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New Variant of Hermite–Hadamard, Fejér and Pachpatte-Type Inequality and Its Refinements Pertaining to Fractional Integral Operator

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Abstract: In order to show novel generalizations of mathematical inequality, fractional integral operators are frequently used. Fractional operators are used to simulate a broad range of scientific as well as engineering phenomena such as elasticity, viscous fluid, fracture mechanics, continuous population, equilibrium, visco-elastic deformation, heat conduction problems, and others. In this manuscript, we introduce some novel notions of generalized preinvexity, namely the (m, tgs)-type s-preinvex function, Godunova–Levin (s, m)-preinvex of the 1st and 2nd kind, and a prequasi m-invex. Furthermore, we explore a new variant of the Hermite–Hadamard (H–H), Fejér, and Pachpatte-type inequality via a generalized fractional integral operator, namely, a non-conformable fractional integral operator (NCFIO). In addition, we explore new equalities. With the help of these equalities, we examine and present several extensions of H–H and Fejér-type inequalities involving a newly introduced concept via NCFIO. Finally, we explore some special means as applications in the aspects of NCFIO. The results and the unique situations offered by this research are novel and significant improvements over previously published findings.

Keywords: preinvex functions; fractional operator; Hadamard inequality; Fejér inequality; Pachpatte inequality

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

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1. Introduction

Convexity theory has played a remarkable and essential role in the growth of many subfields in modern mathematics, including optimization [1], financial mathematics [2], economics [3], and engineering [4]. This theory provides an excellent framework for constructing arithmetic techniques for addressing and investigating complicated mathematical issues. Many scholars and researchers in the past decade have tried to combine innovative ideas into fractional analysis in order to introduce an additional aspect with various characteristics to the subject of mathematical analysis and numerical methods. The analysis of fractional operators has many applications in transform theory [5], mathematical biology [6], fluid flow [7], epidemiology [8], nanotechnology [9], modeling [10,11], and control systems [12]. Given the aforementioned prevailing perspectives and importance, the analysis of fractional operators has become an appealing one for readers and scholars, who can refer to [13–16]. Numerous researchers have continued to work on the theory of inequalities during the previous century. This theory can be helpful in the subject of statistical issues and quadrature-type formulas. Readers who are interested can refer to [17–22].

Fractional assessment and inequality concepts have co-evolved in the modern era. Fractional inequality assessment is a core principle and a fundamental component in the applied sciences. Scholars encourage learners to consider utilizing and employing the fractional operator to address real-world issues and problems. The H–H integral inequalities [23], H–H–M inequalities [24], Simpson-type inequality [25], and Ostrowski inequality [26] have all been discussed as utilizing the R–L fractional integral operators. The KFIO in [27] was implemented to present the H–H inequality and the Fejér-type integral inequalities, whereas [28] used the ABFO to study the S-M integral inequality. Also, the H–H–M inequality was explored via the CFFIO. The aforementioned analysis demonstrates the close association between fractional integral operators and inequalities. Hanson (see [29]) was the first to introduce invex functions. Mond and Weir (see [30]) investigated the idea that the introduction of preinvex functions results in the generalization of convex functions. Mond and Ben-Israel's [31] investigation and discussion related to the invex theory and preinvexity using the bifunction can be seen as an important addition to the field of optimization. According to Neogy and Mohan's investigation (see [32]), the terms invex theory and preinvexity in the sense of differentiable are equal under the right circumstances. It has been demonstrated by numerous scholars that the characteristics of preinvex functions have useful and relevant applications in the science of mathematical programming and optimization. See references [33,34].

We started working in this area because of recent works on preinvexity and fractional inequality that were stated above. In the near future, many authors will be drawn to develop the concept of inequality and convexity in more creative ways by working with various sorts of preinvexities and fractional operators. In 1985, the class of G–L function was first proposed by Godunova and Levin (see [35]). The concept of a quasi-convex function is more extended than classical convexity. There are numerous applications for quasi-convex functions in economics, mathematical analysis, mathematical optimization, and game theory. In the published articles [36–38], the researchers celebrated and proved NCFIO and CDO, respectively. These terms have a broad range of purpose and approaches; see references [39,40].

The current manuscript is arranged as follows:

Firstly, in Section 2, we recall some well-known concepts and terms that are advantageous in our exploration in the following sections. Further in this section, we introduce some new definitions, namely, the (m, tgs)-type s-preinvex function, the G–L (s, m)-preinvex of the 1st type, the G–L (s, m)-preinvex of the 2nd type, and the prequasi m-invex. In Section 3, we construct a new variant of H–H-type inequality pertaining to NCFIO with some interesting remarks and corollaries. In Section 4, we prove and examine a new lemma and on the basis of this newly introduced lemma with the addition of newly introduced concepts and definitions; some extensions of H–H inequality are also explored. In Section 5, we construct a new variant of Fejér-type inequality pertaining to NCFIO with some interesting remarks and corollaries. In Section 6, we present and investigate a new lemma and, on the basis of this newly introduced lemma, with the addition of newly introduced concepts and definitions, some extensions of the Fejér inequality are explored. In Section 7, we construct a new variant of Pachpatte-type inequality pertaining to NCFIO with some interesting remarks and corollaries. In Section 8, we explore some special means as applications in the aspects of NCFIO. In the final Section 9, we offer a brief outcome and describe some possible and potential subsequent study directions.

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems.

2. Preliminaries

It is best to evaluate and elaborate in this section due to the quantity of theorems, definitions, and remarks in order to ensure completeness, quality, and reader interest. The purpose of this section is to demonstrate and analyze several recognizable definitions and terms that we need for our assessment in later sections. The NCD, NCFIO, invex function, and preinvex function are introduced first. Adding Condition C enhances the appeal of this portion. Further, some generalized form of the preinvex family, namely, the $tgs-$type s-preinvex, the G–L preinvex, the s–G–L preinvex of the 1st kind, and the s–G–L preinvex of the 2nd kind are added. We sum up this portion with recalling the function, namely, prequasi-invex, that is needed in our assessment. We also explore a new concept, namely, the (m, tgs)-type s-preinvex function, the G–L (s, m)-preinvex of the 1st type, the G–L (s, m)-preinvex of the 2nd type, and prequasi m-invex.

Definition 1 ([37])**.** *Assume that Ω is a real valued function on $[0, \infty)$, then NCD of Ω is stated by*

$$ {}_{N_3^\alpha}\Omega(\flat) = \lim_{\epsilon \to 0} \frac{\Omega(\flat + \epsilon \flat^{\alpha}) - \Omega(\flat)}{\epsilon}, $$

where $\alpha \in (0, 1)$ and $\flat \in \mathbb{X}$.

Theorem 1 ([38])**.** *Let Ω_1 and Ω_2 be two α–differentiable functions at $\wp$ and $\alpha \in (0, 1)$, $\wp > 0$ then*

(1) ${}_{N_3^\alpha}(\nu\Omega_1 + \mu\Omega_2)(\wp) = \nu_{N_3^\alpha}(\Omega_1)(\wp) + \mu_{N_3^\alpha}(\Omega_2)(\wp), \forall \nu, \mu \in \mathbb{R}$,

(2) ${}_{N_3^\alpha}(\Omega_1\Omega_2)(\wp) = \Omega_2(\vartheta)_{N_3^\alpha}(\Omega_1)(\wp) + \Omega_2(\vartheta)_{N_3^\alpha}\Omega_2(\wp)$,

(3) ${}_{N_3^\alpha}(\frac{\Omega_1}{\Omega_2})(\wp) = \frac{\Omega_2(\vartheta)_{N_3^\alpha}(\Omega_1)(\wp) - \Omega_1(\vartheta)_{N_3^\alpha}(\Omega_2)(\wp)}{\Omega_2(\wp)^2}$,

(4) ${}_{N_3^\alpha}(x) = 0$, *for all* $x \in \mathbb{R}$,

(5) ${}_{N_3^\alpha}(\frac{1}{1-\alpha}\wp^{1-\alpha}) = 1$.

Definition 2 ([41])**.** *Let $\alpha, \flat_1, \flat_2 \in \mathbb{R}$ and $\flat_1 < \flat_2$. We define the following linear spaces:*

$$ L_{\alpha,0}[\flat_1, \flat_2] = \left\{ \Omega : [\flat_1, \flat_2] \to \mathbb{R} \,|\, |\vartheta - u|^{-\alpha}\Omega(\vartheta) \in L^1[\flat_1, \flat_2] \text{ for every } u \in [\flat_1, \flat_2] \right\}, $$
$$ L_{\alpha}[\flat_1, \flat_2] = \left\{ \Omega : [\flat_1, \flat_2] \to \mathbb{R} \,|\, (\vartheta - \flat_1)^{-\alpha}\Omega(\vartheta), (\flat_2 - \vartheta)^{-\alpha}\Omega(\vartheta) \in L[\flat_1, \flat_2] \right\}. $$

Note that, if $\alpha \le 0$, then $L_\alpha[\flat_1, \flat_2] = L^1[\flat_1, \flat_2]$.

Definition 3 ([41])**.** *For each $\Omega \in L[\flat_1, \flat_2]$ and $0 < \flat_1 < \flat_2$, then NCFIO is given by*

$$ {}_{N_3}J_u^\alpha \Omega(x) = \int_u^x \wp^{-\alpha}\Omega(\wp)d\wp, $$

for every $x, u \in [\flat_1, \flat_2]$ and $\alpha \in \mathbb{R}$.

Definition 4 ([41])**.** *For each function $\Omega \in L[\flat_1, \flat_2]$, then left and right NCFIO are stated by*

$$ {}_{N_3}J_{\flat_1^+}^\alpha \Omega(x) = \int_{\flat_1}^x (x - \wp)^{-\alpha}\Omega(\wp)d\wp, $$

$$ {}_{N_3}J_{\flat_2^-}^\alpha \Omega(x) = \int_x^{\flat_2} (\wp - x)^{-\alpha}\Omega(\wp)d\wp, $$

for every $x \in [\flat_1, \flat_2]$ and $\alpha \in \mathbb{R}$.

Remark 1. *If $\alpha = 0$, then NCFIO collapses to the classical integrals, i.e., ${}_{N_3}J_{\flat_1^+}^\alpha \Omega(x) = {}_{N_3}J_{\flat_2^-}^\alpha \Omega(x) = \int_{\flat_1}^{\flat_2} \Omega(\wp)d\wp$.*

Definition 5 ([42])**.** *$\mathbb{X} \subset \mathbb{R}^n$ is invex w.r.t $\Phi(.,.)$, if*

$$ \flat_1 + \wp\Phi(\flat_2, \flat_1) \in \mathbb{X}, $$

$\forall\, \flat_1, \flat_2 \in \mathbb{X}$ and $\wp \in [0, 1]$.

The term invexity has numerous uses in variational inequalities, nonlinear optimization, and in the different areas of applied and pure sciences.

Definition 6 ([43])**.** *Assume that* $\Phi : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}^n$ *and* $\mathbb{X} \subseteq \mathbb{R}^n$*. Then let* $\mathbb{X}$ *be m–invex w.r.t.* Φ *, if*

$$m\flat_2 + \wp\Phi(\flat_1, \flat_2, m) \in \mathbb{X}$$

holds $\forall\, \flat_1, \flat_2 \in \mathbb{X}$, $m \in (0,1]$ *and* $\wp \in [0,1]$.

Example 1 ([43])**.** *Assume that* $m = \frac{1}{4}$, $\mathbb{X} = [\frac{-\pi}{2}, 0) \bigcup (0, \frac{1}{2}]$ *and*

$$\Phi(\flat_2, \flat_1, m) = \begin{cases} m\cos(\flat_2 - \flat_1) & if\ \flat_1 \in (0, \frac{\pi}{2}], \flat_2 \in (0, \frac{\pi}{2}]; \\ -m\cos(\flat_2 - \mu_1) & if\ \flat_1 \in [\frac{-\pi}{2}, 0), \flat_2 \in [\frac{-\pi}{2}, 0); \\ m\cos(\flat_1) & if\ \flat_1 \in (0, \frac{\pi}{2}], \flat_2 \in [\frac{-\pi}{2}, 0); \\ -m\cos(\flat_1) & if\ \flat_1 \in [-\frac{\pi}{2}, 0), \flat_2 \in (0, \frac{\pi}{2}]. \end{cases}$$

Then, $\mathbb{X}\ \forall\ \wp \in [0,1]$ *is an m–invex set but not convex.*

Weir and Mond [30], for the first time in 1988, utilized the concept of invex set and elaborated on the concept of the preinvex function.

Definition 7 ([30])**.** *Assume that* $\Phi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^n$ *and* $\mathbb{X} \subseteq \mathbb{R}^n$*. Then* $\Omega : \mathbb{X} \to \mathbb{R}$ *is preinvex w.r.t* Φ *if*

$$\Omega(\flat_2 + \wp\Phi(\flat_1, \flat_2)) \leq \wp\,\Omega(\flat_1) + (1-\wp)\,\Omega(\flat_2)\,, \qquad \forall \flat_1,\ \flat_2 \in \mathbb{X}\,,\ \ \wp \in [0,1].$$

For the author's excellent work and relevance, see the published articles [44–46].

Over the last decade, a great number of scholars have ended up working on refining the concept of preinvexity in various directions. Kalsoom [47] examined and investigated generalized m-preinvexity, which is stated by

Definition 8. *Assume that* $\Phi : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}^n$ *and* $\mathbb{X} \subseteq \mathbb{R}^n$*. Then* $\Omega : \mathbb{X} \to \mathbb{R}$ *is generalized m-preinvex w.r.t.* Φ *if*

$$\Omega(m\flat_2 + \wp\Phi(\flat_1, \flat_2, m)) \leq \wp\Omega(\flat_1) + m(1-\wp)\Omega(\flat_2), \tag{1}$$

holds for every $\flat_1, \flat_2 \in \mathbb{X}$, $m \in (0,1]$ *and* $\wp \in [0,1]$.

Matłoka [48] was the first to investigate and examine the idea of h-preinvexity in 2013, which is defined by:

Definition 9. *Assume that* $h : [0,1] \to \mathbb{R}$*. Then inequality of the form*

$$\Omega(\flat_2 + \wp\Phi(\flat_2, \flat_1)) \leq h(1-\wp)\Omega(\flat_2) + h(\wp)\Omega(\flat_1),$$

$\forall\, \flat_1, \flat_2 \in \Omega$ *and* $\wp \in [0,1]$ *is said to be h-preinvex with respect to* Φ.

Definition 10 ([49])**.** *Assume that* $\Phi : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}$, $\mathbb{X} \subset \mathbb{R}$ *and* $h : [0,1] \to \mathbb{R}$*. Then an inequality of the form*

$$\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \leq h(\wp)\Omega(\flat_2) + mh(1-\wp)\Omega(\flat_1), \tag{2}$$

$\forall\, \flat_1, \flat_2 \in \mathbb{X}$ *and* $\wp \in [0,1]$ *is said to be generalized* (m,h)*-preinvex function.*

Condition C: Assume that $\Phi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and $\mathbb{X} \subset \mathbb{R}$. Then let $\mathbb{X}$ be an open invex subset w.r.t. Φ. For any $\flat_1, \flat_2 \in \mathbb{X}$ and $\wp \in [0,1]$,

$$\begin{aligned} \Phi(\flat_1, \flat_1 + \wp\, \Phi(\flat_2, \flat_1)) &= -\wp\, \Phi(\flat_2, \flat_1) \\ \Phi(\flat_2, \flat_1 + \wp\, \Phi(\flat_2, \flat_1)) &= (1-\wp)\, \Phi(\flat_2, \flat_1). \end{aligned}$$

For any $\flat_1, \flat_2 \in \mathcal{H}$, $\wp_1, \wp_2 \in [0,1]$, then according to the above equations, we have

$$\Phi(\flat_1 + \wp_2\, \Phi(\flat_2, \flat_1),\ \flat_1 + \wp_1\, \Phi(\flat_2, \flat_1)) = (\wp_2 - \wp_1)\Phi(\flat_2, \flat_1).$$

The above Condition C plays a crucial part in the creation of the theory of inequalities and optimization (see [50,51] and references therein).

The following extended Condition C regarding an aspect of m-preinvex function was also introduced and investigated by Du in [52].

Extended Condition C: Assume that $\Phi : \mathbb{X} \times \mathbb{X} \times (0,1] \to \mathbb{R}$ and $\mathbb{X} \subset \mathbb{R}$. Then let $\mathbb{X}$ be an open invex subset w.r.t. Φ. For any $\flat_1, \flat_2 \in \mathbb{X}$, $\wp \in [0,1]$, then we have

$$\begin{aligned} \Phi(\flat_2, m\flat_2 + \wp\, \Phi(\flat_1, \flat_2, m), m) &= -\wp\, \Phi(\flat_1, \flat_2, m) \\ \Phi(\flat_1, m\flat_2 + \wp\, \Phi(\flat_1, \flat_2, m), m) &= (1-\wp)\, \Phi(\flat_1, \flat_2, m) \\ \Phi(\flat_1, \flat_2, m) &= -\Phi(\flat_2, \flat_1, m). \end{aligned}$$

Definition 11 ([44])**.** *A function $\Omega : \mathbb{X} \to \mathbb{R}$ is tgs-type s-preinvex if*

$$\Omega(\flat_1 + \wp\Phi(\flat_2, \flat_1)) \le \wp^s (1-\wp)^s [\Omega(\flat_1) + \Omega(\flat_2)], \tag{3}$$

holds $\forall\ \wp \in [0,1]$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

Definition 12 ([49])**.** *A function $\Omega : \mathbb{X} \to \mathbb{R}$ is (m,tgs)-type preinvex if*

$$\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \le \wp(1-\wp)[m\Omega(\flat_1) + \Omega(\flat_2)], \tag{4}$$

holds $\forall\ \wp \in [0,1],\ m \in [0,1]$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

Next, we explore and investigate the new definition, namely, the (m, tgs)-type s-preinvex function.

Definition 13. *A function $\Omega : \mathbb{X} \to \mathbb{R}$ is (m, tgs)-type s-preinvex if*

$$\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \le \wp^s (1-\wp)^s [m\Omega(\flat_1) + \Omega(\flat_2)], \tag{5}$$

holds $\forall\ \wp \in [0,1],\ m \in [0,1]$, $s \in [0,1]$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

Remark 2. *(1) If we set m=1, then we get Definition 11.*
(2) If we set s=1, then we get Definition 12.
(3) If we set m=s=1, then we get Definition 7.

Definition 14 ([45])**.** *A real-valued function Ω is G–L preinvex or Q-preinvex if*

$$\Omega(\flat_1 + \wp\Phi(\flat_2, \flat_1)) \le \frac{\Omega(\flat_1)}{1-\wp} + \frac{\Omega(\flat_2)}{\wp}, \tag{6}$$

holds $\forall\ \wp \in (0,1)$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

In 2014, Noor (see [53]) was the first to explore the families of s-G–L preinvex functions of the 1st and 2nd kind.

Definition 15. *A real-valued function Ω is s-G–L preinvex of the 1st kind with $s \in (0,1]$, if*

$$\Omega(\flat_1 + \wp\Phi(\flat_2, \flat_1)) \leq \frac{\Omega(\flat_1)}{1-\wp^s} + \frac{\Omega(\flat_2)}{\wp^s}, \tag{7}$$

holds $\forall\ \wp \in (0,1)$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

Definition 16. *A real-valued function Ω is s-G–L preinvex of the 2nd kind with $s \in [0,1]$, if*

$$\Omega(\flat_1 + \wp\Phi(\flat_2, \flat_1)) \leq \frac{\Omega(\flat_1)}{(1-\wp)^s} + \frac{\Omega(\flat_2)}{\wp^s}, \tag{8}$$

holds $\forall\ \wp \in (0,1)$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

Inspired and motivated by the above literature, here we introduce new definitions, namely, the G–L m-preinvex, the G–L (s,m)-preinvex of the 1st kind, and the G–L (s,m)-preinvex of the 2nd kind.

Definition 17. *A real-valued function Ω is G–L m-preinvex if*

$$\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \leq \frac{m\Omega(\flat_1)}{1-\wp} + \frac{\Omega(\flat_2)}{\wp}, \tag{9}$$

holds $\forall\ \wp \in (0,1)\ \ \flat_1, \flat_2 \in \mathbb{X}$ *and* $m \in [0,1]$.

Definition 18. *A real-valued function $\Omega : \mathbb{X} \to \mathbb{R}$ is said to be G–L (s,m)-preinvex of the 1st kind with $s \in (0,1]$ and $m \in [0,1]$ if*

$$\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \leq \frac{m\Omega(\flat_1)}{1-\vartheta^s} + \frac{\Omega(\flat_2)}{\wp^s}, \tag{10}$$

holds $\forall\ \wp \in (0,1)$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

Definition 19. *A real-valued function $\Omega : \mathbb{X} \to \mathbb{R}$ is said to be G–L (s,m)-preinvex of the 2nd kind with $s \in [0,1]$ and $m \in [0,1]$ if*

$$\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \leq \frac{m\Omega(\flat_1)}{(1-\wp)^s} + \frac{\Omega(\flat_2)}{\wp^s}, \tag{11}$$

holds $\forall\ \wp \in (0,1)$ *and* $\flat_1, \flat_2 \in \mathbb{X}$.

Pini (see [46]) was the first to explore prequasi-invex functions. This function is not quasi–convex but satisfies Condition C generally (Example 1.1, see [46]).

Definition 20. *Assume that $\mathbb{X} \subseteq \mathbb{R}^n$ is an invex set w.r.t. $\Phi(.,.)$. Then $\Omega : \mathbb{X} \to \mathbb{R}$ is prequasi-invex on $\mathbb{X}$ if*

$$\Omega(\flat_1 + \wp\Phi(\flat_2, \flat_1)) \leq \max\{\Omega(\flat_1),\ \Omega(\flat_2)\}, \tag{12}$$

holds $\forall\ \flat_1, \flat_2 \in \mathbb{X}$ *and* $\wp \in [0,1]$.

Remark 3. *Taking $\Phi(\flat_2, \flat_1) = \flat_2 - \flat_1$, then the above Definition 20 collapses to a quasi-convex function.*

Here we investigate a new definition of the prequasi m-invex function, which is defined by:

Definition 21. *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be an m-invex set w.r.t. $\Phi(.,.)$. Then, the real-valued function $\Omega : \mathbb{X} \to \mathbb{R}$ is prequasi m-invex on $\mathbb{X}$ if*

$$\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \leq \max\{\Omega(\flat_1),\ \Omega(\flat_2)\}, \tag{13}$$

holds $\forall\ \flat_1, \flat_2 \in \mathbb{X}$ *and* $\wp \in [0,1]$ *and* $m \in [0,1]$.

3. Hermite–Hadamard Inequality via Non-Conformable Fractional Integral Operator

Since the notion of convex analysis was initiated several decades earlier, numerous important inequalities for the family of convex functions have been presented. In the field of inequalities, one of the most significant and remarkable inequalities is Hadamard's inequality. Hermite and Hadamard [54] were the initial researchers to define this inequality. It has an incredibly fascinating geometric interpretation and a broad range of uses. The H–H inequalities are a modification of the idea of convexity, and they follow Jensen's inequality. The premise of this inequality impressed several mathematicians to evaluate and inspect classical inequalities using various senses of convexity. For example, Kirmaci [55], Mehreen [56], and Xi [57] proved some new variants of this inequality via convex functions. Ozcan [58], Dragomir [59], and Hudzik [60] worked on the idea of s-convexity and explored a new kind of this inequality. Rashid [61] and Butt [62] modified this inequality pertaining to a new family of convexity in the polynomial sense.

The main objective of this section is to present a new version of the H–H-type inequality for a preinvex function via NCFIO.

Theorem 2. *Assume that $\Phi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and $\mathbb{X} \subseteq \mathbb{R}$ with $\flat_1, \flat_2 \in \mathbb{X}$, such that $\mathbb{X}$ is an open invex subset w.r.t. Φ. If $\Omega : [m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)] \to (0, \infty)$ is a preinvex function such that $\Omega \in L_{\alpha,0}[m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)]$ and satisfies Condition C, then the following inequalities for NCFIO are given as:*

$$\begin{aligned}
&\Omega\left(\frac{2m\flat_1 + \Phi(\flat_2, \flat_1, m)}{2}\right) \\
&\leq \frac{1-\alpha}{2\Phi^{1-\alpha}(\flat_2, \flat_1, m)}\left[{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \ + \ {}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\right] \\
&\leq \frac{[\Omega(m\flat_1) + \Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))]}{2} \leq \frac{\Omega(m\flat_1) + \Omega(\flat_2)}{2},
\end{aligned}$$

with $\alpha \leq 0$.

Proof. Since $\flat_1, \flat_2 \in \mathbb{X}$ and $\mathbb{X}$ is an open invex set w.r.t. Φ, $\forall\ \wp \in [0,1]$, we have $m\flat_1 + \wp\Phi(\flat_2, \flat_1, m) \in \mathbb{X}$. By preinvexity of Ω, we have for every $\mu, \nu \in [m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)]$ with $\wp = \frac{1}{2}$,

$$\Omega\left(\mu + \frac{\Phi(\nu, \mu)}{2}\right) \leq \frac{\Omega(\mu) + \Omega(\nu)}{2}.$$

If we choose

$$\mu = m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m)$$

and

$$\nu = m\flat_1 + \wp\Phi(\flat_2, \flat_1, m),$$

by using Condition C, we have

$$
\begin{aligned}
&2\Omega\left(m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m) + \frac{\Phi(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m), m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m))}{2}\right) \\
&= 2\Omega\left(m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m) + \frac{(2\wp-1)\Phi(\flat_2,\flat_1,m)}{2}\right) \\
&= 2\Omega\left(\frac{2m\flat_1 + \Phi(\flat_2,\flat_1,m)}{2}\right) \\
&\le \Omega(m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m)) + \Omega(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m))
\end{aligned}
$$

and multiplying by $\wp^{-\alpha}$, we can write

$$
\begin{aligned}
&2\wp^{-\alpha}\Omega\left(\frac{2m\flat_1 + \Phi(\flat_2,\flat_1,m)}{2}\right) \\
&\le \wp^{-\alpha}\Omega(m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m)) + \wp^{-\alpha}\Omega(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m)).
\end{aligned} \tag{14}
$$

Now, by integrating the inequality (14) w.r.t. $\wp$ over [0,1], we obtain

$$
\begin{aligned}
&\frac{2}{1-\alpha}\Omega\left(\frac{2m\flat_1 + \Phi(\flat_2,\flat_1,m)}{2}\right) \\
&\le \int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m))d\wp + \int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m))d\wp \\
&= \frac{1}{\Phi^{1-\alpha}(\flat_2,\flat_1,m)}\left[\int_{m\flat_1}^{m\flat_1+\Phi(\flat_2,\flat_1,m)} (m\flat_1 + \Phi(\flat_2,\flat_1,m) - x)^{-\alpha}\Omega(x)dx \right. \\
&\quad \left. + \int_{m\flat_1}^{m\flat_1+\Phi(\flat_2,\flat_1,m)} (x - m\flat_1)^{-\alpha}\Omega(x)dx\right] \\
&= \frac{1}{\Phi^{1-\alpha}(\flat_2,\flat_1,m)}\left[{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m)) \;+\; {}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\right].
\end{aligned}
$$

For the proof of the 2nd inequality, we first note that Ω is preinvex on $[m\flat_1, m\flat_1 + \Phi(\flat_2,\flat_1,m)]$ and the mapping of Φ satisfies Condition C; then, for every $\wp \in [0,1]$, Condition C yields

$$
\begin{aligned}
\Omega(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m)) &= \Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m) + (1-\wp)\Phi(m\flat_1, m\flat_1 + \Phi(\flat_2,\flat_1,m))) \\
&\le \wp\Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m)) + (1-\wp)\Omega(m\flat_1),
\end{aligned} \tag{15}
$$

and similarly

$$
\begin{aligned}
&\Omega(m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m)) = \Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m) + \wp\Phi(m\flat_1, m\flat_1 + \Phi(\flat_2,\flat_1,m))) \\
&\le (1-\wp)\Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m)) + \wp\Omega(m\flat_1).
\end{aligned} \tag{16}
$$

Adding the inequalities (15) and (16), we obtain

$$
\Omega(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m)) + \Omega(m\flat_1 + (1-\wp)\Phi(\flat_2,\flat_1,m)) \le \Omega(m\flat_1) + \Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m)).
$$

Now, multiplying by $\wp^{-\alpha}$ and integrating w.r.t. $\wp$ over [0,1], we obtain the following required inequality

$$
\begin{aligned}
&\frac{1}{\Phi^{1-\alpha}(\flat_2,\flat_1,m)}\left[{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m)) \;+\; {}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\right] \\
&\le \frac{1}{1-\alpha}[\Omega(m\flat_1) + \Omega(m\flat_1 + \Phi(\flat_2,\flat_1,m))],
\end{aligned}
$$

which completes the proof. □

Remark 4. *Assuming that $\alpha = 0$ and $m = 1$, we obtain the H–H inequality in the sense of the preinvex function that is investigated by Noor (see* [63]*).*

Remark 5. *If $\alpha = 0$ and $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, then we obtain the H–H inequality for m–convexity, which is explored by Dragomir and Toader* [64].

Remark 6. *If $\alpha = 0$, $m = 1$, and $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, then we retrieve the classical H–H inequality proved by Hadamard (see* [54]*) in the aspect of the convex function.*

Corollary 1. *If $\alpha = 0$, then we obtain the H–H inequality for m-preinvexity, given as:*

$$\Omega\left(\frac{2m\flat_1 + \Phi(\flat_2, \flat_1, m)}{2}\right) \leq \frac{1}{\Phi(\flat_2, \flat_1, m)} \int_{m\flat_1}^{m\flat_1 + \Phi(\flat_2, \flat_1, m)} \Omega(x)dx \leq \frac{[\Omega(m\flat_1) + \Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))]}{2}.$$

Corollary 2. *If we set $m = 1$, then we obtain the H–H inequality via NCFIO for preinvexity, given as:*

$$\Omega\left(\frac{2\flat_1 + \Phi(\flat_2, \flat_1)}{2}\right) \leq \frac{1-\alpha}{2\Phi^{1-\alpha}(\flat_2, \flat_1)} \left[{}_{N_3}J^{\alpha}_{\flat_1^+}\Omega(\flat_1 + \Phi(\flat_2, \flat_1)) \;+\; {}_{N_3}J^{\alpha}_{\flat_1 + \Phi(\flat_2, \flat_1)^-}\Omega(\flat_1)\right] \leq \frac{[\Omega(\flat_1) + \Omega(\flat_1 + \Phi(\flat_2, \flat_1))]}{2}.$$

Corollary 3. *If we set $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, then we obtain the H–H inequality via NCFIO in the aspects of convexity, given as:*

$$\Omega\left(\frac{m\flat_1 + \flat_2}{2}\right) \leq \frac{1-\alpha}{2(\flat_2 - m\flat_1)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(\flat_2) \;+\; {}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(m\flat_1)\right] \leq \frac{\Omega(m\flat_1) + \Omega(\flat_2)}{2}.$$

Corollary 4. *If we set $m = 1$, then we obtain the H–H inequality via NCFIO in the aspects of convexity, given as:*

$$\Omega\left(\frac{\flat_1 + \flat_2}{2}\right) \leq \frac{1-\alpha}{2(\flat_2 - \flat_1)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{\flat_1^+}\Omega(\flat_2) \;+\; {}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(\flat_1)\right] \leq \frac{\Omega(\flat_1) + \Omega(\flat_2)}{2}.$$

4. Generalizations of H–H-Type Inequality via Non-Conformable Fractional Integral Operator

Numerous researchers have started collaborating on new thinking pertaining to this issue from various perspectives in the sphere of convex theory. Many investigators have established new expansions, estimations, and refinements of this inequality in the form of various versions of preinvexity. It is also worth noting that certain classical inequalities with special means as applications can be retrieved from the H–H inequality using the convenience of unique convex functions. Inequalities for convex functions are essential in reviewing and in other aspects of applied and pure mathematics. H–H inequalities in the mode of convexity have gained considerable interest in recent decades and, as a result, a considerable number of incremental improvements and assertions have been obtained.

The intent of this section is to investigate and demonstrate a new equality. On the basis of this newly investigated equality, we acquire some new improvements to the H–H-type inequalities involving an NCFIO. We add some comments to enhance the content and to pique the interest of readers. We begin with a lemma involving NCFIO.

Lemma 1. *Let $\alpha \le -1, m\flat_1 < \flat_2$ and $\Omega : [m\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function. If $\Omega \in L_{\alpha-1}[m\flat_1, \flat_2]$, then*

$$\begin{aligned}
&\left[\frac{\Omega(m\flat_1) + \Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))}{2}\right] - \frac{1-\alpha}{2(\Phi(\flat_2, \flat_1, m))^{-\alpha}} \\
&\times \left\{ \alpha \left[{}_{N_3}J^{\alpha+1}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1) + {}_{N_3}J^{\alpha+1}_{m\flat_1^+}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right] \right. \\
&\left. + \frac{(2-\alpha)}{\Phi(\flat_2, \flat_1, m)} \left[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1) + {}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right] \right\} \\
&= \frac{(\Phi(\flat_2, \flat_1, m))^2}{2}(I_1 + I_2),
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_1, \flat_1, m))d\wp, \\
I_2 &= \int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp.
\end{aligned}$$

Proof. It is obvious that

$$\begin{aligned}
I_1 &= \int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \\
&= \int_0^1 \wp^{1-\alpha}(1-\wp)\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \\
&= \int_0^1 (\wp^{1-\alpha} - \wp^{2-\alpha})\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp.
\end{aligned}$$

Now, integrating by parts, we get

$$\begin{aligned}
I_1 &= \frac{\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))}{(\Phi(\flat_2, \flat_1, m))^2} + \frac{1-\alpha}{(\Phi(\flat_2, \flat_1, m))^2}\left[-\alpha\int_0^1 \wp^{-\alpha-1}\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \right. \\
&\quad \left. -(2-\alpha)\int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \right] \\
&= \frac{\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))}{(\Phi(\flat_2, \flat_1, m))^2} + \frac{1-\alpha}{(\Phi(\flat_2, \flat_1, m))^{2-\alpha}}\left[\alpha\int_{m\flat_1}^{m\flat_1+\Phi(\flat_2,\flat_1,m)}(x - m\flat_1)^{-\alpha-1}\Omega(x)dx \right. \\
&\quad \left. -\frac{(2-\alpha)}{(\Phi(\flat_2, \flat_1, m))}\int_{m\flat_1}^{m\flat_1+\Phi(\flat_2,\flat_1,m)}(x - m\flat_1)^{-\alpha}\Omega(x)dx \right] \\
&= \frac{\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))}{(\Phi(\flat_2, \flat_1, m))^2} - \frac{1-\alpha}{(\Phi(\flat_2, \flat_1, m))^{2-\alpha}}\left[\alpha\, {}_{N_3}J^{\alpha+1}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1) \right. \\
&\quad \left. + \frac{2-\alpha}{\Phi(\flat_2, \flat_1, m)}\, {}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \\
&= \int_0^1 \wp^{1-\alpha}(1-\wp)\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \\
&= \int_0^1 (\wp^{1-\alpha} - \wp^{2-\alpha})\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp.
\end{aligned}$$

Again, applying integration by parts, we find

$$
\begin{aligned}
I_2 &= \frac{\wp^{1-\alpha} - \wp^{2-\alpha}}{\Phi(\flat_2, \flat_1, m)} \Omega'(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))|_0^1 \\
&\quad - \frac{1}{\Phi(\flat_2, \flat_1, m)} \int_0^1 ((1-\alpha)\wp^{-\alpha} - (2-\alpha)\wp^{1-\alpha})\Omega'(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \\
&= -\frac{1}{\Phi(\flat_2, \flat_1, m)} \left[\frac{(1-\alpha)\wp^{-\alpha} - (2-\alpha)\wp^{1-\alpha}}{(\Phi(\flat_2, \flat_1, m))} \Omega(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))|_0^1 \right. \\
&\quad \left. - \frac{1}{\Phi(\flat_2, \flat_1, m)} \int_0^1 (\alpha(1-\alpha)\wp^{-\alpha-1} - (1-\alpha)(2-\alpha)\wp^{-\alpha})\Omega(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \right] \\
&= \frac{\Omega(m\flat_1)}{(\Phi(\flat_2, \flat_1, m))^2} + \frac{(1-\alpha)}{(\Phi(\flat_2, \flat_1, m))^2} \left[-\alpha \int_0^1 \wp^{-\alpha-1}\Omega(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \right. \\
&\quad \left. -(2-\alpha) \int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \right] \\
&= \frac{\Omega(m\flat_1)}{(\Phi(\flat_2, \flat_1, m))^2} + \frac{(1-\alpha)}{(\Phi(\flat_2, \flat_1, m))^2} \\
&\quad \times \left[\frac{-\alpha}{(\Phi(\flat_2, \flat_1, m))^{-\alpha}} \int_{m\flat_1}^{m\flat_1 + \Phi(\flat_2, \flat_1, m)} (m\flat_1 + \Phi(\flat_2, \flat_1, m) - x)^{-\alpha-1}\Omega(x)dx \right. \\
&\quad \left. - \frac{2-\alpha}{(\Phi(\flat_2, \flat_1, m))^{-\alpha+1}} \int_{m\flat_1}^{\flat_1 + \Phi(\flat_2, \flat_1, m)} (m\flat_1 + \Phi(\flat_2, \flat_1, m) - x)^{-\alpha}\Omega(x)dx \right] \\
&= \frac{\Omega(m\flat_1)}{(\Phi(\flat_2, \flat_1, m))^2} - \frac{(1-\alpha)}{(\Phi(\flat_2, \flat_1, m))^{2-\alpha}} \\
&\quad \times \left[\alpha \left({}_{N_3}J_{\flat_1^+}^{\alpha+1}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right) + \frac{2-\alpha}{\Phi(\flat_2, \flat_1)} \left({}_{N_3}J_{m\flat_1^+}^{\alpha}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right) \right]
\end{aligned}
$$

From the above developments, we get

$$
\begin{aligned}
I_1 + I_2 &= \frac{\Omega(m\flat_1) + \Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))}{(\Phi(\flat_2, \flat_1, m))^2} - \frac{1-\alpha}{(\Phi(\flat_2, \flat_1, m))^{2-\alpha}} \\
&\quad \times \left\{ \alpha \left[{}_{N_3}J_{m\flat_1 + \Phi(\flat_2, \flat_1, m)^-}^{\alpha+1}\Omega(m\flat_1) + {}_{N_3}J_{m\flat_1^+}^{\alpha+1}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right] \right. \\
&\quad \left. + \frac{(2-\alpha)}{\Phi(\flat_2, \flat_1, m)} \left[{}_{N_3}J_{m\flat_1 + \Phi(\flat_2, \flat_1, m)^-}^{\alpha}\Omega(\flat_1) + {}_{N_3}J_{m\flat_1^+}^{\alpha}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right] \right\}.
\end{aligned}
$$

Multiplying both sides of the above equality by $\frac{\Phi^2(\flat_2 \flat_1, m)}{2}$, we obtain the proof of Lemma 1. □

Theorem 3. *Assume that $\mathbb{X}$ is defined as in Theorem 2. Let $\Omega : \mathbb{X} \to \mathbb{R}$ be a differentiable function such that $\Omega'' \in L[m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)]$. If the function $\left|\Omega''\right|$ is $(m, tgs)-$type s-preinvex on $[m\flat_1, \flat_1 + \Phi(\flat_2, \flat_1, m)]$, then the fractional integral inequality $\forall\ \alpha \le -1$ and $s \in [0, 1)$ is given as:*

$$
|U| \le \frac{(\Phi(\flat_2, \flat_1, m))^2}{2} \mathcal{B}(s+2, s-\alpha+2) \left(m\left|\Omega''(\flat_1)\right| + \left|\Omega''(\flat_2)\right| \right),
$$

where

$$
\begin{aligned}
U &= \left[\frac{\Omega(m\flat_1) + \Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))}{2} \right] - \frac{1-\alpha}{2(\Phi(\flat_2, \flat_1, m))^{-\alpha}} \\
&\quad \times \left\{ \alpha \left[{}_{N_3}J_{m\flat_1 + \Phi(\flat_2, \flat_1, m)^-}^{\alpha+1}\Omega(m\flat_1) + {}_{N_3}J_{m\flat_1^+}^{\alpha+1}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right] \right. \\
&\quad \left. + \frac{(2-\alpha)}{\Phi(\flat_2, \flat_1, m)} \left[{}_{N_3}J_{m\flat_1 + \Phi(\flat_2, \flat_1, m)^-}^{\alpha}\Omega(m\flat_1) + {}_{N_3}J_{m\flat_1^+}^{\alpha}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right] \right\},
\end{aligned}
$$

and $\mathcal{B}(x,y) = \int_0^1 \wp^{x-1}(1-\wp)^{y-1}d\wp,\ x>0,\ y>0$.

Proof. Employing Lemma 1, we have

$$\begin{aligned}
&\left|\int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m))d\wp\right.\\
&\quad\left.+\int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp\right|\\
&\le \int_0^1 \wp(1-\wp)^{1-\alpha}\left|\Omega''(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m))\right|d\wp\\
&\quad+\int_0^1 \wp(1-\wp)^{1-\alpha}\left|\Omega''(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\right|d\wp.
\end{aligned}$$

Employing the property of (m,tgs)-type s-preinvexity, we have

$$\begin{aligned}
&\int_0^1 \wp(1-\wp)^{1-\alpha}\left|\Omega''(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m))\right|d\wp\\
&\le \int_0^1 \wp(1-\wp)^{1-\alpha}\wp^s(1-\wp)^s\left(m\left|\Omega''(\flat_1)\right|+\left|\Omega''(\flat_2)\right|\right)d\wp\\
&= \mathcal{B}(s+2,s-\alpha+2)\left(m\left|\Omega''(\flat_1)\right|+\left|\Omega''(\flat_2)\right|\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \wp(1-\wp)^{1-\alpha}\left|\Omega''(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\right|d\wp\\
&= \mathcal{B}(s+2,s-\alpha+2)\left(m\left|\Omega''(\flat_1)\right|+\left|\Omega''(\flat_2)\right|\right).
\end{aligned}$$

Hence, we have

$$|U| \le \frac{(\Phi(\flat_2,\flat_1,m))^2}{2}\mathcal{B}(s+2,s-\alpha+2)\left(m|\Omega''(\flat_1)|+|\Omega''(\flat_2)|\right).$$

□

Remark 7. *If we take $s=1$, then fractional integral inequality is given as:*

$$|U| \le \frac{(\Phi(\flat_2,\flat_1,m))^2}{2}\mathcal{B}(3,3-\alpha)\left(m\left|\Omega''(\flat_1)\right|+\left|\Omega''(\flat_2)\right|\right),$$

where U is explored in Theorem 3.

Remark 8. *If we take $\Phi(\flat_2,\flat_1,m)=\flat_2-m\flat_1$, then the fractional integral inequality is given as*

$$|U_x| \le \frac{(\flat_2-m\flat_1)^2}{2}\mathcal{B}(s+2,s-\alpha+2)\left(m\left|\Omega''(\flat_1)\right|+\left|\Omega''(\flat_2)\right|\right),$$

where

$$\begin{aligned}
U_x &= \left[\frac{\Omega(m\flat_1)+\Omega(\flat_2)}{2}\right]-\frac{1-\alpha}{2(\flat_2-m\flat_1)^{-\alpha}}\\
&\quad\times\left\{\alpha\left[{}_{N_3}J^{\alpha+1}_{\flat_2^-}\Omega(m\flat_1)+{}_{N_3}J^{\alpha+1}_{m\flat_1^+}\Omega(\flat_2)\right]\right.\\
&\quad\left.+\frac{(2-\alpha)}{(\flat_2-m\flat_1)}\left[{}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(m\flat_1)+{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(\flat_2)\right]\right\}.
\end{aligned}$$

Remark 9. *If we take Condition C, then the fractional integral inequality is given as*

$$|U| \leq \frac{(\Phi(\flat_2, \flat_1, m))^2}{2} \mathcal{B}(s+2, s-\alpha+2) \left(m \left| \Omega''(\flat_1) \right| + \left| \Omega''(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \right| \right),$$

where U is explored in Theorem 3.

Theorem 4. *Assume that $\mathbb{X}$ is defined as in Theorem 2 and Ω is defined as in Theorem 3. If $\left|\Omega''\right|$ is a G–L (s, m)-preinvex function on $[m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)]$, then the fractional integral inequality $\forall\, \alpha \leq -1$ and $s \in [0, 1)$ is given as*

$$|U| \leq \frac{(\Phi(\flat_2, \flat_1, m))^2}{2} [\mathcal{B}(-s+2, -\alpha+2) + \mathcal{B}(2, -s-\alpha+2)] \left(m \left| \Omega''(\flat_1) \right| + \left| \Omega''(\flat_2) \right| \right),$$

where U is defined in Theorem 3 and $\mathcal{B}(x, y) = \int_0^1 \wp^{x-1}(1-\wp)^{y-1} d\wp,\ x > 0,\ y > 0$.

Proof. Employing Lemma 1, we have

$$\begin{aligned}
&\left| \int_0^1 \wp(1-\wp)^{1-\alpha} \Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m)) d\wp \right. \\
&\left. + \int_0^1 \wp(1-\wp)^{1-\alpha} \Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) d\wp \right| \\
\leq &\int_0^1 \wp(1-\wp)^{1-\alpha} \left| \Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m)) \right| d\wp \\
&+ \int_0^1 \wp(1-\wp)^{1-\alpha} \left| \Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \right| d\wp.
\end{aligned}$$

Since $\left|\Omega''\right|$ is G–L (s, m)-preinvexity, we get

$$\begin{aligned}
&\int_0^1 \wp(1-\wp)^{1-\alpha} \left| \Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m)) \right| d\wp \\
\leq &\ m \left| \Omega''(\flat_1) \right| \int_0^1 \wp^{1-s}(1-\wp)^{1-\alpha} d\wp + \left| \Omega''(\flat_2) \right| \int_0^1 \wp(1-\wp)^{1-\alpha-s} d\wp \\
= &\left[m \left| \Omega''(\flat_1) \right| \mathcal{B}(-s+2, -\alpha+2) + \left| \Omega''(\flat_2) \right| \mathcal{B}(2, -s-\alpha+2) \right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \wp(1-\wp)^{1-\alpha} \left| \Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) \right| d\wp \\
= &\left[m \left| \Omega''(\flat_1) \right| \mathcal{B}(2, -s-\alpha+2) + \left| \Omega''(\flat_2) \right| \mathcal{B}(-s+2, -\alpha+2) \right].
\end{aligned}$$

Hence, we get

$$|U| \leq \frac{(\Phi(\flat_2, \flat_1, m))^2}{2} [\mathcal{B}(-s+2, -\alpha+2) + \mathcal{B}(2, -s-\alpha+2)] \left(m \left| \Omega''(\flat_1) \right| + \left| \Omega''(\flat_2) \right| \right).$$

□

Remark 10. *If we take $s = 1$, then the fractional integral inequality is given as*

$$|U| \leq \frac{(\Phi(\flat_2, \flat_1, m))^2}{2} [\mathcal{B}(1, -\alpha+2) + \mathcal{B}(2, -\alpha+1)] \left(m \left| \Omega''(\flat_1) \right| + \left| \Omega''(\flat_2) \right| \right),$$

where U is discussed in Theorem 3.

Remark 11. *If we take $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, then the fractional integral inequality is given as*

$$|U_x| \leq \frac{(\flat_2 - m\flat_1)^2}{2}[\mathcal{B}(-s+2, -\alpha+2) + \mathcal{B}(2, -s-\alpha+2)]\left(m\left|\Omega''(\flat_1)\right| + \left|\Omega''(\flat_2)\right|\right),$$

where U_x is discussed in Remark 8.

Remark 12. *If we take $s = 1$ and $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, then the fractional integral inequality is given as*

$$|U_x| \leq \frac{(\flat_2 - m\flat_1)^2}{2}[\mathcal{B}(1, -\alpha+2) + \mathcal{B}(2, -\alpha+1)]\left(m\left|\Omega''(\flat_1)\right| + \left|\Omega''(\flat_2)\right|\right),$$

where U_x is discussed in Remark 8.

Remark 13. *If we take Condition C, then the fractional integral inequality is given as*

$$|U| \leq \frac{(\Phi(\flat_2, \flat_1, m))^2}{2} \times [\mathcal{B}(-s+2, -\alpha+2) + \mathcal{B}(2, -s-\alpha+2)]\left(m\left|\Omega''(\flat_1)\right| + \left|\Omega''(m\flat_1 + \Phi(\flat_2, \flat_1, m))\right|\right),$$

where U is discussed in Theorem 3.

Theorem 5. *Assume that $\mathbb{X}$ is defined as in Theorem 2 and Ω is defined as in Theorem 3. If $\left|\Omega''\right|$ is prequasi m-invex on $[m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)]$, then for all $q \geq 1$, $\wp \in [0,1]$ and $\alpha \leq -1$, the fractional integral inequality is given as*

$$|U| \leq \frac{(\Phi(\flat_2, \flat_1, m))^2}{(2-\alpha)(3-\alpha)}\left[\max\left(m\left|\Omega''(\flat_1)\right|^q, \left|\Omega''(\flat_2)\right|^q\right)\right]^{\frac{1}{q}},$$

where U is explored in Theorem 3.

Proof. Employing Lemma 1 and utilizing the power mean inequality, we get

$$|U| \leq \frac{(\Phi(\flat_2, \flat_1, m))^2}{2}\left[\int_0^1 \wp(1-\wp)^{1-\alpha}\left|\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))\right|d\wp + \int_0^1 \wp(1-\wp)^{1-\alpha}\left|\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))\right|d\wp\right].$$

It is obvious that

$$\begin{aligned} I_1 &= \int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \\ &= \int_0^1 \wp^{1-\alpha}(1-\wp)\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \\ &= \int_0^1 (\wp^{1-\alpha} - \wp^{2-\alpha})\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_2, m))d\wp \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1 \wp(1-\wp)^{1-\alpha}\Omega''(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \\ &= \int_0^1 \wp^{1-\alpha}(1-\wp)\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \\ &= \int_0^1 (\wp^{1-\alpha} - \wp^{2-\alpha})\Omega''(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp. \end{aligned}$$

Hence

$$|U| \leq \frac{(\Phi(\flat_2,\flat_1,m))^2}{2}\Bigg[\bigg(\int_0^1 \wp(1-\wp)^{1-\alpha}\Big|\Omega''(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m))\Big|^q d\wp\bigg)^{\frac{1}{q}}$$
$$+\bigg(\int_0^1 \wp(1-\wp)^{1-\alpha}\Big|\Omega''(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\Big|^q d\wp\bigg)^{\frac{1}{q}}\Bigg]$$
$$=\frac{(\Phi(\flat_2,\flat_1,m))^2}{(2-\alpha)(3-\alpha)}\Big[\max\Big(m\Big|\Omega''(\flat_1)\Big|^q,\ |\Omega''(\flat_2)|^q\Big)\Big]^{\frac{1}{q}}.$$

This is the required desired proof. □

Remark 14. *Taking $\Phi(\flat_2,\flat_1,m)=\flat_2-m\flat_1$, the fractional integral inequality is given as*

$$|U_x| \leq \frac{(\flat_2-m\flat_1)^2}{(2-\alpha)(3-\alpha)}\Big[\max\Big(m\Big|\Omega''(\flat_1)\Big|^q,\Big|\Omega''(\flat_2)\Big|^q\Big)\Big]^{\frac{1}{q}},$$

where U_x is discussed in Remark 8.

Remark 15. *Taking Condition C in Theorem 5, the fractional integral inequality is given as*

$$|U| \leq \frac{(\Phi(\flat_2,\flat_1,m))^2}{(2-\alpha)(3-\alpha)}\Big[\max\Big(m\Big|\Omega''(\flat_1)\Big|^q,\Big|\Omega''(m\flat_1+\Phi(\flat_2,\flat_1,m))\Big|^q\Big)\Big]^{\frac{1}{q}},$$

where U is explored in Theorem 3.

5. Fejér-Type Inequality via Non-Conformable Fractional Integral Operator

Integral inequalities are pertinent and have implementations in orthogonal polynomials, combinatory and linear programming, dynamics, number theory, quantum theory, and optimization theory. This topic has drawn a lot of consideration from mathematicians and other researchers. In the literature, this inequality is the most widely recognized one related to the subject of convex analysis. The weighted version of the H–H-type inequality is the Fejér-type inequality. In 1906, Fejér [65] was the first to examine and study this inequality. Varosanec and Bombardelli [66] presented the aspects of h-convexity in this inequality in 2009, stating that

$$\frac{\int_{\flat_1}^{\flat_2}\mathcal{W}(x)dx}{2h\left(\frac{1}{2}\right)}\Omega\left(\frac{\flat_1+\flat_2}{2}\right) \leq \int_{\flat_1}^{\flat_2}\Omega(x)\mathcal{W}(x)dx$$
$$\leq (\flat_2-\flat_1)(\Omega(\flat_1)+\Omega(\flat_2))\int_0^1 h(\wp)\mathcal{W}(\wp\flat_1+(1-\wp)\flat_2)d\wp.$$

where $\mathcal{W}:[\flat_1,\flat_2]\to\mathbb{R}, \mathcal{W}\geq 0$, and symmetric w.r.t. $\frac{\flat_1+\flat_2}{2}$.

Matłoka in [67] examined and explored a new variant of the Fejér inequality in the sense of h-preinvexity in 2014, which states that

$$\frac{\int_{\flat_1}^{\flat_1+\Phi(\flat_2,\flat_1)}\mathcal{W}(x)dx}{2h\left(\frac{1}{2}\right)}\Omega\left(\flat_1+\frac{1}{2}\Phi(\flat_2,\flat_1)\right) \leq \int_{\flat_1}^{\flat_1+\Phi(\flat_2,\flat_1)}\Omega(x)\mathcal{W}(x)dx$$
$$\leq \Phi(\flat_2,\flat_1)(\Omega(\flat_1)+\Omega(\flat_2))\cdot\int_0^1 h(\wp)\mathcal{W}(\flat_1+\wp\Phi(\flat_2,\flat_1))d\wp.$$

In 2017, Matłoka [67] presented the Fejér inequality via RLFIO in the sense of h-preinvexity utilizing Condition C, which is given as

$$\frac{\Gamma(\alpha)}{2\cdot h\left(\frac{1}{2}\right)\cdot\Phi(\flat_2,\flat_1)^{\alpha}}\Omega\left(\flat_1+\frac{1}{2}\Phi(\flat_2,\flat_1)\right)\left[\mathrm{I}^{\alpha}_{(\flat_1+\Phi(\flat_2,\flat_1))}\mathcal{W}(\flat_1)+\mathrm{I}^{\alpha}_{\flat_1^+}\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))\right]$$
$$\leq\frac{\Gamma(\alpha)}{\Phi(\flat_2,\flat_1)^{\alpha}}\left[\mathrm{I}^{\alpha}_{(\flat_1+\Phi(\flat_2,\flat_1))^{\alpha}}\mathcal{W}(\flat_1)\Omega(\flat_1)+\mathrm{I}^{\alpha}_{\flat_1^+}\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))\Omega(\flat_1+\Phi(\flat_2,\flat_1))\right]$$
$$\leq[\Omega(\flat_1)+\Omega(\flat_2)]\cdot\int_0^1\wp^{\alpha-1}[h(\wp)+h(1-\wp)]\mathcal{W}(\flat_1+\wp\Phi(\flat_2,\flat_1))d\wp.$$

Matłoka [67] presented the Fejér inequality via RLFIO in the sense of h-preinvexity, which is given as

$$\Bigg|\frac{\Gamma(\alpha+1)}{\Phi(\flat_2,\flat_1)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{\flat_1^+}\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))\Omega(\flat_1+\Phi(\flat_2,\flat_1))+\mathrm{I}^{\alpha}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\mathcal{W}(\flat_1)\Omega(\flat_1)$$
$$-\mathrm{I}^{\alpha+1}_{\flat_1^+}\mathcal{W}'(\flat_1+\Phi(\flat_2,\flat_1))\Omega(\flat_1+\Phi(\flat_2,\flat_1))+\mathrm{I}^{\alpha+1}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\mathcal{W}'(\flat_1)\Omega(\flat_1)\Big]$$
$$-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1+\Phi(\flat_2,\flat_1))\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))+\Omega(\flat_1)\mathcal{W}(\flat_1)]\Bigg|$$
$$\leq\left[\left|\Omega'(\flat_1)\right|+\left|\Omega'(\flat_2)\right|\right]\cdot\int_0^1\wp^{\alpha}\mathcal{W}(\flat_1+\wp\Phi(\flat_2,\flat_1))[h(\wp)+h(1-\wp)]d\wp.$$

Matłoka [67] presented the Fejér inequality via RLFIO in the sense of h-preinvexity utilizing the power mean inequality, which is given as

$$\Bigg|\frac{\Gamma(\alpha+1)}{\Phi(\flat_2,\flat_1)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{\flat_1^+}\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))\Omega(\flat_1+\Phi(\flat_2,\flat_1))+\mathrm{I}^{\alpha}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\mathcal{W}(\flat_1)\Omega(\flat_1)$$
$$-\mathrm{I}^{\alpha+1}_{\flat_1^+}\mathcal{W}'(\flat_1+\Phi(\flat_2,\flat_1))\Omega(\flat_1+\Phi(\flat_2,\flat_1))+\mathrm{I}^{\alpha+1}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\mathcal{W}'(\flat_1)\Omega(\flat_1)\Big]$$
$$-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1+\Phi(\flat_2,\flat_1))\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))+\Omega(\flat_1)\mathcal{W}(\flat_1)]\Bigg|$$
$$\leq\left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left[\left|\Omega'(\flat_1)\right|^q+\left|\Omega'(\flat_2)\right|^q\right]\int_0^1\wp^{\alpha}[\mathcal{W}(\flat_1+\wp\Phi(\flat_2,\flat_1))]^q[h(\wp)+h(1-\wp)]d\wp\right)^{\frac{1}{q}}.$$

Matłoka [67] presented the Fejér inequality via RLFIO in the sense of h-preinvexity utilizing the Hölder inequality, which is given as

$$\Bigg|\frac{\Gamma(\alpha+1)}{\Phi(\flat_2,\flat_1)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{\flat_1^+}\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))\Omega(\flat_1+\Phi(\flat_2,\flat_1))+\mathrm{I}^{\alpha}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\mathcal{W}(\flat_1)\Omega(\flat_1)$$
$$-\mathrm{I}^{\alpha+1}_{\flat_1^+}\mathcal{W}'(\flat_1+\Phi(\flat_2,\flat_1))\Omega(\flat_1+\Phi(\flat_2,\flat_1))+\mathrm{I}^{\alpha}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\mathcal{W}'(\flat_1)\Omega(\flat_1)\Big]$$
$$-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1+\Phi(\flat_2,\flat_1))\mathcal{W}(\flat_1+\Phi(\flat_2,\flat_1))+\Omega(\flat_1)\mathcal{W}(\flat_1)]\Bigg|$$
$$\leq\frac{2}{(\alpha\rho+1)^{\frac{1}{p}}}\left(\left[\left|\Omega'(\flat_1)\right|^q+\left|\Omega'(\flat_2)\right|^q\right]\cdot\int_0^1[\mathcal{W}(\flat_1+\wp\Phi(\flat_2,\flat_1))]^q h(\wp)d\wp\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

The purpose of this section is to start investigating and introducing a new variant of the Fejér inequality via NCFIO. Some corollaries and remarks are provided to enhance the objectives of this section.

Theorem 6. *Suppose $\Omega : [m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)] \to \mathbb{R}$ is an (m,h)-preinvex function, Condition C for Φ holds, $\Phi(\flat_2, \flat_1, m) > 0$, $h(\frac{1}{2}) > 0$, and $\mathcal{W} : [m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)] \to \mathbb{R}$, $\mathcal{W} \geq 0$ is symmetric w.r.t. $m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)$. Then the inequality via NCFIO is given as*

$$\begin{aligned}
&\frac{\Omega\left(m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)\right)}{2h\left(\frac{1}{2}\right)\Phi(\flat_2, \flat_1, m)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\mathcal{W}(m\flat_1) + {}_{N_3}J^{\alpha}_{m\flat_1^+}\mathcal{W}(m\flat_1 + \Phi(\flat_2, \flat_1, m))\right] \\
&\leq \frac{1}{\Phi(\flat_2, \flat_1, m)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\mathcal{W}(m\flat_1)\right. \\
&\quad \left. + {}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + \Phi(\flat_2, \flat_1, m))\right] \\
&\leq [\Omega(m\flat_1) + \Omega(\flat_2)]\int_0^1 \wp^{-\alpha}[h(\wp) + h(1-\wp)]\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp.
\end{aligned}$$

Proof. Employing the property of (m,h)-preinvexity and Condition C for Φ, we have

$$\Omega\left(m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)\right) \leq h\left(\frac{1}{2}\right)[\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) + \Omega(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))].$$

Multiplying both sides by

$$\wp^{-\alpha}\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) = \wp^{-\alpha}\mathcal{W}(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m)),$$

and then integrating the resulting inequality w.r.t. $\wp$ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \Omega\left(m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)\right)\wp^{-\alpha}\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \\
&\leq h\left(\frac{1}{2}\right)\left[\int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp\right. \\
&\quad \left. + \int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp\right].
\end{aligned}$$

Since

$$\int_0^1 \wp^{-\alpha}\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp = \frac{1}{\Phi(\flat_2, \flat_1, m)^{1-\alpha}}{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\mathcal{W}(m\flat_1),$$

$$\begin{aligned}
&\int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \\
&= \frac{1}{\Phi(\flat_2, \flat_1, m)^{1-\alpha}}{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\mathcal{W}(m\flat_1),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \wp^{-\alpha}\Omega(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + (1-\wp)\Phi(\flat_2, \flat_1, m))d\wp \\
&= \frac{1}{\Phi(\flat_2, \flat_1, m)^{1-\alpha}}{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + \Phi(\flat_2, \flat_1, m)),
\end{aligned}$$

from the above simplifications, we have

$$\begin{aligned}&\frac{\Omega\left(m\flat_1+\frac{1}{2}\Phi(\flat_2,\flat_1,m)\right)}{\Phi(\flat_2,\flat_1,m)^{1-\alpha}}\,{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\mathcal{W}(m\flat_1)\\&\le h\left(\frac{1}{2}\right)\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\mathcal{W}(m\flat_1)\right.\\&\left.+{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\right].\end{aligned} \tag{18}$$

Similarly, we also have

$$\begin{aligned}&\frac{\Omega\left(m\flat_1+\frac{1}{2}\Phi(\flat_2,\flat_1,m)\right)}{\Phi(\flat_2,\flat_1,m)^{1-\alpha}}\,{}_{N_3}J^{\alpha}_{m\flat_1^+}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\\&\le h\left(\frac{1}{2}\right)\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\mathcal{W}(m\flat_1)\right.\\&\left.+{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\right].\end{aligned} \tag{19}$$

After the addition of the above inequalities (18) and (19), we have the proof of the 1st inequality. For the 2nd inequality, we employ the property of (m,h)-preinvexity, which is given as

$$\Omega\big(m\flat_1+\wp\Phi(\flat_2,\flat_1,m)\big)\le h(1-\wp)\Omega(m\flat_1)+h(\wp)\Omega(\flat_2),$$
$$\Omega\big(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m)\big)\le h(\wp)\Omega(m\flat_1)+h(1-\wp)\Omega(\flat_2).$$

By adding these inequalities, we have

$$\Omega\big(m\flat_1+\wp\Phi(\flat_2,\flat_1,m)\big)+\Omega\big(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m)\big)\le h(1-\wp)+h(\wp)[\Omega(m\flat_1)+\Omega(\flat_2)].$$

If we multiply both sides by

$$\wp^{-\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))=\wp^{-\alpha}\mathcal{W}(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m)),$$

and then integrate the obtained inequality over $[0,1]$, we find

$$\begin{aligned}&\int_0^1\wp^{-\alpha}\Omega\big(m\flat_1+\wp\Phi(\flat_2,\flat_1,m)\big)\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp\\&+\int_0^1\wp^{-\alpha}\Omega\big(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m)\big)\mathcal{W}(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m))d\wp\\&\le[\Omega(m\flat_1)+\Omega(\flat_2)]\int_0^1\wp^{-\alpha}[h(\wp)+h(1-\wp)]\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp,\end{aligned}$$

which readily follows

$$\begin{aligned}&\frac{1}{\Phi(\flat_2,\flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\mathcal{W}(m\flat_1)\right.\\&\left.+{}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\right]\\&\le[\Omega(m\flat_1)+\Omega(\flat_2)]\int_0^1\wp^{-\alpha}[h(\wp)+h(1-\wp)]\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp.\end{aligned}$$

This completes the proof of the desired Theorem 6. □

Remark 16. *If we take $\alpha = 0$ and $m = 1$, then we obtain the Fejér inequality in the aspects of h-preinvexity proved by Małoka (see [67]).*

Remark 17. *If we take $\alpha = 0$, $m = 1$ and $\mathcal{W}(x) = 1$, then we obtain the H–H-type inequality via h-preinvexity examined by Małoka (see [48]).*

Remark 18. *If we take $\alpha = 0$, $m = 1$, $\mathcal{W}(x) = 1$, and $h(\wp) = \wp$, then we obtain the H–H-type inequality in the aspects of preinvexity investigated by Noor (see [63]).*

Remark 19. *If we take $\alpha = 0$, $m = 1$, and $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, then we obtain the Fejér-type inequality involving h–convexity explored by Varosanec and Bombardelli (see [66]).*

Remark 20. *If we take $\alpha = 0$, $m = 1$, $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, and $\mathcal{W}(x) = 1$, then we obtain the H–H-type inequality via h–convexity proved by Sarikaya (see [68]).*

Remark 21. *If we take $\alpha = 0$, $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, $\mathcal{W}(x) = 1$, and $h(\wp) = \wp$, then we obtain the H–H-type inequality in the aspects of the m-convex function that were first investigated and explored by Toader and Dragomir [64].*

Remark 22. *If we take $\alpha = 0$, $m = 1$, $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, $\mathcal{W}(x) = 1$, and $h(\wp) = \wp$, then we obtain the H–H-type inequality in the aspects of convexity examined by Hadamard (see [54]).*

Remark 23. *If we take $\alpha = 0$, $m = 1$, $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, $\mathcal{W}(x) = 1$, and $h(\wp) = \wp^s$, then we obtain the H–H inequality in the aspects of s–convexity examined and investigated by Fitzpatrick and Dragomir (see [59]).*

Corollary 5. *If we take $\alpha = 0$, then we obtain the Fejér-type inequality in the aspects of (m, h)-preinvexity, which is given by*

$$\begin{aligned} &\frac{\Omega\left(m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)\right)}{2h\left(\frac{1}{2}\right)\Phi(\flat_2, \flat_1, m)} \int_{m\flat_1}^{m\flat_1 + \Phi(\flat_2, \flat_1, m)} \mathcal{W}(\wp) d\wp \\ &\leq \frac{1}{\Phi(\flat_2, \flat_1, m)} \int_{m\flat_1}^{m\flat_1 + \Phi(\flat_2, \flat_1, m)} \Omega(\wp)\mathcal{W}(\wp) d\wp \\ &\leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{2} \int_0^1 [h(\wp) + h(1-\wp)]\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m)) d\wp. \end{aligned}$$

Corollary 6. *If we take $\alpha = 0$ and $\mathcal{W}(x) = 1$, then we obtain the H–H-type inequality in the aspects of (m, h)-preinvexity, which is given by*

$$\begin{aligned} &\frac{\Omega\left(m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)\right)}{2h\left(\frac{1}{2}\right)} \leq \frac{1}{\Phi(\flat_2, \flat_1, m)} \int_{m\flat_1}^{m\flat_1 + \Phi(\flat_2, \flat_1, m)} \Omega(\wp) d\wp \\ &\leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{2} \int_0^1 [h(\wp) + h(1-\wp)] d\wp. \end{aligned}$$

Corollary 7. *If we take $\alpha = 0$, $\mathcal{W}(x) = 1$, and $h(\wp) = \wp$, then we obtain the H–H-type inequality in the aspects of m-preinvexity, which is given by*

$$\Omega\left(m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)\right) \leq \frac{1}{\Phi(\flat_2, \flat_1, m)} \int_{m\flat_1}^{m\flat_1 + \Phi(\flat_2, \flat_1, m)} \Omega(\wp) d\wp \leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{2}.$$

Corollary 8. *If we take $\alpha = 0$, $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, and $\mathcal{W}(x) = 1$, then we obtain the H–H-type inequality in the aspects of (m, h)–convexity, which is given by*

$$\frac{\Omega\left(\frac{m\flat_1 + \flat_2}{2}\right)}{2h\left(\frac{1}{2}\right)} \leq \frac{1}{\flat_2 - m\flat_1} \int_{m\flat_1}^{\flat_2} \Omega(\wp) d\wp \leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{2} \int_0^1 [h(\wp) + h(1-\wp)] d\wp.$$

Corollary 9. *If we take $\alpha = 0$ and $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, then we obtain the Fejér-type inequality in the aspects of (m, h)–convexity, which is given by*

$$\begin{aligned}
&\frac{\Omega\left(\frac{m\flat_1+\flat_2}{2}\right)}{2h\left(\frac{1}{2}\right)(\flat_2 - m\flat_1)} \int_{m\flat_1}^{\flat_2} \mathcal{W}(\wp)d\wp \\
&\leq \frac{1}{\flat_2 - m\flat_1} \int_{m\flat_1}^{\flat_2} \Omega(\wp)\mathcal{W}(\wp)d\wp \\
&\leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{2} \int_0^1 [h(\wp) + h(1-\wp)]\mathcal{W}(\wp\flat_2 + m(1-\wp)\flat_1)d\wp.
\end{aligned}$$

Corollary 10. *Choosing $\alpha = 0$, $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, $\mathcal{W}(x) = 1$, and $h(\wp) = \wp^s$, then we obtain the H–H-type inequality via (s, m)–convexity, which is given by*

$$2^{s-1}\Omega\left(\frac{m\flat_1 + \flat_2}{2}\right) \leq \frac{1}{\flat_2 - m\flat_1} \int_{m\flat_1}^{\flat_2} \Omega(\wp)d\wp \leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{s+1}.$$

Corollary 11. *Choosing $h = 1$, we obtain the Fejér inequality via NCFIO in the aspects of m-preinvexity, which is given by*

$$\begin{aligned}
&\frac{\Omega\left(m\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1, m)\right)}{\Phi(\flat_2, \flat_1, m)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\mathcal{W}(m\flat_1) + {}_{N_3}J^{\alpha}_{m\flat_1{}^+}\mathcal{W}(m\flat_1 + \Phi(\flat_2, \flat_1, m))\right] \\
&\leq \frac{1}{\Phi(\flat_2, \flat_1, m)^{1-\alpha}} \bigg[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega(m\flat_1)\mathcal{W}(m\flat_1) \\
&\quad + {}_{N_3}J^{\alpha}_{m\flat_1{}^+}\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + \Phi(\flat_2, \flat_1, m))\bigg] \\
&\leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{\Phi(\flat_2, \flat_1, m)^{1-\alpha}} {}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\mathcal{W}(x).
\end{aligned}$$

Corollary 12. *Choosing $h = 1$ and $m = 1$, we obtain the Fejér inequality via NCFIO in the aspects of m-preinvexity, which is given by*

$$\begin{aligned}
&\frac{\Omega\left(\flat_1 + \frac{1}{2}\Phi(\flat_2, \flat_1)\right)}{\Phi(\flat_2, \flat_1)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{\flat_1+\Phi(\flat_2,\flat_1)^-}\mathcal{W}(\flat_1) + {}_{N_3}J^{\alpha}_{\flat_1{}^+}\mathcal{W}(\flat_1 + \Phi(\flat_2, \flat_1))\right] \\
&\leq \frac{1}{\Phi(\flat_2, \flat_1)^{1-\alpha}} \bigg[{}_{N_3}J^{\alpha}_{\flat_1+\Phi(\flat_2,\flat_1)^-}\Omega(\flat_1)\mathcal{W}(\flat_1) \\
&\quad + {}_{N_3}J^{\alpha}_{\flat_1{}^+}\Omega(\flat_1 + \Phi(\flat_2, \flat_1))\mathcal{W}(\flat_1 + \Phi(\flat_2, \flat_1))\bigg] \\
&\leq \frac{[\Omega(\flat_1) + \Omega(\flat_2)]}{\Phi(\flat_2, \flat_1)^{1-\alpha}} {}_{N_3}J^{\alpha}_{\flat_1+\Phi(\flat_2,\flat_1)^-}\mathcal{W}(x).
\end{aligned}$$

Corollary 13. *Choosing $m = 1$ and $\Phi(\flat_2, \flat_1) = \flat_2 - \flat_1$, we obtain the Fejér inequality via NCFIO in the aspects h–convexity, which is given by*

$$\begin{aligned}
&\frac{\Omega\left(\frac{\flat_1+\flat_2}{2}\right)}{2h(\frac{1}{2})(\flat_2 - \flat_1)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{\flat_2^-}\mathcal{W}(\flat_1) + {}_{N_3}J^{\alpha}_{\flat_1{}^+}\mathcal{W}(\flat_2)\right] \\
&\leq \frac{1}{(\flat_2 - \flat_1)^{1-\alpha}} \left[{}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(\flat_1)\mathcal{W}(\flat_1) + {}_{N_3}J^{\alpha}_{\flat_1{}^+}\Omega(\flat_2)\mathcal{W}(\flat_2)\right] \\
&\leq \frac{[\Omega(\flat_1) + \Omega(\flat_2)]}{(\flat_2 - \flat_1)^{1-\alpha}} \int_0^1 \wp^{-\alpha}[h(\wp) + h(1-\wp)]\mathcal{W}(\flat_1 + \wp(\flat_2 - \flat_1))d\wp.
\end{aligned}$$

Corollary 14. *Choosing $h = 1$ and $\Phi(\flat_2, \flat_1, m) = \flat_2 - m\flat_1$, we obtain the Fejér inequality via NCFIO in the aspects of m–convexity, which is given by*

$$\begin{aligned}
&\frac{\Omega\left(\frac{m\flat_1+\flat_2}{2}\right)}{(\flat_2 - m\flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_2^-}\mathcal{W}(m\flat_1) + {}_{N_3}J^{\alpha}_{m\flat_1^+}\mathcal{W}(\flat_2)\right] \\
&\leq \frac{1}{(\flat_2 - m\flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(m\flat_1)\mathcal{W}(m\flat_1) + {}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega(\flat_2)\mathcal{W}(\flat_2)\right] \\
&\leq \frac{[\Omega(m\flat_1) + \Omega(\flat_2)]}{(\flat_2 - m\flat_1)^{1-\alpha}}{}_{N_3}J^{\alpha}_{\flat_2^-}\mathcal{W}(x).
\end{aligned}$$

Corollary 15. *Choosing $h = 1$, $m = 1$, and $\Phi(\flat_2, \flat_1) = \flat_2 - \flat_1$, we attain the Fejér inequality via NCFIO in the aspects of convexity, which is given by*

$$\begin{aligned}
&\frac{\Omega\left(\frac{\flat_1+\flat_2}{2}\right)}{(\flat_2 - \flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_2^-}\mathcal{W}(\flat_1) + {}_{N_3}J^{\alpha}_{\flat_1^+}\mathcal{W}(\flat_2)\right] \\
&\leq \frac{1}{(\flat_2 - \flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(\flat_1)\mathcal{W}(\flat_1) + {}_{N_3}J^{\alpha}_{\flat_1^+}\Omega(\flat_2)\mathcal{W}(\flat_2)\right] \\
&\leq \frac{[\Omega(\flat_1) + \Omega(\flat_2)]}{(\flat_2 - \flat_1)^{1-\alpha}}{}_{N_3}J^{\alpha}_{\flat_2^-}\mathcal{W}(x).
\end{aligned}$$

6. Refinements of Fejér Inequality via Non-Conformable Fractional Integral Operator

Several academics and mathematicians have recently been working on fresh ideas related to this issue from various angles in the convex analysis field. Several new Fejér type inequalities were established in the literature by using different kinds of convexity and different kinds of fractional operators. The main goal of this section is to explore and investigate a new lemma. By utilizing this newly introduced lemma, we obtain some extensions, estimations, and generalizations of the Fejér inequality via NCFIO. In order to obtain the results, we utilize the idea of the (m, h)-preinvex function with the help of the power mean and the Hölder inequality. Several corollaries are offered to illustrate this section.

Lemma 2. *Assume that $\Phi : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and $\mathbb{X} \subseteq \mathbb{R}$, such that $\mathbb{X}$ is an open m-invex subset w.r.t. Φ and $\flat_1, \flat_2 \in \mathbb{X}$ with $\Phi(\flat_2, \flat_1, m) > 0$. Suppose that $\Omega : \mathcal{H} \to \mathcal{R}$ is differentiable mapping on $\mathbb{X}$ such that $\Omega' \in L([m\flat_1, m\flat_1 + \Phi(\flat_2, \flat_1, m)])$. If $\mathcal{W} : \mathbb{X} \to [0, \infty)$ is differentiable, then the following equality holds*

$$\begin{aligned}
&\int_0^1 \left[(1-\wp)^{-2\alpha} + \wp^{-2\alpha}\right]\mathcal{W}(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))\Omega'(m\flat_1 + \wp\Phi(\flat_2, \flat_1, m))d\wp \\
= &\frac{2\alpha}{\Phi(\flat_2, \flat_1, m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\mathcal{W}(m\flat_1 + \Phi(\flat_2, \flat_1, m))\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \\
&+ {}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Big) \\
&- \frac{1}{\Phi(\flat_2, \flat_1, m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\flat_1^+}\mathcal{W}'(m\flat_1 + \Phi(\flat_2, \flat_1, m))\Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m)) \\
&+ {}_{N_3}J^{2\alpha}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Big) \\
&- \frac{1}{\Phi(\flat_2, \flat_1, m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1) - \Omega(m\flat_1 + \Phi(\flat_2, \flat_1, m))\mathcal{W}(m\flat_1 + \Phi(\flat_2, \flat_1, m))].
\end{aligned}$$

Proof. Integrating by parts,

$$\begin{aligned}
&\int_0^1 \Big[(1-\wp)^{-2\alpha}+\wp^{-2\alpha}\Big]\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\Omega'(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp \\
&= \frac{1}{\Phi(\flat_2,\flat_1,m)}\Omega(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\Big[(1-\wp)^{-2\alpha}+\wp^{-2\alpha}\Big]_0^1 \\
&+\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)}\int_0^1\Big[(1-\wp)^{-2\alpha-1}+\wp^{-2\alpha-1}\Big]\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m)) \\
&\times\Omega(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp \\
&-\int_0^1\Big[(1-\wp)^{-2\alpha}+\wp^{-2\alpha}\Big]\mathcal{W}'(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp \\
&= -\frac{1}{\Phi(\flat_2,\flat_1,m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))] \\
&+\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m)) \\
&+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Big) \\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\flat_1^+}\mathcal{W}'(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m)) \\
&+{}_{N_3}J^{2\alpha}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Big),
\end{aligned}$$

which completes the proof. □

Theorem 7. *Assume that $\Phi:\mathbb{X}\times\mathbb{X}\to\mathbb{R}$ and $\mathbb{X}\subseteq\mathbb{R}$, such that $\mathbb{X}$ is an open m-invex subset w.r.t. Φ and $\flat_1,\flat_2\in\mathbb{X}$ with $\Phi(\flat_2,\flat_1,m)>0$. Suppose that $\Omega:\mathbb{X}\to\mathbb{R}$ is a differentiable mapping on $\mathbb{X}$ and $\mathcal{W}:\mathbb{X}\to[0,\infty)$ is differentiable and symmetric to $m\flat_1+\frac{1}{2}\Phi(\flat_2,\flat_1,m)$. If $|\Omega'|$ is generalized (m,h)-preinvex on $\mathbb{X}$, we have the following inequality via NCFIO given as*

$$\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m)) \\
&+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Big) \\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\flat_1^+}\mathcal{W}'(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m)) \\
&+{}_{N_3}J^{2\alpha}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Big) \\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Bigg| \\
&\le\big[m|\Omega'(\flat_1)|+|\Omega'(\flat_2)|\big]\cdot\int_0^1\wp^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))[h(\wp)+h(1-\wp)]d\wp.
\end{aligned}$$

Proof. Employing Lemma 2 and utilizing the property of (m,h)-preinvexity of $|\Omega'|$, we have

$$
\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Bigg({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\\
&+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Bigg)\\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-2\alpha}}\Bigg({}_{N_3}J^{2\alpha}_{m\flat_1^+}\mathcal{W}'(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\\
&+{}_{N_3}J^{2\alpha}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Bigg)\\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Bigg|\\
\le&\int_0^1\left|(1-\wp)^{-2\alpha}+\wp^{-2\alpha}\right|\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\left|\Omega'(m\flat_1+\Phi(\flat_2,\flat_1,m))\right|d\wp\\
\le&\int_0^1\left[(1-\wp)^{-2\alpha}+\wp^{-2\alpha}\right]\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\left[mh(1-\wp)\left|\Omega'(\flat_1)\right|+h(\wp)\left|\Omega'(\flat_2)\right|\right]d\wp\\
=&\,m\left|\Omega'(\flat_1)\right|\cdot\int_0^1(1-\wp)^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))h(1-\wp)d\wp\\
&+m\left|\Omega'(\flat_1)\right|\int_0^1\wp^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))h(1-\wp)d\wp\\
&+\left|\Omega'(\flat_2)\right|\cdot\int_0^1(1-\wp)^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))h(\wp)d\wp\\
&+\left|\Omega'(\flat_2)\right|\int_0^1\wp^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))h(\wp)d\wp\\
=&\left[m\left|\Omega'(\flat_1)\right|+\left|\Omega'(\flat_2)\right|\right]\int_0^1\wp^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\cdot h(\wp)d\wp\\
&+\left[m\left|\Omega'(\flat_1)\right|+\left|\Omega'(\flat_2)\right|\right]\int_0^1\wp^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\cdot h(1-\wp)d\wp\\
&=\left[m\left|\Omega'(\flat_1)\right|+\left|\Omega'(\flat_2)\right|\right]\cdot\int_0^1\wp^{-2\alpha}\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))[h(\wp)+h(1-\wp)]d\wp,
\end{aligned}
$$

which completes the proof. □

Corollary 16. *Choosing $h(\wp)=\wp$ and $\mathcal{W}(x)\equiv 1$, Theorem 7 via NCFIO in the aspects of m-preinvexity is given as*

$$
\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Bigg({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\Omega(m\flat_1)\Bigg)\\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Bigg|\\
\le&\frac{m|\Omega'(\flat_1)|+|\Omega'(\flat_2)|}{1-2\alpha}.
\end{aligned}
$$

Corollary 17. *Choosing $h(\wp)=\wp$, $m=1$, and $\mathcal{W}(x)\equiv 1$, Theorem 7 via NCFIO in the aspects of preinvexity is given as*

$$
\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1)^{-2\alpha}}\Bigg({}_{N_3}J^{2\alpha-1}_{\flat_1^+}\Omega(\flat_1+\Phi(\flat_2,\flat_1))+{}_{N_3}J^{2\alpha-1}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\Omega(\flat_1)\Bigg)\\
&-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1)-\Omega(\flat_1+\Phi(\flat_2,\flat_1))]\Bigg|\\
\le&\frac{|\Omega'(\flat_1)|+|\Omega'(\flat_2)|}{1-2\alpha}.
\end{aligned}
$$

Corollary 18. *Choosing $h(\wp) = \wp^s$ and $\mathcal{W}(x) \equiv 1$, Theorem 7 via NCFIO in the aspects of (s,m)-preinvexity is given as*

$$\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m)) + {}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\Omega(m\flat_1)\Big)$$
$$-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Big|$$
$$\leq [m|\Omega'(\flat_1)|+|\Omega'(\flat_2)|]\cdot\bigg[\frac{1}{1+s-2\alpha}+\frac{\Gamma(1-2\alpha)\Gamma(s+1)}{\Gamma(s-2\alpha-2)}\bigg].$$

Corollary 19. *Choosing $h(\wp) = \wp^s$, $m = 1$, and $\mathcal{W}(x) \equiv 1$, Theorem 7 via NCFIO in the aspects of s-preinvexity is given as*

$$\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{\flat_1^+}\Omega(\flat_1+\Phi(\flat_2,\flat_1)) + {}_{N_3}J^{2\alpha-1}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\Omega(\flat_1)\Big)$$
$$-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1)-\Omega(\flat_1+\Phi(\flat_2,\flat_1))]\Big|$$
$$\leq [|\Omega'(\flat_1)|+|\Omega'(\flat_2)|]\cdot\bigg[\frac{1}{1+s-2\alpha}+\frac{\Gamma(1-2\alpha)\Gamma(s+1)}{\Gamma(s-2\alpha-2)}\bigg].$$

Theorem 8. *Assume that $\mathbb{X}$ and $\mathcal{W}$ are defined as in Theorem 7. If $|\Omega'|^q$, $q > 1$, is generalized (h,m)-preinvex on $\mathbb{X}$, then one has*

$$\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))$$
$$+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Big)$$
$$-\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\flat_1^+}\mathcal{W}'(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))$$
$$+{}_{N_3}J^{2\alpha}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Big)$$
$$-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Big|$$
$$\leq\Big(\frac{2}{1-2\alpha}\Big)^{1-\frac{1}{q}}\Big(\Big[m|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q\Big]$$
$$\times\int_0^1\wp^{-2\alpha}[\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))]^q[h(\wp)+h(1-\wp)]d\wp\Big)^{\frac{1}{q}}.$$

Proof. Employing Lemma 2 and utilizing the property of the (h,m)-preinvex function and power mean inequality, we have

$$\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))$$
$$+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Big)$$
$$-\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\flat_1^+}\mathcal{W}'(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))$$
$$+{}_{N_3}J^{2\alpha}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Big)$$
$$-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Big|$$
$$\leq\Big(\int_0^1\Big[(1-\wp)^{-2\alpha}+\wp^{-2\alpha}\Big]d\wp\Big)^{1-\frac{1}{q}}\cdot$$
$$\cdot\Big(\int_0^1\Big[(1-\wp)^{-2\alpha}+\wp^{-2\alpha}\Big][\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))]^q|\Omega'(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))|^q d\wp\Big)^{\frac{1}{q}}$$
$$\leq\Big(\frac{2}{1-2\alpha}\Big)^{1-\frac{1}{q}}\Big(\Big[m|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q\Big]$$
$$\times\int_0^1\wp^{-2\alpha}[\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))]^q[h(\wp)+h(1-\wp)]d\wp\Big)^{\frac{1}{q}},$$

which completes the proof. □

Corollary 20. *Choosing $h(\wp) = \wp$ and $w(x) \equiv 1$, Theorem 8 via NCFIO in the aspects of m-preinvexity is given as*

$$\begin{aligned}&\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\Omega(m\flat_1)\Big)\\&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[m\Omega(\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Big|\\&\leq\left(\frac{2}{1-2\alpha}\right)^{1-\frac{1}{q}}\left(\frac{|m\Omega'(\flat 1)|^q+|\Omega'(\flat_2)|^q}{1-2\alpha}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 21. *Choosing $h(\wp) = \wp$, $m = 1$ and $w(x) \equiv 1$, then the Theorem 8 via NCFIO in the aspects of preinvexity is given as:*

$$\begin{aligned}&\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{\flat_1^+}\Omega(\flat_1+\Phi(\flat_2,\flat_1))+{}_{N_3}J^{2\alpha-1}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\Omega(\flat_1)\Big)\\&-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1)-\Omega(\flat_1+\Phi(\flat_2,\flat_1))]\Big|\\&\leq\left(\frac{2}{1-2\alpha}\right)^{1-\frac{1}{q}}\left(\frac{|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q}{1-2\alpha}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 22. *Choosing $h(\wp) = \wp^s$, and $w(x) \equiv 1$, Theorem 8 via NCFIO in the aspects of (s,m)-preinvexity is given as*

$$\begin{aligned}&\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}m\Omega(\flat_1)\Big)\\&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[m\Omega(\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Big|\\&\leq\left(\frac{2}{1-2\alpha}\right)^{1-\frac{1}{q}}\left(\Big[m|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q\Big]\cdot\Big[\frac{1}{1+s-2\alpha}+\frac{\Gamma(1-2\alpha)\Gamma(s+1)}{\Gamma(2+s-2\alpha)}\Big]\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 23. *Choosing $h(\wp) = \wp^s$, $m = 1$, and $w(x) \equiv 1$, Theorem 8 via NCFIO in the aspects of s-preinvexity is given as*

$$\begin{aligned}&|\frac{2\alpha}{\Phi(\flat_2,\flat_1)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{\flat_1^+}\Omega(\flat_1+\Phi(\flat_2,\flat_1))+{}_{N_3}J^{2\alpha-1}_{(\flat_1+\Phi(\flat,\flat_1))^-}\Omega(\flat_1)\Big)\\&-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1)-\Omega(\flat_1+\Phi(\flat_2,\flat_1))]\ |\\&\leq\left(\frac{2}{1-2\alpha}\right)^{1-\frac{1}{q}}\left(\Big[|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q\Big]\cdot\Big[\frac{1}{1+s-2\alpha}+\frac{\Gamma(1-2\alpha)\Gamma(s+1)}{\Gamma(2+s-2\alpha)}\Big]\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 9. *Assume that $\mathbb{X}$ and $\mathcal{W}$ are defined as in Theorem 7. If $|\Omega'|^q, q > 1$, is generalized (h,m)-preinvex on $\mathbb{H}$, then the following inequality via NCFIO holds*

$$\begin{aligned}&\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\flat_1^+}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\\&+{}_{N_3}J^{2\alpha-1}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Big)\\&-\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\flat_1^+}\mathcal{W}'(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\\&+{}_{N_3}J^{2\alpha}_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Big)\\&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Big|\\&\leq\frac{2}{(1-2\alpha\rho)^{\frac{1}{p}}}\left(\Big[m|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q\Big]\cdot\int_0^1[\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))]^q h(\wp)d\wp\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Employing Lemma 2 and utilizing the property of the (h,m)-preinvex function and Hölder inequality, we have

$$\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J_{m\flat_1^+}^{2\alpha-1}\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\\
&+{}_{N_3}J_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}^{2\alpha-1}\mathcal{W}(m\flat_1)\Omega(m\flat_1)\Big)\\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)^{1-2\alpha}}\Big({}_{N_3}J_{m\flat_1^+}^{2\alpha}\mathcal{W}'(m\flat_1+\Phi(\flat_2,\flat_1,m))\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\\
&+{}_{N_3}J_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}^{2\alpha}\mathcal{W}'(m\flat_1)\Omega(m\flat_1)\Big)\\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\mathcal{W}(m\flat_1)\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))\mathcal{W}(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Bigg|\\
&\leq\left(\int_0^1(1-\wp)^{-2\alpha p}d\wp\right)^{\frac{1}{p}}\left(\int_0^1[\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))]^q\left|\Omega'(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\right|^q d\wp\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\wp^{-2\alpha p}d\wp\right)^{\frac{1}{p}}\cdot\left(\int_0^1[\mathcal{W}(m\flat_1+\wp\eta(\flat_2,\flat_1,m))]^q\left|\Omega'(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))\right|^q d\wp\right)^{\frac{1}{q}}\\
&\leq\frac{2}{(1-2\alpha p)^{\frac{1}{p}}}\left(\left[m\left|\Omega'(\flat_1)\right|^q+\left|\Omega'(\flat_2)\right|^q\right]\cdot\int_0^1[\mathcal{W}(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))]^q h(\wp)d\wp\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 24. *Choosing $h(\wp)=\wp$ and $w(x)\equiv 1$, Theorem 9 for via NCFIO in the aspects of m-preinvexity is given as:*

$$\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J_{m\flat_1^+}^{2\alpha-1}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))+{}_{N_3}J_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}^{2\alpha-1}\Omega(\flat_1)\Big)\\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[\Omega(m\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Bigg|\\
&\leq\frac{2}{(1-2\alpha\rho)^{\frac{1}{p}}}\left(\frac{m|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 25. *Choosing $h(\wp)=\wp$, $m=1$, and $w(x)\equiv 1$, Theorem 9 via NCFIO in the aspects of preinvexity is given as*

$$\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1)^{-2\alpha}}\Big({}_{N_3}J_{\flat_1^+}^{2\alpha-1}\Omega(\flat_1+\Phi(\flat_2,\flat_1))+{}_{N_3}J_{(\flat_1+\Phi(\flat_2,\flat_1))^-}^{2\alpha-1}\Omega(\flat_1)\Big)\\
&-\frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1)-\Omega(\flat_1+\Phi(\flat_2,\flat_1))]\Bigg|\\
&\leq\frac{2}{(1-2\alpha\rho)^{\frac{1}{p}}}\left(\frac{|\Omega'(\flat_1)|^q+|\Omega'(\flat_2)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 26. *Choosing $h(\wp)=\wp^s$ and $w(x)\equiv 1$, Theorem 9 via NCFIO in the aspects of (m,s)-preinvexity is given as*

$$\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\flat_2,\flat_1,m)^{-2\alpha}}\Big({}_{N_3}J_{m\flat_1^+}^{2\alpha-1}\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))+{}_{N_3}J_{(m\flat_1+\Phi(\flat_2,\flat_1,m))^-}^{2\alpha-1}\Omega(m\flat_1)\Big)\\
&-\frac{1}{\Phi(\flat_2,\flat_1,m)}[m\Omega(\flat_1)-\Omega(m\flat_1+\Phi(\flat_2,\flat_1,m))]\Bigg|\\
&\leq\frac{2}{(1-2\alpha\rho)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left(m\left|\Omega'(\flat_1)\right|^q+\left|\Omega'(\flat_2)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 27. *Choosing* $h(\wp) = \wp^s$, $m = 1$, *and* $w(x) \equiv 1$, *Theorem 9 via NCFIO in the aspects of s-preinvexity is given as*

$$\Big|\frac{2\alpha}{\Phi(\flat_2,\flat_1)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{\flat_1^+}\Omega(\flat_1+\Phi(\flat_2,\flat_1)) + {}_{N_3}J^{2\alpha-1}_{(\flat_1+\Phi(\flat_2,\flat_1))^-}\Omega(\flat_1)\Big) - \frac{1}{\Phi(\flat_2,\flat_1)}[\Omega(\flat_1)-\Omega(\flat_1+\Phi(\flat_2,\flat_1))]\Big| \leq \frac{2}{(1-2\alpha\rho)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\Big(|\Omega'(\flat_1)|^q + |\Omega'(\flat_2)|^q\Big)^{\frac{1}{q}}.$$

7. Pachpatte-Type Inequality via Non-Conformable Fractional Integral Operator

Currently, the subject of convex analysis has gained popularity due to the fact that it is connected to the topic of inequality. Different inequalities are commonly documented as a result of convexity applications in practical sciences. The term preinvexity has been elaborated by a lot of researchers and scientists, and numerous papers have been produced on the subject that offer fresh estimates, extensions, generalizations, and importance. Many investigations have been done on the famous inequality, namely, the Pachpatte-type inequality pertaining to fractional integral operators. The concept of preinvexity has been crucial to the advancement of generalized convex programming. Many improvements to and expansions of this inequality were discovered in the literature. In this section we study and explore this inequality via NCFIO. We enhance this section's utility through the notes that are provided.

Theorem 10. *Assume that* $\mathbb{X}$ *is defined as in Theorem 2. Suppose that* $\Omega_1, \ \Omega_2 : \mathbb{X} \to \mathbb{R}$ *are differentiable functions such that* $\Omega_1, \Omega_2 \in L_{\alpha,0}[m\flat_1, m\flat_1 + \Phi(\flat_2,\flat_1,m)]$. *If* Ω_1, Ω_2 *are preinvex functions on* $[m\flat_1, m\flat_1 + \Phi(\flat_2,\flat_1,m)]$, *then fractional integral inequality for* $\alpha < 0$ *is given as*

$$\frac{1}{(\Phi(\flat_2,\flat_1,m))^{1-\alpha}}\Big[{}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}\Omega_1\Omega_2(m\flat_1) + {}_{N_3}J^{\alpha}_{m\flat_1^+}\Omega_1\Omega_2(m\flat_1+\Phi(\flat_2,\flat_1,m))\Big]$$
$$\leq \Big(\frac{1}{3-\alpha} + \frac{-2}{\alpha^3-6\alpha^2+11\alpha-6}\Big)m^2(\Omega_1\Omega_2)(\flat_1) + \Big(\frac{1}{3-\alpha} + \frac{-2}{\alpha^3-6\alpha^2+11\alpha-6}\Big)(\Omega_1\Omega_2)(\flat_2) + \frac{2m}{\alpha^2-5\alpha+6}(\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)).$$

Proof. Since Ω_1, Ω_2 are m-preinvex, then by the definitions of m-preinvexity, we have

$$\Omega_1(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m)) \leq m(1-\wp)\Omega_1(\flat_1) + \wp\Omega_1(\flat_2)$$

and

$$\Omega_2(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m)) \leq m(1-\wp)\Omega_2(\flat_1) + \wp\Omega_2(\flat_2).$$

If we multiply these inequalities, we get

$$\Omega_1(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m))\Omega_2(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m)) \leq m^2(1-\wp)^2(\Omega_1\Omega_2)(\flat_1) + \wp^2(\Omega_1\Omega_2)(\flat_2) + m\wp(1-\wp)[\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)].$$

Multiplying both sides by $\wp^{-\alpha}$, we obtain

$$\wp^{-\alpha}(\Omega_1\Omega_2)(m\flat_1 + \wp\Phi(\flat_2,\flat_1,m)) \leq \wp^{-\alpha}m^2(1-\wp)^2(\Omega_1\Omega_2)(\flat_1) + \wp^{2-\alpha}(\Omega_1\Omega_2)(\flat_2) + \wp^{1-\alpha}m(1-\wp)[\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)].$$

Now, by integrating the resulting inequality w.r.t $\wp$ over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 \wp^{-\alpha}(\Omega_1\Omega_2)(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp \\
&\leq \int_0^1 \Big(\wp^{-\alpha}m^2(1-\wp)^2(\Omega_1\Omega_2)(\flat_1)+\wp^{2-\alpha}(\Omega_1\Omega_2)(\flat_2)\Big)d\wp \\
&\quad+\int_0^1 \Big(\wp^{1-\alpha}m(1-\wp)[\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)]\Big)d\wp.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\int_0^1 \wp^{-\alpha}(\Omega_1\Omega_2)(m\flat_1+\wp\Phi(\flat_2,\flat_1,m))d\wp \\
&\leq m^2(\Omega_1\Omega_2)(\flat_1)\int_0^1 \wp^{-\alpha}(1-\wp)^2 d\wp+(\Omega_1\Omega_2)(\flat_2)\int_0^1 \wp^{2-\alpha}d\wp \\
&\quad+m[\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)]\int_0^1 \wp^{1-\alpha}(1-\wp)d\wp.
\end{aligned}$$

By computing the above process, we obtain

$$\begin{aligned}
&\frac{1}{(\Phi(\flat_2,\flat_1,m))^{1-\alpha}}\ {}_{N_3}J^{\alpha}_{m\flat_1+\Phi(\flat_2,\flat_1,m)^-}(\Omega_1\Omega_2)(m\flat_1) \\
&\leq \frac{-2m^2}{\alpha^3-6\alpha^2+11\alpha-6}(\Omega_1\Omega_2)(\flat_1)+\frac{1}{3-\alpha}(\Omega_1\Omega_2)(\flat_2) \\
&\quad+\frac{m}{\alpha^2-5\alpha+6}(\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)).
\end{aligned} \tag{20}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^1 \wp^{-\alpha}(\Omega_1\Omega_2)(m\flat_1+(1-\wp)\Phi(\flat_2,\flat_1,m))d\wp \\
&\leq m^2(\Omega_1\Omega_2)(\flat_1)\int_0^1 \wp^{2-\alpha}d\wp+(\Omega_1\Omega_2)(\flat_2)\int_0^1 \wp^{-\alpha}(1-\wp)^2 d\wp \\
&\quad+m[\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)]\int_0^1 \wp^{1-\alpha}(1-\wp)d\wp.
\end{aligned}$$

Therefore, by computing, we derive

$$\begin{aligned}
&\frac{1}{(\Phi(\flat_2,\flat_1,m))^{1-\alpha}}\ {}_{N_3}J^{\alpha}_{m\flat_1^+}(\Omega_1\Omega_2)(m\flat_1+\Phi(\flat_2,\flat_1,m)) \\
&\leq \frac{m^2}{3-\alpha}(\Omega_1\Omega_2)(\flat_1)+\frac{-2}{\alpha^3-6\alpha^2+11\alpha-6}(\Omega_1\Omega_2)(\flat_2) \\
&\quad+\frac{m}{\alpha^2-5\alpha+6}[\Omega_1(\flat_1)\Omega_2(\flat_2)+\Omega_1(\flat_2)\Omega_2(\flat_1)].
\end{aligned} \tag{21}$$

By adding the inequalities (20) and (21), we get the proof. □

Remark 24. *If $\alpha = 0$ and $m = 1$, then we obtain Theorem 3.4 in* [69].

Remark 25. *If $\alpha = 0$, $m = 1$, and $\Phi(\flat_2,\flat_1,m) = \flat_2 - m\flat_1$, then we obtain Theorem 1 in* [70].

8. Applications

This section's primary goal is to establish several novel kinds of inequalities for the harmonic and arithmetic means. Due to their significant performance and outstanding utilization in statistics, probability, numerical approximation, and machine learning, the following methods are well-known and well-liked. A particular instance of the power mean is the harmonic mean. Since this mean is the most suitable measurement for rates and ratios, it equalizes the weights of each data point.

The purpose of this section is to demonstrate and analyze some special means in the aspects of NCFIO for positive numbers $\flat_1, \flat_2$ with $\flat_1 < \flat_2$:

(1) The arithmetic mean

$$A = A(\flat_1, \flat_2) = \frac{\flat_1 + \flat_2}{2}.$$

(2) The harmonic mean

$$H = H(\flat_1, \flat_2) = \frac{2\flat_1\flat_2}{\flat_1 + \flat_2}.$$

However, in the literature, the simple connection between harmonic and arithmetic mean is given by

$$H(\flat_1, \flat_2) \leq G(\flat_1, \flat_2) \leq A(\flat_1, \flat_2).$$

Proposition 1. *Assuming that* $0 < \flat_1 < \flat_2$*; then*

$$A(\flat_1, \flat_2) \leq \frac{1-\alpha}{2(\flat_2 - \flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_1^+}\Omega(\flat_2) + {}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(\flat_1)\right] \leq A(\flat_1, \flat_2). \tag{22}$$

Proof. Taking $\Omega(x) = x$ for $x > 0$ in Corollary 4, Proposition 1 is easily obtained. □

Proposition 2. *Let* $0 < \flat_1 < \flat_2$*; then*

$$A^n(\flat_1, \flat_2) \leq \frac{1-\alpha}{2(\flat_2 - \flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_1^+}\Omega(\flat_2) + {}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(\flat_1)\right] \leq A(\flat_1^n, \flat_2^n). \tag{23}$$

Proof. Taking $\Omega(x) = x^n$ for $x > 0$ in Corollary 4, Proposition 2 is easily obtained. □

Proposition 3. *Let* $0 < \flat_1 < \flat_2$*; then*

$$A^{-1}(\flat_1, \flat_2) \leq \frac{1-\alpha}{2(\flat_2 - \flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_1^+}\Omega(\flat_2) + {}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(\flat_1)\right] \leq H^{-1}(\flat_1, \flat_2). \tag{24}$$

Proof. Taking $\Omega(x) = x^{-1}$ for $x > 0$ in Corollary 4, Proposition 3 is easily obtained. □

Proposition 4. *Let* $0 < \flat_1 < \flat_2$*; then*

$$\frac{1}{A^2(\flat_1, \flat_2)} \leq \frac{1-\alpha}{2(\flat_2 - \flat_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\flat_1^+}\Omega(\flat_2) + {}_{N_3}J^{\alpha}_{\flat_2^-}\Omega(\flat_1)\right] \leq \frac{1}{H^2(\flat_1, \flat_2)}. \tag{25}$$

Proof. Taking $\Omega(x) = \frac{1}{x^2}$ for $x > 0$ in Corollary 4, Proposition 4 is easily obtained. □

9. Conclusions

Fractional calculus has grown to be a key area of research as a consequence of its applications in the mathematical modeling of several complex and nonlocal nonlinear systems. It is very important while researching optimization issues because it has a range of beneficial inequalities. Many authors and investigators from many different disciplines have expressed interest in fractional calculus. With the help of convexity theory, we may develop novel frameworks for numerical models that may be employed to challenge and overcome a broad spectrum of challenges in both the applied and pure sciences. Integral inequalities have applications in physics, functional analysis, optimization theory, and statistical theory. Convex analysis and inequalities have therefore gradually grown in popularity among scholars and attracted attention as a result of several developments, variations, extensions, widely held opinions, and applications.

In this work:

(1) First, we defined new notions of the preinvex family, namely, the G–L m-preinvex function, the G–L (s,m)-preinvex function of the 1st type, the G–L (s,m)-preinvex function of the 2nd type, and the prequasi m-invex function.

(2) We constructed a novel sort of H–H inequality via NCFIO with some amazing corollaries and remarks.
(3) We investigated and explored a new integral identity and, on the basis of this new integral identity with the newly developed concept and definitions, some novel versions and extensions of H–H inequality were examined.
(4) We constructed a novel sort of Fejér inequality via NCFIO with some amazing corollaries and remarks.
(5) We investigated and explored a new integral identity and, on the basis of this new integral identity with the newly developed concept and definitions, some novel versions and extensions of Fejér inequality were examined.
(6) We constructed a new variant of the generalized fractional Pachpatte-type inequality via a newly introduced concept.
(7) We added some special means as applications in the frame of the fractional operator.

It is possible to apply the paper's interesting techniques and practical notions to study Raina functions. The aforementioned inequalities can be discussed in the field of interval analysis and quantum calculus. The field of integral inequalities is one of the most rapidly developing fields of study. Every scientist should be intrigued to understand how different versions of quantum calculus and interval-valued analysis can be implemented for integral inequalities.

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Abbreviations

The following abbreviations are used in this manuscript:

NCFIO	Non-conformable fractional integral operator
NCD	Non-conformable derivative
CDO	Conformable derivative operator
G–L	Godunova–Levin
KFIO	Katugampola fractional integral operator
ABFO	Atangana–Baleanu fractional operator
CFFIO	Caputo–Fibrizio fractional integral operator
H–H	Hermite–Hadamard
H–H–M	Hermite–Hadamard–Mercer
R–L	Riemann–Liouville
RLFIO	Riemann–Liouville fractional integral operator
w.r.t.	with respect to

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