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Existence Results for a Differential Equation Involving the Right Caputo Fractional Derivative and Mixed Nonlinearities with Nonlocal Closed Boundary Conditions

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Abstract: In this study, we present a new notion of nonlocal closed boundary conditions. Equipped with these conditions, we discuss the existence of solutions for a mixed nonlinear differential equation involving a right Caputo fractional derivative operator, and left and right Riemann–Liouville fractional integral operators of different orders. We apply a decent and fruitful approach of fixed point theory to establish the desired results. Examples are given for illustration of the main results. The paper concludes with some interesting observations.

Keywords: right Caputo fractional derivative; Riemann–Liouville fractional integrals; nonlocal closed boundary conditions; existence; fixed point

MSC: 34A08; 34B10; 34B15

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1. Introduction

Closed boundary conditions play a pivotal role in dealing with the fluid dynamics problems when there is no fluid flow along the boundary or through it [1]. The free slip condition is also a type of the closed boundary conditions that ensure the flow along the boundary, but no flow perpendicular to it. These conditions are also important in handling gravity, radiation in diffusion approximation, an elastic wavefield on a closed free surface, radiation heat transfer modeling, CFdesign, deblurring problems, honeycomb lattice, etc. For further details, see the articles [2–7].

Boundary value problems involving different nonlocal fractional differential operators and boundary data have been investigated in the literature. A recent text [8] on the topic presents a nice exposure of such problems. For more examples, see the papers [9–13]. In addition, one can witness a significant development in the area of nonlocal boundary value problems containing ψ, ψ-Hilfer and (k,ψ) Hilfer type fractional derivative operators; for example, see [14–17]. For some results on coupled and hybrid fractional differential equations, see the articles [18–22].

Let us now recall some works on closed boundary conditions. In [23], Setukha studied a three-dimensional Neumann problem in a region with a smooth closed boundary. Ahmad et al. [24] considered fractional differential equations and inclusions complemented with open and closed boundary conditions. In the article [25], the authors studied fractional differential equations with impulse and closed boundary conditions.

In this paper, we propose a nonlocal variant of closed boundary conditions and study an integro-differential equation containing a right Caputo fractional derivative and mixed (usual and Riemann–Liouville type integral) nonlinearities supplemented with these newly introduced conditions. Specifically, we explore the criteria ensuring existence of solutions for the following problem:

$$^{C}D^{\mu}_{T-}y(t) + \lambda I^{\rho}_{T-}I^{\sigma}_{0+}\varphi_1(t, y(t)) = \varphi_2(t, y(t)), \ t \in J := [0, T], \tag{1}$$

$$y(T) = p_1 y(\xi) + p_2 T y'(\xi), \quad Ty'(T) = q_1 y(\xi) + q_2 T y'(\xi), \ 0 < \xi < T, \tag{2}$$

where $^{C}D^{\mu}_{T-}$ represents the right Caputo fractional derivative operator of order $\mu \in (1, 2]$, I^{ρ}_{T-} and I^{σ}_{0+}, respectively, designate the right and left Riemann–Liouville operators of fractional orders $\rho, \sigma > 0$, $\varphi_1, \varphi_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\lambda, p_1, p_2, q_1, q_2 \in \mathbb{R}$.

Here, it worthwhile to mention that our problem is new in the sense of fractional integro-differential equation and nonlocal closed boundary conditions. We can interpret the boundary conditions (2) as the values of the unknown function y and its derivative y' at the right end-point of the domain $[0, T]$ are proportional to a linear combination of these values at a nonlocal position $\xi \in (0, T)$. Physically, the nonlocal closed boundary conditions provide a flexible mechanism to close the boundary at an arbitrary position in the given domain instead of the left end-point of the domain (the usual closed boundary conditions considered in [24]). We first convert the given boundary value problem into an equivalent fixed point problem. Afterward, we establish our main results by applying the fixed point theorems due to Krasnosel'skiĭ and Banach, and the Leray–Schauder nonlinear alternative. It is well known that the methods of modern analysis offer an effective approach to develop the existence theory for initial and boundary value problems. We emphasize that our results are not only new in the given setting but also specialize to some new ones.

The rest of the paper is arranged as follows: Section 2 contains a subsidiary lemma, which facilitates the transformation of the given nonlinear problem into an equivalent fixed-point problem. The main results are presented in Section 3, while examples demonstrating application of these results are constructed in Section 4. The last section addresses some interesting findings.

2. Preliminaries

First of all, we present some preliminary definitions of fractional calculus from the book by Kilbas et al. [26].

Definition 1. *For $g \in L_1[a, b]$, we respectively define the left and right Riemann–Liouville fractional integrals of order $\varrho > 0$ as*

$$I^{\varrho}_{a+}g(t) = \int_a^t \frac{(t-s)^{\varrho-1}}{\Gamma(\varrho)} g(s)ds \ \ \textit{and} \ \ I^{\varrho}_{b-}g(t) = \int_t^b \frac{(s-t)^{\varrho-1}}{\Gamma(\varrho)} g(s)ds.$$

Definition 2. *The right Caputo fractional derivative for a function $g \in AC^m[a, b]$ of order $\varrho \in (m-1, m]$, $m \in \mathbb{N}$ is defined by*

$$^{C}D^{\varrho}_{b-}g(t) = (-1)^m \int_t^b \frac{(s-t)^{m-\varrho-1}}{\Gamma(m-\varrho)} g^{(m)}(s)ds.$$

Next, we solve a linear version of the problems (1) and (2), which is of fundamental importance in obtaining the main results.

Lemma 1. *For $\mathcal{H}, \mathcal{F} \in C(J)$ and $\gamma \neq 0$, the solution of the linear problem*

$$\begin{cases} ^{C}D^{\mu}_{T-}y(t) + \lambda I^{\rho}_{T-}I^{\sigma}_{0+}\mathcal{H}(t) = \mathcal{F}(t), \ t \in J, \\ y(T) = p_1 y(\xi) + p_2 T y'(\xi), \quad Ty'(T) = q_1 y(\xi) + q_2 T y'(\xi), \quad 0 < \xi < T, \end{cases} \tag{3}$$

is given by

$$
\begin{aligned}
y(t) \;=\;& \frac{1}{\Gamma(\mu)}\int_t^T (s-t)^{\mu-1}\Big[\mathcal{F}(s)-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}\mathcal{H}(s)\Big]\,ds \\
&+\frac{a_1(t)}{\Gamma(\mu)}\int_\xi^T (s-\xi)^{\mu-1}\Big[\mathcal{F}(s)-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}\mathcal{H}(s)\Big]\,ds \\
&+\frac{a_2(t)}{\Gamma(\mu-1)}\int_\xi^T (s-\xi)^{\mu-2}\Big[\lambda I_{T-}^{\rho}I_{0+}^{\sigma}\mathcal{H}(s)-\mathcal{F}(s)\Big]\,ds,
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
a_1(t) &= \frac{1}{\gamma}\Big[(p_2q_1-p_1q_2+p_1-q_1)T+q_1t\Big],\\
a_2(t) &= \frac{T}{\gamma}\Big[(p_1q_2-p_2q_1)\xi+(p_2-q_2)T-(p_1q_2-p_2q_1-q_2)t\Big],\\
\gamma &= (p_1q_2-p_2q_1-p_1+q_1-q_2+1)T-q_1\xi\neq 0.
\end{aligned}
\tag{5}
$$

Proof. Applying the operator I_{T-}^{μ} to both sides of the fractional integro-differential equation in (3), we obtain

$$
y(t)=I_{T-}^{\mu}\mathcal{F}(t)-\lambda I_{T-}^{\mu+\rho}I_{0+}^{\sigma}\mathcal{H}(t)-c_0-c_1t,
\tag{6}
$$

where c_0 and c_1 are unknown arbitrary constants. Using (6) in the nonlocal closed boundary conditions: $y(T)=p_1y(\xi)+p_2Ty'(\xi)$ and $Ty'(T)=q_1y(\xi)+q_2Ty'(\xi)$, we find a system of algebraic equations in c_0 and c_1:

$$
\begin{cases}
(p_1-1)c_0+(p_1\xi+p_2T-T)c_1 &= p_1A+p_2TB,\\
q_1c_0+(q_1\xi+q_2T-T)c_1 &= q_1A+q_2TB,
\end{cases}
\tag{7}
$$

where

$$
A=I_{T-}^{\mu}\mathcal{F}(\xi)-\lambda I_{T-}^{\mu+\rho}I_{0+}^{\sigma}\mathcal{H}(\xi),\quad B=\lambda I_{T-}^{\mu+\rho-1}I_{0+}^{\sigma}\mathcal{H}(\xi)-I_{T-}^{\mu-1}\mathcal{F}(\xi).
$$

Solving the system (7), we find that

$$
\begin{aligned}
c_0 &= \frac{-1}{\gamma}\Big\{\big[(q_1p_2-p_1q_2+p_1-q_1)\big]TA+\big[(q_2p_1-p_2q_1)\xi+(p_2-q_2)T\big]TB\Big\},\\
c_1 &= \frac{1}{\gamma}\Big[-q_1A+(p_1q_2-q_1p_2-q_2)TB\Big].
\end{aligned}
$$

Inserting the above values in (6) together with the notation (5) yields the solution (4). The converse follows by direct computation. □

In view of Lemma 1, an operator $\mathcal{G}:\mathcal{X}\to\mathcal{X}$, related to the problems (1) and (2), is defined by

$$
\begin{aligned}
\mathcal{G}y(t) \;=\;& \frac{1}{\Gamma(\mu)}\int_t^T (s-t)^{\mu-1}\Big[\varphi_2(s,y(s))-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}\varphi_1(s,y(s))\Big]\,ds \\
&+\frac{a_1(t)}{\Gamma(\mu)}\int_\xi^T (s-\xi)^{\mu-1}\Big[\varphi_2(s,y(s))-\lambda I_{T-}^{\rho}I_{0+}^{\sigma}\varphi_1(s,y(s))\Big]\,ds \\
&+\frac{a_2(t)}{\Gamma(\mu-1)}\int_\xi^T (s-\xi)^{\mu-2}\Big[\lambda I_{T-}^{\rho}I_{0+}^{\sigma}\varphi_1(s,y(s))-\varphi_2(s,y(s))\Big]\,ds,
\end{aligned}
\tag{8}
$$

where $\mathcal{X}$ stands for the Banach space consisting of all continuous functions from $J\to\mathbb{R}$ equipped with the norm $\|y\|=\sup\{|y(t)|:t\in J\}$. Notice that the problems (1) and (2) is equivalent to the fixed point problem: $\mathcal{G}y=y$, where $\mathcal{G}$ is defined in (8).

Now, we enlist the hypotheses needed in the sequel. $\forall t \in J$, $x, y \in \mathbb{R}$, and the following conditions hold:

($\mathcal{E}_1$) For a Lipschitz constant $\mathcal{K} > 0$, we have $|\varphi_1(t,x) - \varphi_1(t,y)| \leq \mathcal{K}|x-y|$;
($\mathcal{E}_2$) For a Lipschitz constant $\mathcal{L} > 0$, we have $|\varphi_2(t,x) - \varphi_2(t,y)| \leq \mathcal{L}|x-y|$;
($\mathcal{E}_3$) There exist $\beta, \theta \in C(J, \mathbb{R}^+)$ such that $|\varphi_1(t,y)| \leq \theta(t)$ and $|\varphi_2(t,y)| \leq \beta(t)$;
($\mathcal{E}_4$) Let $\phi, \psi : [0,\infty) \to (0,\infty)$ be continuous nondecreasing functions such that

$$|\varphi_1(t,y)| \leq \omega_1(t)\psi(\|y\|),\ |\varphi_2(t,y)| \leq \omega_2(t)\phi(\|y\|),\ \omega_1, \omega_2 \in C(J, \mathbb{R}^+);$$

($\mathcal{E}_5$) $M(\|\omega_2\|\phi(M)\Omega_1 + \|\omega_1\|\psi(M)\Omega_2)^{-1} > 1$, where M is a positive constant and Ω_1 and Ω_2 are given by

$$\begin{aligned} \Omega_1 &= \frac{1}{\Gamma(\mu+1)}\left[T^\mu + \bar{a}_1(T-\xi)^\mu + \bar{a}_2(\mu)(T-\xi)^{\mu-1}\right], \\ \Omega_2 &= \frac{|\lambda|T^\sigma}{\Gamma(\sigma+1)\Gamma(\mu+\rho+1)} \\ &\times \left[T^{\mu+\rho} + \bar{a}_1(T-\xi)^{\mu+\rho} + \bar{a}_2(\mu+\rho)(T-\xi)^{\mu+\rho-1}\right], \end{aligned} \tag{9}$$

with

$$\bar{a}_1 = \max_{t \in J}|a_1(t)|,\quad \bar{a}_2 = \max_{t\in J}|a_2(t)|.$$

3. Main Results

This section is devoted to the existence and uniqueness results for the problems (1) and (2). In the first theorem, we prove the existence of solutions to the given problem by means of Krasnosel'skiǐ's fixed point theorem [27], while Leray–Schauder nonlinear alternative [28] is applied to establish the second existence result. Theorem 3, dealing with the uniqueness of solutions for the problem at hand, is based on Banach's contraction mapping principle [28].

Theorem 1. (**Existence result I**) *If the assumptions* ($\mathcal{E}_1$)–($\mathcal{E}_3$) *are satisfied, then problems* (1) *and* (2) *has at least one solution on* J, *provided that*

$$L\gamma_1 + K\gamma_2 < 1,\ \textit{where}\ \gamma_1 = \frac{T^\mu}{\Gamma(\mu+1)},\ \gamma_2 = \frac{|\lambda|T^{\mu+\rho+\sigma}}{\Gamma(\sigma+1)\Gamma(\mu+\rho+1)}. \tag{10}$$

Proof. Define a closed ball $B_\zeta = \{y \in \mathcal{X} : \|y\| \leq \zeta\}$ with

$$\zeta \geq \|\beta\|\Omega_1 + \|\theta\|\Omega_2. \tag{11}$$

Then, we decompose the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ on B_ζ as $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$, where

$$\begin{aligned} \mathcal{G}_1 y(t) &= \frac{1}{\Gamma(\mu)}\int_t^T (s-t)^{\mu-1}\Big[\varphi_2(s,y(s)) - \lambda I^\rho_{T-} I^\sigma_{0+}\varphi_1(s,y(s))\Big]\, ds, \\ \mathcal{G}_2 y(t) &= \frac{a_1(t)}{\Gamma(\mu)}\int_\xi^T (s-\xi)^{\mu-1}\Big[\varphi_2(s,y(s)) - \lambda I^\rho_{T-} I^\sigma_{0+}\varphi_1(s,y(s))\Big]\, ds \\ &\quad + \frac{a_2(t)}{\Gamma(\mu-1)}\int_\xi^T (s-\xi)^{\mu-2}\Big[\lambda I^\rho_{T-} I^\sigma_{0+}\varphi_1(s,y(s)) - \varphi_2(s,y(s))\Big]\, ds. \end{aligned}$$

Next, we show that $\mathcal{G}_1$ and $\mathcal{G}_2$ verify the hypothesis of Krasnosel'skiǐ's fixed point theorem [27] in three steps.

(i) Let $y, x \in B_\zeta$. Then, we have

$$\|\mathcal{G}_1 y + \mathcal{G}_2 x\|$$

$$
\begin{aligned}
&\leq \sup_{t\in J}\left\{\int_t^T \frac{(s-t)^{\mu-1}}{\Gamma(\mu)}\left[|\varphi_2(s,y(s))|+|\lambda|I_{T-}^{\rho}I_{0+}^{\sigma}|\varphi_1(s,y(s))|\right]ds\right.\\
&\quad +|a_1(t)|[\int_\xi^T \frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}\left[|\varphi_2(s,x(s))|+|\lambda|I_{T-}^{\rho}I_{0+}^{\sigma}|\varphi_1(s,x(s))|\right]ds\\
&\quad \left.+|a_2(t)|\left[\int_\xi^T \frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}\left[|\lambda|I_{T-}^{\rho}I_{0+}^{\sigma}|\varphi_1(s,x(s))|+|\varphi_2(s,x(s))|\right]ds\right\}\right.\\
&\leq \|\beta\|\sup_{t\in J}\left\{\int_t^T \frac{(s-t)^{\mu-1}}{\Gamma(\mu)}ds+|a_1(t)|\int_\xi^T\frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}ds+|a_2(t)|\int_\xi^T\frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}ds\right\}\\
&\quad +\|\theta\||\lambda|\sup_{t\in J}\left\{\int_t^T\frac{(s-t)^{\mu+\rho-1}}{\Gamma(\mu+\rho)}I_{0+}^{\sigma}ds+|a_1(t)|\int_\xi^T\frac{(s-\xi)^{\mu+\rho-1}}{\Gamma(\mu+\rho)}I_{0+}^{\sigma}ds\right.\\
&\quad \left.+|a_2(t)|\int_\xi^T\frac{(s-\xi)^{\mu+\rho-2}}{\Gamma(\mu+\rho-1)}I_{0+}^{\sigma}ds\right\}\\
&\leq \|\beta\|\sup_{t\in J}\left\{\frac{(T-t)^{\mu}}{\Gamma(\mu+1)}+|a_1(t)|\frac{(T-\xi)^{\mu}}{\Gamma(\mu+1)}+|a_2(t)|\frac{(T-\xi)^{\mu-1}}{\Gamma(\mu)}\right\}\\
&\quad +\|\theta\||\lambda|\sup_{t\in J}\left\{T^{\sigma}\frac{(T-t)^{\mu+\rho}}{\Gamma(\sigma)\Gamma(\mu+\rho+1)}+|a_1(t)|T^{\sigma}\frac{(T-\xi)^{\mu+\rho}}{\Gamma(\sigma)\Gamma(\mu+\rho+1)}\right.\\
&\quad \left.+|a_2(t)|T^{\sigma}\frac{(T-\xi)^{\mu+\rho-1}}{\Gamma(\sigma)\Gamma(\mu+\rho)}\right\}\\
&\leq \frac{\|\beta\|}{\Gamma(\mu+1)}\left\{T^{\mu}+\bar{a}_1(T-\xi)^{\mu}+\bar{a}_2(\mu)(T-\xi)^{\mu-1}\right\}\\
&\quad +\frac{\|\theta\||\lambda|T^{\sigma}}{\Gamma(\sigma)\Gamma(\mu+\rho+1)}\left\{T^{\mu+\rho}+\bar{a}_1(T-\xi)^{\mu+\rho}+\bar{a}_2(\mu+\rho)(T-\xi)^{\mu+\rho-1}\right\}\\
&\leq \|\beta\|\Omega_1+\|\theta\|\Omega_2<\zeta,
\end{aligned}
$$

where we used $(\mathcal{E}_3)$ and (11). Thus, $\mathcal{G}_1 y+\mathcal{G}_2 x\in B_\zeta$.

(ii) Using $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$, it is easy to show that

$$
\begin{aligned}
\|\mathcal{G}_1 y-\mathcal{G}_1 x\| &\leq \sup_{t\in J}\left\{\int_t^T\frac{(s-t)^{\mu-1}}{\Gamma(\mu)}|\varphi_2(s,y(s))-\varphi_2(s,x(s))|ds\right.\\
&\quad \left.+|\lambda|\int_t^T\frac{(s-t)^{\mu+\rho-1}}{\Gamma(\mu+\rho)}I_{0+}^{\sigma}|\varphi_1(s,y(s))-\varphi_1(s,x(s))|\right\}ds\\
&\leq (\mathcal{L}\gamma_1+\mathcal{K}\gamma_2)\|y-x\|,
\end{aligned}
$$

which, by the condition $\mathcal{L}\gamma_1+\mathcal{K}\gamma_2<1$ given in (10), verifies that $\mathcal{G}_1$ is a contraction.

(iii) It follows by continuity of φ_1 and φ_2 that $\mathcal{G}_2$ is continuous. Moreover, we have

$$
\begin{aligned}
\|\mathcal{G}_2 y\| &\leq \sup_{t\in J}\left\{|a_1(t)|\int_\xi^T\frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}\left[|\varphi_2(s,y(s))|+|\lambda|I_{T-}^{\rho}I_{0+}^{\sigma}|\varphi_1(s,y(s))|\right]ds\right.\\
&\quad \left.+|a_2(t)|\int_\xi^T\frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}\left[|\lambda|I_{T-}^{\rho}I_{0+}^{\sigma}|\varphi_1(s,y(s))|+|\varphi_2(s,y(s))|\right]ds\right\}\\
&\leq \|\beta\|\sup_{t\in J}\left\{|a_1(t)|\int_\xi^T\frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}ds+|a_2(t)|\int_\xi^T\frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}ds\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+|\lambda|\|\theta\| \sup_{t\in J}\left\{|a_1(t)|\int_{\xi}^{T}\frac{(s-\xi)^{\mu+\rho-1}}{\Gamma(\mu+\rho)}I_{0+}^{\sigma}ds+|a_2(t)|\int_{\xi}^{T}\frac{(s-\xi)^{\mu+\rho-2}}{\Gamma(\mu+\rho-1)}I_{0+}^{\sigma}ds\right\}\\
\leq\ & \|\beta\|(\Omega_1-\gamma_1)+\|\theta\|(\Omega_2-\gamma_2),
\end{aligned}
$$

where Ω_i and γ_i $(i=1,2)$ are respectively given in (9) and (10). This shows that $\mathcal{G}_2$ is uniformly bounded on B_ζ. To establish the compactness of $\mathcal{G}_2$, let $\sup_{(t,y)\in J\times B_\zeta}|\varphi_1(t,y)|=\overline{\varphi}_1$ and $\sup_{(t,y)\in J\times B_\zeta}|\varphi_2(t,y)|=\overline{\varphi}_2$. Then, for $0<t_1<t_2<T$, we have

$$
\begin{aligned}
&|(\mathcal{G}_2y)(t_2)-(\mathcal{G}_2y)(t_1)|\\
\leq\ & |a_1(t_2)-a_1(t_1)|\left\{\int_{\xi}^{T}\frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}\left[|\varphi_2(s,y(s))|+|\lambda|I_{T-}^{\rho}I_{0+}^{\sigma}|\varphi_1(s,y(s))|\right]ds\right\}\\
&+|a_2(t_2)-a_2(t_1)|\left\{\int_{\xi}^{T}\frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}\left[|\lambda|I_{T-}^{\rho}I_{0+}^{\sigma}|\varphi_1(s,y(s))|+|\varphi_2(s,y(s))|\right]ds\right\}\\
\leq\ & \frac{|q_1(t_2-t_1)|}{|\gamma|}\left\{\frac{\overline{\varphi}_2(T-\xi)^{\mu}}{\Gamma(\mu+1)}+\frac{\overline{\varphi}_1|\lambda|T^{\sigma}(T-\xi)^{\mu+\rho}}{\Gamma(\sigma+1)\Gamma(\mu+\rho+1)}\right\}\\
&+\frac{T|(p_1q_2-q_1p_2-q_2)||(t_2-t_1)|}{|\gamma|}\left\{\frac{\overline{\varphi}_1|\lambda|T^{\sigma}(T-\xi)^{\mu+\rho-1}}{\Gamma(\sigma+1)\Gamma(\mu+\rho)}+\frac{\overline{\varphi}_2(T-\xi)^{\mu-1}}{\Gamma(\mu)}\right\}\\
&\longrightarrow 0 \text{ as } t_2\to t_1,
\end{aligned}
$$

independently of $y\in B_\zeta$. Thus, $\mathcal{G}_2$ is equicontinuous. Consequently, $\mathcal{G}_2$ is relatively compact on B_ζ. Hence, $\mathcal{G}_2$ is compact on B_ζ by the Arzelá–Ascoli theorem. We deduce from the steps *(i)*–*(iii)* that the hypothesis of the Krasnosel'skiĭ's fixed point theorem [27] is satisfied. Thus, there exists a fixed point for the operator $\mathcal{G}_1+\mathcal{G}_2=\mathcal{G}$, which is indeed a solution to the problems (1) and (2). □

Remark 1. *Switching the roles of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the last theorem, the assumption $\mathcal{L}\gamma_1+\mathcal{K}\gamma_2<1$ changes to $\mathcal{L}(\Omega_1-\gamma_1)+\mathcal{K}(\Omega_2-\gamma_2)<1$, where Ω_1,Ω_2 and γ_1,γ_2 are respectively given in (9) and (10).*

Theorem 2. (**Existence result II**) *Assume that $(\mathcal{E}_4)$ and $(\mathcal{E}_5)$ are satisfied. Then, problems (1) and (2) has at least one solution on J.*

Proof. We complete the proof in several steps. Let us first establish that the operator $\mathcal{G}$ is completely continuous.

(i) For a fixed number r, let $y\in\mathcal{B}_r=\{y\in\mathcal{X}:\|y\|\leq r\}$. Then, as argued in the proof of Theorem 1, we can find that

$$
\begin{aligned}
\|\mathcal{G}y\| \leq\ & \frac{\|\omega_2\|\phi(r)}{\Gamma(\mu+1)}\left[T^{\mu}+\overline{a}_1(T-\xi)^{\mu}+\overline{a}_2(\mu)(T-\xi)^{\mu-1}\right]\\
&+\frac{|\lambda|T^{\sigma}\|\omega_1\|\psi(r)}{\Gamma(\sigma+1)\Gamma(\mu+\rho+1)}\left[T^{\mu+\rho}+\overline{a}_1(T-\xi)^{\mu+\rho}+\overline{a}_2(\mu+\rho)(T-\xi)^{\mu+\rho-1}\right]\\
=\ & \|\omega_2\|\phi(r)\Omega_1+\|\omega_1\|\psi(r)\Omega_2<\infty.
\end{aligned}
$$

This shows that $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{X}$.

(ii) Bounded sets are mapped into equicontinuous sets by the operator $\mathcal{G}$. For $0<t_1<t_2<T$ and $y\in\mathcal{B}_r$, we have

$$
\begin{aligned}
&|\mathcal{G}y(t_2)-\mathcal{G}y(t_1)|\\
\leq\ & \left|\int_{t_2}^{T}\frac{(s-t_2)^{\mu-1}-(s-t_1)^{\mu-1}}{\Gamma(\mu)}\varphi_2(s,y(s))ds-\int_{t_1}^{t_2}\frac{(s-t_1)^{\mu-1}}{\Gamma(\mu)}\varphi_2(s,y(s))ds\right.
\end{aligned}
$$

$$
\begin{aligned}
& -\lambda \int_{t_2}^{T} \frac{(s-t_2)^{\mu+\rho-1}-(s-t_1)^{\mu+\rho-1}}{\Gamma(\mu+\rho)} I_{0+}^{\sigma}\varphi_1(s,y(s))ds \\
& -\lambda \int_{t_1}^{t_2} \frac{(s-t_1)^{\mu+\rho-1}}{\Gamma(\mu+\rho)} I_{0+}^{\sigma}\varphi_1(s,y(s))ds \Bigg| \\
& +|a_1(t_2)-a_1(t_1)|\left[\left|\int_{\xi}^{T} \frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}\left(\varphi_2(s,y(s))-\lambda I_{T-}^{\rho} I_{0+}^{\sigma}\varphi_1(s,y(s))\right)ds\right|\right] \\
& +|a_2(t_2)-a_2(t_1)|\left[\left|\int_{\xi}^{T} \frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}\left(\lambda I_{T-}^{\rho} I_{0+}^{\sigma}\varphi_1(s,y(s))-\varphi_2(s,y(s))\right)ds\right|\right] \\
\leq\ & \int_{t_2}^{T} \frac{\left|(s-t_2)^{\mu-1}-(s-t_1)^{\mu-1}\right|}{\Gamma(\mu)}|\varphi_2(s,y(s))|ds+\int_{t_1}^{t_2} \frac{(s-t_1)^{\mu-1}}{\Gamma(\mu)}|\varphi_2(s,y(s))|ds \\
& +|\lambda| \int_{t_2}^{T} \frac{\left|(s-t_2)^{\mu+\rho-1}-(s-t_1)^{\mu+\rho-1}\right|}{\Gamma(\mu+\rho)} I_{0+}^{\sigma}|\varphi_1(s,y(s))|ds \\
& +|\lambda| \int_{t_1}^{t_2} \frac{(s-t_1)^{\mu+\rho-1}}{\Gamma(\mu+\rho)} I_{0+}^{\sigma}|\varphi_1(s,y(s))|ds \\
& +|a_1(t_2)-a_1(t_1)|\left[\int_{\xi}^{T} \frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}\left(|\varphi_2(s,y(s))|+|\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|\varphi_1(s,y(s))|\right)ds\right] \\
& +|a_2(t_2)-a_2(t_1)|\left[\int_{\xi}^{T} \frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}\left(|\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|\varphi_1(s,y(s))|+|\varphi_2(s,y(s))|\right)ds\right] \\
\leq\ & \frac{\omega_2(t)\phi(r)}{\Gamma(\mu+1)}\Big\{\left|(T-t_2)^{\mu}-(T-t_1)^{\mu}+(t_2-t_1)^{\mu}\right|+|t_2-t_1|^{\mu} \\
& +\frac{|t_2-t_1|}{|\gamma|}\left[q_1(T-\xi)^{\mu}+|p_1q_2-q_1p_2-q_2|T(\mu)(T-\xi)^{\mu-1}\right]\Big\} \\
& +\frac{|\lambda| T^{\sigma}\omega_1(t)\psi(r)}{\Gamma(\sigma+1)\Gamma(\mu+\rho+1)}\Big\{\left|(T-t_2)^{\mu+\rho}-(T-t_1)^{\mu+\rho}+(t_2-t_1)^{\mu+\rho}\right|+|t_2-t_1|^{\mu+\rho} \\
& +\frac{|t_2-t_1|}{|\gamma|}\left[q_1(T-\xi)^{\mu+\rho}+|p_1q_2-q_1p_2-q_2|T(\mu+\rho)(T-\xi)^{\mu+\rho-1}\right]\Big\} \\
& \longrightarrow 0 \text{ as } t_2 \to t_1,
\end{aligned}
$$

independently of $y \in \mathcal{B}_r$. Thus, $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is completely continuous by the Arzelá–Ascoli theorem.

In order to apply the Leray–Schauder nonlinear alternative [28], we have to show that we can find an open set $\mathcal{V} \subset C(J,\mathbb{R})$ such that $y \neq \varpi\mathcal{G}y$ for $\varpi \in (0,1)$ and $y \in \partial\mathcal{V}$. If it is not so, then $y = \varpi\mathcal{G}y$ for $y \in C(J,\mathbb{R})$ and $\varpi \in (0,1)$. Thus, we have

$$|y(t)| = |\varpi\mathcal{G}y(t)| \leq |\omega_2(t)|\phi(\|y\|)\Omega_1 + |\omega_1(t)|\psi(\|y\|)\Omega_2,$$

which can alternatively be expressed as

$$\frac{\|y\|}{\|\omega_2\|\phi(\|y\|)\Omega_1 + \|\omega_1\|\psi(\|y\|)\Omega_2} \leq 1.$$

By the assumption $(\mathcal{E}_5)$, there exists $\mathcal{M} > 0$ such that $\|y\| \neq \mathcal{M}$. Define a set

$$\mathcal{V} = \{y \in \mathcal{X} : \|y\| < \mathcal{M}\},$$

such that $\partial\mathcal{V}$ contains a solution only if $\|y\| = \mathcal{M}$. Clearly, for some $\varpi \in (0,1)$, there does not exist any solution $y \in \partial\mathcal{V}$ satisfying $y = \varpi\mathcal{G}y$. Therefore, there exists a fixed point $y \in \overline{\mathcal{V}}$ for the operator $\mathcal{G}$, for which problems (1) and (2) correspond to a solution. □

Theorem 3. **(Uniqueness result)** *Let the assumptions* $(\mathcal{E}_1)$ *and* $(\mathcal{E}_2)$ *hold. Then, problems (1) and (2) have a unique solution on* J *if*

$$\mathcal{L}\Omega_1 + \mathcal{K}\Omega_2 < 1, \tag{12}$$

where Ω_1 *and* Ω_2 *are given in (9).*

Proof. Letting $\widehat{\varphi}_1 = \sup_{t\in J} |\varphi_1(t,0)|$, $\widehat{\varphi}_2 = \sup_{t\in J} |\varphi_2(t,0)|$, and

$$\varsigma \geq \frac{\widehat{\varphi}_2\Omega_1 + \widehat{\varphi}_1\Omega_2}{1 - (\mathcal{L}\Omega_1 + \mathcal{K}\Omega_2)},$$

it will be shown that $\mathcal{G}\Theta_\varsigma \subset \Theta_\varsigma$, where $\Theta_\varsigma = \{y \in \mathcal{X} : \|y\| \leq \varsigma\}$ and $\mathcal{G}$ are defined by (8). For $y \in \Theta_\varsigma$, it follows by the assumption $(\mathcal{E}_1)$ that

$$\begin{aligned} |\varphi_1(t,y)| &= |\varphi_1(t,y) - \varphi_1(t,0) + \varphi_1(t,0)| \leq |\varphi_1(t,y) - \varphi_1(t,0)| + |\varphi_1(t,0)| \\ &\leq \mathcal{L}\|y\| + \widehat{\varphi}_1 \leq \mathcal{L}\varsigma + \widehat{\varphi}_1. \end{aligned} \tag{13}$$

Likewise, by the assumption $(\mathcal{E}_2)$, we have

$$|\varphi_2(t,y)| \leq \mathcal{K}\varsigma + \widehat{\varphi}_2. \tag{14}$$

In view of (13) and (14), we obtain

$$\begin{aligned} \|\mathcal{G}y\| &= \sup_{t\in J} |\mathcal{G}y(t)| \\ &\leq \sup_{t\in J}\left\{ \int_t^T \frac{(s-t)^{\mu-1}}{\Gamma(\mu)}\Big[|\varphi_2(s,y(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|\varphi_1(s,y(s))|\Big]ds \right. \\ &\quad +|a_1(t)|\int_\xi^T \frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}\Big[|\varphi_2(s,y(s))| + |\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|\varphi_1(s,y(s))|\Big]ds \\ &\quad +|a_2(t)|\int_\xi^T \frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}\Big[|\lambda| I_{T-}^{\rho} I_{0+}^{\sigma}|\varphi_1(s,y(s))| + |\varphi_2(s,y(s))|\Big]ds \\ &\leq (\mathcal{L}\varsigma + \widehat{\varphi}_2)\sup_{t\in J}\left\{ \int_t^T \frac{(s-t)^{\mu-1}}{\Gamma(\mu)}\,ds + |a_1(t)|\int_\xi^T \frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}ds \right. \\ &\quad \left. +|a_2(t)|\int_\xi^T \frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}ds\right\} \\ &\quad +(\mathcal{K}\varsigma + \widehat{\varphi}_1)|\lambda|\sup_{t\in J}\left\{ \int_t^T \frac{(s-t)^{\mu+\rho-1}}{\Gamma(\mu+\rho)} I_{0+}^{\sigma} ds + |a_1(t)|\int_\xi^T \frac{(s-\xi)^{\mu+\rho-1}}{\Gamma(\mu+\rho)} I_{0+}^{\sigma} ds \right. \\ &\quad \left. +|a_2(t)|\int_\xi^T \frac{(s-\xi)^{\mu+\rho-2}}{\Gamma(\mu+\rho-1)} I_{0+}^{\sigma} ds\right\} \\ &\leq \frac{(\mathcal{L}\varsigma + \widehat{\varphi}_2)}{\Gamma(\mu+1)}\Big[T^\mu + \bar{a}_1(T-\xi)^\mu + \bar{a}_2(\mu)(T-\xi)^{\mu-1}\Big] \\ &\quad + \frac{|\lambda|(\mathcal{K}\varsigma + \widehat{\varphi}_1)T^\sigma}{\Gamma(\sigma+1)\Gamma(\mu+\rho+1)}\Big[T^{\mu+\rho} + \bar{a}_1(T-\xi)^{\mu+\rho} + \bar{a}_2(\mu+\rho)(T-\xi)^{\mu+\rho-1}\Big] \\ &= (\mathcal{L}\varsigma + \widehat{\varphi}_2)\Omega_1 + (\mathcal{K}\varsigma + \widehat{\varphi}_1)\Omega_2 < \varsigma, \end{aligned}$$

which verifies that $\mathcal{G}\Theta_\varsigma \subset \Theta_\varsigma$ since $y \in \Theta_\varsigma$ is arbitrary.

Next, it will be shown that $\mathcal{G}$ is a contraction. For $x, y \in \mathcal{X}$ and $t \in J$, it follows by using the assumptions $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ that

$$
\begin{aligned}
\|\mathcal{G}y-\mathcal{G}x\| &= \sup_{t\in J}\left|(\mathcal{G}y)(t)-(\mathcal{G}x)(t)\right| \\
&\leq \sup_{t\in J}\left\{\int_t^T \frac{(s-t)^{\mu-1}}{\Gamma(\mu)}|\varphi_2(s,y(s))-\varphi_2(s,x(s))|ds\right. \\
&\quad +|\lambda|\int_t^T \frac{(s-t)^{\mu+\rho-1}}{\Gamma(\mu+\rho)}I_{0+}^{\sigma}|\varphi_1(s,y(s))-\varphi_1(s,x(s))|ds \\
&\quad +|a_1(t)|\int_\xi^T \frac{(s-\xi)^{\mu-1}}{\Gamma(\mu)}|\varphi_2(s,y(s))-\varphi_2(s,x(s))|\,ds \\
&\quad +|\lambda||a_1(t)|\int_\xi^T \frac{(s-\xi)^{\mu+\rho-1}}{\Gamma(\mu+\rho)}I_{0+}^{\sigma}|\varphi_1(s,y(s))-\varphi_1(s,x(s))|ds \\
&\quad +|a_2(t)|\int_\xi^T \frac{(s-\xi)^{\mu-2}}{\Gamma(\mu-1)}|\varphi_2(s,y(s))-\varphi_2(s,x(s))|ds \\
&\quad \left.+|\lambda||a_2(t)|\int_\xi^T \frac{(s-\xi)^{\mu+\rho-2}}{\Gamma(\mu+\rho-1)}I_{0+}^{\sigma}|\varphi_1(s,y(s))-\varphi_1(s,x(s))|\,ds\right\} \\
&\leq \|y-x\|\left\{\frac{\mathcal{L}}{\Gamma(\mu+1)}\left[T^{\mu}+\bar{a}_1(T-\xi)^{\mu}+\bar{a}_2(\mu)(T-\xi)^{\mu+1}\right]\right. \\
&\quad \left.+\frac{\mathcal{K}|\lambda|T^{\sigma}}{\Gamma(\sigma+1)\Gamma(\mu+\rho+1)}\left[T^{\mu+\rho}+\bar{a}_1(T-\xi)^{\mu+\rho}+\bar{a}_2(\mu+\rho)(T-\xi)^{\mu+\rho+1}\right]\right\} \\
&= (\mathcal{L}\Omega_1+\mathcal{K}\Omega_2)\|y-x\|,
\end{aligned}
$$

which implies that $\mathcal{G}$ is a contraction as $(\mathcal{L}\Omega_1+\mathcal{K}\Omega_2)<1$ by (12). Therefore, it follows by Banach's contraction mapping principle that $\mathcal{G}$ has a unique fixed point. Consequently, there exists a unique solution to problems (1) and (2) on J. □

4. Examples

Here, we illustrate the results derived in the last section with the aid of examples. Consider the problem:

$$
\begin{cases}
D_{1-}^{7/5}y(t)+2I_{1-}^{6/7}I_{0+}^{9/4}\varphi_1(t,y(t))=\varphi_2(t,y(t)),\ t\in J:=[0,1], \\
y(1)=3y(2/3)-2y'(2/3),\ y'(1)=4y(2/3)-y'(2/3).
\end{cases} \tag{15}
$$

Here, $\mu=7/5,\lambda=2,p_1=3,p_2=-2,q_1=4,q_2=-1,\xi=2/3,\rho=6/7,\sigma=9/4,T=1$. Using the given data, it is found that $\bar{a}_1=\max_{t\in[0,1]}|a_1(t)|=|a_1(t)|_{t=0}\approx 1.125000000$, $\bar{a}_2=\max_{t\in[0,1]}|a_2(t)|=|a_2(t)|_{t=1}\approx 0.6875000000$. Consequently, we obtain

$$\Omega_1\approx 1.498891438,\ \ \Omega_2\approx 0.4534569794,$$

where Ω_1 and Ω_2 are given in (9).

(a) For explaining Theorem 1, let us take

$$\varphi_1(t,y)=\frac{1}{t^2+20}\Big(\tan^{-1}y+\sin y+e^{-t}\Big),\ \varphi_2(t,y)=\frac{1}{3}\Big(\frac{1}{\sqrt{t^2+25}}\frac{|y|}{1+|y|}+\sin t\Big), \tag{16}$$

and observe that

$$|\varphi_1(t,y)|\leq\theta(t)=\frac{\pi+e^{-t}}{t^2+20},\ |\varphi_2(t,y)|\leq\beta(t)=\frac{1}{3}\Big(\frac{1}{\sqrt{t^2+25}}+\sin t\Big).$$

Moreover, $\gamma_1\approx 0.8050432130$, $\gamma_2\approx 0.3055230436$ and $L\gamma_1+K\gamma_2\approx 0.08422185186<1$. Since the hypotheses of Theorem 1 are satisfied, the problem (15) with $\varphi_1(t,y)$ and $\varphi_2(t,y)$ given by (16) has a solution.

(b) For illustrating Theorem 2, consider the problem (15) with

$$\varphi_1(t,y) = \frac{3}{\sqrt{t^2+3600}}\left(\tan^{-1} y + \sin y\right), \ \varphi_2(t,y) = \frac{t^2+1}{10}\left(\frac{|y+1|}{3(1+|y+1|)} + y\cos y\right), \tag{17}$$

and note that $\omega_1(t) = \frac{3}{\sqrt{t^2+3600}}$, $\|\omega_1\| = 1/20$, $\omega_2(t) = (1/10)(t^2+1)$,$\|\omega_2\| = 1/5$, $\phi(\|y\|) = \|y\| + 1/3$ and $\psi(\|y\|) = \|y\| + \pi/2$. Using the condition $(\mathcal{E}_5)$, it is found that $M > 1.351610815$. Clearly, the assumptions of Theorem 2 are verified. Therefore, problem (15) with $\varphi_1(t,y)$ and $\varphi_2(t,y)$ given in (17) has a solution on $[0,1]$.

(c) Clearly, $\varphi_1(t,y)$ and $\varphi_2(t,y)$ (given in (17)) satisfy $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$, respectively, with $\mathcal{K} = 1/10$ and $\mathcal{L} = 4/15$. Moreover, $L\Omega_1 + K\Omega_2 \approx 0.4450500814. < 1$. Thus, the conclusion of Theorem 3 applies to the problem (15) with $\varphi_1(t,y)$ and $\varphi_2(t,y)$ given in (17).

5. Conclusions

Applying the techniques of fixed point theory, we investigated the existence of solutions to a mixed nonlinear differential equation involving a right Caputo fractional derivative operator, and left and right Riemann–Liouville fractional integral operators of different orders, complemented with nonlocal closed boundary conditions. Our results are new in the given setting and specialize to some new ones by taking appropriate values of the parameters involved in the boundary conditions. For instance, our results correspond to nonlocal quasi-periodic boundary conditions: $y(T) = p_1 y(\xi)$, $Ty'(T) = q_2 Ty'(\xi)$ if we take $q_1 = 0 = p_2$ in the present results. Our results interpolate between parametric type periodic conditions $(y(T) = y(\xi),\ y'(T) = y'(\xi))$ for $p_1 = q_2 = 1, p_2 = q_1 = 0$ and parametric anti-periodic conditions $(y(T) = -y(\xi),\ y'(T) = -y'(\xi))$ for $p_1 = q_2 = -1, p_2 = q_1 = 0$. Moreover, the present results become the ones associated with a nonlocal version of Zaremba type boundary conditions: $y(\xi) = 0, y'(T) = 0$ when we take $p_1 \to \infty, q_1 = q_2 = 0$. For more details on Zaremba boundary conditions, see the papers [29,30]. In the future, we plan to study the multivalued and impulsive variants of the problems (1) and (2). We will also extend our present study to a system of coupled right Caputo fractional differential equations subject to nonlocal closed boundary conditions.

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