On Fuzzy Spiral-like Functions Associated with the Family of Linear Operators

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Abstract: At the present time, the study of various classical properties of the geometric function theory using the concept of a fuzzy subset remains limited. In this article, our main aim is to introduce the subclasses of spiral-like functions of complex order in terms of the fuzzy notion and we generalize these subclasses by applying a family of linear operators. The relationships between the newly defined subclasses are examined. In addition, we show that these subclasses are preserved under the well-known Bernardi integral operator.

Keywords: analytic functions; fuzzy subset; fuzzy starlike functions; fuzzy convex functions; Noor integral operator; multiplier transformations

1. Introduction

Let \( A(\Omega) \) denote the class of analytic functions \( f(\zeta) \) in the open unit disk \( \Omega = \{ \zeta : |\zeta| < 1 \} \). The class \( A_n \) contains the functions \( f \in A(\Omega) \), having the series of the form

\[
f(\zeta) = \zeta + a_{n+1}\zeta^{n+1} + a_{n+2}\zeta^{n+2} + \ldots, \quad (\zeta \in \Omega).
\]

For \( n = 1 \), we have \( A_1 = A \), a class of normalized analytic functions in \( \Omega \). We know that \( S, S^* \) and \( C \) denote the subclasses of \( A \) of univalent functions, starlike functions and convex functions, respectively. The class of Caratheodory functions is denoted by \( P \). Let \( f, g \in A \). Then, \( f \prec g \) denotes the subordination of functions \( f \) and \( g \) and is defined as \( f(\zeta) = g(\omega(\zeta)) \), where \( \omega(\zeta) \) is the Schwartz function in \( \Omega \) (see [1]). In [2,3], the authors introduced and studied the concept of differential subordination. Fuzzy subordination and fuzzy differential subordination was first studied by G.I. Oros and Gh. Oros, see [4,5]. Several authors have contributed in the study of fuzzy differential subordination; for example, see [6–10]. Here, we give an overview of some useful basic concepts related to fuzzy differential subordination.

Definition 1 ([11]). Let \( Y \) be a nonempty set. If \( F \) is mapping from \( Y \) to \([0,1]\), then \( F \) is called a fuzzy subset on \( Y \).

Alternatively, a fuzzy subset is also defined as the following:

Definition 2 ([11]). A pair \((M,F_M)\) is called a fuzzy subset on \( Y \), where \( F_M : Y \to [0,1] \) is the membership function of the fuzzy set \((M,F_M)\), and \( M = \{ x \in Y : 0 < F_M(x) \leq 1 \} = \sup(M,F_M) \) is the support of the fuzzy set \((M,F_M)\).
Definition 3 ([11]). Two fuzzy subsets, $(M, F_M)$ and $(N, F_N)$ of $Y$, are equal if and only if $M = N$, whereas $(M, F_M) \subseteq (N, F_N)$ if and only if $F_M(x) \leq F_N(x)$, $x \in Y$.

Definition 4 ([5]). Let $D \subset \mathbb{C}$ and $\xi_0$ be a fixed point in $D$. Then, the analytic function $\tilde{f}$ is subordinate to the analytic function $g$ (written as $\tilde{f} \prec_{F} g$ (or $\tilde{f}(\xi) \prec_{F} g(\xi)$)) if:

$$f(\xi_0) = g(\xi_0) \text{ and } F(\tilde{f}(\xi)) \leq F(g(\xi)), \xi \in D.$$

Remark 1. We can assume a function such as $J_i : \mathbb{C} \to [0, 1]$, $(i = 1, 2, 3, 4)$ as any of the following.

$$J_1(\xi) = \frac{|\xi|}{1 + |\xi|}, \quad J_2(\xi) = \frac{1}{1 + |\xi|}, \quad J_3(\xi) = |\sin|\xi||, \quad J_4(\xi) = |\cos|\xi||.$$

Remark 2. If $D = \Omega$, as in Definition 4, then fuzzy subordination coincides with classical subordination.

The study of linear operators plays a significant role in this field of study. Various well-known operators are defined by using the convolution technique. In [12], the authors introduced the operators as follows:

Let $I_m(\xi) = \xi \frac{\xi}{(1-\xi)^{m+1}}$ and $I_m^{(-1)}(\xi)$ be defined as

$$I_m(\xi) * I_m^{(-1)}(\xi) = \frac{\xi}{(1-\xi)^2}.$$

Then,

$$N_m f(\xi) = I_m^{(-1)}(\xi) * f(\xi), \quad (2)$$

where * denotes the convolution or Hadamard product.

This operator is known as the Noor integral operator of the order $m$. We note that $N_0 f(\xi) = \xi f(\xi)$ and $N_1 f(\xi) = f(\xi)$.

For any real $\delta$, Cho and Kim [13] defined the multiplier transformation of functions $\tilde{f} \in A$ by

$$L^\delta_\gamma \tilde{f}(\xi) = \psi(\xi) * \tilde{f}(\xi) = \xi + \sum_{n=2}^{\infty} \left( \frac{r + \gamma}{r + 1} \right) a_n \xi^n, \quad (3)$$

where $\psi(\xi) = \xi + \sum_{n=2}^{\infty} \left( \frac{r + \gamma}{r + 1} \right) \xi^n$ with $r > -1$.

Now, by making use of (2) and (3), we introduce the operator $N^\delta_{m,r} : A \to A$ as the following.

$$N^\delta_{m,r} f(\xi) = \left( \psi(\xi) * I_m^{(-1)}(\xi) \right) * \tilde{f}(\xi). \quad (4)$$

We use (2)–(4) to obtain the following identities.

$$\xi \left( N^\delta_{m+1,r} f(\xi) \right)' = (m + 1)N^\delta_{m,r} f(\xi) - mN^\delta_{m+1,r} f(\xi). \quad (5)$$

$$\xi \left( N^\delta_{m,r} f(\xi) \right)' = (1 + r)N^\delta_{m+1,r} f(\xi) - rN^\delta_{m,r} f(\xi). \quad (6)$$

The study of operators is an important topic in geometric function theory. After the concept was introduced by an author in [14], many well-known scholars [15–18] investigated this topic using the fuzzy subordination associated with certain operators. We mention a few recent contributions that were published with similar research directions [18–22]. The operators associated with fuzzy differential subordination have applications in various fields of study, such as engineering, biological systems with memory, computer graphics, physics, electric networks, turbulence, etc. In the context of the biological system, Baleanu et al. [23]
proposed a new study on the mathematical modeling of the human liver with the Caputo–Fabrizio fractional derivative. Furthermore, Srivastava et al. [24] examined the transmission dynamics of the dengue infection in terms of fractional calculus. The authors in [25] studied the Korteweg–de Vries equation by using a new integral transform where the fractional derivative was proposed in the Caputo sense. This equation was developed to represent a broad spectrum of physics behaviors in the evolution and association of nonlinear waves. One can see [6, 18, 26, 27] for more applications.

Motivated by this series of research, we defined certain new subclasses $FST(a, \lambda; \varphi), FCV(a, \lambda; \varphi)$ and $FK^{\mu}_\varphi(a, \lambda; \varphi, \phi)$ by using fuzzy subordination.

Denoted by $T$, the class of analytic functions $\varphi(z)$, which are univalent convex functions in $\Omega$ with $\varphi(\zeta) = 1$ and $\Re(\varphi(\zeta)) > 0$ in $\Omega$. For $\varphi, \phi \in T$, $F : \mathbb{C} \rightarrow [0, 1], a, \beta \in \mathbb{R}$ with $|a| < \frac{\pi}{2}, |\beta| < \frac{\pi}{2}$ and $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, we define

$$FST(a, \lambda; \varphi) = \left\{ f \in A : 1 + \frac{e^{ia}}{\lambda \cos \alpha} \left( \frac{\varphi'(\zeta)}{\varphi(\zeta)} - 1 \right) \prec_F \varphi(z) \right\},$$

$$FCV(a, \lambda; \varphi) = \left\{ f \in A : 1 + \frac{e^{ia}}{\lambda \cos \alpha} \left( \frac{\varphi'(\zeta)}{\varphi(\zeta)} - 1 \right) \prec_F \varphi(z) \right\},$$

and

$$FK^{\mu}_\varphi(a, \lambda; \varphi, \phi) = \left\{ f \in A : 1 + \frac{e^{ia}}{\mu \cos \beta} \left( \frac{\varphi'(\zeta)}{\varphi(\zeta)} - 1 \right) \prec_F \phi(z) \right\},$$

where $g \in FST(a, \lambda; \varphi)$.

It is noted that

$$f \in FCV(a, \lambda; \varphi) \Leftrightarrow \varphi' \in FST(a, \lambda; \varphi).$$

In application of the operator given in (4), some new classes are defined, as follows:

**Definition 5.** Let $f \in A, \lambda, \mu \in \mathbb{C} \setminus \{0\}$, $\delta$ be a real, $m \in \mathbb{N}_0$, and $r > -1$. Then,

$$f \in FST^{\delta}_m(a, \lambda; \varphi), \text{ if and only if } N^{\delta}_m f(\zeta) \in FST(a, \lambda; \varphi),$$

$$f \in FCV^{\delta}_m(a, \lambda; \varphi), \text{ if and only if } N^{\delta}_m f(\zeta) \in FCV(a, \lambda; \varphi),$$

and

$$f \in FK^{\delta}_m(a, \lambda; \varphi, \phi), \text{ if and only if } N^{\delta}_m f(\zeta) \in FK^{\mu}_\varphi(a, \lambda; \varphi, \phi).$$

It is clear that

$$f \in FCV^\delta_m(a, \lambda; \varphi), \text{ if and only if } \varphi' \in FST^\delta_m(a, \lambda; \varphi),$$

where $\varphi, \phi \in T, F : \mathbb{C} \rightarrow [0, 1], c$ and $|\beta| < \frac{\pi}{2}$.

Special cases:

(i) If $\delta = 0$ and $m = 1$, then $FST^{\delta}_m(a, \lambda; \varphi) = FST(a, \lambda; \varphi), FCV^{\delta}_m(a, \lambda; \varphi) = FCV(a, \lambda; \varphi)$ and $FK^{\delta}_m(a, \lambda; \varphi, \phi) = FK^{\mu}_\varphi(a, \lambda; \varphi, \phi)$.

(ii) If $\delta = 0$, $m = 1$, $\alpha = 0$ and $\lambda = 1$, then $FST^{\delta}_m(a, \lambda; \varphi)$ and $FCV^{\delta}_m(a, \lambda; \varphi)$ reduce to $FST(\varphi)$ and $FC(\varphi)$, respectively, as shown by Shah et al. [10].

The main investigations of this article consist of the inclusion properties of the classes defined in Definition 5 and applications of the Bernardi integral operator.
2. Main Results

We will use the following lemmas to prove our results.

**Lemma 1 ([7]).** Let \( P : \mathbb{U} \to \mathbb{C} \), with \( \Re(P(\zeta)) > 0 \), \( \zeta \in T \) and \( \Psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C} \), where
\[
\Psi(p(\zeta), x p'(\zeta) + \zeta) = p(\zeta) + P(\zeta) \cdot x p'(\zeta),
\]
is analytic in \( \mathbb{U} \), then
\[
F_{\Psi(\mathbb{C}^2 \times \mathbb{U})}[p(\zeta) + P(\zeta) \cdot x p'(\zeta)] \leq F_{\Phi(\mathbb{I})}(\zeta(\zeta))
\]
implies
\[
F_{\Phi(\mathbb{I})}(p(\zeta)) \leq F_{\Phi(\mathbb{I})}(\zeta(\zeta)), \quad \zeta \in \mathbb{U}.
\]

**Lemma 2 ([8]).** Let \( e, \zeta \in \mathbb{C}, e \neq 0 \), and a convex function \( \zeta \) satisfy
\[
\Re(e \zeta + \zeta) > 0, \quad \zeta \in \mathbb{U}.
\]

If \( p \) is analytic in \( \mathbb{U} \) with \( p(0) = \zeta(0) \) and \( \Psi(p(\zeta), x p'(\zeta) + \zeta) = p(\zeta) + \frac{x p'(\zeta)}{\epsilon p(\zeta) + \zeta} \) is analytic in \( \mathbb{U} \) with \( \Psi(\phi(0), 0) = \zeta(0) \), then
\[
F_{\Psi(\mathbb{C}^2 \times \mathbb{U})}[p(\zeta) + \frac{c p'(\zeta)}{\epsilon p(\zeta) + \zeta}] \leq F_{\Phi(\mathbb{I})}(\zeta(\zeta))
\]
implies
\[
F_{\Phi(\mathbb{I})}(p(\zeta)) \leq F_{\Phi(\mathbb{I})}(\zeta(\zeta)), \quad \zeta \in \mathbb{U}.
\]

2.1. Inclusion Results

**Theorem 1.** If \( \Phi(\zeta) \in T, F : \mathbb{C} \to [0, 1] \), where \( \alpha \in \mathbb{R} \) and \( |\alpha| < \frac{\pi}{4} \), \( 0 \neq \lambda \in \mathbb{C}, \delta \geq 0 \) and \( r, m \in \mathbb{N} \), then
\[
\text{FST}_{m \lambda}^{d+1,r}(\alpha, \lambda; \Phi) \subset \text{FST}_{m \lambda}^{d,r}(\alpha, \lambda; \Phi),
\]
for
\[
\Re\left\{e^{-j\alpha} \lambda \cos(\Phi(\zeta) - 1) + (r + 1)\right\} > 0,
\]
and
\[
\text{FST}_{m \lambda}^{d,r}(\alpha, \lambda; \Phi) \subset \text{FST}_{m \lambda+1}^{d,r}(\alpha, \lambda; \Phi),
\]
for
\[
\Re\left\{e^{-j\alpha} \lambda \cos(\Phi(\zeta) - 1) + (m + 1)\right\} > 0.
\]

**Proof.** First, we have to prove the relation (9).

Let \( f \in \text{FST}_{m \lambda}^{d+1,r}(\alpha, \lambda; \Phi) \). Then, for \( p(\zeta) \) is analytic in \( \mathbb{U} \) with \( p(0) = 1 \), we set
\[
p(\zeta) = \frac{1}{\lambda \cos \alpha} \left\{e^{j\alpha} \left(\frac{H_{m \lambda}^{d+1,1}(\zeta)}{H_{m \lambda}^{d,1}(\zeta)}\right) - (1 - \lambda) \cos \alpha - i \sin \alpha\right\},
\]
Using identity (6) and (11), we find
\[
p(\zeta) = \frac{1}{\lambda \cos \alpha} \left\{e^{j\alpha} \left(1 + r\right) \frac{H_{m \lambda}^{d+1,1}(\zeta)}{H_{m \lambda}^{d,1}(\zeta)} - r\right\} - (1 - \lambda) \cos \alpha - i \sin \alpha\right\},
\]
This implies
\[
(1 + r) \frac{H_{m \lambda}^{d+1,1}(\zeta)}{H_{m \lambda}^{d,1}(\zeta)} = e^{-j\alpha} \lambda \cos(\Phi(\zeta) - 1) + (1 + r).
\]
The logarithmic differentiation and (11) yields
\[
\frac{1}{\lambda \cos \alpha} \left\{ e^{i \lambda (\frac{\lambda^\delta + 1}{m \delta + 1} f(\xi))'} - (1 - \lambda) \cos \alpha - i \sin \alpha \right\} = p(\xi) + \frac{\zeta p'(\xi)}{e^{-i \lambda \cos \alpha (p(\xi) - 1)} + (r + 1)}.
\] (12)

As \( \xi \in FST_{m+1}^{\delta+1,\varphi}(\alpha, \lambda; \varphi) \), from (12), we have
\[
p(\xi) + \frac{\zeta p'(\xi)}{e^{-i \lambda \cos \alpha (p(\xi) - 1)} + (r + 1)} \prec_F \varphi(\xi).
\] (13)

Assuming that
\[\Re \left\{ e^{-i \lambda \cos \alpha (\varphi(\xi) - 1)} + (1 + r) \right\} > 0, \text{ for } \varphi \in T.\]

Then, Lemma 2 and (13) implies that \( p(\xi) \prec_F \varphi(\xi) \). Hence, \( \xi \in FST_{m+1}^{\delta,\varphi}(\alpha, \lambda; \varphi) \).

One can easily prove the relation (10) by using a similar technique, as used in the proof of relation (9), along with the identity (5). \( \square \)

**Theorem 2.** If \( \varphi(\xi) \in T, F : \mathbb{C} \to [0,1], \alpha \in \mathbb{R} \) where \(|\alpha| < \frac{\pi}{2}, 0 \neq \lambda \in \mathbb{C}, \delta \geq 0 \) and \( r, m \in \mathbb{N} \), then
\[
FCV_{m+1,\varphi}^{\delta+1,\varphi}(\alpha, \lambda; \varphi) \subset FCV_{m}^{\delta,\varphi}(\alpha, \lambda; \varphi),
\] (14)
for
\[\Re \left\{ e^{-i \lambda \cos \alpha (\varphi(\xi) - 1)} + (r + 1) \right\} > 0,
\]
and
\[
FCV_{m}^{\delta,\varphi}(\alpha, \lambda; \varphi) \subset FCV_{m+1,\varphi}^{\delta+1,\varphi}(\alpha, \lambda; \varphi),
\] (15)
for
\[\Re \left\{ e^{-i \lambda \cos \alpha (\varphi(\xi) - 1)} + (m + 1) \right\} > 0.
\]

**Proof.** For the proof of (14), we assume that \( \xi \in FCV_{m+1,\varphi}^{\delta+1,\varphi}(\alpha, \lambda; \varphi) \). Then, by (8), \( \zeta \xi_f \in FST_{m+1,\varphi}^{\delta+1,\varphi}(\alpha, \lambda; \varphi) \). This implies, by using Theorem 1, \( \zeta \xi_f \in FST_{m,\varphi}^{\delta,\varphi}(\alpha, \lambda; \varphi) \). Again, by (8), we find \( \xi \in FCV_{m+1,\varphi}^{\delta+1,\varphi}(\alpha, \lambda; \varphi) \).

The relation (15) can be proved by using the same arguments as before. \( \square \)

**Theorem 3.** If \( \varphi, \phi \in T, F : \mathbb{C} \to [0,1], \alpha, \beta \in \mathbb{R}, \) where \(|\alpha| < \frac{\pi}{2}, |\beta| < \frac{\pi}{2}, \lambda, \mu \in \mathbb{C} \setminus \{0\}, \delta \geq 0 \) and \( r, m \in \mathbb{N} \), then
\[
FK_{m,\mu}^{\delta+1,\varphi,\beta}(\alpha, \lambda; \varphi, \phi) \subset FK_{m,\mu}^{\delta,\varphi,\beta}(\alpha, \lambda; \varphi, \phi),
\] (16)
for
\[\Re \left\{ e^{-i \lambda \cos \alpha (\varphi(\xi) - 1)} + (r + 1) \right\} > 0,
\]
and
\[
FK_{m,\mu}^{\delta,\varphi,\beta}(\alpha, \lambda; \varphi, \phi) \subset FK_{m+1,\mu}^{\delta+1,\varphi,\beta}(\alpha, \lambda; \varphi, \phi),
\] (17)
for
\[\Re \left\{ e^{-i \lambda \cos \alpha (\varphi(\xi) - 1)} + (m + 1) \right\} > 0.
\]

**Proof.** First, we prove the relation (16), and let \( \xi \in FK_{m,\mu}^{\delta+1,\varphi,\beta}(\alpha, \lambda; \varphi, \phi) \). Then, by definition, there exists \( g \in FST_{m,\mu}^{\delta+1,\varphi}(\alpha, \lambda; \varphi) \), such that
\[
p(\xi) = \frac{1}{\mu \cos \beta} \left\{ e^{i \beta (\frac{\lambda^\mu + 1}{m^\mu + 1} f(\xi))'} - (1 - \mu) \cos \beta - i \sin \beta \right\} \prec_F \varphi(\xi).
\] (18)
We consider, for \( q(\zeta) \) is analytic in \( \mathcal{U} \) with \( q(0) = 1 \),
\[
q(\zeta) = \frac{1}{\mu \cos \beta} \left\{ e^{i \beta} \left( \frac{\mathcal{N}_{m,r}^\delta f(\zeta)}{\mathcal{N}_{m,r}^\delta g(\zeta)} \right)' - (1 - \mu) \cos \beta - i \sin \beta \right\}. \tag{19}
\]

We use identities (6) and (19), and find
\[
(r + 1) e^{i \beta} \left( \frac{\mathcal{N}_{m,r}^{\delta+1} f(\zeta)}{\mathcal{N}_{m,r}^{\delta} g(\zeta)} \right)' = \mu \cos \beta \zeta q'(\zeta) \left( \frac{\mathcal{N}_{m,r}^\delta g(\zeta)}{\mathcal{N}_{m,r}^\delta f(\zeta)} \right)' + \left( \mu \cos \beta (q(\zeta) - 1) + e^{i \beta} \right) \left( \frac{\mathcal{N}_{m,r}^\delta f(\zeta)}{\mathcal{N}_{m,r}^\delta g(\zeta)} \right)'. \tag{20}
\]

As \( g \in FST_m^{\delta+1,r}(\alpha, \lambda; \phi) \), from Theorem 1, we have \( g \in FST_m^{\delta,r}(\alpha, \lambda; \phi) \). Further, for \( h(\zeta) \) is analytic in \( \mathcal{U} \) with \( h(0) = 1 \), we set
\[
h(\zeta) = \frac{1}{\lambda \cos \alpha} \left\{ e^{i \alpha} \zeta \left( \frac{\mathcal{N}_{m,r}^\delta g(\zeta)}{\mathcal{N}_{m,r}^\delta f(\zeta)} \right) - (1 - \lambda) \cos \alpha - i \sin \alpha \right\}.
\]

Again, we use (6) in the above equation to obtain
\[
e^{-i \alpha} \lambda \cos \alpha (h(\zeta) - 1) + (1 + r) = (1 + r) \zeta \left( \frac{\mathcal{N}_{m,r}^{\delta+1} g(\zeta)}{\mathcal{N}_{m,r}^\delta f(\zeta)} \right). \tag{21}
\]

Putting (20) and (21) in (18), we find
\[
p(\zeta) = \frac{\zeta q'(\zeta)}{e^{-i \alpha} \lambda \cos \alpha (h(\zeta) - 1) + (1 + r)} + \frac{\mu \cos \beta (q(\zeta) - 1) + e^{i \beta}}{\mu \cos \beta} \left( 1 - \frac{r}{e^{-i \alpha} \lambda \cos \alpha (h(\zeta) - 1) + (1 + r)} \right)
\]
\[
\quad + \frac{r (\mu \cos \beta (q(\zeta) - 1) + e^{i \beta})}{e^{-i \alpha} \lambda \cos \alpha (h(\zeta) - 1) + (1 + r)} - \frac{(1 - \mu) \cos \beta - i \sin \beta}{\mu \cos \beta}.
\]

This implies
\[
q(\zeta) + \frac{\zeta q'(\zeta)}{e^{-i \alpha} \lambda \cos \alpha (h(\zeta) - 1) + (1 + r)} \prec_{f} \phi(\zeta). \tag{22}
\]

As \( h(\zeta) \prec_{f} \phi(\zeta) \) and
\[
\Re \left\{ e^{-i \alpha} \lambda \cos \alpha (\phi(\zeta) - 1) + (r + 1) \right\} > 0.
\]

We have
\[
\Re \left\{ e^{-i \alpha} \lambda \cos \alpha (h(\zeta) - 1) + (r + 1) \right\} > 0.
\]

Therefore, Lemma 1 and (22) imply that \( q(\zeta) \prec_{f} \phi(\zeta) \). Hence, \( f \in FK_m^{\delta,r,\beta}(\alpha, \lambda; \phi, \phi) \).

One can easily prove the relation (17) by using a similar technique as the one used in the proof of relation (16), along with the identity (3). \( \square \)

2.2. The Integral Preserving Property

**Theorem 4.** If \( \phi(\zeta) \in T, F : \mathbb{C} \to [0, 1], a \in \mathbb{R}, \) where \( |a| < \frac{\pi}{2} \), \( 0 \neq \lambda \in \mathbb{C}, \delta \geq 0 \) and \( r, m \in \mathbb{N}, \) and let \( f \in FST_m^{\delta,r}(\alpha, \lambda; \phi) \), then \( F_b(\zeta) \in FST_m^{\delta,r}(\alpha, \lambda; \phi) \), where
\[
F_b(\zeta) = \frac{1 + b}{c^b} \int_0^\zeta t^{b-1} f(t) dt, \tag{23}
\]
for
\[
\Re \left\{ e^{-i \alpha} \lambda \cos \alpha (\phi(\zeta) - 1) + (b + 1) \right\} > 0.
\]
Proof. Let $f \in FST^\delta_m(\alpha, \lambda; \varphi)$. Then, for where $Q(\zeta)$ is analytic in $\Omega$ where $Q(0) = 1$, we set

$$Q(\zeta) = \frac{1}{\lambda \cos \alpha} \left\{ e^{ia} \zeta \left( N^\delta_{m,r} F_b(\zeta) \right)' - (1 - \lambda) \cos \alpha - i \sin \alpha \right\}. \tag{24}$$

From (23), we can write

$$\left( \zeta^b F_b(\zeta) \right)' = (1 + b) \zeta^{b-1} f(\zeta).$$

This implies

$$\zeta F_b(\zeta) = (1 + b)f(\zeta) - bF_b(\zeta).$$

Equivalently,

$$\zeta \left( N^\delta_{m,r} F_b(\zeta) \right)' = (1 + b) \left( N^\delta_{m,r} f(\zeta) \right) - b \left( N^\delta_{m,r} F_b(\zeta) \right). \tag{25}$$

From (24) and (25), we have

$$\lambda \cos \alpha (Q(\zeta) - 1) + (1 + b)e^{ia} = (1 + b)e^{ia} \frac{N^\delta_{m,r} f(\zeta)}{N^\delta_{m,r} F_b(\zeta)}. \tag{26}$$

We take the logarithmic differentiation of both sides, and find

$$Q(\zeta) + \frac{\zeta Q'(\zeta)}{e^{-ia} \lambda \cos \alpha (Q(\zeta) - 1) + (b + 1)} \approx F \varphi(\zeta),$$

where $F \in FST^\delta_m(\alpha, \lambda; \varphi)$ along with (24). As $\varphi \in T$ and we assume that $\Re \left\{ e^{-ia} \lambda \cos \alpha (\varphi(\zeta) - 1) + (b + 1) \right\} > 0$, by using Lemma 2, we obtain that $Q(\zeta) \approx F \varphi(\zeta)$ and this completes the proof. \qed

Remark 3. One can easily prove the invariance of the class $FCV^\delta_m(\alpha, \lambda; \varphi)$ under the Bernardi integral operator, as given by (23), by using Theorem 4 together with (8).

Theorem 5. Let $\varphi, \phi \in T, F : \mathbb{C} \to [0, 1], \alpha, \beta \in \mathbb{R}$ with $|\alpha| < \frac{\pi}{2}, |\beta| < \frac{\pi}{2}, \lambda, \mu \in \mathbb{C}\setminus\{0\}$, $\delta \geq 0$ and $r, m \in \mathbb{N}, f \in FK^\alpha_{m,r}(\alpha, \lambda; \varphi, \phi)$, then, if

$$\Re \left\{ e^{-ia} \lambda \cos \alpha (\varphi(\zeta) - 1) + (b + 1) \right\} > 0,$$

It follows that $F_b \in FK^\alpha_{m,r}(\alpha, \lambda; \varphi, \phi)$, where $F_b$ is given by (23).

Proof. Let $f \in FK^\alpha_{m,r}(\alpha, \lambda; \varphi, \phi)$. Then, for $R(\zeta)$ is analytic in $\Omega$ with $R(0) = 1$, we set

$$R(\zeta) = \frac{1}{\mu \cos \beta} \left\{ e^{ib} \zeta \left( N^\delta_{m,r} G_b(\zeta) \right)' - (1 - \mu) \cos \beta - i \sin \beta \right\}. \tag{27}$$

where

$$G_b(\zeta) = \frac{1 + b}{c^b} \int_0^\zeta f^{b-1} s(t) dt, \tag{28}$$

with $g \in FST^\delta_m(\alpha, \lambda; \varphi)$.
We use (25) in (27), and find

\[
(1 + b)e^{i\beta \xi} \left( \mathcal{N}_{m,r}^\delta f(\xi) \right)' = \mu \cos \beta \xi R'(\xi) \mathcal{N}_{m,r}^\delta G_b(\xi) + (\mu \cos \beta (R(\xi) - 1) + b) e^{i\beta \xi} \left( \mathcal{N}_{m,r}^\delta F_b(\xi) \right) + e^{i\beta \xi} \xi \mathcal{N}_{m,r}^\delta G_b(\xi).
\] (29)

Assume that, for \( H(\xi) \) to be analytic in \( \Omega \) with \( H(0) = 1 \),

\[
H(\xi) = \frac{1}{\lambda \cos \alpha} \left\{ e^{i\delta \xi} \left( \mathcal{N}_{m,r}^\delta G_b(\xi) \right)' - (1 - \lambda) \cos \alpha - i \sin \alpha \right\}.
\] (30)

From (28) and (30), we can write

\[
e^{-i\alpha \lambda \cos \alpha} (H(\xi) - 1) + (1 + b) = (1 + b) \frac{\xi \mathcal{N}_{m,r}^\delta G_b(\xi)}{\mathcal{N}_{m,r}^\delta G_b(\xi)}.
\] (31)

We use (27), (29) and (31) with the fact that \( f \in FK_{\mathcal{N}_{m,r}^\delta}(\alpha, \lambda; \varphi, \phi) \), to conclude

\[
R(\xi) + \frac{\xi R'(\xi)}{e^{-i\alpha \lambda \cos \alpha} (H(\xi) - 1) + (1 + b)} \prec_F \varphi(\xi).
\]

As \( H(\xi) \prec_F \varphi(\xi) \), due to the conditions in the theorems and Lemma 1, we obtain our required result. \( \square \)

3. Conclusions

In this article, we examined the applications of fuzzy differential subordination in geometric function theory of complex analysis. We used the convolution technique to define a new operator \( \mathcal{N}_{m,r}^\delta \) for analytic functions. The principle of fuzzy subordination was applied to define certain subclasses of univalent functions associated with the family of linear operators. The inclusion relationships between the subclasses of the univalent functions were studied in terms of fuzzy subordination. Moreover, we proved that the Bernardi integral operator preserves these newly defined subclasses. In addition, generalized fuzzy close-to-convex functions were studied. This work will give an advantage to scholars in this field in their generalizations of various linear or convolution operators in terms of fuzzy subordination. Furthermore, one can study the properties discussed in this work for the different fuzzy subclasses of analytic functions.


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