{F}(t)$ and $u(t)$ as above. If $y_1, ..., y_l \in D(A) \cap D(B)$, then $u(t)$ is a strong solution of problem* (14) *on $[0, T]$. The uniqueness of strong solutions of problem* (14) *holds if we suppose additionally that the function $k(t)$ satisfies* (P2).

An application of Theorem 5 can be given to the impulsive degenerate Volterra integral equations associated with the following degenerate fractional Cauchy problem in $X = L^p(\mathbb{R}^2)$, for example:

$$(P) : \begin{cases} \mathbf{D}_t^{\alpha}[u_{xx} + u_{xy} + u_{yy} - u] = e^{-i\alpha\frac{\pi}{2}}\left[(-1)^{l+1}\frac{\partial^{2l}}{\partial x^{2\alpha}}u + u_{yy}\right], & t \geq 0, \\ u(0, x) = \phi(x);\ u_t(0, x) = 0 \ \text{ if } \alpha \geq 1; \end{cases}$$

see ([28], Example 2.2.27(i)) for more details. We can similarly provide certain applications of exponentially bounded (a, k)-regularized C-resolvent families generated by a pair of closed linear operators A, B to the abstract impulsive degenerate Volterra integral equations; see ([28], Subsection 2.3.3) for more details.

We will not consider here the abstract degenerate multi-term Volterra integro-differential equations with impulsive effects; the interested reader may try to reconsider the problems from ([28], Example 2.3.43, Example 2.3.48) by adding certain impulsive effects therein. For the theory of the abstract degenerate Cauchy problems, we also refer the reader to the research monograph [33] by M. V. Plekhanova, V. E. Fedorov and the references quoted therein.

6. Conclusions and Final Remarks

In this paper, we have provided certain applications of (degenerate) (a, k)-regularized C-resolvent families subgenerated by the multivalued linear operators to the abstract impulsive Volterra integro-differential inclusions. This is probably the first research article which considers the possible applications of C-regularized solution operator families, (ultra-)distribution semigroups and (ultra-)distribution cosine functions in the theory of the abstract impulsive Volterra integro-differential inclusions. More to the point, this seems to be the first research article which concerns the well-posedness of the abstract impulsive higher-order Cauchy problems with integer order derivatives.

The almost periodic type solutions of the abstract impulsive Volterra integro-differential inclusions will be investigated in our forthcoming research article [34].

We close the paper by quoting a few important topics not considered in our previous work:

(1). It seems very plausible that we can similarly analyze the well-posedness of the abstract incomplete Cauchy inclusions with impulsive effects; for more details about the subject, we refer the reader to ([28], Section 2.7, Section 3.9).

(2). We have not considered here the abstract (degenerate) impulsive Volterra integro-differential equations of non-scalar type; cf. ([29], pp. 54–56) and ([28], Section 2.9) for further information in this direction.

(3). Let us finally mention that we have not considered here the abstract impulsive Volterra integro-differential inclusions on the line as well as the existence and uniqueness of discontinuous almost periodic (automorphic) type solutions for certain classes of the abstract impulsive Cauchy problems on the line; see e.g., ([29], pp. 51–53) for more details about this subject in non-degenerate case. Our work almost completely belongs to the realm of pure mathematics, and numerical simulations and illustrations will appear somewhere else.

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References

1. Bazhlekova, E. Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, Eindhoven University of Technology, Eindhoven, The Netherlands, 2001.
2. Diethelm, K. *The Analysis of Fractional Differential Equations*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 2010.
3. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North–Holland Mathematics Studies; Elsevier Science B. V.: Amsterdam, The Netherlands, 2006; Volume 204.
4. Kostić, M. *Abstract Volterra Integro-Differential Equations*; CRC Press: Boca Raton, FL, USA, 2015.
5. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
6. Bainov, D.; Simeonov, P. *Impulsive Differential Equations: Periodic Solutions and Applications*; Wiley: New York, NY, USA, 1993.
7. Bainov, D.; Simeonov, P. *Oscillation Theory of Impulsive Differential Equations*; International Publications: Geneva, Switzerland, 1998.
8. Faree, T.A.; Panchal, S.K. Existence of solution for impulsive fractional differential equations via topological degree method. *J. Korean Soc. Ind. Appl. Math.* **2021**, *25*, 16–25.
9. Halanay, A.; Wexler, D. *Qualitative Theory of Impulse Systems*; Mir: Moscow, Russia, 1971.
10. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific Publishing Co. Pte. Ltd.: Singapore, 1989.
11. Samoilenko, A.M.; Perestyuk, N.A. *Impulsive Differential Equations*; World Scientific: Singapore, 1995.
12. Stamov, G.T. *Almost Periodic Solutions of Impulsive Differential Equations*; Springer: Berlin/Heidelberg,Germany, 2012.
13. Wang, J.R.; Xiang, X.; Wei, W.; Chen, Q. Bounded and periodic solutions of semilinear impulsive periodic system on Banach spaces. *Fixed Point Theory Appl.* **2008**, *2008*, 401947. [CrossRef]
14. Wang, J.; Fečkan, M. A general class of impulsive evolution equations. *Topol. Meth. Nonlinear Anal.* **2015**, *46*, 915–934. [CrossRef]
15. Zhang, X.; Liu, Z.; Yang, S.; Peng Z.; He, Y.; Wei, L. The right equivalent integral equation of impulsive Caputo fractional-order system of order $\epsilon \in (1,2)$. *Fractal Fract.* **2023**, *7*, 37. [CrossRef]
16. Zhang, X.; Zhang, X.; Zhang, M. On the concept of general solution for impulsive differential equations of fractional order $q \in (0,1)$. *Appl. Math. Comp.* **2014**, *247*, 72–89. [CrossRef]
17. Zhou, Y.; Jiao, F. Existence of mild solutions for fractional neutral evolution equations. *Comp. Math. Appl.* **2010**, *59*, 1063–1077. [CrossRef]
18. Haloi, R. Some existence results on impulsive differential equations. In *Proceedings of the Sixth International Conference on Mathematics and Computing*; Giri, D., Buyya, R., Ponnusamy, S., De, D., Adamatzky, A., Abawajy, J.H., Eds.; Advances in Intelligent Systems and Computing; Springer: Singapore, 2021; Volume 1262.
19. Ranjinia, M.C.; Angurajb, A. Nonlocal impulsive fractional semilinear differential equations with almost sectorial operators. *Malaya J. Math.* **2013**, *2*, 43–53.
20. Benedetti, I.; Obukhovskii, V.; Taddei, V. On solvability of the impulsive Cauchy problem for integro-differential inclusions with non-densely defined operators. *Phil. Trans. R. Soc. A* **2021**, *379*, 20190384. [CrossRef] [PubMed]
21. Ke, T.D.; Kinh, C.T. Generalized Cauchy problem involving a class of degenerate fractional differential equations. *Dyn. Contin. Discret. Impuls. Syst. Ser. A Math. Anal.* **2014**, *21*, 449–472.
22. Xiao, T.-J.; Liang, J. Abstract degenerate Cauchy problems in locally convex spaces. *J. Math. Anal. Appl.* **2001**, *259*, 398–412.
23. Favini, A.; Yagi, A. *Degenerate Differential Equations in Banach Spaces*; Chapman and Hall/CRC Pure and Applied Mathematics: New York, NY, USA, 1998.

24. Fečkan, M.; Zhou, Y.; Wang, J.R. On the concept and existence of solution for impulsive fractional differential equations. *Commun. Nonlinear Sci. Numer. Simulat.* **2012**, *17*, 3050–3060. [CrossRef]
25. Fečkan, M.; Zhou, Y.; Wang, J.R. Response to "Comments on the concept of existence of solution for impulsive fractional differential equations [Commun Nonlinear Sci Numer Simul 2014;19:401??.]". *Commun. Nonlinear Sci. Numer. Simulat.* **2014**, *19*, 4213–4215. [CrossRef]
26. Wang, J.R.; Fečkan, M.; Zhou, Y. On the new concept of solutions and existence results for impulsive fractional evolution equations. *Dyn. Partial Differ. Equations* **2011**, *8*, 345–361.
27. Wang, J.R.; Li, X.; Wei, W. On the natural solution of an impulsive fractional differential equation of order $q \in (1, 2)$. *Commun. Nonlinear Sci. Numer. Simulat.* **2012**, *17*, 4384–4394. [CrossRef]
28. Kostić, M. *Abstract Degenerate Volterra Integro-Differential Equations*; Mathematical Institute SANU: Belgrade, Serbia, 2020.
29. Kostić, M. *Generalized Semigroups and Cosine Functions*; Mathematical Institute SANU: Belgrade, Serbia, 2011.
30. Kostić, M. *Almost Periodic and Almost Automorphic Type Solutions of Abstract Volterra Integro-Differential Equations*; W. de Gruyter: Berlin, Germany, 2019.
31. Xiao, T.-J.; Liang, J. *The Cauchy Problem for Higher Order Abstract Differential Equations*; Springer: Berlin, Germany, 1998.
32. Prüss, J. *Evolutionary Integral Equations and Applications*; Birkhäuser-Verlag: Basel, Switzerland, 1993.
33. Plekhanova, M.V.; Fedorov, V.E. *Optimal Control of Degenerate Evolution Systems*; Publication Center IIGU Univeristy: Chelyabinsk, Russia, 2013. (In Russian)
34. Du, W.-S.; Kostić, M.; Velinov, D. Almost Periodic Type Solutions of the Abstract Impulsive Volterra Integro-Differential Inclusions. Preprint. Available online: https://www.researchgate.net/publication/366547982 (accessed on 1 December 2022).