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Article

Finite-Interval Stability Analysis of Impulsive Fractional-Delay Dynamical System

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Abstract: Stability analysis over a finite time interval is a well-formulated technique to study the dynamical behaviour of a system. This article provides a novel analysis on the finite-time stability of a fractional-order system using the approach of the delayed-type matrix Mittag-Leffler function. At first, we discuss the solution's existence and uniqueness for our considered fractional model. Then standard form of integral inequality of Gronwall's type is used along with the application of the delayed Mittag-Leffler argument to derive the sufficient bounds for the stability of the dynamical system. The analysis of the system is extended and studied with impulsive perturbations. Further, we illustrate the numerical simulations of our analytical study using relevant examples.

Keywords: fractional delay system; Gronwall's inequality; finite-time stability; delayed Mittag-Leffler function

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1. Introduction

Fractional calculus is the study of the dynamical behaviour of a differential system with the aid of arbitrary-order derivatives and integrals. Under certain practical situations, it is well understood that mathematical models contain certain ineludible errors that affect the performance of the system. In such circumstances, the fractional derivatives provide an effective analysis on the dynamics of the system due to their memory effect. In the recent past, there have been many studies that have used fractional derivatives in the fields of control engineering and mathematical and biological modelling. A few noteworthy applications of fractional calculus are discussed in [1–4]. Moreover, the investigation of impulsive effects on fractional system dynamics has gained a lot of attention in recent decades. Sudden jumps and discontinuity in system dynamics are better modelled with impulsive differential equations. In general, these impulsive perturbations can be distinguished into two different types: instantaneous and noninstantaneous effects. Impulsive effects of the first type are a vital tool in modelling short-term disturbances that affect the dynamics of a system, whereas the second type of impulsive effect, that is, the noninstantaneous type, models the perturbations that occur for a finite time period. The very first study in the literature on the noninstantaneous type of perturbations was by Hernandez and O'Regan in [5]. Zhang et al. in [6] derived the sufficient bounds on which the perturbations have a stabilizing effect on the trajectory of the system. An analysis of the stability of the fractional impulsive model using Gronwall's integral inequality (GI) of the Bellman type is discussed in [7]. The existence of the solution for the fractional model with perturbation of the noninstantaneous type is analysed in [8]. For more knowledge on the analysis of impulsive perturbations and their effect on dynamical systems, one can refer to the research in [9–15].

Usually, when concerned about the stability of a dynamical system, it is more practical to study the stabilization effect over a bounded time interval rather than an infinite one. Therefore, the concept of finite-time stability (FTS) of differential models has a rich background in the literature. There are different methods available in the literature to study the FTS concept of differential models. The authors of [16] developed the method of a fixed-point approach to derive the FTS bounds for fractional systems. A delayed GI method was used in [17] to analyse the stability of a fractional system with time lags. The FTS of a fractional singular model using a GI method is examined in [18]. The FTS of a stochastic differential model is examined in [19]. A weighted norm method, along with the approach of GI, was utilized to study the FTS bounds of a stochastic differential model in [20]. The stability of fractional neural networks was studied using the linear matrix inequality method in [21].

Delay differential equations (DDE) are used to mathematically model a technological or mechanical system that involves time lags in the system dynamics. Fractional delay differential systems provide a more accurate representation of many real-world phenomena characterized by memory effects and nonlocal interactions. They can capture complex dynamics such as hereditary behaviours, long-range interactions, and anomalous diffusion. Examples include biological systems, viscoelastic materials, networked systems, and financial markets. The numerical solution of DDEs is an intricate process, and accuracy of the output is often questionable. In most cases, problems arise when the delay time period is negligibly small and contains an insignificant step size. Therefore, a novel method to solve continuous and discrete DDEs by the method of an exponential matrix function with delayed arguments was introduced by Khusainov and Shuklin in [22]. Therein, rigorous theory was developed by researchers to extend the study of DDEs and represent their solution in the form of delayed-type matrix exponential functions [23–25]. Zijian Luo [26] derived the bounds required for the FTS of a differential system using Coppel's and Jensen's inequality. Chengblin Liang et al. in [27] introduced the sine and cosine delayed matrix function to study the FTS of FDEs with pure delay.

Li and Wang [28] initiated the FTS analysis of linear FDEs using the delayed-type matrix Mittag Leffler (DPMM) function. Later, the analytical study on the FTS of the system was extended to nonlinear arguments in [29]. Thereafter, the authors of [14] explored the FTS of a differential system with impulsive conditions by utilizing a delayed form of exponential function. Some recent noteworthy work on the FTS analysis of dynamical systems can be found in [30–37].

Taking the motivation from the above work, we introduce the application of the (DPMM) function to study the stability analysis of a fractional semilinear differential system for the very first time. Therein, the analysis of the system is extended and studied with impulsive perturbations. The considered system is of the following form:

$$\begin{aligned}&({}^cD^q v)(\varkappa) = \mathcal{B}v(\varkappa - h) + \mathcal{C}u(\varkappa) + \mathfrak{g}(\varkappa, v(\varkappa - h), u(\varkappa)), \varkappa \in \mathcal{J} = [0, \mathcal{T}],\\ &v(\varkappa) = \phi(\varkappa), -h \le \varkappa \le 0, \quad \phi(\varkappa) \in \mathbb{C}_h^1 = \mathbb{C}([-h, 0], \mathfrak{R}^n),\end{aligned} \tag{1}$$

where $q \in (0, 1)$, the real matrix $\mathcal{B} \in \mathfrak{R}^{n\times n}$, and $u \in \mathfrak{R}^m$ is the control input with associated matrix $\mathcal{C} \in \mathfrak{R}^{n\times m}$. The time delay is denoted by the notion h and $\mathcal{T} = m * h$, for a fixed $m \in \Lambda = 1, 2, 3, \ldots$

The novel outcomes of the work can be encapsulated as follows:

1. The solution's existence and its uniqueness for the considered fractional system are discussed using a delayed matrix Mittag-Leffler argument.
2. Stability analysis of the system is studied using the approach of few direct results from functional analysis and using the method of GI inequality.
3. Further, the sufficient bounds for the FTS of our considered system are analysed with impulsive perturbations.

The organization of the article can be summarized as below:

In Section 2, we discuss the predefined lemmas and definitions. The existence of the solution and its uniqueness are given in Section 3. Further, Section 4 provides sufficient bounds required for the system to admit FTS. Using the approach of GI conditions and through the method of DPMMF, the required FTS bounds are obtained. In Section 5, we extend our study and analyse system (1) with impulsive perturbations. In Section 6, we establish the numerical simulations of our obtained results graphically. Finally, in Section 7 we conclude the finite-time stability analysis discussion with final remarks.

2. Prerequisites

In this entire paper, we use the matrix norm $\|\mathcal{B}\| = \max_{1\leq j\leq n}\sum_{i=1}^{n}|\mathcal{B}|$ and vector norm $\|v\| = \sum_{i=1}^{n}|v_i|$. We use the notion $\mathbb{C}(\mathcal{J},\mathfrak{R}^n)$ for the Banach space of the continuous vector-valued function $\mathcal{J}\to\mathfrak{R}^n$, induced by the norm $\|y\| = \max_{t\in\mathcal{J}}\|y(t)\|$. If $(Z,\|\cdot\|_z)$ and $(Y,\|\cdot\|_y)$ are Banach spaces, then $Z\oplus Y = \{(z,y)|z\in Z, y\in Y\}$, with $\|(z,y)\| = \|z\|+\|y\|$. Moreover, we have the estimation $\|\phi\|_{\mathbb{C}} = \max_{\theta\in[-h,0]}\|\phi(\theta)\|$.

Definition 1 ([38]). *The integral operator of fractional order $q\in(0,1)$ of a function $v:[0,\infty)\to\mathfrak{R}$ is given as $I^q v(\varkappa) = \dfrac{1}{\Gamma(q)}\int_0^{\varkappa}\dfrac{v(t)}{(\varkappa-t)^{1-q}}dt$, $\varkappa>0$ and the right-hand side of the integral equation is well defined pointwise on the given interval $[0,\infty)$.*

Definition 2 ([39]). *The differential operator in the sense of Riemann–Liouville with order $q\in(0,1)$ of a function $v:[0,\infty)\to\mathfrak{R}$ is given as $({}^{RL}D^q v)(\varkappa) = \dfrac{1}{\Gamma(1-q)}\dfrac{d}{d\varkappa}\int_0^{\varkappa}\dfrac{v(t)}{(\varkappa-t)^q}dt, \varkappa>0.$*

Definition 3 ([40]). *The differential operator in the sense of Caputo with order $q\in(0,1)$ of a function $v:[0,\infty)\to\mathfrak{R}$ is given as $({}^{C}D^q v)(\varkappa) = ({}^{RL}D^q v)(\varkappa) - \dfrac{v(0)}{\Gamma(1-q)}\varkappa^{-q}, \varkappa>0.$*

Definition 4 ([41]). *The classical form of the delayed-type Mittag-Leffler matrix is defined as $\mathcal{E}_h^{\mathcal{B}\varkappa^q}:\mathfrak{R}\to\mathfrak{R}^{n\times n}$ with*

$$\mathcal{E}_h^{\mathcal{B}\varkappa^q} = \begin{cases} \Theta, -\infty<\varkappa<-h, \\ I, -h\leq\varkappa\leq 0, \\ I+\mathcal{B}\dfrac{\varkappa^q}{\Gamma(q+1)}+\mathcal{B}^2\dfrac{(\varkappa-h)^{2q}}{\Gamma(2q+1)}+\cdots+\mathcal{B}^m\dfrac{(\varkappa-(m-1)h)^{mq}}{\Gamma(mq+1)}, \\ (m-1)h<\varkappa\leq mh, m\in\Lambda, \end{cases}$$

where the notion Θ denotes the matrix with all entries zero, and I is the identity matrix.

Definition 5 ([42]). *The delayed-type Mittag-Leffler matrix of two parameters $\mathcal{E}_{h,\beta}^{\mathcal{B}\varkappa^q}:\mathfrak{R}\to\mathfrak{R}^{n\times n}$ is of the type*

$$\mathcal{E}_{h,\beta}^{\mathcal{B}\varkappa^q} = \begin{cases} \Theta, -\infty<\varkappa<-h, \\ I\dfrac{(h+\varkappa)^{q-1}}{\Gamma(\beta)}, -h\leq\varkappa\leq 0, \\ I\dfrac{(h+\varkappa)^{q-1}}{\Gamma(\beta)}+\mathcal{B}\dfrac{\varkappa^{2q-1}}{\Gamma(q+\beta)}+\mathcal{B}^2\dfrac{(\varkappa-h)^{3q-1}}{\Gamma(2q+\beta)}+\cdots+\mathcal{B}^m\dfrac{(\varkappa-(m-1)h)^{(m+1)q-1}}{\Gamma(mq+\beta)}, \\ \quad (m-1)h<\varkappa\leq mh,\ m\in\Lambda. \end{cases}$$

Lemma 1 ([29]). *If $(m-1)h<\varkappa\leq m\tau$, $0\leq\tau\leq t$ and $m\in\Lambda$ is a number that is fixed, then $\int_{(m-1)h+\tau}^{\varkappa}(\varkappa-t)^{-q}(t-(m-1)h-\tau)^{mq-1}dt = (\varkappa-(m-1)h-\tau)^{(m-1)q}\mathbb{B}[1-q,mq]$, where the notion $\mathbb{B}[u,v] = \int_0^1\tau^{u-1}(1-\tau)^{v-1}$ denotes the beta function.*

Lemma 2 ([29]). *Let $(m-1)h<\varkappa\leq mh$, $0\leq\tau\leq t$; if $m\in\Lambda$ is fixed, we obtain*

$$\int_{\tau}^{\chi}(\chi-t)^{-q}\mathcal{E}_{h,q}^{\mathcal{B}(t-h-\tau)^q}dt=\int_{\tau}^{\chi}(\chi-t)^{-q}I\frac{(t-\tau)^{q-1}}{\Gamma(q)}dt+\int_{h+\tau}^{\chi}(\chi-t)^{-q}\mathcal{B}\frac{(t-h-\tau)^{2q-1}}{\Gamma(2q)}dt+\dots$$
$$+\int_{(m-1)h+\tau}^{\chi}(\chi-t)^{-q}\mathcal{B}^{m-1}\frac{(t-(m-1)h-\tau)^{mq-1}}{\Gamma(mq)}dt.$$

Lemma 3 ([29])**.** *If $\chi\in((m-1)h,mh]$, $m\in\Lambda$ is fixed, then we have the estimation $\left\|\mathcal{E}_h^{\mathcal{B}\chi^q}\right\|\leq\mathcal{E}_h(\|\mathcal{B}\|\chi^q)$.*

Lemma 4 ([29])**.** *If $\chi\in((m-1)h,mh]$, $m\in\Lambda$ is a number that is fixed and $0\leq t<\chi$, we obtain*

(i) If $\chi-(i-1)h\leq t<\chi-(i-2)h$, $i=2,\dots,m$, we obtain

$$\left\|\mathcal{E}_{h,\beta}^{\mathcal{B}(\chi-h-t)^q}\right\|\leq\sum_{r=2}^{i}\|\mathcal{B}\|^{r-2}\frac{(\chi-(r-2)h-t)^{(r-1)q-1}}{\Gamma((r-2)q+\beta)}.$$

(ii) If $0\leq t<\chi-(m-1)h$, we obtain

$$\left\|\mathcal{E}_{h,\beta}^{\mathcal{B}(\chi-h-t)^q}\right\|\leq\sum_{r=1}^{m}\|\mathcal{B}\|^{r-1}\frac{(\chi-(r-1)h-t)^{rq-1}}{((r-1)q+\beta)}.$$

Lemma 5 ([29])**.** *If $\chi\in((m-1)h,mh]$, $m\in\Lambda$ is a number that is fixed, then*

$$\left\|\int_{-h}^{0}\mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q}\phi'(t)\right\|\leq\mathcal{E}_h(\|\mathcal{B}\|\chi^q)\int_{-h}^{0}\left\|\phi'(t)\right\|dt.$$

Lemma 6 ([29])**.** *If $\chi\in((m-1)h,mh]$ for $i=2,\dots,m$, $m\in\Lambda$ and $q\geq\frac{1}{2}$, we obtain*

$$\int_0^{\chi-(m-1)h}\left\|\mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\right\|\|\psi(t)\|dt+\int_{\chi-(m-1)h}^{\chi}\left\|\mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\right\|\|\psi(t)\|dt$$
$$\leq\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\int_0^{\chi}(\chi-t)^{q-1}\|\psi(t)\|dt,\psi\in(\mathcal{I},\mathfrak{R}^n).$$

Lemma 7 ([7])**.** *Let $v(\chi)>0$, $y(\chi)>0$ be locally integrable and the function $\lambda(\chi)>0$ be nondecreasing and continuous on $\chi\in[0,T)$. Now, if $\lambda(\chi)\leq M$, $q>0$ with*

$$v(\chi)\leq\lambda(\chi)+y(\chi)\int_0^{\chi}(\chi-\mu)^{q-1}v(\mu)d\mu,\quad 0\leq\chi<T.$$

Then,

$$v(\chi)\leq\lambda(\chi)+\int_0^{\chi}\left(\sum_{m=1}^{\infty}\frac{(y(\chi)\Gamma(q))^m}{\Gamma(mq)}(\chi-\mu)^{mq-1}\lambda(\mu)\right)d\mu,\quad 0\leq\chi<T.$$

Lemma 8 ([7])**.** *From Lemma 7, if $\lambda(\chi)>0$ is nondecreasing on $[0,T)$, then we have the following inequality:*

$$v(\chi)\leq\lambda(\chi)E_q(y(\chi)\Gamma(q)\chi^q).$$

3. Formulation of Solution

Lemma 9 ([29])**.** *Consider the below form of linear fractional differential equation:*

$$({}^cD^qv)(\chi)=\mathcal{B}v(\chi-h)+\mathfrak{g}(\chi),\chi\in\mathcal{I},$$
$$v(\chi)=\phi(\chi),-h\leq\chi\leq0,\phi\in\mathbb{C}_h^1.$$

For a continuous function $\mathfrak{g} : \mathcal{J} \to \mathfrak{R}^n$, the solution $v \in \mathbb{C}([-h, \mathcal{T}], \mathfrak{R}^n)$ of the above system can be formulated as given below:

$$v(\varkappa) = \mathcal{E}_h^{\mathcal{B}\varkappa^q}\phi(-h) + \int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\varkappa-h-t)^q}\phi'(t)dt + \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q}\mathfrak{g}(t)dt.$$

From Lemma 9, we can formulate the solution of (1) as given in the following definition.

Definition 6. *The solution $v \in \mathbb{C}([-h, \mathcal{T}], \mathfrak{R}^n)$ of (1) can be written in the following form:*

$$v(\varkappa) = \mathcal{E}_h^{\mathcal{B}\varkappa^q}\phi(-h) + \int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\varkappa-h-t)^q}\phi'(t)dt + \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q}\mathcal{C}u(t)dt + \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q}\mathfrak{g}(t, v(t-h), u(t))dt. \tag{2}$$

In relevance to the analytical proof of our main theorems, we consider the subsequent presumptions:

$[\mathcal{A}_1]$ For $\tilde{c} > 0$, we have $\|\mathcal{C}u_1(t) - \mathcal{C}u_2(t)\| \leq \tilde{c}\|u_1 - u_2\|, \forall t \in \mathcal{J}$, and $u_1, u_2 \in \mathfrak{R}^m$.

$[\mathcal{A}_2]$ For $\mathfrak{g} \in \mathbb{C}(\mathcal{J}, \mathfrak{R}^n)$ there exists nonzero values $\tilde{Q}_1 > 0$ and $\tilde{Q}_2 > 0$ in such a way that

$$\begin{aligned}&\|\mathfrak{g}(t, v(t-h), u_1(t)) - \mathfrak{g}(t, z(t-h), u_2(t))\| \\ &\quad \leq \tilde{Q}_1(\|v(t-h) - z(t-h)\|) + \tilde{Q}_2(\|u_1(t) - u_2(t)\|), \forall v, z \in \mathfrak{R}^n.\end{aligned}$$

$[\mathcal{A}_3]$ $\eta = \tilde{Q}\big(\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\mathcal{T} - (r-1)h)^{rq}\big) < 1$, where $\tilde{Q} = \max\{\tilde{Q}_1, \tilde{c} + \tilde{Q}_2\}$.

At this point, we denote $N = \int_{-h}^{0}\left\|\phi'(t)\right\|dt$. Next, we proceed to discuss the existence and uniqueness of our solution (2).

Theorem 1. *On the assumption that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ hold, we can state that system (1) has a solution $v \in \mathbb{C}([-h, \mathcal{T}], \mathfrak{R}^n)$, which is unique.*

Proof. We begin by defining an operator $\Psi : \mathbb{C}([-h, \mathcal{T}], \mathfrak{R}^n) \times \mathbb{C}([0, \mathcal{T}], \mathfrak{R}^m) \to \mathbb{C}([-h, \mathcal{T}], \mathfrak{R}^n) \times \mathbb{C}([0, \mathcal{T}], \mathfrak{R}^m)$ in such a way that

$$\Psi(v, u)(\varkappa) \leq \mathcal{E}_h^{\mathcal{B}\varkappa^q}\phi(-h) + \int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\varkappa-h-t)^q}\phi'(t)dt + \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q}\mathcal{C}u(t)dt + \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q}\mathfrak{g}(t, v(t-h), u(t))dt. \tag{3}$$

Taking the norm we obtain the following inequality:

$$\begin{aligned}\|\Psi(v_1, u_1)(\varkappa) - \Psi(v_2, u_2)(\varkappa)\| \leq{}& \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q}\|\mathcal{C}\|\|u_1 - u_2\| + \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q} \\ &\times \|\mathfrak{g}(t, v_1(t-h), u_1(t)) - \mathfrak{g}(t, v_2(t-h), u_2(t))\|dt.\end{aligned}$$

Considering $\|v(t-h)\| \leq \sup_{t\in\mathcal{J}}\|v(t)\|$, we obtain

$$\|\Psi(v_1, u_1)(\varkappa) - \Psi(v_2, u_2)(\varkappa)\| \leq \int_0^{\varkappa} \mathcal{E}_{h,q}^{\mathcal{B}(\varkappa-h-t)^q}(\tilde{\mathcal{Q}}_1\|v_1 - v_2\| + \tilde{\mathcal{Q}}_2\,\tilde{c}\|u_1 - u_2\|)dt.$$

From Lemma 4, we can write the above inequality as

$$\|\Psi(v_1, u_1)(\varkappa) - \Psi(v_2, u_2)(\varkappa)\| \leq \Big(\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\varkappa - (r-1)h)^{rq}\Big)\tilde{Q}[\|(v_1, u_1) - (v_2, u_2)\|].$$

From $\mathcal{A}_2$ and $\mathcal{A}_3$, it is clear that Ψ is a contraction mapping and is well defined, which implies

$$\|\Psi(v_1, u_1) - \Psi(v_2, u_2)\|_{\mathbb{C}(\mathcal{J})} \leq \eta\|(v_1, u_1) - (v_2, u_2)\|_{\mathbb{C}(\mathcal{J})}, \forall\, v_1, v_2 \in \mathfrak{R}^n.$$

□

At this point, using the local Lipchitz condition, we state another existence and uniqueness theorem.

$[\mathcal{A}_1']$: For all $\hat{r} > 0$ and $\chi_1 > 0$, there exist constants Q_1 and Q_2, where $Q_i = Q_i(\chi_1, \hat{r}), i = 1, 2$, such that

$$\begin{aligned}\|\mathfrak{g}(\chi, v(\chi - h), u_1(\chi)) - \mathfrak{g}(\chi, z(\chi - h), u_2(\chi))\| \leq Q_1(\|v(\chi - h) - z(\chi - h)\|)\\ + Q_2(\|u_1(\chi) - u_2(\chi)\|), \forall \chi \in [0, \chi_1],\\ \forall v, z \in \mathcal{B}_{\hat{r}} = \{v \in \mathfrak{R}^n, u \in \mathfrak{R}^m \ni \|(v, u)\| \leq \hat{r}\}.\end{aligned}$$

Theorem 2. *If $\mathcal{A}_1'$ holds, then the solution $\chi \in \mathbb{C}([-h, \chi_m), \mathfrak{R}^n)$, $\chi_m \leq \infty$ of system (1) is said to be unique. Moreover, if $\chi_m < \infty$, then $\lim_{\chi\to\chi_m}\|v(\chi)\| = \infty$.*

Proof. Consider $\Psi : \mathbb{C}([-h, \chi_m], \mathfrak{R}^n) \times \mathbb{C}([0, \chi_m], \mathfrak{R}^m) \to \mathbb{C}([-h, \chi_m], \mathfrak{R}^n) \times \mathbb{C}([0, \chi_m], \mathfrak{R}^m)$ as stated in (3).

Part 1: Preliminary estimation.

Arbitrarily, for any $\chi \in [-h, 0]$, we obtain $\|v(\chi)\| = \|\phi(\chi)\|$. Now, choose a subinterval $\mathcal{J}_1 = [0, \chi_1]$; if $\gamma(\chi) = \sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\chi - (r-1)h)^{rq}$, where m belongs to the set $\max\{m \in \Lambda : mh \leq \chi_1\}$, $\hat{r_1} = (\|\phi\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi_1{}^q) + 1$, $M_1 = \sup_{\chi_1\in\mathcal{J}_1}\|\mathfrak{g}(\chi, 0, 0)\|$, then for any $v \in \mathbb{C}([0, \chi_1], \mathfrak{R}^n)$ with $\{\|v(\chi)\| + \|u(\chi\|) : \chi \in [0, \chi_1]\} \leq \hat{r}_1$, we obtain

$$\begin{aligned}\|(\Psi v)(\chi)\| \leq{}& \left\|\mathcal{E}_h^{\mathcal{B}\chi^q}\right\|\|\phi(-h)\| + \left\|\int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q}\phi'(t)\right\|dt + \left\|\int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\mathcal{C}u(t)\right\|dt\\ &+ \left\|\int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\right\|\|\mathfrak{g}(t, v(t-h), u(t))\|dt.\end{aligned}$$

$$\begin{aligned}\|(\Psi v)(\chi)\| \leq{}& (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi_1^q) + \int_0^{\chi}\left\|\mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)}\right\|\tilde{c}\|u\|dt + \int_0^{\chi}\left\|\mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)}\right\|\\ &\times [\|\mathfrak{g}(t, v(t-h), u(t)) - \mathfrak{g}(t, 0, 0) + \mathfrak{g}(t, 0, 0)\|]dt.\end{aligned}$$

$$\begin{aligned}\|(\Psi v)(\chi)\| \leq{}& (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi_1{}^q) + \int_0^{\chi}\left\|\mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)}\right\|\tilde{c}\|u\|dt + \int_0^{\chi}\left\|\mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)}\right\|\\ &\times [Q_1\|v\| + Q_2\|u\| + M_1]dt.\end{aligned}$$

$$\|(\Psi v)(\chi)\| \leq (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi_1{}^q) + \sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\chi - (r-1)h)^{rq}[Q_1\|v\| + (Q_2 + \tilde{c})\|u\| + M_1].$$

$$\|(\Psi v)(\chi)\| \leq (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi_1{}^q) + [Q_1\|v\| + (Q_2 + \tilde{c})\|u\| + M_1]\gamma(\chi).$$

$$\|(\Psi v)(\chi)\| \leq (\|\phi\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi_1{}^q) + [Q\hat{r_1} + M_1]\gamma(\chi), \tag{4}$$

where $Q = \max\{Q_1, Q_2 + \tilde{c}\}$. Further, choose $\rho_1 = \min\{\chi_1, \chi_1^*\}$, where χ_1^* is contained in the set satisfying the property $\{\chi_1^* : \gamma(\chi_1^*) = \dfrac{1}{Q\hat{r_1} + M_1}\}$. Then, from (4) we can conclude that $\|(\Psi v)(\chi)\| \leq \hat{r_1}, \forall \chi \in [0, \rho_1]$.

Part 2: Solution existence and uniqueness locally.

For all $\chi \in [0, \rho_1]$ and $v, z \in \mathbb{B}_{\hat{r}}$, using Lemma 4, we derive the following estimation:

$$\|\Psi(v, u_1)(\chi) - \Psi(z, u_2)(\chi)\| \leq \left\| \int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \right\| \|\mathfrak{C}\| \|u_1 - u_2\| dt + \int_0^\chi \left\| \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \right\| \times \|\mathfrak{g}(t, v(t-h), u_1(t)) - \mathfrak{g}(t, z(t-h), u_2(t))\| dt.$$

$$\|\Psi(v, u_1)(\chi) - \Psi(z, u_2)(\chi)\| \leq \left\| \int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \right\| [\tilde{c}(\|u_1 - u_2\|) + Q_1(\|v - z\|) + Q_2(\|u_1 - u_2\|)] dt.$$

$$\|\Psi(v, u_1)(\chi) - \Psi(z, u_2)(\chi)\| \leq Q\Big(\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)} (\rho_1 - (r-1)h)^{rq} \Big) [\|(v, u_1) - (z, u_2)\|].$$

where $m : \{m \in \Lambda : mh \leq \rho_1\}$, choosing ρ_1, satisfying $\sum_{r=1}^{m} \dfrac{\|\mathcal{B}\|^{r-1}}{\Gamma rq + 1} ((\rho_1 - (r-1)h)^{rq}) < \frac{1}{2Q}$.

Hence,

$$\|\Psi(v, u_1) - \Psi(z, u_2)\|_{\mathbb{C}([0,\rho_1])} \leq \frac{1}{2} \|(v, u_1) - (z, u_2)\|_{\mathbb{C}([0,\rho_1])}.$$

Therefore, on the interval $[0, \rho_1]$, Ψ has a fixed point, and this reduces a solution.

Part 3: Solution extension

At this point, we solve the fixed-point problem $z = \Psi(z)$, and for $\chi_1 \geq \rho_1$, we extend the solution.

For $\chi \in [0, \rho_2]$, we have

$$(\Psi z)(\chi) \leq \mathcal{E}_h^{\mathcal{B}\chi^q} \phi(-h) + \int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q} \phi'(t) dt + \int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \mathfrak{C} u(t) dt + \int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \mathfrak{g}(t, z(t-h), u(t)) dt.$$

Then, $\forall\ \chi \in [0, \rho_2]$; we set $z(\chi) = v(\chi)$,, then take $\chi_2 > \rho_2$ and define $\mathcal{J}_2 = [0, \chi_2]$, $\gamma(\chi) = \sum_{r=1}^{m} \dfrac{\|\mathcal{B}\|^{r-1}}{\Gamma rq + 1} ((\rho_1 - (r-1)h)^{rq})$, where m is from the set $\max\{m \in \Lambda : mh \leq \chi_2\}$, $\hat{r_2} = (\|\phi\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi_2^q) + 1$ and $M_2 = \sup_{\chi \in \rho_2} \|\mathfrak{g}(t, 0, 0)\|$. We choose $\rho_2 = \rho_1 + \min\{\chi_2 - \rho_1, \chi_2^*\}$, with the property that χ_2^* belongs to the set $\left\{ \chi_2^* : \gamma(\chi_2)^* = \dfrac{1}{Q\hat{r_2} + M_2} \right\}$. For $\chi \in [\rho_1, \rho_2]$, we obtain $\|(\Psi v)(\chi)\| \leq \hat{r_2}$; then, by repeating the same procedure, we arrive at a maximal interval where the solution exists. Therefore, $v \in \mathbb{C}([-h, \chi_m), \mathfrak{R}^n)$.

Now, we are left to verify that $\lim_{\chi \to \chi_m} \|v(\chi_m)\| = \infty$. If not, there exists $\{\mathcal{J}_n\}$ converging to χ_m, and for a positive value $\hat{r}$, we obtain $\|v(\rho_m)\| \leq \hat{r}, \forall n$. Then, for a large n, which is sufficiently enough, we obtain ρ_n to be infinitesimally close to χ_m; by the above argument, the solution can be extended far beyond χ_m, which results in a contradiction. As a consequence, we have our proof. □

4. Finite-Time Stability Results

Stability analysis over a predefined finite interval of time is a well-established method to study the behaviour of system dynamics. In the wake of its practical applications, it has been extensively analysed by many researchers. Therefore, in this section, we provide the sufficiently required bounds for the considered system (1) to admit FTS.

Definition 7. *For a solution $v(\chi)$ of (1) and for $\|u(t)\| \leq \beta_u$, we can say that the system is stable over a finite time period with respect to $\{0, \mathcal{J}, h, \delta, \epsilon\}$ if and only if $v \in \mathcal{J}$, $\|\phi\| < \delta \implies \|v(\chi)\| < \epsilon$, where $\beta_u, \delta, \epsilon$ are positive real values with the property of $\delta < \epsilon, \forall\, \chi \in \mathcal{J}$.*

At this point, we impose certain presumptions:

[$\mathcal{A}_4$] Let $\varpi(\cdot) \in \mathbb{C}(\mathcal{J}, \mathfrak{R}^+)$ in such a way that $\|\mathfrak{g}(\chi, v(\chi), u(\chi))\| < \|\varpi(\chi)\|, \forall \chi \in \mathcal{J}$ and $v \in \mathfrak{R}^n$.

[$\mathcal{A}_5$] There exists a $\kappa(.) \in \mathbb{L}^{p_1}(\mathcal{J}, \mathfrak{R}^+)$, $\frac{1}{p_1} = 1 - \frac{1}{p_2}$, $p_2 > 1$, such that $\|\mathfrak{g}(\chi, v(\chi - t), u(\chi))\| \leq \kappa_1(\chi) + \kappa_2(\chi)$ for $\chi \in \mathcal{J}$, $v \in \mathfrak{R}^n$, $u \in \mathfrak{R}^m$ and $P_1(\chi) = \big(\int_0^\chi \kappa_1(\chi)^{p_1}\big)^{\frac{1}{p_1}} < \infty$, $P_2(\chi) = \big(\int_0^\chi \kappa_2(\chi)^{p_1}\big)^{\frac{1}{p_1}} < \infty$.

[$\mathcal{A}_6$] Let $\mathcal{H} > 0$ be a positive value, such that $\|\mathfrak{g}(t, v(t), u(t))\| \leq \mathcal{H}(\|v\| + \|u\|), \forall t \in \mathcal{J}$, $v \in \mathfrak{R}^n$ and $u \in \mathfrak{R}^m$.

Theorem 3. *On the assumption that $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_4$ hold, we state for a fixed $m \in \Lambda$ that system (1) is FTS with respect to $\{0, \mathcal{J}, h, \delta, \epsilon\}$ if*

$$(\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\|\varpi\| + \tilde{c}\beta_u)\Big(\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\chi - (r-1)h)^{rq}\Big) < \epsilon.$$

Proof.

$$\|v(\chi)\| \leq \Big\|\mathcal{E}_h^{\mathcal{B}\chi^q}\Big\|\|\phi(-h)\| + \Big\|\int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q}\phi'(t)\Big\|dt + \Big\|\int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\mathcal{C}u(t)\Big\|dt + \Big\|\int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\Big\|\|\mathfrak{g}(t, v(t-h), u(t))\|dt.$$

$$\|v(\chi)\| \leq (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + \Big\|\int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\Big\|\|\mathcal{C}\|\|u\|dt + \Big\|\int_0^\chi \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\Big\|\varpi(t)dt.$$

$$\|v(\chi)\| \leq (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\|\varpi\| + \tilde{c}\beta_u)\int_0^{\chi-(m-1)h}\sum_{r=1}^{m}\|\mathcal{B}\|^{r-1}\frac{(\chi-(r-1)h-t)^{rq-1}}{\Gamma((r-1)q+q)}dt + (\|\varpi\| + \tilde{c}\beta_u)\sum_{i=2}^{m}\int_{\chi-(i-1)h}^{\chi-(i-2)h}\Big(\sum_{r=2}^{m}\|\mathcal{B}\|^{r-2}\frac{(\chi-(r-2)h-t)^{(r-1)q-1}}{\Gamma((r-2)q+q)}\Big)dt.$$

$$\|v(\chi)\| \leq (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\|\varpi\| + \tilde{c}\beta_u)\sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}((\chi-(r-1)h)^{rq} - ((m-r)h)^{rq}) + (\|\varpi\| + \tilde{c}\beta_u)\sum_{i=2}^{m}\Big(\sum_{r=2}^{i}\frac{\|\mathcal{B}\|^{r-2}}{\Gamma((r-1)q+1)}((\chi-r-1)h)^{(r-1)q} - ((m-r)h)^{(r-1)q})\Big).$$

$$\|v(\chi)\| \leq (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\|\varpi\| + \tilde{c}\beta_u)\Big(\sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\chi-(r-1)h)^{rq}\Big) < \epsilon.$$

□

Theorem 4. *If $q > 1 - \frac{1}{p_2}, (p_2 > 1)$, and further, if $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_4$ hold with $m \in \Lambda$, then we state that system (1) is FTS with respect to $\{0, \mathcal{J}, h, \delta, \epsilon\}$ if*

$$(\delta + N)\mathcal{E}_q(\|\mathcal{B}\|x^q) + (\tilde{c}\beta_u)\Big(\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(x-(r-1)h)^{rq}\Big) + \sum_{r=1}^{m}\bigg(\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq)} \times \frac{(x-(r-1)h)^{rq-1+\frac{1}{p_2}}}{(p_2rq-p_2+1)^{\frac{1}{p_2}}}\bigg)(P_1(x)+P_2(x)) < \epsilon.$$

Proof. Utilizing Lemmas 2–5, and further, by implementing the structural properties of the operator norm, we obtain

$$\|v(x)\| \le \left\|\mathcal{E}_h^{\mathcal{B}x^q}\right\|\|\phi(-h)\| + \|\mathcal{E}_q(\|\mathcal{B}\|x^q)\|\int_{-h}^{0}\|\phi'(t)\|dt + \left\|\int_0^x \mathcal{E}_{h,q}^{\mathcal{B}(x-h-t)^q}\mathcal{C}u(t)\right\|dt + \int_0^x \frac{(x-t)^{q-1}}{\Gamma(q)}\|\mathfrak{g}(x,(v-x),u(t))\|dt + \cdots + \int_0^{x-(m-1)h}\|\mathcal{B}\|^{m-1} \times \frac{(x-(m-1)h-t)^{mq-1}}{\Gamma(mq)}\|\mathfrak{g}(x,(v-x),u(t))\|dt.$$

$$\|v(x)\| \le (\phi(-h)+N)\mathcal{E}_q(\|\mathcal{B}\|x^q) + (\tilde{c}\beta_u)\Big(\sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(x-(r-1)h)^{rq}\Big) + \sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(mq)}\int_0^{x-(r-1)h}(x-(r-1)h-t)^{rq-1}\|\mathfrak{g}(x,(v-x),u(t))\|dt.$$

$$\|v(x)\| \le (\delta+N)\mathcal{E}_q(\|\mathcal{B}\|x^q) + (\tilde{c}\beta_u)\Big(\sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(x-(r-1)h)^{rq}\Big) + \sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(mq)}\bigg(\int_0^{x-(r-1)h}(x-(r-1)h-t)^{p_2(rq-1)}dt\bigg)^{\frac{1}{p_2}} \times \bigg(\int_0^{x-(r-1)h}\|\mathfrak{g}(x,(v-x),u(t))\|^{p_1}dt\bigg)^{\frac{1}{p_1}}.$$

$$\|v(x)\| \le (\delta+N)\mathcal{E}_q(\|\mathcal{B}\|x^q) + (\tilde{c}\beta_u)\Big(\sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(x-(r-1)h)^{rq}\Big) + \sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(mq)} \times \bigg(\int_0^{x-(r-1)h}(x-(r-1)h-t)^{p_2(rq-1)}dt\bigg)^{\frac{1}{p_2}}\bigg(\int_0^{x}(\|(v-t)\|+\|u(t)\|)^{p_1}dt\bigg)^{\frac{1}{p_1}}.$$

$$\|v(x)\| \le (\delta+N)\mathcal{E}_q(\|\mathcal{B}\|x^q) + (\tilde{c}\beta_u)\Big(\sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(x-(r-1)h)^{rq}\Big) + \sum_{r=1}^{m}\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(mq)} \times \bigg(\int_0^{x-(r-1)h}(x-(r-1)h-t)^{p_2(rq-1)}dt\bigg)^{\frac{1}{p_2}} \times \bigg(\int_0^{x}(\|(v-t)\|^{p_1}dt)^{\frac{1}{p_1}} + (\int_0^{x}\|u(t)\|^{p_1}dt)^{\frac{1}{p_1}}\bigg).$$

Taking sup over t, we obtain

$$\|v(\chi)\| \leq (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c}\beta_u)\Big(\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\chi - (r-1)h)^{rq}\Big)$$
$$+ \sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(mq)} \left(\int_0^{\chi-(r-1)h} (\chi - (r-1)h - t)^{p_2(rq-1)} dt \right)^{\frac{1}{p_2}} \left(\int_0^{\chi} (\kappa_1(t)^{p_1})^{\frac{1}{p_1}} \right.$$
$$\left. + \int_0^{\chi} (\kappa_2(t)^{p_1} dt)^{\frac{1}{p_1}} \right).$$

$$\|v(\chi)\| \leq (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c}\beta_u)\Big(\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\chi - (r-1)h)^{rq}\Big)$$
$$+ \sum_{r=1}^{m} \left(\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq)} . \frac{(\chi - (r-1)h)^{rq-1+\frac{1}{p_2}}}{(p_2 rq - p_2 + 1)^{\frac{1}{p_2}}} \right) (P_1(\chi) + P_2(\chi)),$$
$$< \epsilon.$$

□

Theorem 5. *On the assumption that $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{A}_6$ hold and $q \geq \frac{1}{2}$, then we state that system (1) is FTS with respect to $\{0, \mathcal{J}, h, \delta, \epsilon\}$ if*

$$(\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\, \beta_u\, \mathcal{E}_{q,q}\Big(\|\mathcal{B}\| \frac{\chi^q}{q}\Big)\mathcal{E}_q(\mathcal{H}\Gamma(q)\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\chi^q) < \epsilon, \forall\, \chi \in \mathcal{J}.$$

Proof. From Lemmas 3–6 and Lemma 9, we obtain

$$\|v(\chi)\| \leq \left\| \mathcal{E}_h^{\mathcal{B}\chi^q} \right\| \|\phi(-h)\| + \left\| \int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q} \phi'(t) \right\| dt + \left\| \int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \mathcal{C}u(t) \right\| dt$$
$$+ \left\| \int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \right\| \|\mathfrak{g}(t, v(t-h), u(t))\| dt.$$

$$\|v(\chi)\| \leq \|\phi(-h)\|\mathcal{E}_q(\|\mathcal{B}\|\chi_q) + N\,\mathcal{E}_q(\|\mathcal{B}\|\chi_q) + \|u\|\tilde{c}\left\| \int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \right\| dt + \mathcal{H}\left\| \int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q} \right\|$$
$$\times (\|v\| + \|u\|) dt.$$

$$\|v(\chi)\| \leq (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c}\beta_u)\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q) \int_0^{\chi} (\chi - t)^{q-1} dt + \mathcal{H}\, \mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)$$
$$\times \int_0^{\chi} (\chi - t)^{q-1}(\|v\| + \|u\|) dt.$$

$$\|v(\chi)\| \leq (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\, \beta_u \frac{\chi^q}{q}\, \mathcal{E}_{q,q}\big(\|\mathcal{B}\|\chi^q\big)$$
$$+ \mathcal{H}\, \mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q) \int_0^{\chi} (\chi - t)^{q-1} \|v(t)\| dt.$$

Then, by the property of GI, we obtain

$$\|v(\chi)\| \leq (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\, \frac{\beta_u \chi^q}{q}\, \mathcal{E}_{q,q}\big(\|\mathcal{B}\|\chi^q\big)\mathcal{E}_q(\mathcal{H}\Gamma(q)\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\chi^q).$$
$$< \epsilon.$$

□

5. Extension to Impulsive Conditions

We extend the analysis of our stability results of Section 4 to the perturbed system.

Let us consider the following form of fractional dynamical system with impulsive perturbation:

$$\begin{aligned}&({}^{c}D^{q}v)(\chi) = \mathcal{B}v(\chi-h) + \mathcal{C}u(\chi) + \mathfrak{g}(\chi, v(\chi-h), u(\chi)), \chi \in \mathcal{J}_0,\\ &\Delta v(\chi_j) = D_j v(\chi_j),\ j = 1,2,\ldots,l,\\ &v(\chi) = \phi(\chi),\ -h \le \chi \le 0,\end{aligned} \tag{5}$$

where $\mathcal{J}_0 = \mathcal{J} - \{\chi_1, \chi_2, \ldots, \chi_l\}$. The function $D_j : \mathfrak{R}^n \to \mathfrak{R}^n$ is termed as the impulsive function and $v \in \mathbb{C}([-h, \mathcal{T}], \mathfrak{R}^n) \cup \mathbb{PC}[\mathcal{J}, \mathfrak{R}^n]$, such that $v(\chi_i^-) = v(\chi_i)$. All other system parameters of (5) are same as in (1).

By following through the procedure given in Lemma 3.1 [14], we can state the solution of system (5) as given below.

Lemma 10. *A solution $v(\chi) \in \mathbb{C}([-h, \mathcal{T}], \mathfrak{R}^n) \cup \mathbb{PC}[\mathcal{J}, \mathfrak{R}^n]$ of the impulsive system (5) is of the following form:*

$$\begin{aligned}v(\chi) = {}&\mathcal{E}_h^{\mathcal{B}\chi^q}\phi(-h) + \int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q}\phi'(t)dt + \int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\mathcal{C}u(t)dt\\ &+ \int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\mathfrak{g}(t, v(t-h), u(t))dt + \sum_{0<\chi_j<\chi} D_j(v(\chi_j))\mathcal{E}_{h,1}^{\mathcal{B}(\chi-\chi_j-h)^q},\ \chi \in (\chi_{j-1}, \chi_j].\end{aligned} \tag{6}$$

Theorem 6. *The impulsive system (5) is finite-time-stable with respect to $\{0, \mathcal{J}, h, \delta, \epsilon\}$ for $q \ge \frac{1}{2}$ if the following condition holds:*

$$\mathcal{Z}_1(\chi)\mathcal{E}_q(\mathcal{Z}_2(\chi)\Gamma(q)\chi^q) + \sum_{0<\chi_j<\chi} \sigma_{max}(D_j)\left\|v(\chi_j)\right\|\mathcal{E}_{q,1}(\|\mathcal{B}\|\chi^q) < \epsilon,$$

where $\mathcal{Z}_1(\chi) = (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\,\frac{\beta_u\chi^q}{q}\,\mathcal{E}_{q,q}\big(\|\mathcal{B}\|\chi^q\big)$ and $\mathcal{Z}_2(\chi) = \mathcal{H}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)$.

Proof.

$$\begin{aligned}\|v(\chi)\| \le {}&\left\|\mathcal{E}_h^{\mathcal{B}\chi^q}\right\|\|\phi(-h)\| + \left\|\int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q}\right\|\left\|\phi'(t)\right\|dt + \left\|\int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\right\|\|\mathcal{C}\|\|u(t)\|dt\\ &+ \left\|\int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\right\|\|\mathfrak{g}(t, v(t-h), u(t))\|dt + \sum_{0<\chi_j<\chi}\left\|D_j\right\|\left\|(v(\chi_j))\right\|\left\|\mathcal{E}_{h,1}^{\mathcal{B}(\chi-\chi_j-h)^q}\right\|.\end{aligned}$$

$$\begin{aligned}\|v(\chi)\| \le {}&(\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c}\beta_u)\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\int_0^{\chi}(\chi-t)^{q-1}dt + \mathcal{H}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\\ &\times \int_0^{\chi}(\chi-t)^{q-1}(\|v\| + \|u\|)dt + \sum_{0<\chi_j<\chi}\sigma_{max}(D_j)\left\|v(\chi_j)\right\|\mathcal{E}_{q,1}(\|\mathcal{B}\|\chi^q).\end{aligned}$$

$$\begin{aligned}\|v(\chi)\| \le {}&(\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\,\frac{\beta_u\chi^q}{q}\,\mathcal{E}_{q,q}\big(\|\mathcal{B}\|\chi^q\big) + \mathcal{H}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\\ &\times \int_0^{\chi}(\chi-t)^{q-1}\|v(t)\|dt + \sum_{0<\chi_j<\chi}\sigma_{max}(D_j)\left\|v(\chi_j)\right\|\mathcal{E}_{q,1}(\|\mathcal{B}\|\chi^q).\end{aligned}$$

Then, by GI, we obtain

$$\begin{aligned}\|v(\chi)\| \le {}&(\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\,\frac{\beta_u\chi^q}{q}\,\mathcal{E}_{q,q}\big(\|\mathcal{B}\|\chi q\big)\mathcal{E}_q(\mathcal{H}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\Gamma q\chi^q)\\ &+ \sum_{0<\chi_j<\chi}\sigma_{max}(D_j)\left\|v(\chi_j)\right\|\mathcal{E}_{q,1}(\|\mathcal{B}\|\chi^q).\end{aligned}$$

If $\mathcal{Z}_1(\chi) = (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\,\frac{\beta_u \chi^q}{q}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)$ and $\mathcal{Z}_2(\chi) = \mathcal{H}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)$,

$$\begin{aligned}\|v(\chi)\| &\leq \mathcal{Z}_1(\chi)\mathcal{E}_q(\mathcal{Z}_2(\chi)\Gamma q\chi^q) + \sum_{0<\chi_j<\chi} \sigma_{max}(D_j)\|v(\chi_j)\|\mathcal{E}_{q,1}(\|\mathcal{B}\|\chi^q),\\ &< \epsilon.\end{aligned}$$

□

Theorem 7. *One can state that the impulsive system (5) admits FTS with respect to* $\{0, \mathcal{J}, h, \delta, \epsilon\}$ *for* $q \geq \frac{1}{2}$ *if the following strict inequality is satisfied:*

$$\mathcal{Z}_1(\chi)\big(1 + \tilde{\mathcal{S}}\sigma_{max} D\mathcal{E}_q(\mathcal{Z}_2\Gamma(q)\chi^q)\big)^j\,\mathcal{E}_q(\mathcal{Z}_2\Gamma(q)\chi^q) < \epsilon,$$

where $\mathcal{Z}_1(\chi) = (\delta + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q)\tilde{\mathcal{S}} + \mathcal{S}(\tilde{c} + \mathcal{H})\frac{\beta_u \mathcal{T}^q}{q}$ *and* $\mathcal{Z}_2(\chi) = \mathcal{H}\mathcal{S}$.

Proof. On taking the norm to the solution $v(\chi)$, we obtain

$$\begin{aligned}\|v(\chi)\| \leq{}& \left\|\mathcal{E}_h^{\mathcal{B}\chi^q}\right\|\|\phi(-h)\| + \left\|\int_{-h}^{0} \mathcal{E}_h^{\mathcal{B}(\chi-h-t)^q}\right\|\left\|\phi'(t)\right\|dt + \left\|\int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\right\|\|\mathcal{C}\|\|u(t)\|dt\\ &+ \left\|\int_0^{\chi} \mathcal{E}_{h,q}^{\mathcal{B}(\chi-h-t)^q}\right\|\|\mathfrak{g}(t, v(t-h), u(t))\|dt + \sum_{0<\chi_j<\chi}\left\|D_j\right\|\left\|(v(\chi_j))\right\|\left\|\mathcal{E}_{h,1}^{\mathcal{B}(\chi-\chi_j-h)^q}\right\|.\end{aligned}$$

$$\begin{aligned}\|v(\chi)\| \leq{}& (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c}\beta_u)\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\int_0^{\chi}(\chi - t)^{q-1}dt + \mathcal{H}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\\ &\times \int_0^{\chi}(\chi - t)^{q-1}(\|v\| + \|u\|)dt + \sum_{0<\chi_j<\chi}\sigma_{max}(D_j)\left\|v(\chi_j)\right\|\mathcal{E}_{q,1}(\|\mathcal{B}\|\chi^q).\end{aligned}$$

$$\begin{aligned}\|v(\chi)\| \leq{}& (\|\phi(-h)\| + N)\mathcal{E}_q(\|\mathcal{B}\|\chi^q) + (\tilde{c} + \mathcal{H})\frac{\beta_u \chi^q}{q}\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q) + \mathcal{H}\,\mathcal{E}_{q,q}(\|\mathcal{B}\|\chi^q)\\ &\times \int_0^{\chi}(\chi - t)^{q-1}\|v\| + \sum_{0<\chi_j<\chi}\sigma_{max}(D_j)\|v(\chi_j)\|\mathcal{E}_{q,1}(\|\mathcal{B}\|\chi^q).\end{aligned}$$

If $\mathcal{E}_{q,q}(\|\mathcal{B}\|{\chi_f}^q) = \mathcal{S}$ and $\mathcal{E}_{q,1}(\|\mathcal{B}\|{\chi_f}^q) = \tilde{\mathcal{S}}$,

$$\begin{aligned}\|v(\chi)\| \leq{}& (\|\phi(-h)\| + N)\tilde{\mathcal{S}} + \mathcal{S}(\tilde{c} + \mathcal{H})\frac{\beta_u {\chi_f}^q}{q} + \mathcal{S}\,\mathcal{H}\int_0^{\chi}(\chi - t)^{q-1}\|v(t)\|dt\\ &+ \sum_{0<\chi_j<\chi}\sigma_{max}(D_j)\|v(\chi_j)\|\tilde{\mathcal{S}}.\end{aligned}$$

If $\mathcal{Z}_1(\chi) = (\delta + N)\tilde{\mathcal{S}} + \mathcal{S}(\tilde{c} + \mathcal{H})\frac{\beta_u {\chi_f}^q}{q}$ and $\mathcal{Z}_2(\chi) = \mathcal{H}\mathcal{S}$,

$$\|v(\chi)\| \leq \mathcal{Z}_1(\chi) + \mathcal{Z}_2(\chi)\int_0^{\chi}(\chi - t)^{q-1}\|v(t)\|dt + \tilde{\mathcal{S}}\sum_{0<\chi_j<\chi}\sigma_{max}(D_j)\|v(\chi_j)\|.$$

By Lemma 2.2 [7], we have

$$\begin{aligned}\|v(\chi)\| &\leq \mathcal{Z}_1(\chi)\big(1 + \tilde{\mathcal{S}}\sigma_{max} D\mathcal{E}_q(\mathcal{Z}_2(\chi)\Gamma(q)\chi^q)\big)^j\,\mathcal{E}_q(\mathcal{Z}_2(\chi)\Gamma(q)\chi^q),\\ &< \epsilon.\end{aligned}$$

□

Remark 1. *In this paper, it is important to note that the inequality in Lemma 6 holds only if the order of the fractional system $q \geq \frac{1}{2}$. Therefore, the results of Theorems 5–7 are applicable only to the fractional system with $q \geq \frac{1}{2}$.*

6. Examples

Example 1. *Consider the following differential system:*

$$({}^{c}D^{0.8}v)(\varkappa) = \mathcal{B}v(\varkappa - 0.2) + \mathcal{C}u(\varkappa) + \mathfrak{g}(\varkappa, v(\varkappa - 0.2), u(\varkappa)), \varkappa \in \mathcal{J} = [0, 0.4],$$
$$v(\varkappa) = [0.2, 0.1]^{\mathsf{T}}, -0.2 \leq \varkappa \leq 0, \tag{7}$$

where

$$\mathcal{B} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}, \ \mathcal{C} = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix}, \ u(\varkappa) = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix},$$

$$\mathfrak{g}(\varkappa, v(\varkappa - 0.2), u(\varkappa)) = \begin{pmatrix} \varkappa^2 \dfrac{|v_1(\varkappa - 0.2)|}{1 + |v_1(\varkappa - 0.2)|} + u_1(\varkappa) \\ \\ \varkappa^2 \dfrac{|v_2(\varkappa - 0.2)|}{1 + |v_2(\varkappa - 0.2)|} + u_2(\varkappa) \end{pmatrix}.$$

Further, using (2), the solution of (7) can be represented as below:

$$v(\varkappa) = \mathcal{E}_{0.2}^{\mathcal{B}\varkappa^{0.8}} \phi(-0.2) + \int_{0.2}^{0} \mathcal{E}_{0.2}^{\mathcal{B}(\varkappa - 0.2 - t)} \phi'(t) + \int_{0}^{\varkappa} \mathcal{E}_{0.2,0.8}^{(\varkappa - 0.2 - t)} \mathcal{C}u(t)dt$$
$$+ \int_{0}^{\varkappa} \mathcal{E}_{0.2,0.8}^{(\varkappa - 0.2 - t)} \mathfrak{g}(t, v(t - 0.2), u(t))dt.$$

On simplification, we obtain

$$\mathcal{E}_{0.2}^{\mathcal{B}\varkappa^{0.8}} \phi(-0.2) = \begin{cases} I, \quad -0.2 \leq \varkappa \leq 0, \\ \\ I + \mathcal{B}\dfrac{\varkappa^{0.8}}{\Gamma(1.8)} \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}, \quad 0 < \varkappa \leq 0.2, \\ \\ I + \mathcal{B}\dfrac{\varkappa^{0.8}}{\Gamma(1.8)} \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix} + \mathcal{B}^2 \dfrac{(\varkappa - 0.2)^{1.6}}{\Gamma(1.6)} \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}, \quad 0.2 < \varkappa \leq 0.4. \end{cases}$$

$$\mathcal{E}_{0.2,0.8}^{\mathcal{B}\varkappa^{0.8}} \begin{cases} I\dfrac{(0.2 + \varkappa)^{-0.2}}{\Gamma(0.8)}, \quad -0.2 < \varkappa \leq 0, \\ \\ I\dfrac{(0.2 + \varkappa)^{-0.2}}{\Gamma(0.8)} + \mathcal{B}\dfrac{\varkappa^{0.6}}{\Gamma(1.6)}, \quad 0 < \varkappa \leq 0.2, \\ I\dfrac{(0.2 + \varkappa)^{-0.2}}{\Gamma(0.8)} + \mathcal{B}\dfrac{\varkappa^{0.6}}{\Gamma(1.6)} + \mathcal{B}^2 \dfrac{(\varkappa - 0.2)^{1.4}}{\Gamma(2.4)}, \quad 0.2 < \varkappa \leq 0.4. \end{cases}$$

Using Mat-lab, we give the graphical representation of solution of system (7) as in Figure 1.

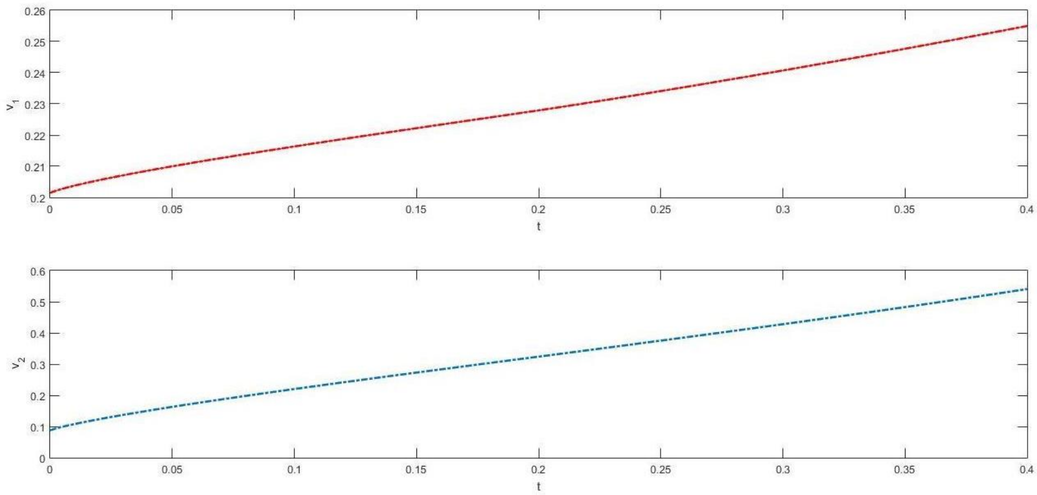

Figure 1. State response $v(t)$ of system (7).

Moreover, on simplification, $\|\mathfrak{g}(\chi, v, u_1) - \mathfrak{g}(\chi, z, u_2)\| \leq 2\chi^2\|v - z\| + \|u_1 - u_2\|, \forall v, z \in \mathfrak{R}^n$ *and* $u_1, u_2 \in \mathfrak{R}^m$. *If* $p_1 = p_2 = 2$, *we have* $N = \int_0^t \|\phi'(t)\| = 0$, $\|\phi\|_{\mathbb{C}} = 0.2$, $\tilde{c} = 0.4$, $\beta_u = 1$, $\eta = 0.74$, $\varpi(\chi) = 0.72$, $\mathcal{E}_{0.8}(\|\mathcal{B}\|\,\chi^{0.8}) = 1.08$, $\mathcal{E}_{0.8,0.8}(\|\mathcal{B}\|\chi^{0.8}) = 0.90$, $\sum_{r=1}^{m} \frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq+1)}(\chi - (r-1)h)^{rq} = 0.53$ *and* $\frac{\|\mathcal{B}\|^{r-1}}{\Gamma(rq)} \cdot \frac{(\chi - (r-1)h)^{rq-1+\frac{1}{p_2}}}{(p_2rq - p_2 + 1)^{\frac{1}{p_2}}} = 0.90$. *On evaluation, we obtain* $\mathcal{P}_1(\chi) = 0.9$, $\mathcal{P}_2(\chi) = 1.38$, $\mathcal{H} = 0.32$ *and* $Q = 1.04$. *For* $\delta = 0.21$, *we obtain Table 1 and Figure 2 represents the estimation of the state vectors.*

Table 1. FTS bounds of system (7) at $\mathcal{T} = 0.4$.

Theorem	$\|\phi\|_{\mathbb{C}}$	**q**	$\mathcal{T}$	**h**	δ	$\|v\|$	ϵ	**(FTS)**
4.1	0.2	0.8	0.4	0.2	0.21	3.053	3.10	yes
4.2	0.2	0.8	0.4	0.2	0.21	2.81	2.82 (optimal)	yes
4.3	0.2	0.8	0.4	0.2	0.21	10.64	10.65	yes

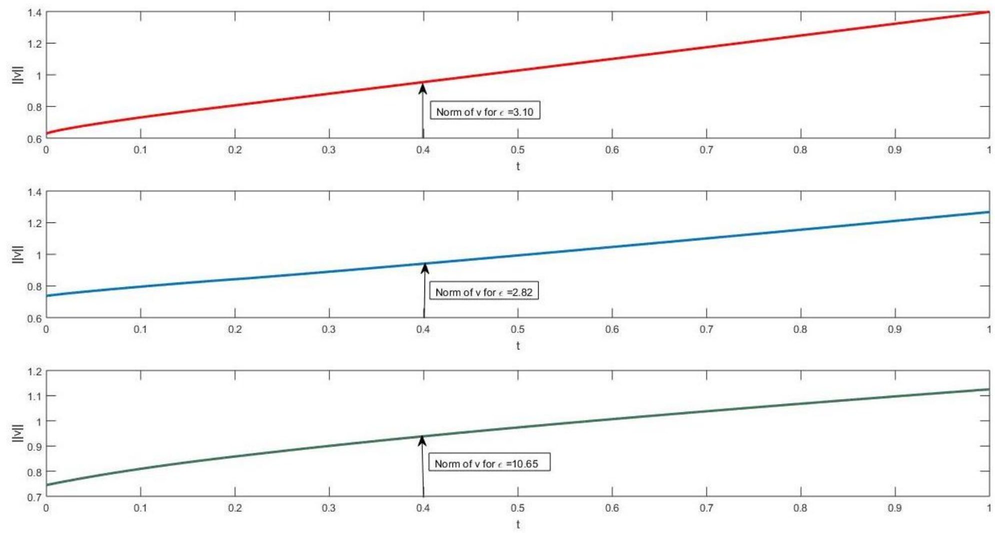

Figure 2. Estimation of the bound ϵ for $\|v\|$ of system (7) when $q = 0.8$, $\mathcal{T} = 0.4$.

Example 2. *Figure 3 represents the electric currents j_1, j_2 flowing through loops 1 and 2. Let e_1, e_2 be the source voltages for the corresponding loops and $\mathcal{L}_1, \mathcal{L}_2$ be the inductances connected to the resistors $\mathcal{R}_1$ and $\mathcal{R}_2$. CB_1 and CB_2 are the circuit breakers connected to loops 1 and 2, which results in a constant time delay h. Further, M_1 and M_2 serve as nonlinear sources, which are usually devices such as switches(generators), batteries, or any nonlinear form of resistors.*

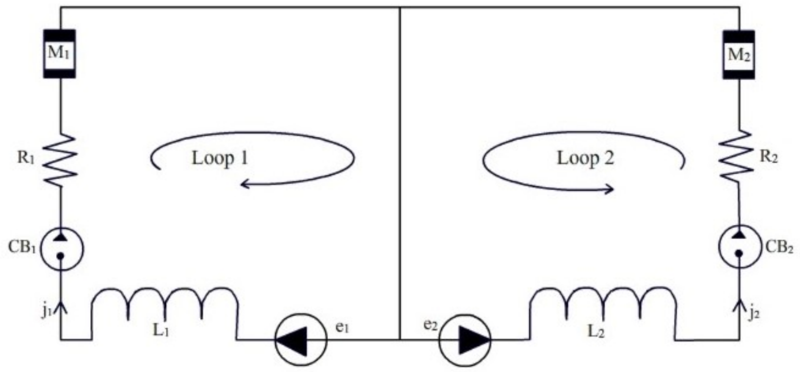

Figure 3. RL-type circuit system.

Applying the voltage law formulated by the physicist Kirchhoff, we obtain

$$e_1(t) = j_1(t-h)\mathcal{R}_1 + \frac{d^q j_1}{dt}\mathcal{L}_1 + M_1(\cdot),$$
$$e_2(t) = j_2(t-h)\mathcal{R}_2 + \frac{d^q j_2}{dt}\mathcal{L}_2 + M_2(\cdot).$$

If the circuit experiences a certain natural shock, which results in the jump of the state equations, then the impulsive model of the circuit system with initial currents $j_1(0)$ and $j_2(0)$ is stated below:

$$\begin{aligned} &({}^c D^q j)(t) = \mathcal{B}j(t-h) + \mathcal{C}e(t) + \mathfrak{g}(t, j(t-h), e(t)),\ t \in \mathcal{J}_0, \\ &\Delta j(t_i) = D_i j(t_i),\ i = 1, 2, \ldots, l, \\ &j(t) = [t, 2t]^T,\ -h \le t \le 0, \end{aligned}$$

with

$$\mathcal{B} = \begin{pmatrix} -\frac{\mathcal{R}_1}{\mathcal{L}_1} & 0 \\ 0 & -\frac{\mathcal{R}_2}{\mathcal{L}_2} \end{pmatrix},\ \mathcal{C} = \begin{pmatrix} \frac{1}{\mathcal{L}_1} & 0 \\ 0 & \frac{1}{\mathcal{L}_2} \end{pmatrix},\ \mathfrak{g}(t, j(t-h), e(t)) = \begin{pmatrix} -\frac{M_1(\cdot)}{\mathcal{L}_1} \\ -\frac{M_2(\cdot)}{\mathcal{L}_2} \end{pmatrix},$$

and

$$j(t) = \begin{pmatrix} j_1(t) \\ j_2(t) \end{pmatrix},\ e(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix}.$$

If $\mathcal{R}_1 = 0.3, \mathcal{R}_2 = 0.4$, $\mathcal{L}_1 = \mathcal{L}_2 = 1$. Let $e = u$. For $j = v$, $\mathcal{J}_0 = [0, 0.4]/\{0.1, 0.3\}$, we have

$$\mathcal{B} = \begin{pmatrix} -0.3 & 0 \\ 0 & -0.4 \end{pmatrix}, \mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}, D_i = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \mathfrak{g} = \begin{pmatrix} t^2 \frac{|v_1(t-0.1)|}{1+|v_1(t-0.1)|} \\ t^2 \frac{|v_2(t-0.1)|}{1+|v_2(t-0.1)|} \end{pmatrix}.$$

Choosing $q = 0.8$, we obtain

$\|\mathfrak{g}(t, y_1) - \mathfrak{g}(t, y_2) \le 2t^2 \|y_1 - y_2\|$ *and* $\|\mathfrak{g}(t, y)\| \le 2t^2$, $\forall t \in \mathcal{J}$. $\|\phi\| = 0.5$, $N = \int_{-0.1}^{0} \|\phi'(t\|$, $\mathcal{S} = 1.10, \tilde{\mathcal{S}} = 1.23$, $\mathcal{Z}_1 = 1.312, \mathcal{Z}_2 = 0.352$. *Then, on choosing $\delta = 0.6$, a direct application of Theorem 7 yields $\epsilon = 2.96$, for which the system is finite-time-stable.*

7. Conclusions

This study presents a systematic investigation on the bounds required for a fractional-order dynamical system to admit FTS. The required conditions are derived using the standard form of integral inequality of the GI type, along with the application of DPMML functions. The analytical discussion is extended to a system with impulsive perturbations of instantaneous form. Further, in future, the analysis can be enhanced to interpret the system with nonlinear disturbances and perturbations of noninstantaneous types. Moreover, we will focus on developing more comprehensive and efficient methods for analysing finite-time stability. This could involve developing novel mathematical techniques to analyse the finite-time stability of fractional switched systems modelling electrical circuits.

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