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Solvability for the ψ-Caputo-Type Fractional Differential System with the Generalized p-Laplacian Operator

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Abstract: In this article, by combining a recent critical point theorem and several theories of the ψ-Caputo fractional operator, the multiplicity results of at least three distinct weak solutions are obtained for a new ψ-Caputo-type fractional differential system including the generalized p-Laplacian operator. It is noted that the nonlinear functions do not need to adapt certain asymptotic conditions in the paper, but, instead, are replaced by some simple algebraic conditions. Moreover, an evaluation criterion of the equation without solutions is also provided. Finally, two examples are given to demonstrate that the ψ-Caputo fractional operator is more accurate and can adapt to deal with complex system modeling problems by changing different weight functions.

Keywords: fractional differential equation; ψ-Caputo fractional derivative; critical point theorem

MSC: 26A33; 34B15; 34A08

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1. Introduction

As a popular research object in recent years, fractional differential equations (FDEs) play an important role in modeling many practical problems of science and engineering, such as fluid flow, anomalous diffusion, viscoelastic mechanics, epidemiology, etc. (see [1–6]). There are various definitions of fractional integration and differentiation, including the most widely used classical definitions of Riemann–Liouville, Caputo, Hadamard and others (see [7–10]). Currently, these classical definitions are employed in many fields, such as fractional boundary and initial value problems (see [11–15]). In order to overcome the inconvenience arising from a large number of definitions, Kilbas et al. advanced a new and more general form, called the ψ-Caputo-type fractional derivative (cf. [7]). By drawing into the weight function $\psi(t)$, different definitional forms of fractional calculus were generalized and unified into a whole expression. In 2017, Almeida [16] investigated the relevant properties of the new operator and provided a theoretical basis for studying ψ-Caputo-type FDEs in depth.

When the weight function $\psi(t)$ is specified as certain functions, the ψ-Caputo fractional derivative can be degenerated into certain classical functions. Therefore, based on ψ-Caputo fractional integration and differentiation, the modeling accuracy of practical problems is greatly improved. Most recently, some existence results for ψ-Caputo FDEs were achieved by applying fixed-point theorems in topological methods (see [17–20]). For instance, ref. [18] considered the solvability of the ψ-Caputo-type FDE by taking advantage of a novel fixed-point theorem. In [19], the authors derived the existence and uniqueness of solutions for a ψ-Caputo fractional initial value problem by applying some standard fixed-point theorems.

However, so far as is known to the authors, there are few studies which have focused on solvability for ψ-Caputo FDEs based on variational methods and critical point theory.

In light of this point, in this paper, we consider a new ψ-Caputo-type fractional differential system, including the generalized p-Laplacian operator.

$$\begin{cases} {}^C D_{b^-}^{\alpha,\psi}(\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t))) + |z(t)|^{p-2}z(t) = \xi g(t,z(t)) + \lambda f(t,z(t)), t \in [a,b], \\ z(a) = z(b) = 0, \end{cases} \tag{1}$$

where $0 < \alpha \leq 1$, $0 \leq a < b < +\infty$, $\lambda > 0$, $\xi \geq 0$, $1 < p < \infty$, and the right and left α-order ψ-Caputo fractional derivatives are ${}^C D_{b^-}^{\alpha,\psi}$ and ${}^C D_{a^+}^{\alpha,\psi}$. The weight function $\psi(t) \in C^1[a,b]$ increases with $\psi'(t) \neq 0$ for all $t \in [a,b]$; the p-Laplacian operator is defined by $\Phi_p(s) = |s|^{p-2}s (s \neq 0)$ with $\Phi_p(0) = 0$, $f, g \in C([a,b] \times \mathbb{R}, \mathbb{R})$ and satisfying $f(t,0) = g(t,0) = 0$ for every $t \in [a,b]$.

What is particularly noteworthy is that the nonlinear functions f and g in this article do not need to adapt certain asymptotic conditions; we can acquire the multiplicity of at least three distinct solutions only by imposing algebraic conditions on the nonlinearities. This work is a generalization of several results reported in the literature which are concerned with classical fractional operators.

2. Fractional Calculus and Critical Point Theorem

In this section, we present the definitions of some kinds of fractional integrals and differentials, as well as related properties, and one effective critical point theorem.

Definition 1 ([7,16]). *Let $-\infty < a < b < +\infty$, $t \in [a,b]$, $z(t)$ is integrable, $\psi(t) \in C^1[a,b]$ is increasing with $\psi'(t) \neq 0$ for all $t \in [a,b]$. The left and right ψ-Riemann–Liouville fractional integrals of a function z are defined, respectively, by*

$$I_{a^+}^{\alpha,\psi} z(t) = \frac{1}{\Gamma(\alpha)} \int_a^t z(\varsigma)(\psi(t) - \psi(\varsigma))^{\alpha-1} \psi'(\varsigma) d\varsigma, \forall \alpha > 0,$$
$$I_{b^-}^{\alpha,\psi} z(t) = \frac{1}{\Gamma(\alpha)} \int_t^b z(\varsigma)(\psi(\varsigma) - \psi(t))^{\alpha-1} \psi'(\varsigma) d\varsigma, \forall \alpha > 0.$$

Let $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$. The left and right ψ-Riemann–Liouville fractional derivatives of a function z are, respectively, defined by

$$\begin{aligned} D_{a^+}^{\alpha,\psi} z(t) =& \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\psi} z(t) \\ =& \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t z(\varsigma)(\psi(t) - \psi(\varsigma))^{n-\alpha-1} \psi'(\varsigma) d\varsigma, \\ D_{b^-}^{\alpha,\psi} z(t) =& \left(\frac{-1}{\psi'(t)} \frac{d}{dt}\right)^n I_{b^-}^{n-\alpha,\psi} z(t) \\ =& \frac{1}{\Gamma(n-\alpha)} \left(-\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_t^b z(\varsigma)(\psi(\varsigma) - \psi(t))^{n-\alpha-1} \psi'(\varsigma) d\varsigma. \end{aligned}$$

Especially, for $0 < \alpha < 1$,

$$\begin{aligned} D_{a^+}^{\alpha,\psi} z(t) =& \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) I_{a^+}^{1-\alpha,\psi} z(t) \\ =& \frac{1}{\Gamma(1-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right) \int_a^t z(\varsigma)(\psi(t) - \psi(\varsigma))^{-\alpha} \psi'(\varsigma) d\varsigma, \end{aligned} \tag{2}$$

$$\begin{aligned} D_{b^-}^{\alpha,\psi} z(t) =& \left(\frac{-1}{\psi'(t)} \frac{d}{dt}\right) I_{b^-}^{1-\alpha,\psi} z(t) \\ =& \frac{1}{\Gamma(1-\alpha)} \left(\frac{-1}{\psi'(t)} \frac{d}{dt}\right) \int_t^b z(\varsigma)(\psi(\varsigma) - \psi(t))^{-\alpha} \psi'(\varsigma) d\varsigma. \end{aligned} \tag{3}$$

Obviously, the classical Riemann–Liouville fractional derivative can be acquired by choosing the weight function $\psi(t) = t$.

Definition 2 ([7,16])**.** *Let* $-\infty < a < b < +\infty$, $z(t), \psi(t) \in C^n[a,b]$, *such that* ψ *is increasing and* $\psi'(t) \neq 0$. *Define the left and right* ψ*-Caputo fractional derivatives of a function* z *by*

$$\begin{aligned}
{}^C D_{a^+}^{\alpha,\psi} z(t) =& I_{a^+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n z(t) \\
=& \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(\varsigma))^{n-\alpha-1} \psi'(\varsigma) \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma} \right)^n z(\varsigma) d\varsigma, \forall \alpha > 0, \\
{}^C D_{b^-}^{\alpha,\psi} z(t) =& I_{b^-}^{n-\alpha,\psi} \left(-\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n z(t) \\
=& \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\psi(\varsigma) - \psi(t))^{n-\alpha-1} \psi'(\varsigma) \left(\frac{1}{\psi'(\varsigma)} \frac{d}{d\varsigma} \right)^n u(\varsigma) d\varsigma, \forall \alpha > 0,
\end{aligned}$$

where $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}$, $n = \alpha$ *for* $\alpha \in \mathbb{N}$. *Especially, for* $0 < \alpha < 1$,

$${}^C D_{a^+}^{\alpha,\psi} z(t) = I_{a^+}^{1-\alpha,\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) z(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t z'(\varsigma)(\psi(t) - \psi(\varsigma))^{-\alpha} d\varsigma, \quad (4)$$

$${}^C D_{b^-}^{\alpha,\psi} z(t) = I_{b^-}^{1-\alpha,\psi} \left(-\frac{1}{\psi'(t)} \frac{d}{dt} \right) z(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b z'(\varsigma)(\psi(\varsigma) - \psi(t))^{-\alpha} d\varsigma. \quad (5)$$

Obviously, the classical Caputo fractional derivative can be acquired by choosing the weight function $\psi(t) = t$.

Property 1 ([16])**.** *If* $z(t) \in C^n[a,b]$, $-\infty < a < b < +\infty$, *we have*

$$\begin{aligned}
{}^C D_{a^+}^{\alpha,\psi} z(t) &= D_{a^+}^{\alpha,\psi} \left[z(t) - \Sigma_{k=0}^{n-1} \frac{1}{k!} (\psi(t) - \psi(a))^k \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^k z(a) \right], \forall \alpha > 0, \\
{}^C D_{b^-}^{\alpha,\psi} z(t) &= D_{b^-}^{\alpha,\psi} \left[z(t) - \Sigma_{k=0}^{n-1} \frac{(-1)^k}{k!} (\psi(b) - \psi(t))^k \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^k z(b) \right], \forall \alpha > 0,
\end{aligned}$$

where $n = [\alpha] + 1$ *for* $\alpha \notin \mathbb{N}$, $n = \alpha$ *for* $\alpha \in \mathbb{N}$.

This paper deals mainly with the Caputo-type fractional derivative with the weight function ψ. In what follows, an important and proper fractional derivative space is defined, which is crucial for the system (1) to establish a variational structure.

Definition 3. *Let* $\frac{1}{p} < \alpha \leq 1$, $1 < p < \infty$. *Define the* ψ*-Caputo fractional derivative space* $H_{(\alpha,\psi,p)}$ *by the closure of* $C_0^\infty([a,b], \mathbb{R})$ *with weighted norm*

$$\|z\|_{(\alpha,\psi,p)} := \left(\int_a^b | z(t) |^p \, dt + \int_a^b \psi'(t) \, | {}^C D_{a^+}^{\alpha,\psi} z(t) |^p \, dt \right)^{\frac{1}{p}}. \quad (6)$$

Apparently, $H_{(\alpha,\psi,p)}$ *is the space of* $z(t) \in L^p[a,b]$ *with an* α *order* ψ*-Caputo fractional derivative* ${}^C D_{a^+}^{\alpha,\psi} z(t) \in L^p[a,b]$ *and* $z(a) = z(b) = 0$. *The Banach space* $H_{(\alpha,\psi,p)}$ *is separable and reflexive, cf.* [21].

Lemma 1. *For any* $0 < \alpha \leq 1$, *we have*

$${}^C D_{a^+}^{\alpha,\psi} z(t) = D_{a^+}^{\alpha,\psi} z(t), \; {}^C D_{b^-}^{\alpha,\psi} z(t) = D_{b^-}^{\alpha,\psi} z(t), \forall z(t) \in H_{(\alpha,\psi,p)}.$$

Proof. Due to Property 1 and $z(a) = z(b) = 0$, we can obtain the desired conclusion directly. □

Lemma 2 ([21]). *For $1 \leq p < \infty$, $0 < \alpha < 1$ and $z \in L^p([a,b],\mathbb{R})$, we have*

$$\| I_{a^+}^{\alpha,\psi} z \|_{L^p[a,t]} \leq \frac{[\psi(t)]^\alpha \max_{a\leq t\leq b}\{\psi'(t)\}}{\Gamma(\alpha+1)} \| z \|_{L^p[a,t]},$$

for all $t \in [a,b]$.

Lemma 3 ([16]). *Let function $z(t) \in C^n[a,b]$ and $\alpha > 0$, then*

$$I_{a^+}^{\alpha,\psi C}D_{a^+}^{\alpha,\psi} z(t) = z(t) - \sum_{k=0}^{n-1} \frac{z_\psi^{[k]}(a)}{k!}(\psi(t)-\psi(a))^k,$$
$$I_{b^-}^{\alpha,\psi C}D_{b^-}^{\alpha,\psi} z(t) = z(t) - \sum_{k=0}^{n-1} (-1)^k \frac{z_\psi^{[k]}(b)}{k!}(\psi(b)-\psi(t))^k,$$

where $z_\psi^{[k]}(t) := \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^k z(t)$. Especially, $I_{a^+}^{\alpha,\psi C}D_{a^+}^{\alpha,\psi} z(t) = z(t) - z(a)$, $I_{b^-}^{\alpha,\psi C}D_{b^-}^{\alpha,\psi} z(t) = z(t) - z(b)$, for $0 < \alpha < 1$.

Lemma 4. *Let $1 < p < \infty$, $\frac{1}{p} < \alpha \leq 1$. For any $z(t) \in H_{(\alpha,\psi,p)}$, we have*

$$\|z\|_{L^p} \leq \frac{[\psi(b)]^\alpha \max_{a\leq t\leq b}\{\psi'(t)\}}{\Gamma(\alpha+1)} \left(\int_a^b |\, {}^C D_{a^+}^{\alpha,\psi} z(t)\, |^p \, dt \right)^{\frac{1}{p}}. \tag{7}$$

Additionally, if $\frac{1}{p}+\frac{1}{q} = 1$, then

$$\|z\|_\infty \leq \frac{(\psi(b)-\psi(a))^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} \left(\int_a^b \psi'(t)\, |\, {}^C D_{a^+}^{\alpha,\psi} z(t)\, |^p \, dt \right)^{\frac{1}{p}}. \tag{8}$$

Denote

$$\widetilde{L} = \frac{(\psi(b)-\psi(a))^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}},\ \widehat{L} = \frac{[\psi(b)]^\alpha \max_{a\leq t\leq b}\{\psi'(t)\}}{\Gamma(\alpha+1)}. \tag{9}$$

Proof. For any $z(t) \in H_{(\alpha,\psi,p)}$ with $z(a) = z(b) = 0$, using the Hölder inequality and Lemma 3, yields

$$\begin{aligned}
|z(t)| =& |I_{a^+}^{\alpha,\psi C}D_{a^+}^{\alpha,\psi} z(t)| = \frac{1}{\Gamma(\alpha)}\left| \int_a^t {}^C D_{a^+}^{\alpha,\psi} z(\varsigma)(\psi(t)-\psi(\varsigma))^{\alpha-1}\psi'(\varsigma)d\varsigma \right| \\
\leq& \frac{1}{\Gamma(\alpha)}\left(\int_a^b \left|(\psi(t)-\psi(\varsigma))^{\alpha-1}(\psi'(\varsigma))^{\frac{1}{q}}\right|^q d\varsigma \right)^{\frac{1}{q}} \left(\int_a^b \left|{}^C D_{a^+}^{\alpha,\psi} z(\varsigma)(\psi'(\varsigma))^{\frac{1}{p}}\right|^p d\varsigma \right)^{\frac{1}{p}} \\
\leq& \frac{(\psi(b)-\psi(a))^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)(q(\alpha-1)+1)^{\frac{1}{q}}} \left(\int_a^b \psi'(t)|{}^C D_{a^+}^{\alpha,\psi} z(t)|^p dt \right)^{\frac{1}{p}}.
\end{aligned}$$

Uniting Lemmas 2 and 3, we can obtain the inequality (7) instantly. □

Based on the inequality (7), we can observe that the norm (6) and norm $\|z\|_{(\alpha,\psi,p)} := \left(\int_a^b \psi'(t)\, |\, {}^C D_{a^+}^{\alpha,\psi} z(t)\, |^p \, dt \right)^{\frac{1}{p}}$ are equal in form.

Lemma 5 ([21])**.** *Let $\frac{1}{p} < \alpha \leq 1$. Suppose that any sequence $\{z_k\}$ converges to z in $H_{(\alpha,\psi,p)}$ weakly. Then, $z_k \to z$ in $C[a,b]$ as $k \to \infty$, i.e., $\|z_k - z\|_\infty \to 0$ as $k \to \infty$.*

Lemma 6. *We mean by a weak solution of the system (1), for any $z(t) \in H_{(\alpha,\psi,p)}$, such that*

$$\begin{aligned} &\int_a^b \psi'(t)\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t))^C D_{a^+}^{\alpha,\psi} y(t) + \Phi_p(z(t))y(t)\psi'(t)dt \\ &= \xi \int_a^b \psi'(t)y(t)g(t,z(t))dt + \lambda \int_a^b \psi'(t)y(t)f(t,z(t))dt, \end{aligned} \tag{10}$$

for any $y(t) \in H_{(\alpha,\psi,p)}$.

Proof. Taking advantage of (3), (4), and the Dirichlet boundary value in system (1), we obtain

$$\begin{aligned} &\int_a^b \Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t))^C D_{a^+}^{\alpha,\psi} y(t)\psi'(t)dt \\ =&\frac{1}{\Gamma(1-\alpha)} \int_a^b \int_a^t \Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t))(\psi(t)-\psi(\varsigma))^{-\alpha} y'(\varsigma)\psi'(t)d\varsigma dt \\ =&\frac{1}{\Gamma(1-\alpha)} \int_a^b \left[\int_t^b \psi'(\varsigma)^C D_{a^+}^{\alpha,\psi} z(\varsigma)(\psi(\varsigma)-\psi(t))^{-\alpha} d\varsigma\right] y'(t)dt \\ =&\frac{1}{\Gamma(1-\alpha)} \left[\int_t^b \psi'(\varsigma)\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(\varsigma))(\psi(\varsigma)-\psi(t))^{-\alpha} d\varsigma\right] y(t) \,\Big|_{t=a}^{t=b} \\ &-\frac{1}{\Gamma(1-\alpha)} \int_a^b \frac{d}{dt}\left[\int_t^b \psi'(\varsigma)\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(\varsigma))(\psi(\varsigma)-\psi(t))^{-\alpha} d\varsigma\right] y(t)dt \\ =&-\frac{1}{\Gamma(1-\alpha)} \int_a^b y(t)\psi'(t)\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)\int_t^b \psi'(\varsigma)\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(\varsigma))(\psi(\varsigma)-\psi(t))^{-\alpha} d\varsigma dt \\ =&\int_a^b D_{b^-}^{\alpha,\psi}(\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t)))\psi'(t)y(t)dt. \end{aligned}$$

Hence, owing to Lemma 1, we get

$$\int_a^b \psi'(t)^C D_{a^+}^{\alpha,\psi} y(t)\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t))dt = \int_a^b y(t)\psi'(t)^C D_{b^-}^{\alpha,\psi}(\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t)))dt. \tag{11}$$

At this point, we multiply both sides of system (1) by $\psi'(t)y(t)$, and then integrate both ends from a to b simultaneously. Following (11), we can obtain the relationship (10). □

Next, we recall an interesting and useful three critical points theorem provided by Bonanno and Candito. This theorem provides the critical theory technology to obtain the multiplicity results for system (1) in our work.

Let H be a nonempty set, and $\Phi, \Psi : H \to \mathbb{R}$ be two functions. For any $\rho, \rho_1, \rho_2 > \inf_H \Phi$, $\rho_2 > \rho_1, \rho_3 > 0$, we define

$$\begin{aligned} \mathcal{A}(\rho) &:= \inf_{z \in \Phi^{-1}(-\infty,\rho)} \frac{\sup_{y \in \Phi^{-1}(-\infty,\rho)} \Psi(y) - \Psi(z)}{\rho - \Phi(z)}, \\ \mathcal{B}(\rho_1,\rho_2) &:= \inf_{z \in \Phi^{-1}(-\infty,\rho_1)} \sup_{y \in \Phi^{-1}(\rho_1,\rho_2)} \frac{\Psi(y) - \Psi(z)}{\Phi(y) - \Phi(z)}, \\ \mathcal{D}(\rho_2,\rho_3) &:= \frac{\sup_{z \in \Phi^{-1}(-\infty,\rho_2+\rho_3)} \Psi(z)}{\rho_3}, \\ \mathcal{G}(\rho_1,\rho_2,\rho_3) &:= \max\{\mathcal{A}(\rho_1), \mathcal{A}(\rho_2), \mathcal{D}(\rho_2,\rho_3)\}. \end{aligned}$$

Theorem 1 ([22], Theorem 3.3)**.** *Let H be a reflexive real Banach space, and $\Phi : H \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on H^* where H^* is the dual space of H. Let $\Psi : H \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

(I) $\inf_H \Phi = \Phi(0) = \Psi(0) = 0$;

(II) *For any $z_1, z_2 \in H$, such that $\Psi(z_1) \geq 0$ and $\Psi(z_2) \geq 0$, one has*

$$\inf_{0 \leq r \leq 1} \Psi(rz_1 + (1-r)z_2) \geq 0.$$

Assume that there are three positive constants ρ_1, ρ_2, ρ_3 with $\rho_1 < \rho_2$, $\rho_3 > 0$, such that

(III) $\mathcal{A}(\rho_1) < \mathcal{B}(\rho_1, \rho_2)$;

(IV) $\mathcal{A}(\rho_2) < \mathcal{B}(\rho_1, \rho_2)$;

(V) $\mathcal{D}(\rho_2, \rho_3) < \mathcal{B}(\rho_1, \rho_2)$.

Then, for each $\lambda \in]\frac{1}{\mathcal{B}(\rho_1,\rho_2)}, \frac{1}{\mathcal{G}(\rho_1,\rho_2,\rho_3)}[$, the functional $\Phi - \lambda\Psi$ exists at three distinct critical points z_1, z_2, z_3, such that $z_1 \in \Phi^{-1}(-\infty, \rho_1)$, $z_2 \in \Phi^{-1}(\rho_1, \rho_2)$ and $z_3 \in \Phi^{-1}(-\infty, \rho_2 + \rho_3)$.

3. Multiplicity Results

Denote $F(t,z) = \int_a^z f(t,\varsigma)d\varsigma$ and $G(t,z) = \int_a^z g(t,\varsigma)d\varsigma$. Firstly, we consider the functionals $\mathcal{F}_1, \mathcal{F}_2 : H_{(\alpha,\psi,p)} \to \mathbb{R}$ with

$$\mathcal{F}_1(z) := \frac{1}{p}\int_a^b |{}^C D_{a^+}^{\alpha,\psi} z(t)|^p \psi'(t)dt + \frac{1}{p}\int_a^b |z(t)|^p \psi'(t)dt, \tag{12}$$

$$\mathcal{F}_2(z) := \int_a^b \psi'(t)F(t,z(t))dt + \frac{\xi}{\lambda}\int_a^b \psi'(t)G(t,z(t))dt. \tag{13}$$

Obviously, $\mathcal{F}_1, \mathcal{F}_2 \in C^1(H_{(\alpha,\psi,p)}, \mathbb{R})$ and

$$\mathcal{F}_1'(z)(y) = \int_a^b \psi'(t)\Phi_p({}^C D_{a^+}^{\alpha,\psi} z(t))\, {}^C D_{a^+}^{\alpha,\psi} y(t)dt + \int_a^b \psi'(t)y(t)\Phi_p(z(t))dt, \tag{14}$$

$$\mathcal{F}_2'(z)(y) = \int_a^b f(t,z(t))\psi'(t)y(t)dt + \frac{\xi}{\lambda}\int_a^b g(t,z(t))\psi'(t)y(t)dt, \tag{15}$$

for any $z(t), y(t) \in H_{(\alpha,\psi,p)}$.

Define $\mathcal{F} = \mathcal{F}_1 - \lambda\mathcal{F}_2$. It is not difficult to see that the critical point of the functional $\mathcal{F}$ is consistent with the weak solution of system (1).

Lemma 7. *The functional $\mathcal{F}_1$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on $H^*_{(\alpha,\psi,p)}$.*

Proof. In fact, consider the inequality in Lemma 4.2 of [23]

$$(|a_1|^{p-2}a_1 - |a_2|^{p-2}a_2)(a_1 - a_2) \geq \begin{cases} |a_1 - a_2|^p, & p \geq 2, \\ \frac{|a_1-a_2|^2}{(|a_1|+|a_2|)^{2-p}}, & 1 < p \leq 2, \end{cases} \tag{16}$$

for any $a_1, a_2 \in \mathbb{R}$. For $p \geq 2$, according to (16), we have

$$\begin{aligned} &\int_a^b \psi'(t)\Big(\Phi_p({}^C D_{a^+}^{\alpha,\psi} z_1(t)) - \Phi_p({}^C D_{a^+}^{\alpha,\psi} z_2(t))\Big){}^C D_{a^+}^{\alpha,\psi}(z_1(t) - z_2(t))dt \\ \geq &\int_a^b |{}^C D_{a^+}^{\alpha,\psi}(z_1(t) - z_2(t))|^p \psi'(t)dt = \|z_1 - z_2\|^p_{(\alpha,\psi,p)}, \end{aligned} \tag{17}$$

and

$$\int_a^b (z_1(t) - z_2(t))\psi'(t)\Big(\Phi_p(z_1(t)) - \Phi_p(z_2(t))\Big)dt \geq \int_a^b \mid z_1(t) - z_2(t) \mid^p \psi'(t)dt. \quad (18)$$

For $1 < p \leq 2$, from the Hölder inequality, this yields

$$\begin{aligned}
&\int_a^b \mid {}^C D_{a^+}^{\alpha,\psi} z_1(t) - {}^C D_{a^+}^{\alpha,\psi} z_2(t) \mid^p \psi'(t)dt \\
\leq &\left(\int_a^b \frac{\mid {}^C D_{a^+}^{\alpha,\psi} z_1(t) - {}^C D_{a^+}^{\alpha,\psi} z_2(t) \mid^2}{(\mid {}^C D_{a^+}^{\alpha,\psi} z_1(t) \mid + \mid {}^C D_{a^+}^{\alpha,\psi} z_2(t) \mid)^{2-p}} \psi'(t)dt\right)^{\frac{p}{2}} \times \\
&\left(\int_a^b (|{}^C D_{a^+}^{\alpha,\psi} z_1(t)| + |{}^C D_{a^+}^{\alpha,\psi} z_2(t)|)^p \psi'(t)dt\right)^{\frac{2-p}{2}} \\
\leq &2^{\frac{p(2-p)}{2}} (\|z_1\|^p_{(\alpha,\psi,p)} + \|z_2\|^p_{(\alpha,\psi,p)})^{\frac{2-p}{2}} \left(\int_a^b \frac{\mid {}^C D_{a^+}^{\alpha,\psi} z_1(t) - {}^C D_{a^+}^{\alpha,\psi} z_2(t) \mid^2}{(\mid {}^C D_{a^+}^{\alpha,\psi} z_1(t) \mid + \mid {}^C D_{a^+}^{\alpha,\psi} z_2(t) \mid)^{2-p}} \psi'(t)dt\right)^{\frac{p}{2}},
\end{aligned}$$

Then, by means of (16), we derive

$$\begin{aligned}
&\int_a^b \psi'(t)\Big(\Phi_p({}^C D_{a^+}^{\alpha,\psi} z_1(t)) - \Phi_p({}^C D_{a^+}^{\alpha,\psi} z_2(t))\Big){}^C D_{a^+}^{\alpha,\psi}(z_1(t) - z_2(t))dt \quad (19)\\
\geq &\int_a^b \psi'(t)\frac{\mid {}^C D_{a^+}^{\alpha,\psi} z_1(t) - {}^C D_{a^+}^{\alpha,\psi} z_2(t) \mid^2}{(\mid {}^C D_{a^+}^{\alpha,\psi} z_1(t) \mid + \mid {}^C D_{a^+}^{\alpha,\psi} z_2(t) \mid)^{2-p}} dt \\
\geq &2^{\frac{p(p-2)}{2}} \|z_1 - z_2\|^2_{(\alpha,\psi,p)} (\|z_1\|^p_{(\alpha,\psi,p)} + \|z_2\|^p_{(\alpha,\psi,p)})^{\frac{p-2}{p}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_a^b \psi'(t)\Big(\Phi_p(z_1(t)) - \Phi_p(z_2(t))\Big)(z_1(t) - z_2(t))dt \quad (20)\\
\geq &\left(\int_a^b \psi'(t) \mid z_1(t) - z_2(t) \mid^p dt\right)^{\frac{2}{p}} \left(\int_a^b \psi'(t)(|z_1(t)| + |z_2(t)|)^p dt\right)^{\frac{p-2}{p}}.
\end{aligned}$$

Consequently, owing to (17)–(20), for all $1 < p < \infty$, we have

$$\begin{aligned}
&\int_a^b \psi'(t)\Big(\Phi_p({}^C D_{a^+}^{\alpha,\psi} z_1(t)) - \Phi_p({}^C D_{a^+}^{\alpha,\psi} z_2(t))\Big){}^C D_{a^+}^{\alpha,\psi}(z_1(t) - z_2(t))dt \quad (21)\\
+ &\int_a^b \psi'(t)\Big(\Phi_p(z_1(t)) - \Phi_p(z_2(t))\Big)(z_1(t) - z_2(t))dt \geq 0.
\end{aligned}$$

Hence, combining (14) with (21), we obtain

$$\begin{aligned}
&(\mathcal{F}_1'(z_1) - \mathcal{F}_1'(z_2))(z_1 - z_2) \\
= &\int_a^b \Phi_p({}^C D_{a^+}^{\alpha,\psi} z_1(t))\psi'(t){}^C D_{a^+}^{\alpha,\psi}(z_1(t) - z_2(t))dt + \int_a^b \Phi_p(z_1(t))\psi'(t)(z_1(t) - z_2(t))dt \\
&- \int_a^b \Phi_p({}^C D_{a^+}^{\alpha,\psi} z_2(t))\psi'(t){}^C D_{a^+}^{\alpha,\psi}(z_1(t) - z_2(t))dt - \int_a^b \Phi_p(z_2(t))\psi'(t)(z_1(t) - z_2(t))dt \\
= &\int_a^b \Big(\Phi_p({}^C D_{a^+}^{\alpha,\psi} z_1(t)) - \Phi_p({}^C D_{a^+}^{\alpha,\psi} z_2(t))\Big)\psi'(t){}^C D_{a^+}^{\alpha,\psi}(z_1(t) - z_2(t))dt \\
&+ \int_a^b \Big(\Phi_p(z_1(t)) - \Phi_p(z_2(t))\Big)\psi'(t)(z_1(t) - z_2(t))dt \geq 0,
\end{aligned}$$

Obviously, the functional $\mathcal{F}_1'$ is strictly monotone. Then, $\mathcal{F}_1'$ possesses an inverse on $H^*_{\alpha,\psi,p}$, which is continuous owing to Theorem 26.A(d) in [24]. $\square$

For simplicity of discussion, we introduce some notations before describing the main theorems.

For any $\vartheta > 0$, denote $\Omega(\vartheta) = \{t \in \mathbb{R} : |t|^p \leq \vartheta\}$, and

$$\widehat{F}_\vartheta = \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta)} F(t,z(t))dt,\ \widehat{G}_\vartheta = \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta)} G(t,z(t))dt,$$

$$\begin{aligned} Q =& \frac{1}{\beta^p(b-a)^p}\Bigg\{ \int_a^{a+\beta(b-a)} (t-a)^{(1-\alpha)p}dt \\ &+ \int_{a+\beta(b-a)}^{b-\beta(b-a)} |(t-a)^{1-\alpha} - (t-(a+\beta(b-a)))^{1-\alpha}|^p dt \\ &+ \int_{b-\beta(b-a)}^{b} |(t-a)^{1-\alpha} - (t-(a+\beta(b-a)))^{1-\alpha} - (t-(b-\beta(b-a)))^{1-\alpha}|^p dt \Bigg\}, \end{aligned} \tag{22}$$

$$\begin{aligned} \sigma_{(\lambda,G)} = \min\Bigg\{ &\min\left\{ \frac{\frac{\vartheta_1}{p\widetilde{L}^p} - \lambda\widehat{F}_{\vartheta_1}}{\widehat{G}_{\vartheta_1}}, \frac{\frac{\vartheta_2}{p\widetilde{L}^p} - \lambda\widehat{F}_{\vartheta_2}}{\widehat{G}_{\vartheta_2}}, \frac{\frac{(\vartheta_3-\vartheta_2)}{p\widetilde{L}^p} - \lambda\widehat{F}_{\vartheta_3}}{\widehat{G}_{\vartheta_3}} \right\}, \\ &\frac{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^p Q - \lambda(\int_{a+\beta(b-a)}^{b-\beta(b-a)} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1})}{\int_{a+\beta(b-a)}^{b-\beta(b-a)} G(t,\Gamma(2-\alpha)\mu)dt - \widehat{G}_{\vartheta_1}} \Bigg\}, \end{aligned} \tag{23}$$

for $0 < \beta < \frac{1}{2}$, $\mu > 0$.

Theorem 2. *Assuming that F is non-negative; there exist positive constants $\vartheta_1, \vartheta_2, \vartheta_3, \mu$ with $\vartheta_1 < \widetilde{L}^p\mu^p Q$, $\vartheta_2 > \widetilde{L}^p\mu^p Q[1+\widetilde{L}^P(b-a)]$ and $\vartheta_2 < \vartheta_3$, such that*

(H_1)

$$\max\left\{ \frac{\widehat{F}_{\vartheta_1}}{\vartheta_1}, \frac{\widehat{F}_{\vartheta_2}}{\vartheta_2}, \frac{\widehat{F}_{\vartheta_3}}{\vartheta_3-\vartheta_2} \right\} < \frac{1}{p\widetilde{L}^p} \frac{\int_{a+\beta(b-a)}^{b-\beta(b-a)} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1}}{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^p Q},$$

Then, for every

$$\lambda \in \left] \frac{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^p Q}{\int_{a+\beta(b-a)}^{b-\beta(b-a)} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1}}, \frac{1}{p\widetilde{L}^p} \min\left\{ \frac{\vartheta_1}{\widehat{F}_{\vartheta_1}}, \frac{\vartheta_2}{\widehat{F}_{\vartheta_2}}, \frac{\vartheta_3-\vartheta_2}{\widehat{F}_{\vartheta_3}} \right\} \right[$$

and every non-negative function G, there exists $\sigma_{(\lambda,G)} > 0$ presented in (23), such that, for each $\xi \in [0, \sigma_{\lambda,G}[$, the system (1) possesses at least three distinct solutions z_1, z_2, z_3 and satisfies $\max_{t\in[a,b]} |z_1(t)|^p < \vartheta_1$, $\max_{t\in[a,b]} |z_2(t)|^p < \vartheta_2$, and $\max_{t\in[a,b]} |z_3(t)|^p < \vartheta_3$.

Proof. Firstly, we consider the functional $\mathcal{F}_1$. It is easy to observe that $\mathcal{F}_1$ is coercive. For any weakly convergent sequence $\{z_k\}_{k=1}^\infty$, which converges to z in $H_{(\alpha,\psi,p)}$. Using Lemma 5, we have $\{z_k\}$ that is convergent uniformly to z in $C([a,b],\mathbb{R})$. That is,

$$\begin{aligned} \liminf_{k\to\infty} \mathcal{F}_1(z_k) &= \liminf_{k\to\infty} \left\{ \frac{1}{p}\|z_k\|^p_{(\alpha,\psi,p)} + \frac{1}{p}\int_a^b \psi'(t)|z_k(t)|^p dt \right\} \\ &\geq \frac{1}{p}\|z\|^p_{(\alpha,\psi,p)} + \frac{1}{p}\int_a^b \psi'(t)|z(t)|^p dt = \mathcal{F}_1(z), \end{aligned}$$

Thus, $\mathcal{F}_1$ is weakly lower semi-continuous. On the other hand, because of $z_k \rightharpoonup z$ in $H_{(\alpha,\psi,p)}$ as $k \to \infty$, i.e., $z_k \to z$ on $[a,b]$ uniformly. Since $F, G \in C^1([a,b]\times\mathbb{R},\mathbb{R})$, then $F(t,z_k) \to F(t,z)(k\to\infty)$ and $G(t,z_k) \to G(t,z)(k\to\infty)$. By means of the Lebesgue control convergence theorem, we have $\mathcal{F}_2(z_k) \to \mathcal{F}_2(z)$, i.e., $\mathcal{F}_2$ is strongly continuous on $H_{\alpha,\psi,p}$. Hence, $\mathcal{F}_2$ is a compact operator.

In view of (8) and (14), we have

$$\begin{aligned}\frac{1}{p}\|z\|^p_{(\alpha,\psi,p)} \leq \mathcal{F}_1(z) \leq & \frac{1}{p}\|z\|^p_{(\alpha,\psi,p)} + \frac{1}{p}\|z\|^p_\infty(\psi(b)-\psi(a)) \\ \leq & \frac{1}{p}[1+\widetilde{L}^P(\psi(b)-\psi(a))]\|z\|^p_{(\alpha,\psi,p)}. \end{aligned} \tag{24}$$

For $0<\beta<\frac{1}{2}$, $\psi(t)=t$, define $\nu(t)$ by setting

$$\nu(t)=\begin{cases}\frac{\Gamma(2-\alpha)\mu}{\beta(b-a)}(t-a),\ t\in[a,a+\beta(b-a)[,\\ \Gamma(2-\alpha)\mu,\ t\in[a+\beta(b-a),b-\beta(b-a)],\\ \frac{\Gamma(2-\alpha)\mu}{\beta(b-a)}(b-t),\ t\in]b-\beta(b-a),b].\end{cases}$$

It can be obtained through simple calculation that

$${}^C D^{\alpha,\psi}_{a^+}\nu(t)=\begin{cases}\frac{\mu}{\beta(b-a)}(t-a)^{1-\alpha}, & t\in[a,a+\beta(b-a)[,\\ \frac{\mu}{\beta(b-a)}[(t-a)^{1-\alpha}-(t-(a+\beta(b-a)))^{1-\alpha}], & \\ & t\in[a+\beta(b-a),b-\beta(b-a)],\\ \frac{\mu}{\beta(b-a)}[(t-a)^{1-\alpha}-(t-(a+\beta(b-a)))^{1-\alpha}-(t-(b-\beta(b-a)))^{1-\alpha}], & \\ & t\in]b-\beta(b-a),b].\end{cases}$$

Then

$$\begin{aligned}&\int_a^b \psi'(t)|{}^C D^{\alpha,\psi}_{a^+}\nu(t)|^p dt\\ =&\frac{\mu^p}{\beta^p(b-a)^p}\bigg\{\int_a^{a+\beta(b-a)}(t-a)^{(1-\alpha)p}dt\\ &+\int_{a+\beta(b-a)}^{b-\beta(b-a)}|(t-a)^{1-\alpha}-(t-(a+\beta(b-a)))^{1-\alpha}|^p dt\\ &+\int_{b-\beta(b-a)}^{b}|(t-a)^{1-\alpha}-(t-(a+\beta(b-a)))^{1-\alpha}-(t-(b-\beta(b-a)))^{1-\alpha}|^p dt\bigg\},\end{aligned}$$

from (22), we can obtain that $\|\nu\|^p_{(\alpha,\psi,p)}=\mu^p Q$. Combining (24) yields

$$\frac{1}{p}\mu^p Q \leq \mathcal{F}_1(\nu) \leq \frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^p Q. \tag{25}$$

Choose $\rho_1=\frac{1}{p\widetilde{L}^p}\vartheta_1$, $\rho_2=\frac{1}{p\widetilde{L}^p}\vartheta_2$, $\rho_3=\frac{1}{p\widetilde{L}^p}(\vartheta_3-\vartheta_2)$. From the conditions $\vartheta_3>\vartheta_2$, $\vartheta_1<\widetilde{L}^p\mu^p Q$ and $\vartheta_2>\widetilde{L}^p\mu^p Q[1+\widetilde{L}^P(b-a)]$, we achieve

$$\rho_1<\mathcal{F}_1(\nu)<\rho_2,\ \rho_3>0. \tag{26}$$

By means of (8) and (24), we derive

$$\begin{aligned}\mathcal{F}_1^{-1}(-\infty,\rho_1)=&\{z\in H_{(\alpha,\psi,p)},\mathcal{F}_1(z)<\rho_1\}\\ \subseteq&\{z\in H_{(\alpha,\psi,p)},\|z\|^p_{(\alpha,\psi,p)}\leq p\rho_1\}\\ \subseteq&\{z\in H_{(\alpha,\psi,p)},\|z\|^p_\infty\leq p\widetilde{L}^P\rho_1\}\\ =&\{z\in H_{(\alpha,\psi,p)},\|z\|^p_\infty\leq\vartheta_1\}.\end{aligned}$$

The same procedure can be easily adapted to obtain

$$\mathcal{F}_1^{-1}(-\infty,\rho_2) \subseteq \{z \in H_{(\alpha,\psi,p)}, \|z\|_\infty^p \leq \vartheta_2\},$$
$$\mathcal{F}_1^{-1}(-\infty,\rho_2+\rho_3) \subseteq \{z \in H_{(\alpha,\psi,p)}, \|z\|_\infty^p \leq \vartheta_3\}.$$

Hence, we get

$$\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_1)} \int_a^b F(t,z(t))\psi'(t)dt \leq \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta_1)} F(t,z(t))dt = \widehat{F}_{\vartheta_1},$$
$$\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_2)} \int_a^b F(t,z(t))\psi'(t)dt \leq \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta_2)} F(t,z(t))dt = \widehat{F}_{\vartheta_2},$$
$$\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_2+\rho_3)} \int_a^b F(t,z(t))\psi'(t)dt \leq \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta_3)} F(t,z(t))dt = \widehat{F}_{\vartheta_3}.$$

Similarly,

$$\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_1)} \int_a^b G(t,z(t))\psi'(t)dt \leq \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta_1)} G(t,z(t))dt = \widehat{G}_{\vartheta_1},$$
$$\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_2)} \int_a^b G(t,z(t))\psi'(t)dt \leq \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta_2)} G(t,z(t))dt = \widehat{G}_{\vartheta_2},$$
$$\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_2+\rho_3)} \int_a^b G(t,z(t))\psi'(t)dt \leq \int_a^b \psi'(t) \max_{z\in\Omega(\vartheta_3)} G(t,z(t))dt = \widehat{G}_{\vartheta_3}.$$

Since $\mathcal{F}_1(0) = \mathcal{F}_2(0) = 0$ and $0 \in \mathcal{F}_1^{-1}(-\infty,\rho_1)$, one has

$$\begin{aligned}
\mathcal{A}(\rho_1) &= \inf_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_1)} \frac{[\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_1)} \mathcal{F}_2(z)] - \mathcal{F}_2(z)}{\rho_1 - \mathcal{F}_1(z)} \\
&\leq \frac{[\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_1)} \mathcal{F}_2(z)] - \mathcal{F}_2(0)}{\rho_1 - \mathcal{F}_1(0)} \\
&= \frac{\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_1)}[\int_a^b F(t,z(t))\psi'(t)dt + \frac{\xi}{\lambda}\int_a^b G(t,z(t))\psi'(t)dt]}{\rho_1} \\
&\leq \frac{p\widetilde{L}^p}{\vartheta_1}(\widehat{F}_{\vartheta_1} + \frac{\xi}{\lambda}\widehat{G}_{\vartheta_1}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{A}(\rho_2) &\leq \frac{\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_2)} \mathcal{F}_2(z)}{\rho_2} \\
&= \frac{\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_2)}[\int_a^b F(t,z(t))\psi'(t)dt + \frac{\xi}{\lambda}\int_a^b G(t,z(t))\psi'(t)dt]}{\rho_1} \\
&\leq \frac{p\widetilde{L}^p}{\vartheta_2}(\widehat{F}_{\vartheta_2} + \frac{\xi}{\lambda}\widehat{G}_{\vartheta_2}),
\end{aligned}$$

and

$$\mathcal{D}(\rho_2,\rho_3) = \frac{\sup_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_2+\rho_3)}[\int_a^b \psi'(t)F(t,z(t))dt + \frac{\xi}{\lambda}\int_a^b \psi'(t)G(t,z(t))dt]}{\rho_3}$$
$$\leq \frac{p\widetilde{L}^p}{\vartheta_3-\vartheta_2}(\widehat{F}_{\vartheta_3} + \frac{\xi}{\lambda}\widehat{G}_{\vartheta_3}).$$

Furthermore, for each $z \in \mathcal{F}_1^{-1}(-\infty,\rho_1)$, from (25) and (26), one has

$$\mathcal{B}(\rho_1,\rho_2) = \inf_{z\in\mathcal{F}_1^{-1}(-\infty,\rho_1)} \sup_{y\in\mathcal{F}_1^{-1}(\rho_1,\rho_2)} \frac{\mathcal{F}_2(y)-\mathcal{F}_2(z)}{\mathcal{F}_1(y)-\mathcal{F}_1(z)}$$
$$\geq \frac{\int_{a+\beta(b-a)}^{b-\beta(b-a)} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1} + \frac{\xi}{\lambda}(\int_{a+\beta(b-a)}^{b-\beta(b-a)} G(t,\Gamma(2-\alpha)\mu)dt - \widehat{G}_{\vartheta_1})}{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^p Q}.$$

Since $\xi < \sigma_{(\lambda,G)}$, we can easily get that

$$\frac{p\widetilde{L}^p}{\vartheta_1}(\widehat{F}_{\vartheta_1} + \frac{\xi}{\lambda}\widehat{G}_{\vartheta_1}) < \frac{1}{\lambda},\ \frac{p\widetilde{L}^p}{\vartheta_2}(\widehat{F}_{\vartheta_2} + \frac{\xi}{\lambda}\widehat{G}_{\vartheta_2}) < \frac{1}{\lambda},\ \frac{p\widetilde{L}^p}{\vartheta_3-\vartheta_2}(\widehat{F}_{\vartheta_3} + \frac{\xi}{\lambda}\widehat{G}_{\vartheta_3}) < \frac{1}{\lambda}, \tag{27}$$

and

$$\frac{\int_{a+\beta(b-a)}^{b-\beta(b-a)} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1} + \frac{\xi}{\lambda}(\int_{a+\beta(b-a)}^{b-\beta(b-a)} G(t,\Gamma(2-\alpha)\mu)dt - \widehat{G}_{\vartheta_1})}{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^p Q} > \frac{1}{\lambda}. \tag{28}$$

Combining (27) with (28), we observe that

$$\mathcal{G}(\rho_1,\rho_2,\rho_3) < \mathcal{B}(\rho_1,\rho_2).$$

Furthermore, assuming that z^* and z^{**} are two local minima of $\mathcal{F}$, then, z^* and z^{**} are critical points of $\mathcal{F}$; namely, they are weak solutions of system (1). Since F and G are assumed to be non-negative, for fixed $\xi,\lambda > 0$, one has $F(t,\tau z^* + (1-\tau)z^{**})dt + \frac{\xi}{\lambda}G(t,\tau z^* + (1-\tau)z^{**}) \geq 0$, which means that $\mathcal{F}_2(\tau z^* + (1-\tau)z^{**}) \geq 0$ for all $0 \leq \tau \leq 1$.

Thus, uniting Lemma 7 and Theorem 1, for every

$$\lambda \in \left]\frac{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^p Q}{\int_{a+\beta(b-a)}^{b-\beta(b-a)} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1}}, \frac{1}{p\widetilde{L}^p}\min\left\{\frac{\vartheta_1}{\widehat{F}_{\vartheta_1}}, \frac{\vartheta_2}{\widehat{F}_{\vartheta_2}}, \frac{\vartheta_3-\vartheta_2}{\widehat{F}_{\vartheta_3}}\right\}\right[$$

and $\xi \in [0,\sigma_{(\lambda,G)}[$, the functional $\mathcal{F}$ has three critical points z_1,z_2,z_3 on $H_{(\alpha,\psi,p)}$ and satisfies $\mathcal{F}_1(z_1) < \rho_1$, $\mathcal{F}_1(z_2) < \rho_2$ and $\mathcal{F}_1(z_3) < \rho_2+\rho_3$. That is, $\max_{t\in[a,b]}|z_1(t)|^p < \vartheta_1$, $\max_{t\in[a,b]}|z_2(t)|^p < \vartheta_2$, and $\max_{t\in[a,b]}|z_3(t)|^p < \vartheta_3$. Then, consider the fact that the critical points of the functional $\mathcal{F}$ are consistent with weak solutions of system (1), we obtain the main conclusion. □

Theorem 3. *Assume $f,g : [a,b]\times\mathbb{R} \to \mathbb{R}$ are non-negative. Then, the weak solutions of system (1) obtained in the Theorem 2 are non-negative.*

Proof. From Theorem 2, there exist at least three weak solutions z_1,z_2,z_3 with

$$\max_{t\in[a,b]}|z_1(t)|^p < \vartheta_1,\ \max_{t\in[a,b]}|z_2(t)|^p < \vartheta_2,\ \max_{t\in[a,b]}|z_3(t)|^p < \vartheta_3,$$

for system (1). We claim that z_1,z_2,z_3 are non-negative. In fact, let $\widehat{z}$ be a nontrivial weak solution of system (1). We assume the set $\Theta = \{t \in (a,b] : \widehat{z}(t) < 0\}$ is non-empty with the positive measure. For any $t \in [a,b]$, define $y^*(t) = \min\{0, \widehat{z}(t)\}$. Obviously, $y^*(t) \in H_{(\alpha,\psi,p)}$ and satisfies

$$\begin{aligned} &\int_a^b \psi'(t)\Phi_p({}^C D_{a^+}^{\alpha,\psi}\widehat{z}(t))\,{}^C D_{a^+}^{\alpha,\psi} y^*(t) + \psi'(t)y^*(t)\Phi_p(\widehat{z}(t))dt \\ &-\lambda\int_a^b y^*(t)f(t,\widehat{z}(t))\psi'(t)dt - \xi\int_a^b y^*(t)g(t,\widehat{z}(t))\psi'(t)dt = 0,\ \forall \widehat{z}(t) \in \Theta. \end{aligned} \tag{29}$$

Since f, g are non-negative, due to (29), one has

$$\begin{aligned} 0 &\geq \int_a^b \psi'(t)\Phi_p({}^C D_{a^+}^{\alpha,\psi}\widehat{z}(t))\,{}^C D_{a^+}^{\alpha,\psi} y^*(t)dt + \int_a^b \psi'(t)y^*(t)\Phi_p(\widehat{z}(t))dt \\ &= \int_a^b \psi'(t)|{}^C D_{a^+}^{\alpha,\psi}\widehat{z}(t)|^p dt + \int_a^b \psi'(t)|\widehat{z}(t)|^p dt \\ &\geq \|\widehat{z}\|^p_{(\alpha,\psi,p)} \geq 0,\ \forall \widehat{z}(t) \in \Theta, \end{aligned}$$

which means that $\widehat{z} \equiv 0$ in Θ, which is a contradiction. Therefore, we get the desired result. □

Theorem 4. *Assume that there exists a constant C_0, such that*

$$\lim_{z\to 0} \frac{\lambda f(t,z) + \xi g(t,z)}{|z|^{p-1}} \leq C_0 \tag{30}$$

uniformly in $z \in \mathbb{R}$, $t \in [a,b]$. Then, the system (1) does not include any nontrivial weak solution.

Proof. We assume that system (1) exists in at least one nontrivial weak solution on $H_{(\alpha,\psi,p)}$. Let $z_0 \in H_{(\alpha,\psi,p)}$ be a nontrivial weak solution. Based on (30), there exists $\varepsilon > 0$, such that

$$\lambda f(t,z) + \xi g(t,z) \leq C_0\varepsilon|z|^{p-1},\ \forall z \in \mathbb{R}. \tag{31}$$

Combining (14), (15) and (31) yields

$$\begin{aligned} 0 =& \mathcal{F}'(z_0)(z_0) = \mathcal{F}_1'(z_0)(z_0) - \lambda\mathcal{F}_2'(z_0)(z_0) \\ =& \int_a^b \psi'(t)|{}^C D_{a^+}^{\alpha,\psi} z_0(t)|^p + \psi'(t)|z_0(t)|^p dt \\ &- \int_a^b [\lambda f(t,z_0(t)) + \xi g(t,z_0(t))]\psi'(t)z_0(t)dt \\ \geq& \|z_0\|^p_{(\alpha,\psi,p)} - \int_a^b C_0\varepsilon|z_0(t)|^p\psi'(t)dt \\ \geq& \|z_0\|^p_{(\alpha,\psi,p)} - \varepsilon C_0[\psi(b)-\psi(a)]\|z_0\|^p_\infty \\ \geq& \left(1 - \varepsilon C_0\widetilde{L}^p[\psi(b)-\psi(a)]\right)\|z_0\|^p_{(\alpha,\psi,p)}. \end{aligned}$$

Choosing ε small enough, such that $\mathcal{F}'(z_0)(z_0) > 0$, we get a contradiction. Therefore, the system (1) does not include any nontrivial weak solution on $H_{(\alpha,\psi,p)}$. □

4. Examples

Example 1. *Let $a = 0, b = 1$, $\alpha = 0.6$, $p = 2$, $\psi(t) = e^t$. Consider the following FDE*

$$\begin{cases} {}^C D_{1^-}^{0.6,e^t}({}^C D_{0^+}^{0.6,e^t} z(t)) + z(t) = \lambda f(t,z) + \xi g(t,z),\ t \in [0,1], \\ z(0) = z(1) = 0. \end{cases} \tag{32}$$

Define $f(t,z) = \frac{1}{10}\frac{1}{z^2}e^{\frac{-1}{z}}$. *Choose* $\beta = \frac{1}{3}, \mu = 1, \vartheta_1 = 0.01, \vartheta_2 = 0.2, \vartheta_3 = 0.3$. *By direct calculation, we obtain that* $F(t,z) = \frac{1}{10}e^{\frac{-1}{z}}$, $\widetilde{L} \approx 0.8$, $Q = 0.143$, $\vartheta_1 < \widetilde{L}^p\mu^pQ = 0.09$, $\vartheta_2 > \widetilde{L}^p\mu^pQ[1+\widetilde{L}^P(b-a)] = 0.15$, $\vartheta_2 < \vartheta_3$, *and*

$$\begin{aligned}
\frac{\widehat{F}_{\vartheta_1}}{\vartheta_1} &= \frac{\int_0^1 e^t \max_{|z|^2\leq\vartheta_1}\{\frac{1}{10}e^{\frac{-1}{z}}\}dt}{0.01} \approx 6\times 10^{-46},\\
\frac{\widehat{F}_{\vartheta_2}}{\vartheta_2} &= \frac{\int_0^1 e^t \max_{|z|^2\leq\vartheta_2}\{\frac{1}{10}e^{\frac{-1}{z}}\}dt}{0.2} \approx 5.7\times 10^{-3},\\
\frac{\widehat{F}_{\vartheta_3}}{\vartheta_3-\vartheta_2} &= \frac{\int_0^1 e^t \max_{|z|^2\leq\vartheta_3}\{\frac{1}{10}e^{\frac{-1}{z}}\}dt}{0.1} \approx 6\times 10^{-2}.
\end{aligned}$$

Then

$$6\times 10^{-2} = \max\left\{\frac{\widehat{F}_{\vartheta_1}}{\vartheta_1}, \frac{\widehat{F}_{\vartheta_2}}{\vartheta_2}, \frac{\widehat{F}_{\vartheta_3}}{\vartheta_3-\vartheta_2}\right\} < \frac{1}{p\widetilde{L}^p}\frac{\int_{\frac{1}{3}}^{\frac{2}{3}} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1}}{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^pQ} = 0.15.$$

Therefore, according to Theorems 2 and 3, for every $\lambda \in]1.02, 13[$, *there exists* $\sigma_{(\lambda,G)} > 0$, *such that, for each* $\xi \in [0, \sigma_{\lambda,G}[$, *the system (33) possesses three distinct non-negative weak solutions* $z_1, z_2, z_3 > 0$ *with* $\max_{t\in[0,1]}|z_1(t)|^2 < 0.01$, $\max_{t\in[0,1]}|z_2(t)|^2 < 0.2$ *and* $\max_{t\in[0,1]}|z_3(t)|^2 < 0.3$.

Example 2. *Let* $a = 0, b = 1$, $\alpha = 0.75$, $p = 3$, $\psi(t) = t^{\frac{1}{2}}$. *Consider the following FDE*

$$\begin{cases} {}^CD_{1^-}^{0.75,t^{\frac{1}{2}}}\Phi_3({}^CD_{0^+}^{0.75,t^{\frac{1}{2}}}z(t)) + |z(t)|z(t) = \lambda f(t,z) + \xi g(t,z),\ t\in(0,1],\\ z(0) = z(1) = 0. \end{cases} \tag{33}$$

Define $f(t,z) = \begin{cases} 4z^3, z\leq 1,\\ \frac{4}{z}, z>1. \end{cases}$ *Then,* $F(t,z) = \begin{cases} z^4, z\leq 1,\\ 4\ln(z), z>1. \end{cases}$ *Choose* $\beta = \frac{1}{3}$, $\mu = 1$, $\vartheta_1 = 0.1$, $\vartheta_2 = 1$, $\vartheta_3 = 1.5$. *By direct calculation, we obtain that* $\widetilde{L} \approx 0.8$, $Q = 1.08$, $\vartheta_1 < \widetilde{L}^p\mu^pQ = 0.512$, $\vartheta_2 > \widetilde{L}^p\mu^pQ[1+\widetilde{L}^P(b-a)] = 0.774$, $\vartheta_2 < \vartheta_3$, *and*

$$\begin{aligned}
\frac{\widehat{F}_{\vartheta_1}}{\vartheta_1} &= \frac{\frac{1}{2}\int_0^1 t^{-\frac{1}{2}}\max_{|z|^3\leq\vartheta_1}\{z^4\}dt}{0.1} = 0.64,\\
\frac{\widehat{F}_{\vartheta_2}}{\vartheta_2} &= \frac{1}{2}\int_0^1 t^{-\frac{1}{2}}\max_{|z|^3\leq\vartheta_2}\{z^4\}dt = 1,\\
\frac{\widehat{F}_{\vartheta_3}}{\vartheta_3-\vartheta_2} &= \frac{\frac{1}{2}\int_0^1 t^{-\frac{1}{2}}\max_{|z|^3\leq\vartheta_3}\{4\ln(z)\}dt}{0.5} \approx 1.08.
\end{aligned}$$

Then,

$$1.08 = \max\left\{\frac{\widehat{F}_{\vartheta_1}}{\vartheta_1}, \frac{\widehat{F}_{\vartheta_2}}{\vartheta_2}, \frac{\widehat{F}_{\vartheta_3}}{\vartheta_3-\vartheta_2}\right\} < \frac{1}{p\widetilde{L}^p}\frac{\int_{\frac{1}{3}}^{\frac{2}{3}} F(t,\Gamma(2-\alpha)\mu)dt - \widehat{F}_{\vartheta_1}}{\frac{1}{p}[1+\widetilde{L}^P(b-a)]\mu^pQ} = 2.9.$$

Therefore, according to Theorem 2, for every $\lambda \in]0.2, 0.65[$, *there exists* $\sigma_{(\lambda,G)} > 0$, *such that, for each* $\xi \in [0, \sigma_{\lambda,G}[$, *the system (33) possesses three distinct weak solutions* z_1, z_2, z_3, *satisfying* $\max_{t\in[0,1]}|z_1(t)|^3 < 0.1$, $\max_{t\in[0,1]}|z_2(t)|^3 < 1$, *and* $\max_{t\in[0,1]}|z_3(t)|^3 < 1.5$.

5. Conclusions

This paper considered a new ψ-Caputo-type fractional differential system including the generalized p-Laplacian operator. By means of a three critical points theorem given by Bonanno and Candito, and several properties of the ψ-Caputo fractional operator, the existence of at least three distinct non-negative weak solutions was studied. Due to a mild condition, an evaluation criterion for the equation without solutions was given. What is noteworthy is that the nonlinear functions f, g do not need to adapt certain asymptotic conditions—the multiplicity results were established only by imposing algebraic conditions on nonlinear functions. This work represents a generalization of several results reported in the literature which concern classical fractional operators.

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