The Extended Direct Algebraic Method for Extracting Analytical Solitons Solutions to the Cubic Nonlinear Schrödinger Equation Involving Beta Derivatives in Space and Time

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Abstract: In the field of nonlinear optics, quantum mechanics, condensed matter physics, and wave propagation in rigid and other nonlinear instability phenomena, the nonlinear Schrödinger equation has significant applications. In this study, the soliton solutions of the space-time fractional cubic nonlinear Schrödinger equation with Kerr law nonlinearity are investigated using an extended direct algebraic method. The solutions are found in the form of hyperbolic, trigonometric, and rational functions. Among the established solutions, some exhibit wide spectral and typical characteristics, while others are standard. Various types of well-known solitons, including kink-shape, periodic, V-shape, and singular kink-shape solitons, have been extracted here. To gain insight into the internal formation of these phenomena, the obtained solutions have been depicted in two- and three-dimensional graphs with different parameter values. The obtained solitons can be employed to explain many complicated phenomena associated with this model.

Keywords: beta derivative; the space-time fractional cubic nonlinear Schrödinger equation; soliton solutions; the extended direct algebraic method

1. Introduction

Diverse real-world phenomena have been explained using nonlinear models, leading to the revelation of important information. Fractional nonlinear evolution equations represent an advanced class of differential equations that yield improved results. These equations help to illustrate intricate physical phenomena, attracting many researchers to work in this field due to their significant applications. Within the realm of fractional nonlinear evolution equations, the nonlinear Schrödinger equation plays a crucial role and finds applications in various areas such as quantum mechanics, optical fiber, plasma physics, fluid mechanics, biology, the dispersion of chemically reactive materials, electricity, shallow water wave phenomena, heat flow, finance, and fractal dynamics.

The relationship between the nonlinearity and dispersion components of medium solitons is uncovered, and as they travel through the medium, their undulation structure remains unaltered. The soliton solutions derived from FNLEE have practical and commercial applications in various fields such as optical fiber technology, telecommunications, signal processing, image processing, system identification, water purification, plasma physics, medical device sterilization, chemistry, and other related domains [1,2]. Various dynamic approaches have been introduced and implemented in the literature to solve nonlinear fractional differential equations (NFDES) and obtain analytical traveling wave solutions, for example, the exp-function method [3], the Modified Exp-function method [4], the inverse scattering transformation method [5,6], the Bäcklund transformation method [7], the homogenous balance method [8,9], the Jacobi elliptic function method [10], the unified algebraic method [11], the sine-cosine method [12,13], the tanh-coth method [14,15],
improved modified extended tanh-function method [16,17], the Lie symmetry analysis method [18], the extended generalized \((G'/G)\)-expansion method [19], the modified simple equation method [20], the generalized Kudryashov method [21,22], the sine-Gordon expansion method [23], the Riccati–Bernoulli equation method [24,25], the new extended direct algebraic method [26,27], and the new auxiliary equation method [28].

Fractional derivatives have been widely applied in diverse scientific and engineering fields, including physics, mechanics, signal processing, control systems, biomedical engineering, finance and economics, electromagnetism, and fluid mechanics. For instance, the mathematical modelling of viscoelastic food ingredients experiencing stress and relaxation can be accomplished using fractional calculus [29]. These applications showcase the adaptability and practical value of fractional derivatives across a range of scientific and engineering disciplines, enabling improved modelling capabilities and deeper comprehension of intricate phenomena.

In this article, we consider the space and time fractional cubic NLSE with the Kerr law nonlinearity with spatial and time fraction in the following form [30].

\[
\frac{\partial^\alpha U}{\partial t^\alpha} + r \frac{\partial^{\alpha+\beta} U}{\partial x^{\alpha+\beta}} + s \frac{\partial^{\beta} U}{\partial x^{2\beta}} + z |U|^2 U = 0, \tag{1}
\]

where \(U(x,t)\) is a complex-valued wave profile which is related to spatial co-ordinate \(x\), and temporal variable \(t\). In addition, \(r, s, \text{and} z\) are real coefficients with fractional parameters \(0 < \alpha \leq 1\) and \(0 < \beta \leq 1\). The cubic nonlinear Schrödinger equation involving beta derivatives in space and time is used to model certain nonlinear optical phenomena. For example, it can describe the propagation of ultrashort optical pulses in nonlinear media with anomalous dispersion.

Here, by utilizing the equations \(\frac{\partial^\alpha U}{\partial t} = D_t^\alpha U\), \(\frac{\partial^{\alpha+\beta} U}{\partial x^{\alpha+\beta}} = D_x^{\alpha+\beta} U\), \(\frac{\partial^{\beta} U}{\partial x^{2\beta}} = D_x^{2\beta} U\), where \(i = \sqrt{-1}\), and assuming \(\alpha = \beta\) with beta fractional derivatives, Equation (1) is transformed into the following form

\[
i D_t^\alpha U + r D_t^\alpha D_x^\alpha U + s D_x^{2\alpha} U + z |U|^2 U = 0. \tag{2}
\]

The model has been investigated in the previous literature using various methods, including Nucci’s reduction method and the simplest equation method [31], the fractional Riccati expansion method [32], the fractional mapping expansion method [33], the \((G'/G)\)-expansion method [34], and the Adomain decomposition method [35].

To the best of our current knowledge, the extended direct algebraic method has not yet been applied to the model represented by Equation (1) to evaluate soliton solutions. The application of this method extends to various fields of nonlinear sciences, including mathematical physics, quantum physics, and engineering. However, the extended direct algebraic method is modified and implemented on a nonlinear space-time fractional model in Equation (1). By doing so, advanced, fresh, and wide-ranging soliton solutions are obtained. In this study, our primary focus is to establish advanced and widely applicable soliton solutions for the space-time fractional cubic nonlinear Schrödinger equation using the recommended method. The obtained soliton solutions exhibit wave-like behavior and are expressed in trigonometric, hyperbolic, and exponential forms. This research will also provide valuable insight into the internal formation of the travelling wave phenomena by depicting the obtained solutions in two- and three-dimensional graphs with different parameter values. Furthermore, the soliton solutions derived from this study will also contribute to the interpretation of complex phenomena associated with this particular space-time fractional model.

This article organizes its contents as follows: Section 2 presents the properties of the beta derivative. The algorithm of the proposed method is explained in Section 3. Section 4 provides a mathematical analysis. In Section 5, graphical representation and discussion are presented. The comparison scheme is outlined in Section 6, and finally, Section 7 concludes the article.
2. Definition of Beta Derivative and Its Properties

Several definitions of fractional derivatives, such as Riemann Liouville, the modified Riemann Liouville, the Caputo, the Caputo–Fabrizio, the conformable fractional derivative the Atangana–Baleanu derivatives, have been developed recently by many researchers [36,37]. Most of the fractional derivatives do not agree with the well-known properties of classical calculus such as the chain rule, the Leibnitz rule, and the derivative of a constant is zero. Atangana et al. [38] launched a new crucial and progressive definition of fractional derivatives called beta derivative, which follows the fundamental properties of classical calculus.

Definition 1: Let \( \alpha \in \mathbb{R} \) and the function \( h = h(x) : [a, \infty) \to \mathbb{R} \), then the beta derivative of order \( \alpha \) with respect to \( x \) is defined as follows [39]:

\[
D_{\alpha}^x h(x) = \lim_{\varepsilon \to 0} \frac{h(x + \varepsilon (x + \frac{1}{\Gamma(\alpha)})^{1-\alpha}) - h(x)}{\varepsilon}, \quad \text{for all } x \geq 0 \text{ and } \alpha \in (0, 1].
\]

where \( \Gamma \) is the gamma function. \( D_{\alpha}^x h(x) = \frac{d}{dx} h(x) \) for \( \alpha = 1 \).

Properties: If \( h(x) \) and \( u(x) \) are \( \alpha \)-order differentiable for all \( x > 0 \), and \( d_1, d_2 \) are real constants, then the beta derivative encompasses the following properties [39]:

i. \( D_{\alpha}^x (d_1 h(x) + d_2 u(x)) = d_1 D_{\alpha}^x h(x) + d_2 D_{\alpha}^x u(x) \).
ii. \( D_{\alpha}^x (k) = 0 \), here \( k \) is a constant.
iii. \( D_{\alpha}^x (h(x)u(x)) = h(x)D_{\alpha}^x u(x) + u(x)D_{\alpha}^x h(x) \).
iv. \( D_{\alpha}^x \left( \frac{h(x)}{u(x)} \right) = \frac{u(x)D_{\alpha}^x (h(x)) - h(x)D_{\alpha}^x (u(x))}{u^2(x)} \).
v. \( D_{\alpha}^x \left( h(u(x)) \right) = D_{\alpha}^x (h(u(x)))u'(x) \).
vi. \( D_{\alpha}^x (h^{-1}(x)) = \frac{1}{h'(x)} \).
vii. \( D_{\alpha}^x h(x) = \left( x + \frac{1}{\Gamma(\alpha)} \right)^{1-\alpha} \frac{dh(x)}{dx} \).

By using these properties of beta derivative, fractional differential equations simply turn into ordinary differential equations. As of now, the beta derivative has not been found to have any limitations and it fulfills all the properties associated with integer-order derivatives. Furthermore, it exhibits the property of yielding a derivative of zero for constant functions [40–42]. The beta derivative is a non-local derivative that exhibits its distinctiveness when applied to functions that embody the entire characteristic of the function itself. It serves as a generalized version of the Caputo and Riemann–Liouville derivatives. In comparison to other derivatives, the beta derivative offers greater flexibility and can accurately model complex systems. Its applications are widespread, ranging from electrochemical systems and complex geometries to modeling electromagnetic waves in dielectric media and cancer treatment [43–45]. Numerous scientific studies have reported the utilization of the beta derivative in diverse fields, further enhancing its appeal and prompting its application to real-world problems [40,41,46,47].

3. Algorithm of the Extended Direct Algebraic Method

In this section, we have presented the extended direct algebraic method as an effective technique. This method enables us to obtain fresh and wide-ranging analytical solutions for model (1). By employing this technique, fractional partial differential equations can be transformed into ordinary differential equations, simplifying the calculation process. The algorithm is narrated below:

Step 1: Let the general form of the fractional order nonlinear evolution equation be

\[
\mathcal{F}(u, u_x, u_t, D_t^\alpha u, D_x^\alpha u, D_{tt}^\alpha u, D_{xx}^{2\alpha} u, D_{tx}^{2\alpha} u, \ldots) = 0
\]

(3)
where \( \mathcal{F} \) is a polynomial of \( u(x, t) \), and \( D^\alpha_t \) be fractional derivative of \( \alpha \)-order and \( u(x, t) \) is the travelling wave variable, where subscripts denote partial derivatives.

Let us hypothesize about the travelling wave solution

\[
U = u(\xi)e^{i\theta} \tag{4}
\]

where \( u(\xi) \) as a function of \( \xi \) with

\[
\xi = \frac{1}{\alpha} \left( x + \frac{1}{\Gamma(\alpha)} \right) \theta - \frac{\nu}{\alpha} \left( t + \frac{1}{\Gamma(\alpha)} \right), \quad \theta = -\frac{k}{\alpha} \left( x + \frac{1}{\Gamma(\alpha)} \right) + \frac{\omega}{\Gamma(\alpha)} \left( t + \frac{1}{\Gamma(\alpha)} \right) \tag{5}
\]

In Equation (5) \( v, \kappa \) are respectively velocity and soliton frequency, \( \omega \) is wave number and \( \theta \) is the soliton phase component.

Inserting the above transformation into Equation (3), we obtain the following ordinary differential equation of integer order:

\[
H(u, u', u'', u''', \ldots) = 0, \tag{6}
\]

where \( H \) be the polynomial of function \( u(\xi) \), and prime denotes the derivative with respect to \( \xi \).

According to the new extended algebraic method, the solution of Equation (6) can be expressed in the form

\[
u(\xi) = \sum_{j=0}^{N} c_j H_j(\xi); \quad c_N \neq 0, \tag{7}
\]

where \( c_j (0 \leq j \leq N) \) are constant coefficients to be evaluated later and \( H(\xi) \) satisfies the following ordinary differential equation,

\[
H'(\xi) = \mu + \gamma H(\xi) + \lambda H(\xi)^2, \tag{8}
\]

where prime denotes derivative with respect to \( \xi \), and \( \mu, \gamma, \lambda \) are constant coefficients. The general solutions of Equation (8) (adequate solutions) are given in [27].

By substituting Equations (7) and (8) into Equation (6), we obtain a polynomial of \( H(\xi) \). By extracting the coefficient terms of different powers of \( H^j(\xi) \) where \( j = 0, 1, 2, \ldots \) and setting them equal to zero, then we obtain a system of algebraic equations with various parametric values such as such as \( c_j (j = 0, 1, 2, \ldots) \), \( \mu, \gamma, \lambda, \omega, \) and \( \kappa \). By solving these algebraic equations, we can determine the values of the unknown parameters. Substituting these values of the parameters along with Equation (8) into Equation (7), as the broadspectrum solutions of Equation (8) are known, we obtain new and more general solutions.

For several values of \( \mu, \gamma, \lambda \) and their correlation, Equation (8) gives disparate general solutions of NLSEs.

4. Mathematical Analysis

In this subsection, we studied the space and time fractional cubic NLSEs to find more general and standard exact wave solutions using an extended direct algebraic method. Furthermore, we discuss the mathematical analysis of the wave solutions. The fractional transformation in Equation (5) converts Equation (2) into the following ordinary differential equation, comprising both real and imaginary parts.

\[
-(k^2 s + \omega - kr\omega)u + zu^3 + (s - rv)u'' = 0, \tag{9}
\]

\[
v = \frac{(2ks - rv)}{(kr - 1)}. \tag{10}
\]
Now balancing between the highest order derivatives and highest power of the nonlinear term in Equation (9), we obtain \( N = 1 \). Therefore, the solution of Equation (9) is of the form

\[
u = c_0 + c_1 H(\xi).\tag{11}\]

By substituting the results from (8) into Equation (9) along with Equation (10), we obtain a polynomial equation in \( H(\xi) \), where \( 0 \leq j \leq N \). Taking zero the resemble coefficient power of \( H^j(\xi) \), we achieve a set of algebraic equations with \( c_0, c_1, \mu, \gamma, \lambda \). Calculating this set of algebraic equations with the software Mathematica, we obtain the values of the parameter as follows:

\[
c_0 = \pm \frac{\sqrt{\gamma}}{\sqrt{-2z+4kr^2+2r^2\gamma^2+4r^2z\lambda}}, \quad c_1 = \pm \frac{2\sqrt{\gamma}}{\sqrt{-2z(2-4kr+2r^2\gamma^2-r^2\gamma^2+4r^2z\lambda)}}, \quad \omega = \frac{\sqrt{-2z+4kr^2+2r^2\gamma^2+4r^2z\lambda}}{2z-4kr+2r^2\gamma^2-r^2\gamma^2+4r^2z\lambda},\tag{12}\]

where \( \mu, \gamma, \lambda \) and \( r, s, k, z \) are free parameters.

Now, embedding the values of (12) into (11) and the hypothesis of the auxiliary equation for different conditions, we establish the travelling wave solutions of (1) which are given below.

**Case 1:** When \( \gamma^2 - 4\mu \lambda < 0 \) and \( \lambda \neq 0 \), the solutions are as follows:

\[
u_1(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} + \frac{\sqrt{\gamma M}}{2\lambda} \tan \left( \frac{\sqrt{\gamma M}}{2} \xi \right) \right) \right\} \exp[-i\theta] \tag{13}\]

\[
u_2(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} + \frac{\sqrt{\gamma M}}{2\lambda} \cot \left( \frac{\sqrt{\gamma M}}{2} \xi \right) \right) \right\} \exp[-i\theta] \tag{14}\]

\[
u_3(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{\gamma M}}{2\lambda} \tan \left( \frac{\sqrt{\gamma M}}{2} \xi \right) \pm \sec \left( \frac{\sqrt{\gamma M}}{2} \xi \right) \right) \right\} \exp[-i\theta] \tag{15}\]

\[
u_4(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{\gamma M}}{2\lambda} \cot \left( \frac{\sqrt{\gamma M}}{2} \xi \right) \pm \csc \left( \frac{\sqrt{\gamma M}}{2} \xi \right) \right) \right\} \exp[-i\theta] \tag{16}\]

\[
u_5(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{\gamma M}}{2\lambda} \left( \tan \left( \frac{\sqrt{\gamma M}}{4} \xi \right) \pm \cot \left( \frac{\sqrt{\gamma M}}{4} \xi \right) \right) \right) \right\} \exp[-i\theta] \tag{17}\]

where \( M = \gamma^2 - 4\mu \lambda, B = \frac{\sqrt{\gamma}}{\sqrt{-2z(2 - 4kr + 2r^2\gamma^2 - r^2\gamma^2 + 4r^2z\lambda)}} \), under the condition \( (2 - 4kr + 2r^2\gamma^2 - r^2\gamma^2 + 4r^2z\lambda)) > 0 \) for the existence of the solutions.

**Case 2:** For \( \gamma^2 - 4\mu \lambda > 0 \) and \( \gamma \neq 0 \), we obtain

\[
u_6(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{M}}{2\lambda} \tanh \left( \frac{\sqrt{M}}{2} \xi \right) \right) \right\} \exp[-i\theta] \tag{18}\]

\[
u_7(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{M}}{2\lambda} \coth \left( \frac{\sqrt{M}}{2} \xi \right) \right) \right\} \exp[-i\theta] \tag{19}\]

\[
u_8(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{M}}{2\lambda} \left( \tanh \left( \sqrt{M} \xi \right) \pm \sech \left( \sqrt{M} \xi \right) \right) \right) \right\} \exp[-i\theta] \tag{20}\]
\[ u_9(\xi) = \pm B \left\{ \gamma + 2\lambda \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{M}}{2\lambda} \left( \coth \left( \sqrt{M} \xi \right) \pm \text{csch} \left( \sqrt{M} \xi \right) \right) \right) \right\} \exp[-i\theta] \quad (21) \]

\[ u_{10}(\xi) = \pm B \left\{ \gamma + 2\Omega \left( -\frac{\gamma}{2\lambda} - \frac{\sqrt{M}}{2\lambda} \left( \tanh \left( \sqrt{M} \xi \right) + \coth \left( \frac{\sqrt{M}}{4} \xi \right) \right) \right) \right\} \exp[-i\theta] \quad (22) \]

where \( M = \gamma^2 - 4\mu \lambda, B = \frac{\sqrt{\xi}}{\sqrt{-z(2 - 4kr + 2k^2r^2 - r^2\gamma^2 + 4r^2z\lambda)}} \), with conditions \((zs(2 - 4kr + 2k^2r^2 - r^2\gamma^2 + 4r^2z\lambda)) > 0\) for the existing solution.

**Case 3:** While \( \mu \lambda > 0 \) and \( \gamma = 0 \), we obtain

\[ u_{11}(\xi) = \pm B \left\{ 2\lambda \left( \sqrt{\frac{\mu}{\lambda}} \tan \left( \sqrt{\mu\lambda} \xi \right) \right) \right\} \exp[-i\theta] \quad (23) \]

\[ u_{12}(\xi) = \pm B \left\{ 2\lambda \left( -\sqrt{\frac{\mu}{\lambda}} \cot \left( \sqrt{\mu\lambda} \xi \right) \right) \right\} \exp[-i\theta] \quad (24) \]

\[ u_{13}(\xi) = \pm B \left\{ 2\lambda \left( \sqrt{\frac{\mu}{\lambda}} \left( \tan \left( 2\sqrt{\mu\lambda} \xi \right) \pm \sec \left( 2\sqrt{\mu\lambda} \xi \right) \right) \right) \right\} \exp[-i\theta] \quad (25) \]

\[ u_{14}(\xi) = \pm B \left\{ 2\lambda \left( \sqrt{\frac{\mu}{\lambda}} \left( -\cot \left( 2\sqrt{\mu\lambda} \xi \right) \pm \csc \left( 2\sqrt{\mu\lambda} \xi \right) \right) \right) \right\} \exp[-i\theta] \quad (26) \]

\[ u_{15}(\xi) = \pm B \left\{ 2\lambda \left( \frac{1}{2} \sqrt{\frac{\mu}{\lambda}} \left( \tan \left( \frac{\sqrt{\mu\lambda}}{2} \xi \right) - \cot \left( \frac{\sqrt{\mu\lambda}}{2} \xi \right) \right) \right) \right\} \exp[-i\theta] \quad (27) \]

where \( B = \frac{\sqrt{\xi}}{\sqrt{-z(2 - 4kr + 2k^2r^2 + 4r^2z\lambda)}} \), with the condition \((zs(2 - 4kr + 2k^2r^2 + 4r^2z\lambda)) > 0\).

**Case 4:** When \( \mu \lambda < 0 \) and \( \gamma = 0 \), we obtain

\[ u_{16}(\xi) = \pm B \left\{ 2\lambda \left( -\sqrt{\frac{-\mu}{\lambda}} \tanh \left( \sqrt{-\mu\lambda} \xi \right) \right) \right\} \exp[-i\theta] \quad (28) \]

\[ u_{17}(\xi) = \pm B \left\{ 2\lambda \left( -\sqrt{\frac{-\mu}{\lambda}} \coth \left( \sqrt{-\mu\lambda} \xi \right) \right) \right\} \exp[-i\theta] \quad (29) \]

\[ u_{18}(\xi) = \pm B \left\{ 2\lambda \left( \sqrt{-\frac{\mu}{\lambda}} \left( -\tanh \left( 2\sqrt{-\mu\lambda} \xi \right) \pm \text{sech} \left( 2\sqrt{-\mu\lambda} \xi \right) \right) \right) \right\} \exp[-i\theta] \quad (30) \]

\[ u_{19}(\xi) = \pm B \left\{ 2\lambda \left( \sqrt{-\frac{\mu}{\lambda}} \left( -\coth \left( 2\sqrt{-\mu\lambda} \xi \right) \pm \text{csch} \left( 2\sqrt{-\mu\lambda} \xi \right) \right) \right) \right\} \exp[-i\theta] \quad (31) \]

\[ u_{20}(\xi) = \pm B \left\{ 2\lambda \left( -\sqrt{-\frac{\mu}{\lambda}} \left( \tanh \left( \frac{\sqrt{-\mu\lambda}}{2} \xi \right) + \coth \left( \frac{\sqrt{-\mu\lambda}}{2} \xi \right) \right) \right) \right\} \exp[i\theta] \quad (32) \]

where \( B = \frac{\sqrt{\xi}}{\sqrt{-z(2 - 4kr + 2k^2r^2 + 4r^2z\lambda)}} \), with conditions \((zs(2 - 4kr + 2k^2r^2 + 4r^2z\lambda)) > 0\) for the existence of the solutions.
Case 5: For $\gamma = 0$ and $\lambda = \mu$

$$u_{21}(\xi) = \pm B \{2\lambda (\tan(\mu \xi))\} \exp[-i\theta],$$  \hphantom{.} (33)

$$u_{22}(\xi) = \pm B \{2\lambda (-\cot(\mu \xi))\} \exp[-i\theta],$$  \hphantom{.} (34)

$$u_{23}(\xi) = \pm B \{2\lambda (\tan(2\mu \xi) \pm \sec(2\mu \xi))\} \exp[-i\theta],$$  \hphantom{.} (35)

$$u_{24}(\xi) = \pm B \{2\lambda (-\cot(2\mu \xi) \pm \csc(2\mu \xi))\} \exp[-i\theta],$$  \hphantom{.} (36)

$$u_{25}(\xi) = \pm B \left\{2\lambda \left(\frac{1}{2} (\tan(\frac{\mu \xi}{2}) - \cot(\frac{\mu \xi}{2}))\right)\right\} \exp[-i\theta].$$  \hphantom{.} (37)

where $B = \sqrt[-2]{-z(2 - 4kr + 2k^2r^2 + 4r^2z\mu)}$, with conditions $(zs(2 - 4kr + 2k^2r^2 + 4r^2z\mu)) > 0$ for the existence of the solutions.

Case 6: While $\gamma = 0$ and $\lambda = -\mu$

$$u_{26}(\xi) = \pm B \{2\lambda (-\tanh(\mu \xi))\} \exp[-i\theta],$$  \hphantom{.} (38)

$$u_{27}(\xi) = \pm B \{2\lambda (-\coth(\mu \xi))\} \exp[-i\theta],$$  \hphantom{.} (39)

$$u_{28}(\xi) = \pm B \{2\lambda (-\tanh(2\mu \xi) \pm \sech(2\mu \xi))\} \exp[-i\theta],$$  \hphantom{.} (40)

$$u_{29}(\xi) = \pm B \{2\lambda (-\coth(2\mu \xi) \pm \csch(2\mu \xi))\} \exp[i\theta],$$  \hphantom{.} (41)

$$u_{30}(\xi) = \pm B \left\{2\lambda \left(-\frac{1}{2} (\tanh(\frac{\mu \xi}{2}) - \coth(\frac{\mu \xi}{2}))\right)\right\} \exp[i\theta]$$  \hphantom{.} (42)

where $B = \sqrt[-2]{-z(2 - 4kr + 2k^2r^2 - 4r^2z\mu)}$, with conditions $(zs(2 - 4kr + 2k^2r^2 - 4r^2z\mu)) > 0$.

Case 7: For $\gamma = \lambda = 0$, we achieve the following soliton solutions:

$$u_{31}(\xi) = \pm B \{\gamma + 2\lambda (\mu \xi)\} \exp[-i\theta].$$  \hphantom{.} (43)

Case 8: For $\gamma = \mu = 0$, the solution converts into:

$$u_{32}(\xi) = \pm B \left\{2\lambda \left(-\frac{1}{\lambda \xi}\right)\right\} \exp[-i\theta].$$  \hphantom{.} (44)

$$B = \sqrt{-z(2 - 4kr + 2k^2r^2 + 4r^2z\lambda)}, \text{with conditions } (zs(2 - 4kr + 2k^2r^2 + 4r^2z\lambda)) > 0.$$

Case 9: While $\mu = 0$ and $\gamma \neq 0$, we obtain

$$u_{33}(\xi) = \pm B \left\{\gamma + 2\lambda \left(-\frac{\gamma}{\lambda (\cosh(\gamma \xi) - \sinh(\gamma \xi) + 1)}\right)\right\} \exp[-i\theta]$$  \hphantom{.} (45)

$$u_{34}(\xi) = \pm B \left\{\gamma + 2\lambda \left(-\frac{\gamma (\cosh(\gamma \xi) + \sinh(\gamma \xi))}{\lambda (\cosh(\gamma \xi) + \sinh(\gamma \xi) + 1)}\right)\right\} \exp[-i\theta]$$  \hphantom{.} (46)

where $B = \pm \sqrt{-z(2 - 4kr + 2k^2r^2 - r^2\gamma^2 + 4r^2z\Omega)}$, with conditions $(zs(2 - 4kr + 2k^2r^2 - r^2\gamma^2 + 4r^2z\Omega)) > 0$. 


The soliton solutions obtained in this study are diverse and novel, originating from the general solutions.

5. Physical Significance and Explanations
In this section, attained soliton solutions of the space and time fractional cubic NLSEs are presented in Figures 1–5 and discussed the nature of these solitons for several values of unknown parameters through the software Mathematica.

![Figure 1. Three-dimensional and two-dimensional plot of kink shape soliton solution of $u_6$.](image1)

$\alpha = 0.15, \alpha = 0.5, \alpha = 0.99$

![Figure 2. The 3D and 2D plots of the periodic travelling wave solution $u_6$.](image2)

$\alpha = 0.15, \alpha = 0.5, \alpha = 0.99$
The results that were established earlier by several approaches and some of them are fresh. It is noticed that from the attained results few of them are analogous to the equation obtained through the extended algebraic equation method with Abdelwahed et al. [30] solutions. We compare the results of the space-time fractional cubic nonlinear Schrodinger equation obtained through the extended algebraic equation method with Abdelwahed et al. [30] solutions. It is noticed that from the attained results few of them are analogous to the results that were established earlier by several approaches and some of them are fresh. 

Figure 3. The 3D and 2D plots of V shape soliton solution $u_{26}$.

Figure 4. The 3D and 2D plot of singular bell shape soliton solution corresponds to $u_{27}$.

Figure 5. The 3D and 2D plot of the soliton solution corresponds to $u_{35}$. 

Portraits have been shown at $t = 1$ with values of arbitrary parameters $s = 2$, $\gamma = 0$, $r = 1.8$, $\mu = 1$, $\alpha = 0.25, \alpha = 0.55, \alpha = 0.99$. 

Portraits of 3D have been showed within the interval $0 \leq x \leq 5$ and for different value of $\alpha = 4.1$.

The 3D and 2D plots of V shape soliton solution and the value of fractional order, wave velocity, wave number and other wave variables.

The solution $u_{35}$ exhibits soliton solution for modulus part with velocity $\Omega = 1$, for the value of $\alpha = 5$, $\gamma = 0.9$, $\mu = 1$, $\Omega = 1.518, k = 1.82$, $\nu = 2.79$.

The modulus part of the solution $u_{26}$ exhibits soliton solution for modulus part with velocity $\Omega = 1$, for the value of $\alpha = 0.99$, $\gamma = 0.99$, $\mu = 0.5$, $\Omega = 0.25, 0.45, 0.99$, $\nu = 0.004$.

The solution $u_{27}$ exhibits V shaped soliton solution with velocity $\Omega = 1$ for the value of $\alpha = 0.99$, $\gamma = 0.99$, $\mu = 0.5$, $\Omega = 0.25, 0.45, 0.99$, $\nu = 0.004$. 

Taking other various values of free parameters this model provides the same type of soliton solutions, repeat solitons have been neglected here and solitons profile depend on the interval $0 \leq x \leq 5$, and 2D. Portraits have been shown at $t = 1$ with values of arbitrary parameters $s = 2$, $\gamma = 0$, $r = 1.8$, $\mu = 1$. 

The 3D and 2D plots of V shape soliton solution $u_{26}$ and those graphs-
The accomplished solutions are related to two parts including the real part and the imaginary part. The solutions provide various types of solitons such as kink shape soliton, singular kink shape soliton, V shape soliton, periodic soliton, flat kink shape soliton, anti-singular kink shape soliton, soliton solutions and in such manners. The wave velocity and wave number have significant effects on the travelling wave profile.

The solution \( u_6 \) exhibits kink-shaped soliton solution for the modulus part with velocity \( v = -1.443 \) depicted in Figure 1, and different values of \( \alpha = 0.99, 0.5, 0.15 \) are shown below in the graph. Portraits of 3D are shown within the interval \( 0 \leq x \leq 5 \) and \( 0 \leq t \leq 5 \) and 2D Portraits are shown at \( t = 1 \) with values of arbitrary parameters \( \gamma = 3, s = -0.105, r = 0.482, \mu = 1.389, \Omega = 1, \) and \( k = -1.27. \)

The solution \( u_6 \) exhibits periodic shaped soliton solution for the real part with soliton speed \( v = 1.84 \) depicted in Figure 2. Several values of \( \alpha = 0.15, \alpha = 0.5, \alpha = 0.99 \) portraits of 3D have been shown within the interval \( 0 \leq x \leq 5 \) and \( 0 \leq t \leq 5 \), and 2D graphical representation has been shown for \( t = 0 \) and with values of arbitrary parameter \( \gamma = 4.066, s = -0.345, r = 0.044, \mu = 1.546, \Omega = 1.446, k = -1.27. \)

The modulus part of the solution \( u_{26} \) exhibits V shaped soliton solution with velocity \( v = -4.79 \) and for different value of \( \alpha = 0.25, 0.45, 0.99 \) it has depicted in Figure 3. Portraits of 3D have been showed within the interval \( 0 \leq x \leq 5 \) and \( 0 \leq t \leq 5 \), and 2D Portraits have been shown at \( t = 1 \) with values of arbitrary parameters \( s = 2, \gamma = 0, r = 1.888, \mu = 1, \Omega = 1, k = -5. \)

The solution \( u_{27} \) exhibits singular bell shape soliton for modulus part with velocity \( v = 4.1 \) for the value of \( \alpha = 0.9, k = 3 \) depicted in Figure 4, but while decreasing the value \( \alpha = 0.55, 0.25, k = 1.68 \) then the wave velocity become \( v = 2.79 \) and those graphical representation are provided in Figure 4b,c. Portraits of 3D are shown within the interval \( 0 \leq x \leq 5 \) and \( 0 \leq t \leq 5 \), and 2D. Portraits are shown at \( t = 1 \) with values of arbitrary parameters \( s = 2, \gamma = 0, r = 1.8, \mu = 1. \)

The solution \( u_{35} \) exhibits soliton solution for modulus part with velocity \( v = 0.004 \) and the value of \( \alpha = 0.99 \) depicted in Figure 5. Portraits of 3D have been showed within the interval \( 0 \leq x \leq 5 \) and \( 0 \leq t \leq 5 \), and 2D Portraits have been shown at \( t = 1 \) with values of arbitrary parameters \( s = 0.01, \gamma = 0.315, r = 1.816, \mu = 0, \Omega = 1.518, k = 1.82. \)

Taking other various values of free parameters this model provides the same type of soliton solutions, repeat solitons have been neglected here and solitons profile depend on the value of fractional order, wave velocity, wave number and other wave variables.

6. Comparison

We compare the results of the space-time fractional cubic nonlinear Schrodinger equation obtained through the extended algebraic equation method with Abdelwahed et al. [30] solutions. It is noticed that from the attained results few of them are analogous to the results that were established earlier by several approaches and some of them are fresh.
Solution Using the Simplest Equation Method

\[ q(x, t) = -\frac{2\delta u_1}{\sqrt{1 - x^2 - 2\delta + \delta^2}} \times \tanh[\sqrt{-\lambda} \alpha (x + \frac{1}{\Gamma})^2 - \frac{d(\xi + 1)}{2}] \times e^{-\frac{1}{2}(x + \frac{1}{\Gamma})^2 \times \frac{1}{\Gamma} t} \]

The Attained Solutions

\[ u_1\left(\xi \right) = \pm \frac{2\lambda \gamma_0}{\sqrt{-\alpha}(2 - 4x^2 + 2\rho^2)^2 - \rho^2 - \rho \delta} \times \tanh[\sqrt{-\alpha}(x + \frac{1}{\Gamma})^2 - \frac{d(\xi + 1)}{2}] \times e^{-\frac{1}{2}(x + \frac{1}{\Gamma})^2 \times \frac{1}{\Gamma} t} \]

\[ u_1\left(\xi \right) = \pm \frac{2\lambda \gamma_0}{\sqrt{-\alpha}(2 - 4x^2 + 2\rho^2)^2 - \rho^2 - \rho \delta} \times \tanh[\sqrt{-\alpha}(x + \frac{1}{\Gamma})^2 - \frac{d(\xi + 1)}{2}] \times e^{-\frac{1}{2}(x + \frac{1}{\Gamma})^2 \times \frac{1}{\Gamma} t} \]
In the above table, we discussed the solutions obtained in this paper with the previous study. Hashemi et al. [30] have given more than two solutions which are not homologous with the attained results.

7. Conclusions

The extended direct algebraic method has been used to derive novel exact analytical soliton solutions of the cubic nonlinear Schrödinger equation with fractional space-time terms. In this study, Kerr’s law nonlinearities are utilized, which arise from the non-harmonic motion of bound electrons when light pulses propagate in optical fibers. All solutions are expressed in terms of trigonometric and hyperbolic functions. Computational calculations and graphical representations of the solutions are plotted using the Wolfram Mathematica software. The graphical representations of these solutions help us to visualize and understand the internal features of the system more accurately. Among these solutions, some are new and have not been reported previously in the literature.

Author Contributions: F.T.: Conceptualization, Methodology, Software, Validation, Resources, Writing-original draft., Software. M.A.A.: Supervision, Project administration, Funding acquisition. M.S.O.: Data curation, Writing-original draft, Writing-review editing, Formal analysis. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their sincere thanks to the anonymous referees and editor for their valuable comments and suggestions.

Conflicts of Interest: The authors have no conflict of interest to declare that are relevant to the content of this article.

References
1. Esen, H.; Ozisik, M.; Secer, A.; Bayram, M. Optical soliton perturbation with Fokas–Lenells equation via enhanced modified extended tanh-expansion approach. Optik 2022, 267, 169615. [CrossRef]
2. Siddique, I.; Jaradat, M.M.; Zafar, A.; Mehdiz, K.B.; Osman, M.S. Exact traveling wave solutions for two prolific conformable M-Fractional differential equations via three diverse approaches. Results Phys. 2021, 28, 104557. [CrossRef]
4. Shakeel, M.; Attaullah; Shah, N.A.; Chung, J.D. Modified Exp-Function Method to Find Exact Solutions of Microtubules Nonlinear Dynamics Models. Symmetry 2023, 15, 360. [CrossRef]
30. Wu, G.-Z.; Yu, L.-J.; Wang, Y.-Y. Fractional optical solitons of the space-time fractional nonlinear Schrödinger equation. *Optik* **2020**, *207*, 164405. [CrossRef]
40. Ismael, H.F.; Bulut, H.; Baskonus, H.M.; Geo, W. Dynamical behaviors to the coupled Schrodinger-Boussinesq system with the beta derivative. *AIMS Math.* **2021**, *6*, 7909–7928. [CrossRef]


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