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Nonlocal Problems for Hilfer Fractional q-Difference Equations

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Abstract: In the paper, we investigate a kind of Hilfer fractional q-difference equations with nonlocal condition. Firstly, the existence and uniqueness results of solutions are obtained by using topological degree theory and Banach fixed point theorem. Subsequently, the existence of extremal solutions in an ordered Banach space is discussed by monotone iterative method. In that following, we consider the Ulam stability results for equations. Finally, two examples are given to illustrate the effectiveness of theory results.

Keywords: Hilfer fractional q-difference; topological degree theory; extremal solution; Ulam stability; nonlocal condition

1. Introduction

Fractional differential equations have a deep physical background (see [1–4]), which are more accurate in describing many natural phenomena as compared to integer ones. Many scholars have been devoted to the study of nonlocal Hilfer fractional differential equations (see [5–13]). Fractional q-calculus theory is an important branch of discrete mathematics. With the increasing popularity and development of information technology, q-differential is increasingly applied to natural science and engineering, especially in mathematical physical models, dynamical systems, quantum physics and economics, the literatures [14–17] were first detailed the definition and introduction of the q-calculus.

With the attention of many experts and scholars on fractional q-difference, rich results have been achieved on fractional q-difference equations via q-Gronwall equality (see [18]), the existence and stability of the solutions for Riemann–Liouville fractional q-difference equations (see [19–33]), Caputo fractional q-difference initial boundary value problems (see [34–39]). In [40], Boutiara explored the mixed multi-term fractional q-difference equations with q-integral boundary conditions by using topological degree theory. However, there are few studies on the problem of Hilfer fractional q-difference equations. In [41], Ahmed et al. introduced the definition of Hilfer fractional q-derivative, and discussed the uniqueness of solution for Hilfer fractional hybrid q-integro-difference equation of variable order:

$$\begin{cases} {}_qD_t^{\alpha(t),\beta}[x(t)-f(t,x(t))] = g(t,x(t),{}_qI_t^{\beta}x(t)), \quad t\in(0,T], \\ {}_qI_t^{1-\gamma(t)}x(0)=x_0, \quad {}_qI_t^{1-\gamma(t)}f(0,x(0))=f_0, \quad \gamma(t)=\alpha(t)+\beta-\alpha(t)\beta, \end{cases}$$

where $0<\alpha(t)<1, 0\leq\beta\leq 1, 0<q<1$.

Based on the above discussion, the main objective of the paper is to study the nonlocal initial value problem and Ulam stability for Hilfer fractional q-difference equation as follows.

$$\begin{cases} {}_{a^+}D_q^{\alpha,\beta}u(t) = f(t,u(t),Hu(t),Gu(t)), \quad t\in I=(a,T], \\ {}_{a^+}I_q^{(1-\alpha)(1-\beta)}u(a^+) = \sum\limits_{i=1}^{m}\lambda_i u(\pi_i), \quad \pi_i\in(a,T], \end{cases} \tag{1}$$

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Citation: Tian, C.; Gu, H.; Yang, Z. Nonlocal Problems for Hilfer Fractional q-Difference Equations. *Fractal Fract.* **2023**, *7*, 101. https://doi.org/10.3390/fractalfract7020101

Academic Editors: Sotiris K. Ntouyas, Jessada Tariboon and Bashir Ahmad

Received: 17 October 2022
Revised: 13 January 2023
Accepted: 14 January 2023
Published: 17 January 2023

where ${}_{a^+}D_q^{\alpha,\beta}$ denotes the Hilfer fractional q-derivative of order α and type β with lower limit a, ${}_{a^+}I_q^{(1-\alpha)(1-\beta)}$ is the Riemann–Liouville fractional q-integral as presented in Section 2. $0 < \alpha < 1, 0 \le \beta \le 1, 0 < q < 1, 0 < \lambda_i < 1$, the given function $f : I \times R^3 \to R$, and $\{\pi_i, i = 1, 2, \cdots, m\}$ are fixed points satisfying $a < \pi_1 \le \cdots \le \pi_m \le T$. The operators H and G are give by

$$Hu(t) = \int_a^t (t - qs)_q^{v-1} h(t, s, u(s)) d_q s, \qquad Gu(t) = \int_a^t (t - qs)_q^{w-1} g(t, s, u(s)) d_q s,$$

$h, g : I^2 \times R \to R$, $0 \le v, w \le 1$. In this work, different from the previous results, we mainly studied the existence and uniqueness of solution in a weighted space of continuous functions for the nonlocal problem (1). Moreover, we give the sufficient conditions to discuss the existence of extremal solution in an ordered Banach space, and the solution of nonlocal problem (1) is Ulam–Hyers stable under some certain conditions.

The remaining structure of paper is organized as follows. In Section 2, we introduce some notations and recall some definitions and lemmas. Subsequently, we obtain the existence and uniqueness results of solutions to nonlocal problem (1) in Section 3. Moreover, in Section 4, we discuss the existence of extremal solutions of nonlocal problem (1) in an ordered Banach space. In Section 5, we also consider the Ulam–Hyers stability and Ulam–Hyers–Rassias stability results for nonlocal problem (1). Two examples are given to illustrate theory results in Section 6. Finally, some conclusions are given.

2. Preliminaries

In the section, we will give some notations, definitions and lemmas for fractional q-calculus.

Let $0 < a < T < +\infty$, $I' = [a, T]$, $X = C(I', R)$ be the Banach space of all continuous functions from I' to R with the norm $\|u\|_C = max\{|u(t)| : t \in I'\}$. $C^n(I', R)$ denotes the space of n times continuously differentiable functions on I', $\mathfrak{D}_C$ represents the class of all bounded mapping in $C(I', R)$.

For $\gamma = \alpha + \beta - \alpha\beta$, we have $0 < \gamma \le 1$, $\gamma \ge \alpha$, $\gamma \ge \beta$, $(1 - \alpha)(1 - \beta) = 1 - \gamma < 1 - \beta(1 - \alpha)$.

We consider the weighted space of continuous functions:

$$C_\gamma(I') = \{u : I \to R |\ (t - a)_q^\gamma\ u(t) \in C(I', R)\}, \quad 0 \le \gamma < 1$$

and

$$C_\gamma^n(I') = \{u \in C^{n-1}(I, R) |\ u^{(n)} \in C_\gamma(I')\}, \quad C_\gamma^0(I') = C_\gamma(I'), \quad n \in N.$$

with the norms

$$\|u\|_{C_\gamma} = \|(t - a)_q^\gamma u(t)\|_C \quad \text{and} \quad \|u\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \|u^{(k)}\|_C + \|u^{(k)}\|_{C_\gamma}.$$

Moreover,

$$C_{1-\gamma}^\gamma(I') = \{u \in C_{1-\gamma}(I'), {}_{a^+}D_q^\gamma u \in C_{1-\gamma}(I')\}, \quad C_{1-\gamma}^{\alpha,\beta}(I') = \{u \in C_{1-\gamma}(I'), {}_{a^+}D_q^{\alpha,\beta} u \in C_{1-\gamma}(I')\},$$

these spaces satisfy the following properties:

$$C_{1-\gamma}^\gamma(I') \subset C_{1-\gamma}^{\alpha,\beta}(I'), \quad C_{\gamma_1}(I') \subset C_{\gamma_2}(I'), \quad 0 \le \gamma_1 \le \gamma_2.$$

Let $B_r^{C_{1-\gamma}}(I') = \{u \in C_{1-\gamma}(I') : \|u\|_{C_{1-\gamma}} \le r\}$ is bounded convex and closed subset of $C_{1-\gamma}(I')$.

Definition 1 ([14]). *For $a \in R$, let $[a]_q = \frac{1-q^a}{1-q}$. The q-analogue of the power $(t-s)_q^n$ is*

$$(t-s)_q^0 = 1, \quad (t-s)_q^n = \prod_{k=0}^{n-1}(t-q^k s), \quad n \in N,$$

$$(t-s)_q^\alpha = t^\alpha \prod_{k=0}^{\infty} \frac{t-q^k s}{t-q^{k+\alpha} s}, \quad 0 \le s \le t, \quad \alpha \in \mathbb{C}\backslash\{\pm n\}, \ (\mathbb{C} \text{ be the set of complex numbers}).$$

Definition 2 ([14,17]). *For $|\ q\ | < 1$, the q-Gamma function is defined as*

$$\Gamma_q(\alpha) = \frac{(1-q)^{1-\alpha}}{(1-q)_q^{1-\alpha}}, \ \alpha \in \mathbb{C} \setminus \{-n : n \in N_0\}.$$

Notice that the q-Gamma function satisfies

$$\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha) = \frac{1-q^\alpha}{1-q}\Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad \alpha > 0.$$

Definition 3 ([14]). *For any $\alpha, \beta > 0$, $q \in (0,1)$, the q-Beta function is defined by*

$$B_q(\alpha, \beta) = \int_0^1 t_q^{\alpha-1}(1-qt)_q^{\beta-1} d_q t,$$

in particular $B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$.

Let a typical q-geometry set be the time scale set defined by $T_q = \{0, q^n : n \in Z\}$, where $0 < q < 1$, $Z = \{0, \pm 1, \pm 2, \cdots\}$.

Definition 4 ([14]). *For $u : T_q \to R$, the q-integral of u is defined as:*

$$I_q u(t) = \int_0^t u(s) d_q s = (1-q)t \sum_{i=0}^{\infty} q^i u(tq^i),$$

for $0 \le a \in T_q$, $\int_a^t u(s) d_q s = \int_0^t u(s) d_q s - \int_0^a u(s) d_q s$.

Definition 5 ([14]). *The q-derivative of function $u : T_q \to R$ is defined as:*

$$D_q u(t) = \frac{d_q u}{d_q t} = \frac{u(t) - u(qt)}{(1-q)t}, \quad t \in T_q \backslash \{0\},$$
$$D_q u(0) = \frac{d_q u}{d_q t}|_{t=0} = \lim_{n\to\infty} \frac{u(tq^n) - u(0)}{tq^n}, \quad t \neq 0.$$

the higher order q-derivatives $D_q^n u(t)$ is defined by $D_q^0 u(t) = u(t)$, $D_q^n u(t) = D_q(D_q^{n-1} u)(t)$, $n \ge 1$.

In particular $D_q I_q f(t) = f(t)$, if f is continuous at 0, then $I_q D_q f(t) = f(t) - f(0)$.

Definition 6 ([17]). *Let $u : T_q \to R$, $\alpha \neq -k$, $k \in N$, then Riemann–Liouville fractional q-integral of order $\alpha > 0$ is defined as*

$${}_{a^+}I_q^\alpha u(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} u(s) d_q s,$$

in particular ${}_aI_q^\alpha(1) = \frac{(t-a)_q^\alpha}{\Gamma_q(\alpha+1)}$, $t > 0$.

Definition 7 ([17]). *Let $u : T_q \to R$, $\alpha > 0$, $n = \lceil \alpha \rceil$ is a minimum integer greater than or equal to the α, then the Riemann–Liouville fractional q-derivative of order α of function u is defined by $D_q^0 u(t) = u(t)$, and*

$$ {}_{a^+}D_q^{\alpha} u(t) = D_q^n \, {}_{a^+}I_q^{n-\alpha} u(t) = \frac{1}{\Gamma_q(n-\alpha)} \left(\frac{d_q}{d_q t}\right)^n \int_a^t (t-qs)_q^{n-\alpha-1} u(s) d_q s. $$

Definition 8 ([16]). *The Caputo fractional q-derivative of order $\alpha > 0$ of function $u : T_q \to R$ is defined as ${}^C D_q^0 u(t) = u(t)$, and*

$$ {}_{a^+}^{C}D_q^{\alpha} u(t) = {}_{a^+}I_q^{n-\alpha} D_q^n u(t), \; t \in I. $$

Definition 9 ([41]). *The Hilfer fractional q-derivative of order $0 < \alpha < 1$ and $0 \le \beta \le 1$ with lower limit a of function $u(t)$ is defined by*

$$ {}_{a^+}D_q^{\alpha,\beta} u(t) = ({}_{a^+}I_q^{\beta(1-\alpha)} D_q ({}_{a^+}I_q^{(1-\beta)(1-\alpha)} u))(t), $$

where $D_q = \frac{d_q}{d_q t}$.

Remark 1 ([41]). *The Hilfer fractional q-derivative can be viewed as a generalization of the Riemann–Liouville and Caputo q-derivative:*

(i) The operator ${}_{a^+}D_q^{\alpha,\beta}$ also can be rewritten as

$$ {}_{a^+}D_q^{\alpha,\beta} u(t) = ({}_{a^+}I_q^{\beta(1-\alpha)} D_q ({}_{a^+}I_q^{(1-\beta)(1-\alpha)} u))(t) = {}_{a^+}I_q^{\beta(1-\alpha)} \, {}_{a^+}D_q^{\gamma} u(t), \; \gamma = \alpha + \beta - \alpha\beta. $$

(ii) Let $\beta = 0$, the Riemann–Liouville fractional q-derivative can be presented as ${}_{a^+}D_q^{\alpha} := {}_{a^+}D_q^{\alpha,0}$.

(iii) Let $\beta = 1$, the Caputo fractional q-derivative can be presented as ${}_{a^+}^{C}D_q^{\alpha} := {}_{a^+}I_q^{1-\alpha} D_q$.

Lemma 1 ([17]). *Let $0 < \alpha < 1$, $0 \le \gamma < 1$. If $u \in C_\gamma(I')$ and ${}_{a^+}I_q^{1-\alpha} u \in C_\gamma^1(I')$, then*

$$ ({}_{a^+}I_q^{\alpha} \, {}_{a^+}D_q^{\alpha} u)(t) = u(t) - \frac{{}_{a^+}I_q^{1-\alpha} u(a^+)}{\Gamma_q(\alpha)} (t-a)_q^{\alpha-1}, \quad t \in I. $$

Lemma 2 ([17]). *Let $\alpha > 0$, $\beta > 0$, $\gamma = \alpha + \beta - \alpha\beta$, If $u \in C_{1-\gamma}^{\gamma}(I')$, then*

$$ {}_{a^+}I_q^{\gamma} \, {}_{a^+}D_q^{\gamma} u(t) = {}_{a^+}I_q^{\alpha} \, {}_{a^+}D_q^{\alpha,\beta} u(t), \quad {}_{a^+}D_q^{\gamma} \, {}_{a^+}I_q^{\alpha} u(t) = {}_{a^+}D_q^{\beta(1-\alpha)} u(t). $$

Lemma 3. *Let $u \in L^1(I')$ and ${}_{a^+}D_q^{\beta(1-\gamma)} u \in L^1(I')$ existed, then*

$$ {}_{a^+}D_q^{\alpha,\beta} \, {}_{a^+}I_q^{\alpha} u = {}_{a^+}I_q^{\beta(1-\alpha)} \, {}_{a^+}D_q^{\beta(1-\alpha)} u. $$

Proof. By the literature [17], we have ${}_{a+}D_q^{\alpha} \, {}_{a+}I_q^{\beta} u(t) = {}_{a+}I_q^{\beta-\alpha} u(t)$ $(\beta \ge \alpha \ge 0)$, which obtains

$$ {}_{a+}D_q^{\alpha,\beta} \, {}_{a+}I_q^{\alpha} u = {}_{a+}I_q^{\beta(1-\alpha)} D_q ({}_{a+}I_q^{(1-\beta)(1-\alpha)} \, {}_{a+}I_q^{\alpha}) u = {}_{a+}I_q^{\beta(1-\alpha)} D_q ({}_{a+}I_q^{1-\beta(1-\alpha)}) u = {}_{a+}I_q^{\beta(1-\alpha)} \, {}_{a+}D_q^{\beta(1-\alpha)} u. $$

□

Lemma 4 ([34,38]). *Suppose $\alpha > 0$, $\varphi(t)$ is a nonnegative function locally integrable on $a \le t < T$(some $T \le +\infty$) and $m(t)$ is a nonnegative, nondecreasing continuous function defined on $a \le t < T$, $w(t) \le K$(constant), and suppose $u(t)$ is nonnegative and locally integrable on $a \le t < T$ with $u(t) \le w(t) + m(t) {}_{a^+}I_q^{\alpha} u(t)$, then $u(t) \le w(t) + \sum\limits_{i=0}^{\infty} (m(t)\Gamma_q(\alpha))^i \, {}_{a^+}I_q^{i\alpha} w(t)$.*

Next, we introduce some propositions and definitions about the Kuratowski non-compactness measure $\varpi : \mathfrak{D}_C \to [0, \infty)$ in [42,43].

$\varpi(U) = inf\{\epsilon > 0 : \text{U can be covered by finitely many sets with diameter} \leq \epsilon\}.$

Proposition 1 ([42,43])**.** *The Kuratowski measure of non-compactness ssatisfies some proposition:*

(i) $U \subset V \Rightarrow \varpi(U) \leq \varpi(V),\ U, V \subset X,$

(ii) $\varpi(U) = 0$ *if and only if* U *is relatively compact,*

(iii) $\varpi(U) = \varpi(\bar{U}) = \varpi(conv(U))$*, where* $\bar{U}$ *and* $conv(U)]$ *represent the closure and the convex hull of* U*, respectively,*

(iv) $\varpi(U + V) \leq \varpi(U) + \varpi(V),$

(v) $\varpi(\lambda U) = |\lambda|\varpi(U),\ \lambda \in R.$

Definition 10 ([42,43])**.** *Let* $\mathcal{U} : U \to X$ *be a bounded continuous map and* $U \subset X$*. The operator* $\mathcal{U}$ *is* ϖ*-Lipschitz if there exists constant* $l \geq 0$ *such that* $\varpi(\mathcal{U}(V)) \leq l\varpi(V),\ V \subset U$*. Moreover,* $\mathcal{U}$ *is called a strict* ϖ*-contraction if* $l < 1$*.*

Definition 11 ([42,43])**.** *For bounded and non-precompact subset* V *of* U*, the mapping* $\mathcal{U}$ *is said to be* ϖ*-condensing if* $\varpi(\mathcal{U}(V)) < \varpi(V)$*. That is,* $\varpi(\mathcal{U}(V)) \geq \varpi(V)$ *implies* $\varpi(V) = 0$*.*

Proposition 2 ([44])**.** *(i) If* $\mathcal{U}, \mathcal{V} : U \to X$ *are* ϖ*-Lipschitz with constants* l_1 *and* l_2 *respectively, then* $\mathcal{U} + \mathcal{V} : U \to X$ *is* ϖ*-Lipschitz with constant* $l_1 + l_2$*.*

(ii) If $\mathcal{U} : U \to X$ *is compact, then* $\mathcal{U}$ *is* ϖ*-Lipschitz with* $l = 0$*.*

(iii) If $\mathcal{U} : U \to X$ *is Lipschitz with constant* l*, then* $\mathcal{U}$ *is* ϖ*-Lipschitz with constant* l*.*

Lemma 5 ((Topological degree theory) [44])**.** *Let* $\mathbb{F} : U \to X$ *be* ϖ*-condensing and*

$$\Omega = \{u \in U :\ \text{there exists } \xi \in [0, 1] \text{ such that } u = \xi\mathbb{F}u\}.$$

If Ω *is a bounded set in* X*, then there exists* $r > 0$ *such that* $\Omega \subset B_r(0)$*, and the degree*

$$deg(I - \xi\mathbb{F}, B_r(0), 0) = 1, \forall\xi \in [0, 1].$$

Consequently, $\mathbb{F}$ *has at least one fixed point and the set of the fixed points of* $\mathbb{F}$ *lies in* $B_r(0)$*.*

3. Existence and Uniqueness of Solution

In the section, we obtain the equivalent Volterra integral equation corresponding to the nonlocal problem (1), and further obtain the existence results of solution. Throughout the article, we let $f_u(t) = f(t, u(t), Hu(t), Gu(t))$.

Now, we introduce some hypotheses as follows.

(H_1) $f : I \times R^3 \to R$ satisfies $f \in C^{\beta(1-\alpha)}_{1-\gamma}(I')$ for any $u \in C_{1-\gamma}(I')$. For each $u, v \in C_{1-\gamma}(I')$, there exists constant $L > 0$ such that

$$|f(t, u, Hu, Gu) - f(t, v, Hv, Gv)| \leq L(|u - v| + |Hu - Hv| + |Gu - Gv|).$$

(H_2) For $u, v \in X$, $h, g : I^2 \times R \to R$ are continuous functions, and there exist constants $\theta_i, l_i > 0, (i = 1, 2)$, such that

$$|h(t, s, u(s)| \leq \theta_1|u(t)|, \quad |g(t, s, u(s)| \leq \theta_2|u(t)|,$$
$$|h(t, s, u) - h(t, s, v)| \leq l_1|u - v|, \quad |g(t, s, u) - g(t, s, v)| \leq l_2|u - v|.$$

Lemma 6. *Let* $f_u(t) \in C_{1-\gamma}(I')$ *for any* $u \in C_{1-\gamma}(I')$*.* $u \in C^{\gamma}_{1-\gamma}(I')$ *is a solution of the nonlocal initial value problem (1) is equivalent to* u *satisfies the following Volterra integral equation*

$$u(t) = \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1} f_u(s) d_q s + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1} f_u(s) d_q s, \quad (2)$$

where $P = \frac{1}{\Gamma_q(\gamma) - \sum_{i=1}^{m} \lambda_i (\pi_i - a)_q^{\gamma-1}}$.

Proof. Referring to [41], We obtain the following procedure.

Since ${}_{a+}I_q^{1-\gamma} u \in C[a,T]$ and ${}_{a+}D_q^{\gamma} u = D_{q\,a+}I_q^{1-\gamma} \in C_{1-\gamma}(I')$, then ${}_{a+}I_q^{1-\gamma} u \in C_{1-\gamma}^{1}(I')$, by Lemma 1

$$ {}_{a+}I_q^{\gamma}\,{}_{a+}D_q^{\gamma} u(t) = u(t) - \frac{{}_{a+}I_q^{1-\gamma} u(a^+)}{\Gamma_q(\gamma)} (t-a)_q^{\gamma-1}, \quad t \in I. \tag{3} $$

By ${}_{a+}D_q^{\gamma} u \in C_{1-\gamma}(I')$, and Lemma 2 we have

$$ {}_{a+}I_q^{\gamma}\,{}_{a+}D_q^{\gamma} u = {}_{a+}I_q^{\alpha}\,{}_{a+}D_q^{\alpha,\beta} u = {}_{a+}I_q^{\alpha} f, \quad t \in I, \tag{4} $$

by (3) and (4) we obtain

$$ u(t) = \frac{{}_{a+}I_q^{1-\gamma} u(a^+)}{\Gamma_q(\gamma)} (t-a)_q^{\gamma-1} + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f(s,u(s),Fu(s),Gu(s)) d_q s. \tag{5} $$

Substituting $t = \pi_i$ into (5), we have

$$ u(\pi_i) = \frac{{}_{a+}I_q^{1-\gamma} u(a^+)}{\Gamma_q(\gamma)} (\pi_i - a)_q^{\gamma-1} + \frac{1}{\Gamma_q(\alpha)} \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1} f(s,u(s),Fu(s),Gu(s)) d_q s $$

Thus, we have

$$ \begin{aligned} & {}_{a+}I_q^{1-\gamma} u(a^+) = \sum_{i=1}^{m} \lambda_i u(\pi_i) \\ = \; & \frac{{}_{a+}I_q^{1-\gamma} u(a^+)}{\Gamma_q(\gamma)} \sum_{i=1}^{m} \lambda_i (\pi_i - a)_q^{\gamma-1} + \sum_{i=1}^{m} \lambda_i \frac{1}{\Gamma_q(\alpha)} \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1} f(s,u(s),Fu(s),Gu(s)) d_q s, \end{aligned} $$

then it implies

$$ {}_{a+}I_q^{1-\gamma} u(a^+) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\alpha)} P \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1} f(s,u(s),Fu(s),Gu(s)) d_q s, \tag{6} $$

where $P = \frac{1}{\Gamma_q(\gamma) - \sum_{i=1}^{m} \lambda_i (\pi_i - a)_q^{\gamma-1}}$.

Submitting (6) to (5), we obtain

$$ u(t) = \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1} f_u(s) d_q s + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f_u(s) d_q s. $$

□

By Lemma 2, we consider the following operators $A, B : X \to X$:

$$ \begin{aligned} Au(t) &= \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1} f_u(s) d_q s, \quad t \in I, \\ Bu(t) &= \frac{1}{\Gamma_q(\alpha)} \int_a^t (t-qs)_q^{\alpha-1} f_u(s) d_q s, \quad t \in I, \end{aligned} $$

then the right side of integral Equation (2) can be written as the operator

$$ Fu(t) = Au(t) + Bu(t), \; t \in I. $$

Obviously, by the continuity of f, operator F is well defined.

Let $\eta_{h_1} = \frac{\theta_1(T-a)_q^v}{[v]_q}$, $\eta_{h_2} = \frac{l_1(T-a)_q^v}{[v]_q}$, $\eta_{g_1} = \frac{\theta_2(T-a)_q^w}{[w]_q}$, $\eta_{g_2} = \frac{l_2(T-a)_q^w}{[w]_q}$, $\hat{\lambda} = \max\{\lambda_i, i = 1, 2, \cdots, m\}$.

Lemma 7. *Under the assumptions (H_1)–(H_2), A is continuous in the weighted space $C_{1-\gamma}(I')$, and satisfies the inequality as below*

$$\|Au\|_{C_{1-\gamma}} \leq \rho_1[L(1+\eta_{h_1}+\eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}], \quad u \in X,$$

where $\rho_1 = \frac{\Gamma_q(\gamma)|P|}{\Gamma_q(\gamma+\alpha)} \sum_{i=1}^{m} \lambda_i(\pi_i - a)_q^{\alpha+\gamma-1}, \quad f_0(s) = f(s,0,0,0).$

Proof. We choose a bounded subset $B_r^{C_{1-\gamma}}(I') = \{u \in C_{1-\gamma}(I') : \|u\|_{C_{1-\gamma}} \leq r\} \subset X$, and consider a sequence $\{u_n\} \in B_r^{C_{1-\gamma}}(I')$ such that $\lim_{n\to\infty} u_n = u \in B_r^{C_{1-\gamma}}(I')$, we need to prove that $\|Au_n - Au\|_{C_{1-\gamma}} \to 0$, $n \to \infty$.

By the continuity of $f_{u_n}(t)$ and $f_u(t)$, it follows that $|f_{u_n}(s) - f_u(s)| \to 0$, as $n \to \infty$, also

$$\left|(t-a)_q^{1-\gamma}(f_{u_n}(s) - f_u(s))\right| \leq (T-a)_q^{1-\gamma}|f_{u_n}(s) - f_u(s)|,$$

which implies that the left term is bounded and integrable, by the Lebesgue dominated convergent theorem, we obtain

$${}_{a^+}I_q^{\alpha}\|f_{u_n} - f_u\|_{C_{1-\gamma}} \to 0, \quad n \to \infty,$$

so we have $\|Au_n - Au\|_{C_{1-\gamma}} \to 0$ as $n \to \infty$, which implies the continuity of the operator A.

Let $f_0(s) = f(s,0,0,0)$, by the assumption (H_1) and (H_2), we have

$$\begin{aligned}
&\left|(t-a)_q^{1-\gamma}Au(t)\right| \\
\leq\ & \frac{|P|}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1}|f_u(s)|d_qs \\
\leq\ & \frac{|P|}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1}(|f_u(s) - f_0(s)| + |f_0(s)|)d_qs \\
\leq\ & \frac{|P|}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1}[L(|u(s)| + |Hu(s)| + |Gu(s)|) + |f_0(s)|]d_qs \\
\leq\ & \frac{|P|}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i(\pi_i - a)_q^{\alpha+\gamma-1}B_q(\gamma,\alpha)\left[L\left(\|u\|_{C_{1-\gamma}} + \|Hu\|_{C_{1-\gamma}} + \|Gu\|_{C_{1-\gamma}}\right) + \|f_0(s)\|_{C_{1-\gamma}}\right] \\
\leq\ & \frac{\Gamma_q(\gamma)|P|}{\Gamma_q(\gamma+\alpha)} \sum_{i=1}^{m} \lambda_i(\pi_i - a)_q^{\alpha+\gamma-1}\left[L(1+\eta_{h_1}+\eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\right],
\end{aligned}$$

where

$$\begin{aligned}
\int_a^t (t-qs)_q^{\alpha-1}|u(s)|d_qs \leq\ & \left(\int_a^t (t-qs)_q^{\alpha-1}(s-a)_q^{\gamma-1}d_qs\right)\|u\|_{C_{1-\gamma}} \\
=\ & (t-a)_q^{\alpha+\gamma-1}B_q(\gamma,\alpha)\|u\|_{C_{1-\gamma}}.
\end{aligned}$$

Let $\rho_1 = \frac{\Gamma_q(\gamma)|P|}{\Gamma_q(\gamma+\alpha)} \sum_{i=1}^{m} \lambda_i(\pi_i - a)_q^{\alpha+\gamma-1}$, then

$$\|Au\|_{C_{1-\gamma}} \leq \rho_1\left[L\big(1+\eta_{h_1}+\eta_{g_1}\big)\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\right].$$

□

Lemma 8. *Operator $A : X \to X$ is compact. In addition, A is ϖ-Lipschitz.*

Proof. Taking a bounded subset $\Omega \subset B_r^{C_{1-\gamma}}(I')$. For any $u \in \Omega$, by the Lemma 7, we know $A(\Omega)$ is uniformly bounded. Next, we will show that the operator A is equicontinuity. Taking $t_1, t_2 \in I,\ u \in \Omega$, we have

$$\begin{aligned} |Au(t_2) - Au(t_1)| \leq & \frac{|P|}{\Gamma_q(\alpha)}\Big[(t_2-a)_q^{\gamma-1} - (t_1-a)_q^{\gamma-1}\Big]\sum_{i=1}^m \lambda_i \int_a^{\pi_i}(\pi_i - qs)_q^{\alpha-1}|f_u(s)|d_qs \\ \leq & \frac{|P|\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha)}\Big[(t_2-a)_q^{\gamma-1} - (t_1-a)_q^{\gamma-1}\Big]\sum_{i=1}^m \lambda_i(\pi_i - a)_q^{\alpha+\gamma-1}\|f_u(s)\|_{C_{1-\gamma}}, \end{aligned}$$

so

$$\|Au(t_2) - Au(t_1)\|_{C_{1-\gamma}} \to 0, \quad t_1 \to t_2$$

So, by Ascoli–Arzelà theorem, operator A is compact. Moreover, by Proposition 2, A is ϖ-Lipschitz . □

Lemma 9. *Assume that the hypothese (H_1) holds, then B is ϖ-Lipschitz with constant $\delta = \frac{L(T-a)_q^\alpha}{\Gamma_q(\alpha)}(1 + \eta_{h_2} + \eta_{g_2})$. Moreover*

$$\|Bu\|_{C_{1-\gamma}} \leq \rho_2\Big[L(1 + \eta_{h_1} + \eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\Big],$$

where $\rho_2 = \frac{\Gamma_q(\alpha)}{\Gamma_q(\gamma+\alpha)}(T-a)_q^\alpha$.

Proof. Take $u, v \in X$, we have

$$\begin{aligned} \Big|(t-a)_q^{1-\gamma}(Bu(t) - Bv(t))\Big| \leq & (t-a)_q^{1-\gamma}\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}|f_u(s) - f_v(s)|d_qs \\ \leq & (t-a)_q^{1-\gamma}\frac{L(t-a)_q^\alpha}{\Gamma_q(\alpha)}(|u-v| + |Hu - Hv| + |Gu - Gv|) \\ \leq & \frac{L(T-a)_q^\alpha}{\Gamma_q(\alpha)}\Big[(t-a)_q^{1-\gamma}(1 + \eta_{h_2} + \eta_{g_2})|u-v|\Big] \\ \leq & \frac{L(T-a)_q^\alpha}{\Gamma_q(\alpha)}(1 + \eta_{h_2} + \eta_{g_2})\|u-v\|_{C_{1-\gamma}}, \end{aligned}$$

Set $\delta = \frac{L(T-a)_q^\alpha}{\Gamma_q(\alpha)}(1 + \eta_{h_2} + \eta_{g_2})$, it obtain that $\|Bu - Bv\|_{C_{1-\gamma}} \leq \delta\|u-v\|_{C_{1-\gamma}}$.

Hence, operator B is Lipschitz on X with constant δ, by Proposition 2, B is ϖ-Lipschitz with constant δ. Moreover, we obtain

$$\begin{aligned} & \Big|(t-a)_q^{1-\gamma}Bu(t)\Big| \\ \leq & (t-a)_q^{1-\gamma}\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}(|f_u(s) - f_0(s)| + |f_0(s)|)d_qs \\ \leq & (t-a)_q^{1-\gamma}\frac{\Gamma_q(\alpha)}{\Gamma_q(\gamma+\alpha)}(t-a)_q^{\alpha+\gamma-1}\Big[L\Big(\|u\|_{C_{1-\gamma}} + \|Hu\|_{C_{1-\gamma}} + \|Gu\|_{C_{1-\gamma}}\Big) + \|f_0(s)\|_{C_{1-\gamma}}\Big] \\ \leq & \frac{\Gamma_q(\alpha)}{\Gamma_q(\gamma+\alpha)}(T-a)_q^\alpha\Big[L(1 + \eta_{h_1} + \eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\Big], \end{aligned}$$

Set $\rho_2 = \frac{\Gamma_q(\alpha)}{\Gamma_q(\gamma+\alpha)}(T-a)_q^\alpha$, so

$$\|Bu\|_{C_{1-\gamma}} \leq \rho_2\Big[L(1 + \eta_{h_1} + \eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\Big]$$

□

Theorem 1. *Under hypotheses (H_1) and (H_2), nonlocal problem (1) has at least one solution $u \in B_r^{C_\gamma^{1-\gamma}}(I)$ if $\delta < 1$.*

Proof. Firstly, operators A, B, F are clearly bounded and continuous and, by Lemma 8, we obtain that A is ϖ-Lipschitz with constant 0. By Lemma 9, B is ϖ-Lipschitz with constant δ. Thus, F is ϖ-Lipschitz with constant δ, and F is strict ϖ-constraction with constant δ since $\delta < 1$, we obtain that F is ϖ-condensing.

Next, considering set

$$\Omega = \{u \in X : \text{ there exists } \xi \in [0,1] \text{ such that } u = \xi F u\},$$

we prove that Ω is bounded.

Let $u \in \Omega$, then $u = \xi F u = \xi(Au + Bu)$, and t

$$\begin{aligned}
& \|u\|_{C_{1-\gamma}} \\
\leq\ & \xi(\|Au\|_{C_{1-\gamma}} + \|\mathrm{B}u\|_{C_{1-\gamma}}) \\
\leq\ & \rho_1\Big[L(1+\eta_{h_1}+\eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\Big] + \rho_2\Big[L(1+\eta_{h_1}+\eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\Big] \\
\leq\ & (\rho_1+\rho_2)\Big[L(1+\eta_{h_1}+\eta_{g_1})\|u\|_{C_{1-\gamma}} + \|f_0(s)\|_{C_{1-\gamma}}\Big].
\end{aligned}$$

So, we obtain

$$\|u\|_{C_{1-\gamma}} \leq \frac{\rho_1+\rho_2}{1-L(\rho_1+\rho_2)(1+\eta_{h_1}+\eta_{g_1})}\|f_0(s)\|_{C_{1-\gamma}},$$

thus, we conclude that Ω is bounded.

Finally, by Lemma 3 and repeating the same process of proof in [6] (Lemma 2), since $u \in C^{\gamma}_{1-\gamma}(I')$ and by the definition of $C^{\gamma}_{1-\gamma}(I')$, we have ${}_{a^+}D^{\gamma}_q u \in C_{1-\gamma}(I')$, so we obtain $u \in B_r^{C_\gamma^{1-\gamma}}(I)$, it show that the solution of nonlocal problem (1) is actually in $B_r^{C_\gamma^{1-\gamma}}(I)$. This completes the proof. □

Theorem 2. *Assume that (H_1) and (H_2) hold, then nonlocal problem (1) has a unique solution if $\mathcal{L} < 1$, where $\mathcal{L} = \frac{L(1+\eta_{h_2}+\eta_{g_2})}{\Gamma_q(\alpha+1)}\Big[|P|m\hat{\lambda}(T-a)_q^{\alpha+\gamma-1} + (T-a)_q^{\alpha}\Big]$.*

Proof. Taking $t \in I$ and $u, v \in C_{1-\gamma}(I)$, we obtain

$$\begin{aligned}
& \Big|(t-a)_q^{1-\gamma}(Fu(t)-Fv(t))\Big| \\
\leq\ & \frac{|P|}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}|f_u(s)-f_v(s)|d_qs + (t-a)_q^{1-\gamma}\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-qs)_q^{\alpha-1}|f_u(s)-f_v(s)|d_qs \\
\leq\ & \frac{|P|}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}L(|u-v|+\eta_{h_2}|u-v|+\eta_{g_2}|u-v|)d_qs \\
& +(t-a)_q^{1-\gamma}\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-qs)_q^{\alpha-1}L(|u-v|+\eta_{h_2}|u-v|+\eta_{g_2}|u-v|)d_qs \\
\leq\ & (T-a)_q^{\alpha+\gamma-1}|P|m\hat{\lambda}\frac{1}{\Gamma_q(\alpha+1)}L(1+\eta_{h_2}+\eta_{g_2})\|u-v\|_{C_{1-\gamma}} + \frac{L(T-a)_q^{\alpha}}{\Gamma_q(\alpha+1)}(1+\eta_{h_2}+\eta_{g_2})\|u-v\|_{C_{1-\gamma}} \\
=\ & \frac{L(1+\eta_{h_2}+\eta_{g_2})}{\Gamma_q(\alpha+1)}\Big[|P|m\hat{\lambda}(T-a)_q^{\alpha+\gamma-1} + (T-a)_q^{\alpha}\Big]\|u-v\|_{C_{1-\gamma}}.
\end{aligned}$$

Let $\mathcal{L} = \frac{L(1+\eta_{h_2}+\eta_{g_2})}{\Gamma_q(\alpha+1)}\Big[|P|m\hat{\lambda}(T-a)_q^{\alpha+\gamma-1} + (T-a)_q^{\alpha}\Big]$, we have

$$\|Fu(t)-Fv(t)\|_{C_{1-\gamma}} \leq \mathcal{L}\|u-v\|_{C_{1-\gamma}}.$$

Since $\mathcal{L} < 1$, It follows that operator F is strict contraction. By Banach's fixed point theorem, we know that F has a unique fixed point, which implies that nonlocal problem (1) has a unique solution. This completes the proof. □

4. The Existence of Extremal Solutions

Let J is an ordered Banach space with the norm $\|\cdot\|_{C_{1-\gamma}}$ and partial order "$\leq$" (i.e., for any $u, v \in J$, $u \leq v$ is equivalent to $u - v \leq 0$). In the following, we will investigate the existence of extremal solutions for nonlocal problem (1).

Definition 12. *If a function $\mu_0 \in C_{1-\gamma}(I')$ satisfies*

$$\begin{cases} {}_{a^+}D_q^{\alpha,\beta}\mu_0(t) \leq f(t, \mu_0(t), H\mu_0(t), G\mu_0(t)), \quad t \in I', \\ {}_{a^+}I_q^{(1-\alpha)(1-\beta)}\mu_0(a^+) \leq \sum_{i=1}^{m} \lambda_i \mu_0(\pi_i), \quad \pi_i \in I', \end{cases} \tag{7}$$

then it is called a lower solution of nonlocal problem (1); if all the inequalities in (7) are reversed, it is called an upper solution of nonlocal problem (1).

Theorem 3. *If nonlocal problem (1) has a lower solution $\mu_0 \in C_{1-\gamma}(I')$ and an upper solution $\tau_0 \in C_{1-\gamma}(I')$ with $\mu_0 \leq \tau_0$. Suppose that conditions (H_3)–(H_5) are satisfied:*

(H_3) There exists a constant $\Upsilon > 0$ such that

$$f(t, u_1, v_1, z_1) - f(t, u_2, v_2, z_2) \leq \Upsilon(u_2 - u_1), \quad t \in I'$$

where $\mu_0(t) \leq u_1 \leq u_2 \leq \tau_0(t)$, $H\mu_0(t) \leq v_1 \leq v_2 \leq H\tau_0(t)$, $G\mu_0(t) \leq z_1 \leq z_2 \leq G\tau_0(t)$.

(H_4) $(1-q)^{1-\gamma} - \sum_{i=1}^{m} \lambda_i \pi_i > 0$.

(H_5) There is a positive constant $\hbar$ such that

$$\varpi(\{f(t, u_n, v_n, z_n)\}) \leq \hbar(\varpi(\{u_n\}) + \varpi(\{v_n\}) + \varpi(\{z_n\})), \quad t \in I'$$

and decreasing or increasing monotonic sequences $\{u_n\} \subset [\mu_0(t), \tau_0(t)]$, $\{v_n\} \subset [H\mu_0(t), H\tau_0(t)]$, $\{z_n\} \subset [G\mu_0(t), G\tau_0(t)]$.

Then the nonlocal problem (1) has minimal and maximal solutions $\underline{u}$ and $\bar{u}$ between μ_0 and τ_0.

Proof. We define operator $S : [\mu_0, \tau_0] \to C_{1-\gamma}(I')$ as follows

$$Su(t) = \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)} \sum_{i=1}^{m} \lambda_i \int_a^{\pi_i} (\pi_i - qs)_q^{\alpha-1}[f_u(s) + \Upsilon u(s)]d_q s + \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - qs)_q^{\alpha-1}[f_u(s) + \Upsilon u(s)]d_q s \tag{8}$$

for all $t \in I$. Since f_u is continuous, it is obvious that the operator S is continuous. Next, the proof process is divided into three steps.

Step 1. We show the operator S is an increasing monotonic operator.

Firstly, by (H_4) we know $P > 0$ by the below

$$\begin{aligned} &\Gamma_q(\gamma) - \sum_{i=1}^{m} \lambda_i (\pi_i - a)_q^{\gamma-1} = (1-q)^{\gamma-1}(1-q)_q^{\gamma-1} - \sum_{i=1}^{m} \lambda_i \pi_i (1 - \tfrac{a}{\pi_i})_q^{\gamma-1} \\ &\geq [(1-q)^{1-\gamma} - \sum_{i=1}^{m} \lambda_i \pi_i] \min_i \{(1-q)_q^{\gamma-1}, (1 - \tfrac{a}{\pi_i})_q^{1-\gamma}\} \geq 0. \end{aligned}$$

In fact, for $\forall t \in I$, $\mu_0(t) \leq u_1 \leq u_2 \leq \tau_0(t)$, by (H_3) we have

$$f_{u_2}(s) + \Upsilon u_2(s) \geq f_{u_1}(s) + \Upsilon u_1(s),$$

$${}_{a^+}I_q^{\alpha}(f_{u_2}(s) + \Upsilon u_2(s)) \geq {}_{a^+}I_q^{\alpha}(f_{u_1}(s) + \Upsilon u_1(s)).$$

So from Equation (7) we obtain that $Su_2 \geq Su_1$.

Step 2. We first show that $\mu_0(t) \leq S\mu_0(t),\ S\tau_0(t) \leq \mu_0(t)$. Let $\aleph(t) = {}_{a^+}D_q^{\alpha,\beta}\mu(t) + \Upsilon\mu(t),\ \aleph \in X$, and $\aleph(t) \leq f_{\mu_0}(s)(t, \mu_0(t), H\mu_0(t), G\mu_0(t)) + \Upsilon\mu_0(t),\ t \in I$. By Definition (7) we have

$$
\begin{aligned}
&\mu_0(t)\\
= &\ \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i - qs)_q^{\alpha-1}\aleph(s)d_qs + \frac{1}{\Gamma_q(\alpha)}\int_a^t(t-qs)_q^{\alpha-1}\aleph(s)d_qs\\
\leq &\ \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i - qs)_q^{\alpha-1}(f_{\mu_0}(s) + \Upsilon\mu_0(t))d_qs + \frac{1}{\Gamma_q(\alpha)}\int_a^t(t-qs)_q^{\alpha-1}(f_{\mu_0}(s) + \Upsilon\mu_0(s))d_qs\\
= &\ S\mu_0(t), \quad t \in I,
\end{aligned}
$$

which implies that $\mu_0 \leq S\mu_0$. Similarly, it can be shown that $S\tau_0 \leq \tau_0$. So the operator S is a continuous increasing monotonic operator.

Now, we define two iterative sequences $\{\mu_n\}$ and $\{\tau_n\}$ in $[\mu_0, \tau_0]$

$$
\mu_n = S\mu_{n-1}, \quad \tau_n = S\tau_{n-1}, \quad n = 1, 2, \cdots. \tag{9}
$$

By the monotonicity of S, we have

$$
\mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots \leq \tau_n \leq \cdots \leq \tau_2 \leq \tau_1 \leq \tau_0 \tag{10}
$$

Step 3. We show that the sequences μ_n and τ_n are convergent in I.

Here, we obtain $R = \{\mu_n : n \in N\}$ and $R_0\{\mu_{n-1} : n \in N\}$, we have $R = S(R_0)$. From $R_0 = R \cup \{\mu_0\}$ it follows that

$$
\varpi(R_0(t)) = \varpi(\{\mu_n(t)\}_{n=0}^{\infty}) = \varpi(\{\mu_0(t)\} \cup \{\mu_n(t)\}_{n=1}^{\infty}) = \varpi(\{\mu_n(t)\}_{n=1}^{\infty}) = \varpi(R(t)),\ t \in I.
$$

Let $\phi(t) := \varpi(R(t)) = \varpi(SR_0(t)),\ t \in I$, we will show that $\phi(t) \equiv 0$ in I.

$$
\begin{aligned}
&\phi(t) = \varpi(R(t)) = \varpi(S(R_0)(t))\\
= &\ \varpi\Bigg(\Bigg\{\frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i - qs)_q^{\alpha-1}[f_{\mu_{n-1}}(s) + \Upsilon\mu_{n-1}(s)]d_qs\\
&+ \frac{1}{\Gamma_q(\alpha)}\int_a^t(t-qs)_q^{\alpha-1}[f_{\mu_{n-1}}(s) + \Upsilon\mu_{n-1}(s)]d_qs\Bigg\}\Bigg)\\
\leq &\ \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i - qs)_q^{\alpha-1}[L(\varpi(\{\mu_{n-1}(s)\}) + \varpi(\{H\mu_{n-1}(s)\}) + \varpi(\{G\mu_{n-1}(s)\})) + \Upsilon\\
&\varpi(\{\mu_{n-1}(s)\})]d_qs + {}_{a^+}I_q^{\alpha}[L(\varpi(\{\mu_{n-1}(t)\}) + \varpi(\{H\mu_{n-1}(t)\}) + \varpi(\{G\mu_{n-1}(t)\})) + \Upsilon\varpi(\{\mu_{n-1}(t)\})]\\
\leq &\ P(T-a)_q^{\gamma-1}\sum_{i=1}^{m}\lambda_i\frac{(\pi_i - a)_q^{\alpha}}{\Gamma_q(\alpha+1)}\big(L(1+\eta_{h_1}+\eta_{g_1}) + \Upsilon\big)\phi(t) + \big(L(1+\eta_{h_1}+\eta_{g_1}) + \Upsilon\big)_{a^+}I_q^{\alpha}\phi(t)\\
\leq &\ \frac{Pm\hat{\lambda}(T-a)_q^{\alpha+\gamma-1}}{\Gamma_q(\alpha+1)}\big(L(1+\eta_{h_1}+\eta_{g_1}) + \Upsilon\big)\phi(t) + \big(L(1+\eta_{h_1}+\eta_{g_1}) + \Upsilon\big)_{a^+}I_q^{\alpha}\phi(t),
\end{aligned}
$$

we have

$$
\phi(t) \leq \frac{L(1+\eta_{h_1}+\eta_{g_1}) + \Upsilon}{1 - \frac{1}{\Gamma_q(\alpha+1)}P(T-a)_q^{\alpha+\gamma-1}m\hat{\lambda}\big(L(1+\eta_{h_1}+\eta_{g_1}) + \Upsilon\big)}\,{}_{a^+}I_q^{\alpha}\phi(t).
$$

Hence, by Lemma 4, $\phi(t) \equiv 0$ in I', then for any $t \in I'$, $\mu_n(t)$ is precompact. From (10), the sequence $\{\mu_n(t)\}$ is convergent, that is, $\lim_{n\to\infty}\mu_n(t) = \underline{u}(t),\ t \in I'$. In the same way, $\lim_{n\to\infty}\tau_n(t) = \bar{u}(t),\ t \in I'$.

Obviously, $\{\mu_n(t)\} \in C_{1-\gamma}(I')$, and $\underline{u}(t)$ is bounded integrable on I'. By (8) and (9), using the Lebesgue dominated convergence theorem, letting $n \to \infty$, we obtain that $\underline{u}(t) = S(\underline{u})(t)$, therefore $\underline{u}(t) \in C_{1-\gamma}(I')$ and $\underline{u} = S\underline{u}$. Similarly, $\bar{u}(t) \in C_{1-\gamma}(I'),\ \bar{u} = S\bar{u}$. That is, $\mu_0 \leq \underline{u} \leq \bar{u} \leq \tau_0$.

From the monotonicity of S, it is easy to obtain that $\underline{u}$ and $\bar{u}$ are the minimal and maximal solution of the problem (1) in $[\mu_0, \tau_0]$. □

5. Ulam Stability

In the section, we will discuss the Ulam–Hyers stability and Ulam–Hyers–Rassias stability of nonlocal problem (1).

Considering the q-difference nonlocal problem (1) and the following inequalities:

$$\left|{}_{a^+}D_q^{\alpha,\beta}v(t) - f(t, v(t), Hv(t), Gv(t))\right| \le \epsilon, \quad t \in I, \tag{11}$$

$$\left|{}_{a^+}D_q^{\alpha,\beta}v(t) - f(t, v(t), Hv(t), Gv(t))\right| \le \epsilon\varphi(t), \quad t \in I. \tag{12}$$

Definition 13 ([1,45]). *Nonlocal problem (1) is said to be Ulam–Hyers stable if there is a constant $d_f > 0$ such that for all $\epsilon > 0$, and for each solution $v \in C$ of inequalities (11), there is a solution $u \in C$ of Equation (1) such that*

$$\|v - u\|_{C_{1-\gamma}} \le \epsilon d_f, \quad t \in I. \tag{13}$$

Remark 2 ([1,45]). *If the constant ϵd_f in inequality (13) is replaced with the function $d_f(\epsilon) \in C(R_+, R_+)$ with $d_f(0) = 0$, then the nonlocal problem (1) is Generalized Ulam–Hyers stable.*

Definition 14 ([1,45]). *Nonlocal problem (1) is said to be Ulam–Hyers–Rassias stable with respect to φ if there is a constant $d_{f,\varphi} > 0$ such that for all $\epsilon > 0$ and each solution $v \in C$ of inequalities (12), there is a solution $u \in C$ of Equation (1) such that*

$$\|v - u\|_{C_{1-\gamma}} \le \epsilon d_{f,\varphi}\varphi(t), \quad t \in I. \tag{14}$$

Remark 3 ([1,45]). *If the funtion $\epsilon\varphi(t)$ in inequality (12) and (14) is replaced with the function $\varphi(t)$, then nonlocal problem (1) is Generalized Ulam–Hyers–Rassias stable.*

Remark 4 ([1,45]). *If $\varphi(t)$ is a constant in the Definition 14, then nonlocal problem (1) is Ulam–Hyers stable.*

Theorem 4. *Suppose that (H_1)–(H_2) hold, then nonlocal problem (1) is Ulam–Hyers stable.*

Proof. Let $v(t)$ be a solution of the inequality (11), $u(t)$ be a solution of nonlocal problem (1). By inequality (11), we obtain

$$-\epsilon \le {}_{a^+}D_q^{\alpha,\beta}v(t) - f_v(t) \le \epsilon,$$

$$f_v(t) - \epsilon \le {}_{a^+}D_q^{\alpha,\beta}v(t) \le f_v(t) + \epsilon,$$

we obtain

$$\begin{aligned}
&\left|(t-a)_q^{1-\gamma}\left(v(t) - \frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i - qs)_q^{\alpha-1}f_v(s)d_qs + {}_{a^+}I_q^{\alpha}f_v(s)d_qs\right)\right| \\
&\le \epsilon(t-a)_q^{1-\gamma}\left(|P|(t-a)_q^{\gamma-1}\sum_{i=1}^{m}\lambda_i\frac{(\pi_i-a)_q^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{(t-a)_q^{\alpha}}{\Gamma_q(\alpha+1)}\right) \\
&\le \epsilon\frac{|P|m\hat{\lambda}(T-a)_q^{\alpha} + (T-a)_q^{\alpha-\gamma+1}}{\Gamma_q(\alpha+1)} := \epsilon N_1,
\end{aligned}$$

then

$$
\begin{aligned}
& \left|(t-a)_q^{1-\gamma}(v(t)-u(t))\right| \\
= & \left|(t-a)_q^{1-\gamma}\left(v(t)-\frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}f_u(s)d_qs+{}_{a^+}I_q^{\alpha}f_u(s)d_qs\right)\right| \\
\leq & \left|(t-a)_q^{1-\gamma}\left(v(t)-\frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}f_v(s)d_qs+{}_{a^+}I_q^{\alpha}f_v(s)d_qs\right)\right| \\
& +\left|\frac{P}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}(f_v(s)-f_u(s))d_qs\right|+\left|(t-a)_q^{1-\gamma}{}_{a^+}I_q^{\alpha}(f_v(s)-f_u(s))d_qs\right| \\
\leq & \epsilon N_1+\frac{|P|m\hat{\lambda}(T-a)_q^{\alpha+\gamma-1}+(T-a)_q^{\alpha}}{\Gamma_q(\alpha+1)}L(1+\eta_{h_2}+\eta_{g2})\|v-u\|_{C_{1-\gamma}}:=\epsilon N_1+M\|v-u\|_{C_{1-\gamma}},
\end{aligned}
$$

so, we obtain

$$\|v-u\|_{C_{1-\gamma}}\leq\frac{\epsilon N_1}{1-M}:=\epsilon d_f.$$

Hence, by the Definition 13, the nonlocal problem (1) is Ulam–Hyers stable. □

Remark 5. *Using the similar proof procedure in the Theorem 4, we can also obtain that the problem (1) is Generalized Ulam–Hyers stable.*

Theorem 5. *In addition to assumptions (H_1)–(H_2), assume that*
(H_6) there is a continuous function $\varphi(t):I\to R_+$ and $\beta_i\in R_+$ such that ${}_{a^+}I_q^{\pi_i}\varphi(t)\leq\beta_i\varphi(t)$, let $\bar{\beta}=\max_i\{\beta_i,i=1,2,\cdots,m\}$.

Then nonlocal problem (1) is Ulam–Hyers–Rassias stable with respect to φ.

Proof. Let $v(t)$ be a solution of the inequality (12), $u(t)$ be the solution of nonlocal problem (1). By inequality (12), we obtain

$$
\begin{aligned}
& \left|(t-a)_q^{1-\gamma}\left(v(t)-\frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}f_v(s)d_qs+{}_{a^+}I_q^{\alpha}f_v(s)d_qs\right)\right| \\
\leq & \epsilon(|P|m\hat{\lambda}\bar{\beta}+\bar{\beta}(T-a)_q^{1-\gamma})\varphi(t):=\epsilon N_2\varphi(t),
\end{aligned}
$$

then

$$
\begin{aligned}
& \left|(t-a)_q^{1-\gamma}(v(t)-u(t))\right| \\
\leq & \left|(t-a)_q^{1-\gamma}\left(v(t)-\frac{P(t-a)_q^{\gamma-1}}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}f_v(s)d_qs+{}_{a^+}I_q^{\alpha}f_v(s)d_qs\right)\right| \\
& +\left|\frac{P}{\Gamma_q(\alpha)}\sum_{i=1}^{m}\lambda_i\int_a^{\pi_i}(\pi_i-qs)_q^{\alpha-1}(f_v(s)-f_u(s))d_qs\right|+\left|(t-a)_q^{1-\gamma}{}_{a^+}I_q^{\alpha}(f_v(s)-f_u(s))d_qs\right| \\
\leq & \epsilon N_2\varphi(t)+M\|v-u\|_{C_{1-\gamma}},
\end{aligned}
$$

so, we obtain

$$\|v-u\|_{C_{1-\gamma}}\leq\frac{\epsilon\varphi(t)N_2}{1-M}:=\epsilon d_{f,\varphi}\varphi(t).$$

So, by the Definition 14, nonlocal problem (1) is Ulam–Hyers–Rassias stable. □

Remark 6. *Similarly, problem (1) is also Generalized Ulam–Hyers–Rassias stable with respect to φ in the Theorem 5.*

6. Examples

Example 1. *Considering the Hilfer fractional q-difference equation with nonlocal condition as follows.*

$$\begin{cases} {}_{2^+}D_q^{\alpha,\beta}u(t) = f(t,u(t),Hu(t),Gu(t)), \quad t \in I = (2,3], \\ {}_{2^+}I_q^{(1-\alpha)(1-\beta)}u(2^+) = 2u(\frac{5}{2}), \end{cases} \tag{15}$$

where $\alpha = \frac{3}{5}$, $\beta = \frac{1}{3}$, $\gamma = \frac{11}{15}$, $q = \frac{1}{2}$, $\nu = \frac{9}{10}$, $w = \frac{3}{4}$, $f(t,u(t),Hu(t),Gu(t)) = (t-2)^{-\frac{1}{8}} + \frac{t-2}{12}\sin u + \frac{1}{22}(Hu + Gu)$.

For each $f_u(t) \in C_{\frac{4}{15}}[2,3]$, it is obvious that $(t-2)_q^{4/15} f_u(t) = (t-2)_{1/2}^{4/15}((t-2)^{-\frac{1}{8}} + \frac{t-2}{12}\sin u + \frac{1}{22}(Hu+Gu)) \in C([2,3],R)$, $f_u(t) \in C_{4/15}(I')$, also $(t-2)_q^{4/15}\,{}_{2^+}D_q^{2/15} f_u(t) \in C([2,3],R)$, that is $f \in C_{4/15}^{2/15}(I')$. Moreover

$$\begin{aligned} |f_u(t) - f_v(t)| \leq &\ \tfrac{1}{12}|\sin u - \sin v| + \tfrac{1}{22}(|Hu - Hv| + |Gu - Gv|) \\ \leq &\ \tfrac{1}{12}(|\sin u - \sin v| + |Hu - Hv| + |Gu - Gv|), \end{aligned}$$

Set $h(t,s,u(s)) = \frac{\cos u(t)}{(t-\frac{1}{4})s}$, $g(t,s,u(s)) = \frac{\sin u(t)}{\frac{2}{3}(t-\frac{1}{5})s}$, then

$$|h(t,s,u(s))| \leq \frac{2}{7}|u(t)|,\ |g(t,s,u(s))| \leq \frac{5}{12}|u(t)|,$$

$$|h(t,s,u(s)) - h(t,s,v(s))| \leq \frac{2}{7}|u-v|,\ |g(t,s,u(s)) - g(t,s,v(s))| \leq \frac{5}{12}|u-v|,$$

that is, $\theta_1 = l_1 = \frac{2}{7}$, $\theta_2 = l_2 = \frac{5}{12}$, $L = \frac{1}{12}$, $\lambda = 2$, we obtain

$$\eta_{h_1} = \eta_{h_2} \approx 0.3, \eta_{g_1} = \eta_{g_2} \approx 0.52,$$

$$|P| = \left|\frac{1}{\Gamma_q(\gamma) - 2(\frac{5}{2}-2)_q^{\gamma-1}}\right| \approx \left|\frac{1}{1.18 - 2 \times 8.1}\right| = \frac{1}{15},$$

$$\Gamma_q(\alpha+1) = (1-q)^{1-\alpha}(1-q)_q^{\alpha-1} = (1-q)^{2/5}(1-q)_q^{\alpha-1} \approx 1.37,$$

so

$$\begin{aligned} \mathcal{L} = &\ \tfrac{L(1+\eta_{h_2}+\eta_{g_2})}{\Gamma_q(\alpha+1)}\left[|P|m\hat{\lambda}(T-a)_q^{\alpha+\gamma-1} + (T-a)_q^{\alpha}\right] \\ = &\ \tfrac{\frac{1}{12}\times(1+0.3+0.52)}{1.37} \times (\tfrac{1}{15} \times 2 + 1) = 0.13 < 1. \end{aligned}$$

Moreover, let $\varphi(t) = \mathrm{e}^{\arctan t}$ for each $t \in I'$, there exists a real number $0 < \epsilon_i < 1$ such that

$${}_{2^+}I_q^{\pi_i}\varphi(t) \leq \frac{e^{\arctan t}}{\epsilon_i^2(1+q+q^2)} \leq \frac{1}{\epsilon_i^2}\varphi(t) \leq \beta_i\varphi(t).$$

Hence, the conditions of Theorems 2 and 4 are satisfied, which implies nonlocal problem (15) has a unique solution and is Ulam–Hyers–Rassias stable with respect to φ.

Example 2. *We consider the following nonlocal Hilfer fractional q-difference extremal solutions:*

$$\begin{cases} {}_{2^+}D_q^{\alpha,\beta}u(t) = f(t,u(t),Hu(t),Gu(t)), \quad t \in I' = [2,3], \\ {}_{2^+}I_q^{(1-\alpha)(1-\beta)}u(2^+) = \frac{1}{10}u(\frac{23}{10}) + \frac{1}{5}u(\frac{13}{5}), \quad \pi_i \in I', \end{cases} \tag{16}$$

where $\alpha = \frac{3}{5}$, $\beta = \frac{1}{3}$, $\gamma = \frac{11}{15}$, $q = \frac{1}{2}$, $\nu = \frac{9}{10}$, $w = \frac{3}{4}$, $h(t,s,u(s)) = \frac{e^{\arctan u(s)}}{(t-\frac{1}{4})s}$, $g(t,s,u(s)) = \frac{e^{\arctan u(s)}}{\frac{2}{3}(t-\frac{1}{5})s}$.

From the above Example 1, we know that (H_1)–(H_2) hold, and for $t \in I'$,

$$Hu_2 - Hu_1 \leq \eta_{h_2}(u_2 - u_1), \quad Gu_2 - Gu_1 \leq \eta_{g_2}(u_2 - u_1),$$

there exists a constant Υ, it have

$$f(t,u_1(t),Hu_1(t),Gu_1(t)) - f(t,u_2(t),Hu_2(t),Gu_2(t)) \le \Upsilon(u_2-u_1).$$

In addition, when $\mu_0(t) \le u_1(t) \le u_2(t) \le \tau_0(t)$, then

$$H\mu_0(t) \le Hu_1(t) \le Hu_2(t) \le H\tau_0(t),$$

$$G\mu_0(t) \le Gu_1(t) \le Gu_2(t) \le G\tau_0(t),$$

the monotonic sequences $\{u_n\} \subset [\mu_0(t), \tau_0(t)]$, $\{v_n\} \subset [H\mu_0(t), H\tau_0(t)]$, $\{z_n\} \subset [G\mu_0(t), G\tau_0(t)]$. and there will exists a constant $\hbar > 0$ has

$$\varpi(\{f(t,u_n,v_n,z_n)\}) \le \hbar(\varpi(\{u_n\}) + \varpi(\{v_n\}) + \varpi(\{z_n\})), \quad t \in I'$$

In addition, about H_4, $(1-q)^{1-\gamma} - \sum_{i=1}^{m} \lambda_i \pi_i > 0 \approx 0.83 - 0.75 = 0.08 > 0$ holds.

Therefore, by Theorem 3, the nonlocal problem (16) have minimal and maximal solutions $\underline{u}$ and $\bar{u}$ between μ_0 and τ_0.

7. Conclusions

In this paper, we consider a kind of Hilfer fractional q-difference-integral equations with nonlocal condition in a weighted space of continuous functions. Firstly, the existence and uniqueness results of solutions are obtained by using topological degree theory and Banach fixed point theorem. Subsequently, the existence of extremal solutions in an ordered Banach space is discussed by monotone iterative method. We then consider the Ulam stability results for a nonlocal problem (1). Finally, two examples are given to illustrate the effectiveness of the theory results.

In the next work, we will continue to study the Hilfer fractional q-difference equation in-depth, considering the equations with impulsive effects, the controllability of the equations and so on.

Author Contributions: Conceptualization, C.T. and H.G.; formal analysis, Z.Y.; writing—original draft preparation, C.T. and H.G.; writing—review and editing, Z.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by National Natural Science Foundation of China (11961069), Outstanding Young Science and technology Training program of Xinjiang (2019Q022).

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors would like to thank the editor and referees for their careful reading of this paper.

Conflicts of Interest: The authors declare that they have no competing interest.

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