Synchronization of Discrete-Time Fractional-Order Complex-Valued Neural Networks with Distributed Delays

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Abstract: This research investigates the synchronization of distributed delayed discrete-time fractional-order complex-valued neural networks. The necessary conditions have been established for the stability of the proposed networks using the theory of discrete fractional calculus, the discrete Laplace transform, and the theory of fractional-order discrete Mittag–Leffler functions. In order to guarantee the global asymptotic stability, adequate criteria are determined using Lyapunov’s direct technique, the Lyapunov approach, and some novel analysis techniques of fractional calculation. Thus, some sufficient conditions are obtained to guarantee the global stability. The validity of the theoretical results are finally shown using numerical examples.

Keywords: fractional order; synchronization; complex-valued; discrete-time; neural networks

1. Introduction

The 19th century witnessed the majority of the development of fractional calculus theory. More than 300 years ago, in Leibniz’s letter to L’Hospital from 1695, fractional calculus was first introduced. The distinct advantage of fractional-order systems over traditional integer-order systems is that they provide an ideal instrument for describing the memory and hereditary features of diverse materials and processes. Fractional calculus did not receive much attention for a very long time due to the complexity and lack of application for the background. Fractional-order differential equations have recently been demonstrated to be useful modelling tools in a variety of scientific and engineering domains, as shown in [1–3]. The dynamical characteristics of neural networks have drawn significant attention during the last few decades. Due to neural networks’ effective application in optimization, signal processing, associative memory, parallel computing, pattern recognition, artificial intelligence, etc., their dynamical features have come under intense scrutiny during the past few decades. As fractional calculus advanced quickly, several researchers astonishingly found that fractional calculus could be implemented into neural networks [4–10].

Despite the significant progress made by continuous fractional calculus, discrete fractional calculus research is still in its early stages. In order to explore discrete fractional calculus, Diaz and Osler introduced an infinite series in 1974. However, continuous-time and discrete-time systems are two complementary characteristics in real-world applications, therefore the question of whether discrete-time systems have similar dynamical behaviors to their continuous-time counterparts has emerged. It is crucial to study the dynamic
behavior of discrete fractional calculus since not all discrete operators in theoretical research have the same properties as continuous operators. Researchers often take continuous-time systems into consideration when simulating and analyzing dynamic behavior on computers. However, in a digital network, signal reception and operation are based on discrete time rather than continuous time [11–18].

The networks that process complex-valued input by employing complex-valued parameters and variables are known as complex-valued neural networks (CVNNs) [19–27]. In comparison to real-valued neural networks, complex-valued neural networks have a favoured superiority in easier network layout, quicker training times, and increased power throughout complex signal processing. However, according to Liouville’s theorem, every bounded and smooth activation function in CVNNs simplifies to a constant. Therefore, it is more difficult and important to understand the dynamical behaviors of CVNNs. Additionally, they have better solutions than real-valued neural networks for resolving several challenging real-life problems, such as the XOR problem. A variety of techniques have been used to evaluate the stability of CVNNs based on the outcomes so far. Some scholars provided numerous significant results in recent years to guarantee the dynamics of complex-valued neural networks with temporal delays [28–38].

Synchronization for time-delayed neural networks has received particular attention due to their numerous potential applications in the areas of image processing, signal processing, associative memory, and secure communication. Synchronization has grown in popularity as a neural network research issue during the past decade. There are several different types of fractional neural network synchronization issues in use today [39,40]. These synchronization analyses are carried out using a singular Gronwall inequality and Filippov solution theorem [41–43].

The broad field of science and engineering known as stability theory examines how dynamical structures affect both linear and nonlinear systems. Most stability studies conducted in recent decades have focused on stability in the Lyapunov sense, including asymptotic, exponential, and uniform stability. The well-known methods for time-domain stability analysis for systems with integer orders, such as the Lyapunov functional method and those combined with Razumikhin-type techniques, cannot be easily generalised to FO systems with time delay because it is challenging to calculate the FO derivatives of Lyapunov functions. The Caputo definition is used. A numerical example is used to demonstrate the accuracy of the suggested procedure. The novelties of the study are given below:

1. We studied the global synchronization of discrete-time fractional-order complex-valued neural networks with distributed delays.
2. Unlike the previous literature, this paper explicitly examines the stability for discrete fractional-order complex-valued neural networks using the stability theory in complex fields as opposed to breaking down complex-valued systems into real-valued systems.
3. Using the Lyapunov direct technique, the synchronization condition of FOCVNNs with temporal delays is determined. In light of the definition of the Caputo fractional difference, it is simple to calculate the first-order backward difference of the Lyapunov function that we design, which includes discrete fractional sum terms.
4. Some conditions regarding the global Mittag-Leffler stability of fractional-order CVNNs are established using fractional derivative inequalities and fractional-order appropriate Lyapunov functions.
5. It is necessary to investigate the essential characteristics of the discrete Mittag–Leffler function and the Nabla discrete Laplace transform.
6. Finally, numerical illustrations are provided.

2. Preliminaries

Let \( \Delta \varphi(h) := \varphi(h) - \varphi(h - 1) \) be a backward difference operator and \( \nabla^m \varphi(h) := \nabla (\nabla^{m-1} \varphi(h)) \) be the operator, where \( m \in N^+ \).
Definition 1 ([44]). The Nabla discrete fractional sum of order $\beta > 0$ is defined as:
\[
\nabla_{a}^{-\beta} q(h) = \frac{1}{\Gamma(\beta)} \sum_{s=a}^{h} (h - s)^{-\beta} q(s),
\]
where $a \in R$, $q(s) = s - 1$, $h \in N_a = \{a, a + 1, a + 2, \ldots\}$.

Definition 2 ([45]). The Riemann–Liouville fractional difference of order $\beta > 0$ is defined as
\[
\nabla_{a}^{\beta} q(h) = \nabla_{a}^{m} (\nabla_{a}^{-(m-n)} q(h)),
\]
where $m - 1 < \beta \leq m, m \in N^+, t \in N_{a+m}$.

Definition 3 ([46]). (Global Mittag–Leffler stability) The origin of System (1) is Mittag–Leffler stable if
\[
||x(h)|| \leq \left\{ R(x(h_0))E_q \left(-\delta (h - h_0)^{\sigma}\right) \right\}^{\sigma},
\]
where $t_0$ denotes the initial instant, $q \in (0, 1), \delta > 0, \sigma > 0, R(0) = 0, R(x) \geq 0$, and $R(x)$ is locally Lipschitz on $x \in R$ with respect to the Lipschitz constant $R_0$.

Lemma 1 ([47]). Let $q(h) = (q_1(h), \ldots, q_m(h))^T \in R^m$ be a positive definite matrix, which, if $H \in R^{m \times m}$ is a positive definite matrix, implies
\[
\nabla_{0}^{\beta} q(h) H q(h) \leq 2q^{T}(h) H^{c} \nabla_{0}^{\beta} q(h), \beta \in (0, 1).
\]

Lemma 2 ([48]). For $0 < \beta \leq 1, h = a + n$,
\[
\nabla_{a}^{\beta} q^2(h) \leq 2q(h) \nabla_{a}^{\beta} q(h).
\]

Lemma 3 ([49]). Suppose that $V(h) \in R$ is a continuous, differentiable, and non-negative function satisfying
\[
D^\beta V(h) \leq -bV(h) + cV(h - \omega), \ 0 < \beta < 1,
\]
\[
V(h) = \varphi(h) \geq 0, \ h \in [-\omega, 0].
\]
If $b > \sqrt{2c}$ and $c > 0$, then for all $\varphi(h) \geq 0, \omega > 0, \lim_{h \rightarrow +\infty} V(h) = 0$.

Lemma 4 ([50]). Let $V(h)$ be a continuous function on $[0, +\infty)$ satisfying
\[
D^\beta V(h) \leq \delta V(h), \beta \in (0, 1)
\]
and let $\delta$ be a constant. Then,
\[
V(h) \leq V(0)E_\beta(\delta h^\beta).
\]

3. Main Results

We consider the following discrete-time fractional-order complex-valued neural networks with time-varying delays:
\[
\nabla_{0}^{\beta} \gamma_{\phi}(h) = -c_\phi \gamma_{\phi}(h) + \sum_{\phi=1}^{m} a_{\phi} \phi f_{\phi}(\gamma_{\phi}(h)) + \sum_{\phi=1}^{m} b_{\phi} \phi f_{\phi}(\gamma_{\phi}(h - \omega)) + \sum_{\phi=1}^{m} d_{\phi} \int_{0}^{\infty} K_{\phi}(s) f_{\phi}(\gamma_{\phi}(s)) ds + I_{\phi}, t \in N_1,
\]

(1)
\[
\gamma(h) = \Phi_\psi(h), t \in N_1,
\]

where \( C^\beta_0 \) denotes the Caputo fractional difference operator with the order \( \beta (0 < \beta < 1) \). \( \gamma(h) = [\gamma_\psi(h), \gamma_\psi_2(h), \ldots, \gamma_\psi_m(h)]^T \in R^n \) denotes the state vector, \( f(\gamma(h)) = [f_1(\gamma_\psi(h)), f_2(\gamma_\psi_2(h)), \ldots, f_m(\gamma_\psi_m(h))]^T: C^m \rightarrow C^m \) are vector-valued activation functions, \( c_\psi, d_\psi, b_\psi, d_\psi \) are connection weight matrices, and \( \tau, \sigma(h) \) are time-varying delays.

The response system is designed as

\[
C^\beta_0 \delta_\psi(h) = -c_\psi \delta_\psi(h) + \sum_{\Phi=1}^m a_\psi \Phi f_\psi(\delta_\psi(h)) + \sum_{\Phi=1}^m b_\psi \Phi f_\psi(\delta_\psi(h - \omega)) + \sum_{\Phi=1}^m d_\psi \int_0^\infty K_\psi(s)f_\psi(\gamma(s))ds + l_\psi + u_\psi(h), \tag{2}
\]

\[
\delta_\psi(h) = \Phi_\psi(h), h \in N_1,
\]

where \( \delta_1(h) = (\delta_{11}(h), \delta_{12}(h), \ldots, \delta_{1m}(h))^T \in C^n, u_\psi(h) = [u_{\psi_1}(h), u_{\psi_2}(h), \ldots, u_{\psi_m}(h)]^T \in R^n \) is the control input.

Here, we study their solutions using Filippov regularization. Then, System (1) can be expressed as

\[
C^\beta_0 \gamma_\psi(h) = -c_\psi \gamma_\psi(h) + \sum_{\Phi=1}^m a_\psi \Phi F[f_\psi(\gamma_\psi(h))] + \sum_{\Phi=1}^m b_\psi \Phi F[f_\psi(\gamma_\psi(h - \omega))] + \sum_{\Phi=1}^m d_\psi \int_0^\infty K_\psi(s)f_\psi(\gamma_\psi(s))ds,
\]

\[
\gamma_\psi(h) = F[\Phi_\psi(h)], t \in N_1. \tag{3}
\]

If there exist \( p_1(h) \in F[f_1(x)] \), then

\[
C^\beta_0 \gamma_\psi(t) = -c_\psi \gamma_\psi(t) + \sum_{\Phi=1}^m a_\psi \Phi p_\psi(\gamma_\psi(t)) + \sum_{\Phi=1}^m b_\psi \Phi p_\psi(\gamma_\psi(h - \omega)) + \sum_{\Phi=1}^m d_\psi \int_0^\infty K_\psi(s)p_\psi(\gamma_\psi(s))ds,
\]

\[
\gamma_\psi(h) = p_\psi(t), h \in N_1. \tag{4}
\]

Similarly, from System (2), we have

\[
C^\beta_0 \delta_\psi(h) = -c_\psi \delta_\psi(h) + \sum_{\Phi=1}^m a_\psi \Phi p_\psi(\delta_\psi(h)) + \sum_{\Phi=1}^m b_\psi \Phi p_\psi(\delta_\psi(h - \omega)) + \sum_{\Phi=1}^m d_\psi \int_0^\infty K_\psi(s)p_\psi(\delta_\psi(s))ds + u_\psi(h),
\]

\[
\delta_\psi(h) = \bar{p}_\psi(h), t \in N_1. \tag{5}
\]

Assumptions:

(H1): Let \( \gamma(h) = \xi(h) + i\chi(h) \) and \( \delta(h) = \xi(h) + i\chi(h) \); then, we have

\[
p_\psi(\gamma_j(h - \omega_s)) = p_\psi(\gamma_j(h - \omega_s), \delta_j(h - \omega_s)) + ip_\psi(\gamma_j(h - \omega_s), \delta_j(h - \omega_s))
\]

(H2): \( |\phi^R_j(h - \omega_s) - k^R_j(\delta_j(h - \omega_s))| \leq \lambda^R_j |\gamma_j(h - \omega_s) - \delta_j(h - \omega_s)| + \lambda^R_j |\gamma_j(h - \omega_s) - \delta_j(h - \omega_s)|
\]

\[
|\phi^I_j(h - \omega_s) - k^I_j(\delta_j(h - \omega_s))| \leq \lambda^I_j |\gamma_j(h - \omega_s) - \delta_j(h - \omega_s)| + \lambda^I_j |\gamma_j(h - \omega_s) - \delta_j(h - \omega_s)|.
\]

From (H1) and (H2), Systems (4) and (5) can be expressed as
\[ C \nabla_{0}^{\beta} \xi_{\phi}(h) = -c_{\phi} \xi_{\phi}(h) + \sum_{\phi=1}^{m} a^{R}_{\phi \phi} p_{\phi}^{R}(\xi_{\phi}(h), \chi_{\phi}(t)) \sum_{\phi=1}^{m} a^{L}_{\phi \phi} p_{\phi}^{L}(\xi_{\phi}(h), \chi_{\phi}(h)) + \sum_{\phi=1}^{m} b_{\phi \phi}^{R} p_{\phi}^{R}(\xi_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) - \sum_{\phi=1}^{m} b_{\phi \phi}^{L} p_{\phi}^{L}(\xi_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) \]  
\[ + \sum_{\phi=1}^{m} d_{\phi \phi}^{R} \int_{0}^{\infty} K_{\phi \phi}(s) p_{\phi}^{R}(\xi_{\phi}(s), \chi_{\phi}(s)) ds - \sum_{\phi=1}^{m} d_{\phi \phi}^{L} \int_{0}^{\infty} K_{\phi \phi}(s) p_{\phi}^{L}(\xi_{\phi}(s), \chi_{\phi}(s)) ds, \]  
\[ (6) \]

\[ C \nabla_{0}^{\beta} \chi_{\phi}(h) = -c_{\phi} \chi_{\phi}(h) + \sum_{\phi=1}^{m} a^{R}_{\phi \phi} p_{\phi}^{R}(\xi_{\phi}(h), \chi_{\phi}(h)) + \sum_{\phi=1}^{m} a^{L}_{\phi \phi} p_{\phi}^{L}(\xi_{\phi}(h), \chi_{\phi}(h)) + \sum_{\phi=1}^{m} b_{\phi \phi}^{R} p_{\phi}^{R}(\xi_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) + \sum_{\phi=1}^{m} b_{\phi \phi}^{L} p_{\phi}^{L}(\xi_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) \]  
\[ + \sum_{\phi=1}^{m} d_{\phi \phi}^{R} \int_{0}^{\infty} K_{\phi \phi}(s) p_{\phi}^{R}(\xi_{\phi}(s), \chi_{\phi}(s)) ds + \sum_{\phi=1}^{m} d_{\phi \phi}^{L} \int_{0}^{\infty} K_{\phi \phi}(s) p_{\phi}^{L}(\xi_{\phi}(s), \chi_{\phi}(s)) ds, \]  
\[ (7) \]

\[ C \nabla_{0}^{\beta} \xi_{\phi}(h) = -c_{\phi} \xi_{\phi}(h) + \sum_{\phi=1}^{m} a^{R}_{\phi \phi} p_{\phi}^{R}(\xi_{\phi}(h), \chi_{\phi}(h)) - \sum_{\phi=1}^{m} a^{L}_{\phi \phi} p_{\phi}^{L}(\xi_{\phi}(t), \chi_{\phi}(h)) + \sum_{\phi=1}^{m} b_{\phi \phi}^{R} p_{\phi}^{R}(\xi_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) - \sum_{\phi=1}^{m} b_{\phi \phi}^{L} p_{\phi}^{L}(\xi_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) \]  
\[ + \sum_{\phi=1}^{m} d_{\phi \phi}^{R} \int_{0}^{\infty} K_{\phi \phi}(s) p_{\phi}^{R}(\xi_{\phi}(s), \chi_{\phi}(s)) ds - \sum_{\phi=1}^{m} d_{\phi \phi}^{L} \int_{0}^{\infty} K_{\phi \phi}(s) p_{\phi}^{L}(\xi_{\phi}(s), \chi_{\phi}(s)) ds, \]  
\[ (8) \]

\[ C \nabla_{0}^{\beta} \chi_{\phi}(t) = -c_{\phi} \chi_{\phi}(t) + \sum_{\phi=1}^{m} a^{R}_{\phi \phi} p_{\phi}^{R}(\xi_{\phi}(h), \chi_{\phi}(t)) + \sum_{\phi=1}^{m} a^{L}_{\phi \phi} p_{\phi}^{L}(\xi_{\phi}(h), \chi_{\phi}(t)) + \sum_{\phi=1}^{m} b_{\phi \phi}^{R} p_{\phi}^{R}(\xi_{\phi}(h - \omega), \chi_{\phi}(t - \omega)) + \sum_{\phi=1}^{m} b_{\phi \phi}^{L} p_{\phi}^{L}(\xi_{\phi}(h - \omega), \chi_{\phi}(t - \omega)) \]  
\[ + \sum_{\phi=1}^{m} d_{\phi \phi}^{R} \int_{0}^{\infty} K_{\phi \phi}(s) K_{\phi \phi}(s) p_{\phi}^{R}(\xi_{\phi}(s), \chi_{\phi}(s)) ds + \sum_{\phi=1}^{m} d_{\phi \phi}^{L} \int_{0}^{\infty} K_{\phi \phi}(s) K_{\phi \phi}(s) p_{\phi}^{L}(\xi_{\phi}(s), \chi_{\phi}(s)) ds, \]  
\[ (9) \]

We define \( \psi^{R}(h) = \zeta'(h) - \xi(h), \psi^{L}(h) = \chi'(h) - \chi(h) \) as the synchronization errors.
Let \( u_{\phi} = 0 \). Then, the system's error is defined as
\[ C \nabla_{0}^{\beta} [\psi^{R}_{\phi}(h)] = -c_{\phi} \psi^{R}_{\phi}(h) + \sum_{\phi=1}^{m} a^{R}_{\phi \phi} [p_{\phi}^{R}(\zeta'_{\phi}(h), \chi_{\phi}(h)) - p_{\phi}^{R}(\xi_{\phi}(h), \chi_{\phi}(h))] \]  
\[ - \sum_{\phi=1}^{m} a^{L}_{\phi \phi} [p_{\phi}^{L}(\zeta'_{\phi}(h), \chi_{\phi}(t)) - p_{\phi}^{L}(\xi_{\phi}(h), \chi_{\phi}(h))] + \sum_{\phi=1}^{m} b_{\phi \phi}^{R} [p_{\phi}^{R}(\zeta'_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) - p_{\phi}^{R}(\xi_{\phi}(t - \omega), \chi_{\phi}(h - \omega))] \]  
\[ - \sum_{\phi=1}^{m} b_{\phi \phi}^{L} [p_{\phi}^{L}(\zeta'_{\phi}(h - \omega), \chi_{\phi}(h - \omega)) - p_{\phi}^{L}(\xi_{\phi}(h - \omega), \chi_{\phi}(h - \omega))] + \sum_{\phi=1}^{m} d_{\phi \phi}^{R} \int_{0}^{\infty} K_{\phi \phi}(s) [p_{\phi}^{R}(\zeta'_{\phi}(s), \chi_{\phi}(s)) - p_{\phi}^{R}(\xi_{\phi}(s), \chi_{\phi}(s))] ds, \]  
\[ (10) \]
\[- \sum_{\phi=1}^{m} d_\phi^\gamma \int_0^\infty K_{\phi\phi}(s) [p^R_\phi(\xi_0^\phi(s), \chi_0^\phi(s)) - p^I_\phi(\xi_0^\phi(s), \chi_0^\phi(s))] ds. \]

\[
C \nabla_0^\gamma \psi_0^\phi(h) = -c_\phi \psi_0^I_\phi(h) + \sum_{\phi=1}^{m} a_\phi^R [p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h)) - p^I_\phi(\xi_0^\phi(h), \chi_0^\phi(h))] \\
+ \sum_{\phi=1}^{m} a_\phi^R [p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h)) - p^R_\phi(\gamma_\phi(h), \gamma_\phi(h))] \\
+ \sum_{\phi=1}^{m} b_\phi^R [p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h)) - p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h))] \]

(11)

**Theorem 1.** Under Assumptions (H1) and (H2) and Lemma 3, Systems (10) and (11) are globally asymptotically stable and satisfy \( \theta_1 > \sqrt{2\theta_2}, \theta_2 > 0. \)

**Proof.** We construct a Lyapunov functional

\[
V(h) = \sum_{\phi=1}^{m} [||\psi_0^R_\phi(h)|| + ||\psi_0^I_\phi(h)||].
\]

(12)

In the light of Lemma 1, we can calculate the fractional difference of \( V(h) \),

\[
C \nabla_0^\gamma V(h) \leq \sum_{\phi=1}^{m} C \nabla_0^\gamma [||\psi_0^R_\phi(t)|| + ||\psi_0^I_\phi(h)||],
\]

\[
C \nabla_0^\gamma V(h) = \sum_{\phi=1}^{m} |\text{sign}(\psi_0^I_\phi(h))| \left\{ -c_\phi \psi_0^I_\phi(h) + \sum_{\phi=1}^{m} a_\phi^R [p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h)) - p^I_\phi(\xi_0^\phi(h), \chi_0^\phi(h))] \\
- \sum_{\phi=1}^{m} a_\phi^R [p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h)) - p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h))] \\
+ \sum_{\phi=1}^{m} b_\phi^R [p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h)) - p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h))] \\
- \sum_{\phi=1}^{m} b_\phi^R [p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h)) - p^R_\phi(\xi_0^\phi(h), \chi_0^\phi(h))] \right\}
\]

(13)
Under Assumptions (H1) and (H2), Systems (10) and (11) can be expressed as:

\[
C \nabla^R_\phi V(h) \leq \sum_{q=1}^{m} \left\{ -c_p |\psi^R_p(h)| + \sum_{\phi=1}^{m} \left[ a^R_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} |a^R_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} |b^R_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} \left[ a^I_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} |a^I_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} \left[ a^I_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} |a^I_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} \left[ a^I_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
+ \sum_{\phi=1}^{m} |a^I_{q\phi} |\lambda^R_\phi |\psi^R_p(h)| + \lambda^R_\phi |\psi^R_p(h)| \right] \\
\right\}
\]

Under Assumptions (H1) and (H2), Systems (10) and (11) can be expressed as:
\[ + \sum_{\phi=1}^{m} |a_{\phi}^{R}| \int_{0}^{\infty} K_{\phi \psi}(s) \lambda_{\phi}^{RR} |\psi_{\phi}^{R}(s)| ds + \sum_{\phi=1}^{m} |a_{\phi}^{R}| \int_{0}^{\infty} K_{\phi \psi}(s) \lambda_{\phi}^{RI} |\psi_{\phi}^{I}(s)| ds \]

\[ + \sum_{\phi=1}^{m} |a_{\phi}^{I}| \lambda_{\phi}^{RR} |\psi_{\phi}^{R}(s)| ds + \sum_{\phi=1}^{m} |a_{\phi}^{I}| \lambda_{\phi}^{RI} |\psi_{\phi}^{I}(s)| ds \]

\[ + \sum_{\phi=1}^{m} \left\{ - c_{\phi} |\psi_{\phi}^{I}(h)| + \sum_{\phi=1}^{m} |a_{\phi}^{R}| \lambda_{\phi}^{R I} |\psi_{\phi}^{R}(h)| + \sum_{\phi=1}^{m} |a_{\phi}^{I}| \lambda_{\phi}^{R I} |\psi_{\phi}^{I}(h)| \right\} \]

\[ + \sum_{\phi=1}^{m} |a_{\phi}^{I}| \lambda_{\phi}^{RR} |\psi_{\phi}^{R}(h)| + \sum_{\phi=1}^{m} |a_{\phi}^{I}| \lambda_{\phi}^{RI} |\psi_{\phi}^{I}(h)| \]

\[ + \sum_{\phi=1}^{m} |b_{\phi}^{R}| \lambda_{\phi}^{RR} |\psi_{\phi}^{R}(h - \omega)| + \sum_{\phi=1}^{m} |b_{\phi}^{R}| \lambda_{\phi}^{RI} |\psi_{\phi}^{I}(h - \omega)| \]

\[ + \sum_{\phi=1}^{m} |b_{\phi}^{I}| \lambda_{\phi}^{RR} |\psi_{\phi}^{R}(h - \omega)| + \sum_{\phi=1}^{m} |b_{\phi}^{I}| \lambda_{\phi}^{RI} |\psi_{\phi}^{I}(h - \omega)| \]

\[ + \sum_{\phi=1}^{m} |a_{\phi}^{R}| \lambda_{\phi}^{RR} |\psi_{\phi}^{R}(h)| + \sum_{\phi=1}^{m} |a_{\phi}^{I}| \lambda_{\phi}^{R I} |\psi_{\phi}^{I}(h)| \]

\[ + \sum_{\phi=1}^{m} |b_{\phi}^{R}| \lambda_{\phi}^{RR} |\psi_{\phi}^{R}(h)| + \sum_{\phi=1}^{m} |b_{\phi}^{I}| \lambda_{\phi}^{RI} |\psi_{\phi}^{I}(h)| \]
\[
\begin{align*}
&\leq - \theta_{\min}^{(1)} \sum_{\phi=1}^{m} |\psi^R_\phi(h)| - \theta_{\min}^{(2)} \sum_{\phi=1}^{m} |\psi^I_\phi(h)| + \theta_{\max}^{(1)} \sum_{\phi=1}^{m} |\psi^R_\phi(h - \omega)| + \theta_{\max}^{(2)} \sum_{\phi=1}^{m} |\psi^I_\phi(h - \omega)|,
&= - \theta_1 V(h) + \theta_2 V(h - \omega),
\end{align*}
\]

where

\[
\begin{align*}
\theta_{\min}^{(1)} &= \left\{ c_\phi - \sum_{\phi=1}^{m} |a^{R}_{\phi\phi}| \lambda^R_{\phi} - \sum_{\phi=1}^{m} |a^{I}_{\phi\phi}| \lambda^I_{\phi} - \sum_{\phi=1}^{m} |d^{R}_{\phi\phi}| \lambda^R_{\phi} - \sum_{\phi=1}^{m} |d^{I}_{\phi\phi}| \lambda^I_{\phi} \right\},
\theta_{\min}^{(2)} &= \left\{ c_\phi - \sum_{\phi=1}^{m} |a^{I}_{\phi\phi}| \lambda^I_{\phi} - \sum_{\phi=1}^{m} |a^{R}_{\phi\phi}| \lambda^R_{\phi} - \sum_{\phi=1}^{m} |d^{I}_{\phi\phi}| \lambda^I_{\phi} - \sum_{\phi=1}^{m} |d^{R}_{\phi\phi}| \lambda^R_{\phi} \right\},
\theta_{\max}^{(1)} &= \sum_{\phi=1}^{m} \left\{ \sum_{\phi=1}^{m} |a^{R}_{\phi\phi}| \lambda^R_{\phi} + \sum_{\phi=1}^{m} |b^{R}_{\phi\phi}| \lambda^R_{\phi} + \sum_{\phi=1}^{m} |b^{I}_{\phi\phi}| \lambda^I_{\phi} + \sum_{\phi=1}^{m} |b^{I}_{\phi\phi}| \lambda^R_{\phi} \right\},
\theta_{\max}^{(2)} &= \sum_{\phi=1}^{m} \left\{ \sum_{\phi=1}^{m} |b^{R}_{\phi\phi}| \lambda^R_{\phi} + \sum_{\phi=1}^{m} |b^{I}_{\phi\phi}| \lambda^I_{\phi} + \sum_{\phi=1}^{m} |b^{I}_{\phi\phi}| \lambda^R_{\phi} \right\},
\end{align*}
\]

By Lemma 3, \( \theta_1 > \sqrt{2}\theta_2, \theta_2 > 0 \).

Consequently, Systems (10) and (11) are globally asymptotically stable. \( \Box \)

**Theorem 2.** Under Assumptions (H1) and (H2) and Lemma 3, Systems (10) and (11) are globally asymptotically stable and satisfy \( \Theta_1 > \sqrt{2}\Theta_2, \Theta_2 > 0 \).

**Proof.** Consider the auxiliary function

\[
V(h) = \sum_{\phi=1}^{m} \frac{1}{2} (\psi^R_\phi(h))^2 + \sum_{\phi=1}^{m} \frac{1}{2} (\psi^I_\phi(h))^2.
\]

In light of Lemma 2, calculating the fractional difference of \( V(t) \), we have

\[
\begin{align*}
\mathcal{C}^{\psi^R_\phi}_{0} V(h) &\leq \sum_{\phi=1}^{m} \left\{ - c_\phi (\psi^R_\phi(h))^2 + \sum_{\phi=1}^{m} a^{R}_{\phi\phi} \psi^R_\phi(t) [\lambda^R_{\phi} \psi^R_\phi(h) + \lambda^I_{\phi} \psi^I_\phi(h)] \\
&\quad + \sum_{\phi=1}^{m} a^{I}_{\phi\phi} \psi^R_\phi(h) [\lambda^I_{\phi} \psi^R_\phi(h) + \lambda^I_{\phi} \psi^I_\phi(h)] \\
&\quad + \sum_{\phi=1}^{m} b^{R}_{\phi\phi} \psi^R_\phi(h) [\lambda^R_{\phi} \psi^R_\phi(h - \omega) + \lambda^R_{\phi} \psi^I_\phi(h - \omega)] \\
&\quad + \sum_{\phi=1}^{m} b^{I}_{\phi\phi} \psi^R_\phi(h) [\lambda^I_{\phi} \psi^R_\phi(h - \omega) + \lambda^I_{\phi} \psi^I_\phi(h - \omega)] \\
&\quad + \sum_{\phi=1}^{m} a^{R}_{\phi\phi} \psi^R_\phi(h) \int_{0}^{s} K_{\phi\phi}(s) [\lambda^R_{\phi} \psi^R_\phi(s) + \lambda^R_{\phi} \psi^I_\phi(s)] ds \\
&\quad + \sum_{\phi=1}^{m} a^{I}_{\phi\phi} \psi^R_\phi(h) \int_{0}^{s} K_{\phi\phi}(s) [\lambda^I_{\phi} \psi^R_\phi(s) + \lambda^I_{\phi} \psi^I_\phi(s)] ds \\
&\quad + \sum_{\phi=1}^{m} \left\{ - c_\phi (\psi^I_\phi(h))^2 + \sum_{\phi=1}^{m} a^{R}_{\phi\phi} \psi^I_\phi(h) [\lambda^R_{\phi} \psi^R_\phi(h) + \lambda^I_{\phi} \psi^I_\phi(h)] \\
&\quad + \sum_{\phi=1}^{m} a^{I}_{\phi\phi} \psi^I_\phi(h) [\lambda^I_{\phi} \psi^R_\phi(h) + \lambda^I_{\phi} \psi^I_\phi(h)] \\
&\quad + \sum_{\phi=1}^{m} b^{R}_{\phi\phi} \psi^I_\phi(h) [\lambda^R_{\phi} \psi^R_\phi(h - \omega) + \lambda^R_{\phi} \psi^I_\phi(h - \omega)] \\
&\quad + \sum_{\phi=1}^{m} b^{I}_{\phi\phi} \psi^I_\phi(h) [\lambda^I_{\phi} \psi^R_\phi(h - \omega) + \lambda^I_{\phi} \psi^I_\phi(h - \omega)] \\
&\quad + \sum_{\phi=1}^{m} a^{R}_{\phi\phi} \psi^I_\phi(h) \int_{0}^{s} K_{\phi\phi}(s) [\lambda^R_{\phi} \psi^R_\phi(s) + \lambda^R_{\phi} \psi^I_\phi(s)] ds \\
&\quad + \sum_{\phi=1}^{m} a^{I}_{\phi\phi} \psi^I_\phi(h) \int_{0}^{s} K_{\phi\phi}(s) [\lambda^I_{\phi} \psi^R_\phi(s) + \lambda^I_{\phi} \psi^I_\phi(s)] ds \right\}.
\end{align*}
\]
\[
\sum_{\phi=1}^{m} d_{\phi \psi}^{I}(l) \left[ \lambda_{\phi}^{RR} \psi_{\phi}^{S}(h) + \lambda_{\phi}^{RI} \psi_{\phi}^{I}(h) \right] \\
+ \left\{ -c_{\phi}(\psi_{\phi}^{S}(h))^2 + \left[ \sum_{\phi=1}^{m} a_{\phi \psi}^{RR}(h) \lambda_{\phi}^{RR} \psi_{\phi}^{S}(h) + \sum_{\phi=1}^{m} a_{\phi \psi}^{RI}(h) \lambda_{\phi}^{RI} \psi_{\phi}^{I}(h) \right] \right\} \\
+ \left\{ -c_{\phi}(\psi_{\phi}^{I}(h))^2 + \left[ \sum_{\phi=1}^{m} b_{\phi \psi}^{RR}(h) \lambda_{\phi}^{RR} \psi_{\phi}^{I}(h) + \sum_{\phi=1}^{m} b_{\phi \psi}^{RI}(h) \lambda_{\phi}^{RI} \psi_{\phi}^{I}(h) \right] \right\} \\
+ \left\{ -c_{\phi}(\psi_{\phi}^{I}(h))^2 + \left[ \sum_{\phi=1}^{m} d_{\phi \psi}^{RR}(h) \lambda_{\phi}^{RR} \psi_{\phi}^{I}(h) + \sum_{\phi=1}^{m} d_{\phi \psi}^{RI}(h) \lambda_{\phi}^{RI} \psi_{\phi}^{I}(h) \right] \right\}
\]
\[\begin{align*}
+ \left\{ \sum_{\phi=1}^{m} b_{\phi}^{R} \phi^{R} (h) \lambda^{R} \phi \phi (h - \omega) + \sum_{\phi=1}^{m} b_{\phi}^{R} \phi^{R} (h) \lambda^{I} \phi \phi (h - \omega) \right\} \\
+ \left\{ \sum_{\phi=1}^{m} d_{\phi}^{R} \phi^{R} (h) \lambda^{R} \phi \phi (h) + \sum_{\phi=1}^{m} d_{\phi}^{R} \phi^{R} (h) \lambda^{I} \phi \phi (t) \right\} \\
+ \left\{ \sum_{\phi=1}^{m} d_{\phi}^{R} \phi^{R} (h) \lambda^{R} \phi \phi (h) + \sum_{\phi=1}^{m} d_{\phi}^{R} \phi^{R} (h) \lambda^{I} \phi \phi (h) \right\} \\
+ \left\{ \sum_{\phi=1}^{m} a_{\phi}^{R} \phi^{R} (h) \lambda^{R} \phi \phi (h) + \sum_{\phi=1}^{m} a_{\phi}^{R} \phi^{R} (h) \lambda^{I} \phi \phi (h) \right\}
\end{align*}\]

\[
\begin{align*}
+ \sum_{\phi=1}^{m} \left\{ - c_{\phi} (\phi^{R} (h))^{2} + \sum_{\phi=1}^{m} a_{\phi}^{R} \phi^{R} (h) \lambda^{R} \phi \phi (h) + \sum_{\phi=1}^{m} a_{\phi}^{R} \phi^{R} (h) \lambda^{I} \phi \phi (h) \right\}
\end{align*}\]

(16)
\[
+ \left[ \frac{1}{2} \sum_{\psi=1}^{m} d_{\psi}^{R} \lambda_{\psi}^{R} \left( (\phi_{\psi}(h))^2 + (\phi_{\psi}(h))^2 \right) \right] + \left[ \frac{1}{2} \sum_{\psi=1}^{m} d_{\psi}^{L} \lambda_{\psi}^{L} \left( (\phi_{\psi}(h))^2 + (\phi_{\psi}(h))^2 \right) \right]
\leq -\Theta_{1} \min_{m} \frac{1}{2} \sum_{\psi=1}^{m} (\psi_{\psi}(t))^2 - \Theta_{2} \min_{m} \frac{1}{2} \sum_{\psi=1}^{m} (\psi_{\psi}(t))^2 + \Theta_{1} \max_{m} \frac{1}{2} \sum_{\psi=1}^{m} (\psi_{\psi}(h - \omega))^2 + \Theta_{2} \max_{m} \frac{1}{2} \sum_{\psi=1}^{m} (\psi_{\psi}(h - \omega))^2,
\]

where

\[
\begin{align*}
\Theta_{1} & = \left\{ c_{\psi} - \frac{1}{2} \sum_{\phi=1}^{m} a_{\phi}^{R} \lambda_{\phi}^{R} \right\} - \frac{1}{2} \sum_{\phi=1}^{m} a_{\phi}^{R} \lambda_{\phi}^{R} - \frac{1}{2} \sum_{\phi=1}^{m} b_{\phi}^{R} \lambda_{\phi}^{R} - \frac{1}{2} \sum_{\phi=1}^{m} b_{\phi}^{R} \lambda_{\phi}^{R}
\Theta_{2} & = \left\{ a_{\psi}^{R} \lambda_{\psi}^{R} - \frac{1}{2} \sum_{\psi=1}^{m} a_{\psi}^{R} \lambda_{\psi}^{R} \right\} - \frac{1}{2} \sum_{\psi=1}^{m} a_{\psi}^{R} \lambda_{\psi}^{R} - \frac{1}{2} \sum_{\psi=1}^{m} b_{\psi}^{R} \lambda_{\psi}^{R} - \frac{1}{2} \sum_{\psi=1}^{m} b_{\psi}^{R} \lambda_{\psi}^{R}
\Theta_{1} & = \left\{ \sum_{\phi=1}^{m} \left( b_{\phi}^{R} \lambda_{\phi}^{R} + m \sum_{\phi=1}^{m} b_{\phi}^{R} \lambda_{\phi}^{R} + m \sum_{\phi=1}^{m} b_{\phi}^{R} \lambda_{\phi}^{R} \right) \right\},
\Theta_{2} & = \left\{ m \sum_{\phi=1}^{m} \left( b_{\phi}^{R} \lambda_{\phi}^{R} + m \sum_{\phi=1}^{m} b_{\phi}^{R} \lambda_{\phi}^{R} + m \sum_{\phi=1}^{m} b_{\phi}^{R} \lambda_{\phi}^{R} \right) \right\},
\end{align*}
\]

By Lemma 3, \( \Theta_{1} > \sqrt{2} \Theta_{2} \), and \( \Theta_{2} > 0 \).
Consequently, Systems (10) and (11) are globally asymptotically stable. \( \square \)

**Remark 1.** Consider the master system

\[
\mathbf{C} \mathbf{D}_{x}^{\alpha} \gamma_{\phi}(h) = -c_{\phi} \gamma_{\phi}(h) + \sum_{\phi=1}^{m} a_{\phi} f_{\phi}(\gamma_{\phi}(h)) + \sum_{\phi=1}^{m} b_{\phi} f_{\phi}(\gamma_{\phi}(h - \omega)) \tag{17}
\]

By Lemma 3, \( \Theta_{1} > \sqrt{2} \Theta_{2} \), and \( \Theta_{2} > 0 \).
Consequently, Systems (10) and (11) are globally asymptotically stable. \( \square \)

**Remark 1.** Consider the master system

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\]

By Lemma 3, \( \Theta_{1} > \sqrt{2} \Theta_{2} \), and \( \Theta_{2} > 0 \).
Consequently, Systems (10) and (11) are globally asymptotically stable. \( \square \)
Consider the slave system
\[ C \nabla^\beta_0 \tilde{\varphi}(\bar{t}) = -c_\varphi \tilde{\varphi}(\bar{t}) + \sum_{\varphi=1}^{m} a_{\varphi} \tilde{\varphi}(\bar{t}) \tilde{f}_\varphi(\bar{t}) + \sum_{\varphi=1}^{m} b_{\varphi} \tilde{\varphi}(\bar{t}) \tilde{f}_\varphi(\bar{t}) \] 
\[ + \sum_{\varphi=1}^{m} d_{\varphi} \int_{0}^{\infty} K_{\varphi}(s) f_{\varphi}(s) ds + I_{\varphi} + u_{\varphi}(\bar{t}), \quad \bar{t} \in \mathbb{N}_1, \]  
(18)

The error system is defined as
\[ C \nabla^\beta_0 \psi(\bar{t}) = -c_\psi \psi(\bar{t}) + \sum_{\varphi=1}^{m} a_{\varphi} \psi(\bar{t}) \psi(\bar{t}) + \sum_{\varphi=1}^{m} b_{\varphi} \psi(\bar{t}) \psi(\bar{t}) \] 
\[ + \sum_{\varphi=1}^{m} d_{\varphi} \int_{0}^{\infty} K_{\varphi}(s) f_{\varphi}(s) ds + u_{\varphi}(\bar{t}), \quad \bar{t} \in \mathbb{N}_1, \]  
(19)

**Theorem 3.** Under Assumptions (H1) and (H2), the system is globally Mittag–Leffler stable if the activation functions are bounded. Let \( u_{\varphi}(\bar{t}) = 0 \); then, 
\[ 0 < \rho = \rho_1 - \rho_2 < 1, \]

\[ \rho_1 = \min_{1 \leq \varphi \leq m} \left\{ c_\varphi - \sum_{\varphi=1}^{m} |a_{\varphi}| L_\varphi - \sum_{\varphi=1}^{m} |d_{\varphi}| L_\varphi \right\}, \]

\[ \rho_2 = \max_{1 \leq \varphi \leq m} \left\{ \sum_{\varphi=1}^{m} |b_{\varphi}| L_\varphi \right\} > 0 \]

**Proof.** Let us consider the Lyapunov functional
\[ V(t, \xi(t)) = \sum_{\varphi=1}^{m} |\psi(t)|. \]

(21)

By calculating the Nabla–Caputo left-fractional difference of \( V(t) \) along the trajectories of System (1), we obtain
\[ C \nabla^\beta_0 V(t, \psi(h)) \leq \sum_{\varphi=1}^{m} \text{sign}(\psi(h)) \left\{ -c_\psi \psi(h) + \sum_{\varphi=1}^{m} a_{\varphi} \psi(h) \psi(\bar{t}) + \sum_{\varphi=1}^{m} b_{\varphi} \psi(h) \psi(\bar{t}) \right\} \]
\[ + \sum_{\varphi=1}^{m} d_{\varphi} \int_{0}^{\infty} K_{\varphi}(s) f_{\varphi}(\psi(h)) ds \]
\[ \leq \sum_{\varphi=1}^{m} \left\{ -c_\psi |\psi(h)| + \sum_{\varphi=1}^{m} |a_{\varphi}| |\psi(h)| + \sum_{\varphi=1}^{m} |b_{\varphi}| |\psi(h)| \right\} \]
\[ + \sum_{\varphi=1}^{m} |d_{\varphi}| \int_{0}^{\infty} K_{\varphi}(s) f_{\varphi}(\psi(h)) ds \]
\[ \leq \sum_{\varphi=1}^{m} \left\{ -c_\psi |\psi(h)| + \sum_{\varphi=1}^{m} |a_{\varphi}| |\psi(h)| + \sum_{\varphi=1}^{m} |b_{\varphi}| |\psi(h)| + \sum_{\varphi=1}^{m} |d_{\varphi}| |\psi(h)| \right\} \]
\[ + \sum_{\varphi=1}^{m} \sum_{\varphi=1}^{m} |b_{\varphi}| |\psi(h)| \]
\[ \leq -\rho_1 V(h, \psi(h)) + \rho_2 \sup_{h - \tau \leq s \leq h} V(s, \psi(s)) \]
as any solution \( \varphi(t) \) of Error System (22), which satisfies the Razumikhin condition. Hence, on the basis of the Razumikhin technique, one has the criteria

\[
\sup_{\bar{h}-\omega \leq s \leq t} V(s, \varphi(s)) \leq V(h, \varphi(h)) \tag{23}
\]

Next, based on Systems (22) and (23), assume that there exists a constant \( \Delta > 0 \). One can then obtain

\[
D^\beta V(h, \varphi(h)) \leq -(\rho_1 - \rho_2) V(h, \varphi(h)),
\]

\[
\rho_1 - \rho_2 \geq \Delta, \tag{24}
\]

and from (24), one observes that

\[
D^\beta V(h, \varphi(h)) \leq -\Delta V(h, \varphi(h)) \tag{25}
\]

Then, from (25) and Lemma 1, one has

\[
V(h, \varphi(h)) \leq V(0) E_\beta(-\Delta h^\alpha), h \in [0, \infty) \tag{26}
\]

Therefore, one concludes that

\[
||\varphi(h)|| = ||\dot{\gamma}(h) - \gamma(h)||,
\]

\[
= \sum_{\psi=1}^m |\gamma_\psi(h) - \gamma(h)||, \tag{27}
\]

\[
\leq ||\psi_0 - \psi_0|| E_\alpha(-\delta h^\alpha)
\]

According to Definition 3, the fractional-order complex-valued neural network (1) achieves global Mittag–Leffler synchronization with fractional Systems (10) and (11). This completes the proof of Theorem 3. \( \square \)

4. Numerical Examples

Numerical examples are provided to demonstrate the validity of the results in this section.

Example 1. Consider the following discrete-time fractional-order complex-valued neural networks:

\[
\nabla_0^\beta \gamma_\psi(h) = -c_\psi \gamma_\psi(h) + \sum_{\phi=1}^m a_{\psi \phi} f_\phi(\gamma_\phi(h)) + \sum_{\phi=1}^m b_{\psi \phi} f_\phi(\gamma_\phi(h - \omega)) + \sum_{\phi=1}^m d_{\psi \phi} \int_0^\infty f_\phi(\gamma_\phi(s))ds \tag{28}
\]

Suppose \( \beta = 0.95 \) and \( \omega = 0.2 \) and suppose the parameters and the function are defined by

\[
\begin{align*}
C &= \begin{bmatrix} 16 & 8 \\ 4 & 16 \end{bmatrix}, \\
A &= \begin{bmatrix} -0.4 - 0.2i & 0.2 + 0i \\ 0.2 - 0.5i & 0.4 + 0.2i \end{bmatrix}, \\
B &= \begin{bmatrix} 0.7 - 0.7i & 0.4 + 0.4i \\ 0.3 + 0.9i & -0.4 + 0.1i \end{bmatrix}, \\
D &= \begin{bmatrix} 0.6 - 0.8i & 0.3 + 0.4i \\ 0.2 + 0.3i & -0.7 + 0.7i \end{bmatrix}.
\end{align*}
\]

\[
U_R = \begin{bmatrix} 0.1 \tan(t) & 0.4 \cot(t) \end{bmatrix}, \quad U_I = \begin{bmatrix} 0.3 \tan(t) & 0.7 \cot(t) \end{bmatrix}
\]
Example 2. Consider the following discrete-time fractional-order complex-valued neural networks:

\[
\mathcal{C} \nabla^\beta_0 \Gamma_\phi(h) = -c \phi \Gamma_\phi(h) + \sum_{\phi=1}^m a_{\phi \phi} f_\phi(\gamma_\phi(h)) + \sum_{\phi=1}^m b_{\phi \phi} f_\phi(\gamma_\phi(h-\omega)) + \sum_{\phi=1}^m d_{\phi \phi} \int_0^\infty f_\phi(\gamma_\phi(s)) ds \tag{29}
\]

where \( \beta = 0.56 \), \( \gamma_\phi(h) = \gamma_\phi^R(h)i \), \( \gamma_\phi^R(t), \gamma_\phi^I(t) \in \mathbb{R} \), \( f_\phi(\gamma_\phi) = 0.3\tanh(\gamma_\phi^R) + 0.3\tanh(\gamma_\phi^I) \), \( g_\phi(\gamma_\phi) = 0.65\tanh(\gamma_\phi^R) + 0.65\tanh(\gamma_\phi^I)i \), \( \tau = 3 \), and

\[
C = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.9 \end{bmatrix},
\]

\[
A = \begin{bmatrix} 0.1 + 0.97i & 0.6 + 0.3i & 0.5 + 0.2i \\ 0.9 + 0.2i & 0.2 + 0.7i & 0.6 + 0.5i \\ 0.5 + 0.1i & 0.6 + 0.5i & 0.6 + 0.7i \end{bmatrix},
\]

\[
B = \begin{bmatrix} 0.2 + 0.87i & 5.56 + 3.45i & 3.56 + 5.45i \\ 0.56 + 2.45i & 4.56 + 2.45i & 0.56 + 6.45i \\ 1.56 + 0.35i & 2.56 + 4.45i & 0.56 + 8.45i \end{bmatrix},
\]

\[
D = \begin{bmatrix} 0.3 + 0.29i & 3.56 + 4.45i & 4.56 + 2.45i \\ 0.56 + 2.55i & 2.56 + 5.45i & 1.56 + 3.45i \\ 3.56 + 0.45i & 1.56 + 4.35i & 4.56 + 3.45i \end{bmatrix}.
\]

Assumptions \( H_1 \) and \( H_2 \) are satisfied for \( \lambda^R_i, \lambda^I_i, \lambda^R_R, \lambda^I_R = 1 \), \( c_1 = 28.95 \), \( |a_{11}^R| = 0.75 \), \( |a_{11}^I| = 0.97 \), \( |b_{11}^R| = 0.95 \), \( |b_{11}^I| = 0.87 \), \( |d_{11}^R| = 0.28 \), \( |d_{11}^I| = 0.29 \).

From Theorem 1, \( \theta_{\min}^{(1)} = 24.3700, \theta_{\min}^{(2)} = 24.3700, \theta_{\max}^{(1)} = 3.56, \theta_{\max}^{(2)} = 3.64 \), and \( \theta_1 = 48.7400, \theta_2 = 7.2000 \).

For these conditions, we have 48.7400 < 10.1823. Hence, it follows from Theorem 1 that the system can achieve global asymptotic synchronization.

Example 3. Consider the following discrete-time fractional order complex-valued neural networks:

\[
\mathcal{C} \nabla^\beta_0 \Gamma_\phi(h) = -c \phi \Gamma_\phi(h) + \sum_{\phi=1}^m a_{\phi \phi} f_\phi(\gamma_\phi(h)) + \sum_{\phi=1}^m b_{\phi \phi} f_\phi(\gamma_\phi(h-\omega)) + \sum_{\phi=1}^m d_{\phi \phi} \int_0^\infty f_\phi(\gamma_\phi(s)) ds \tag{30}
\]

Suppose \( \beta = 0.98 \) and \( \omega = 0.2 \) and suppose the parameters and the function are defined by \( \lambda^R_i, \lambda^I_i, \lambda^R_R, \lambda^I_R = 1 \), \( c_1 = 19.95 \), \( |a_{11}^R| = 0.72 \), \( |a_{11}^I| = 0.77 \), \( |b_{11}^R| = 0.94 \), \( |b_{11}^I| = 0.27 \), \( |d_{11}^R| = 0.23 \), \( |d_{11}^I| = 0.69 \).

\[
C = \begin{bmatrix} 16 & 8 \\ 4 & 16 \end{bmatrix},
\]

\[
A = \begin{bmatrix} -0.4 - 0.2i & 0.2 + 0i \\ 0.2 + 0.5i & 0.4 + 0.2i \end{bmatrix},
\]

\[
B = \begin{bmatrix} 0.7 - 0.7i & 0.4 + 0.4i \\ 0.3 + 0.9i & -0.4 + 0.1i \end{bmatrix}.
\]
\[
\begin{bmatrix}
0.6 - 0.8i & 0.3 + 0.4i \\
0.2 + 0.3i & -0.7 + 0.7i
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.2 \tan(t) & 0.5 \cot(t) \\
0.3 \tan(t) & 0.7 \cot(t)
\end{bmatrix}
\]

\[\Theta_{\min}^{(1)} = 7.8300, \Theta_{\min}^{(2)} = 8.35, \Theta_{\max}^{(1)} = 2.75, \Theta_{\max}^{(2)} = 2.48.\]  
For these conditions, we have 16,1800 > 7.3963. Hence, it follows from Theorem 2 that this system can achieve global asymptotic synchronization.

**Example 4.** Consider the discrete-time fractional-order complex-valued neural network

\[
\begin{aligned}
C_{0}^{\beta} \gamma_{\phi}(h) &= -c_{\phi} \gamma_{\phi}(h) + \sum_{\phi=1}^{m} a_{\phi} f_{\phi}(\gamma_{\phi}(h)) + \sum_{\phi=1}^{m} b_{\phi} f_{\phi}(\gamma_{\phi}(h - \omega)) + \sum_{\phi=1}^{m} d_{\phi} \int_{0}^{\infty} f_{\phi}(\gamma_{\phi}(s))ds
\end{aligned}
\]  

(31)
Suppose $\beta = 0.98$ and $\omega = 0.2$ and suppose the parameters and the function are defined by

$$C = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.9 \end{bmatrix},$$

$$A = \begin{bmatrix} 0.1 + 0.97i & 0.2 + 0.6i & 0.5 + 0.2i \\ 0.9 + 0.2i & 0.2 + 0.7i & 0.6 + 0.5i \\ 0.5 + 0.1i & 0.6 + 0.5i & 0.6 + 0.7i \end{bmatrix},$$

$$B = \begin{bmatrix} 0.2 + 0.87i & 5.56 + 3.45i & 3.56 + 5.45i \\ 0.56 + 2.45i & 4.56 + 2.45i & 0.56 + 6.45i \\ 1.56 + 0.35i & 2.56 + 4.45i & 0.56 + 8.45i \end{bmatrix},$$

$$D = \begin{bmatrix} 0.3 + 0.29i & 3.56 + 4.45i & 4.56 + 2.45i \\ 0.56 + 2.55i & 2.56 + 3.45i & 1.56 + 3.45i \\ 3.56 + 4.5i & 1.56 + 4.35i & 4.56 + 3.45i \end{bmatrix}.$$

Assumptions $H_1$, $H_2$ are satisfied for $c_1 = 2.14$, $a_{11} = 0.94$, $d_{11} = 0.92$, $b_{11} = 0.87$, $L_1 = 1$.

By Theorem 3, we find that $\rho_1 = 1.28 > 0$, $\rho_2 = 0.87 > 0$.

Hence, $0 < \rho_1 - \rho_2 = 0.41 < 1$, and Theorem 3 holds. Therefore, (31) is globally Mittag–Leffler stable.

**Remark 2.** Many scholars have discussed the uniform stability, global asymptotic stability, and finite-time stability of fractional-order CVNNs with time delays Zhang et al. [30], Rakkiyappan et al. [21], Wang et al. [19], and Song et al. [22]. Most of these scholars considered that the activation functions of complex-valued systems can be separated into their real parts and imaginary parts. Thus, they transformed CVNNs to equivalent RVNNs to analyze their dynamic behavior. However, this method increases the dimension of systems and brings difficulties upon analysis. Compared with the existing literature, regardless of the activity, functions are separable, and the provided existence and finite-time stability criteria for discrete fractional-order CVNNs are valid and feasible in this paper.

**Remark 3.** Many authors studied the dynamics prosperities of discrete fractional difference equations in a real field. However, there are very few results about discrete fractional-order system in complex fields. Different from the existing literature, we first investigated discrete fractional-order CVNNs and analyzed their dynamic behavior.

**Remark 4.** In the aforementioned works, it is noted that only the discrete constant delays are involved in the network models. In this situation, discrete delays cannot well characterize the neural networks since the signal propagation is no longer instantaneous. Consequently, the distributed delays should also be taken into account in the description of neural network models. In recent decades, many researchers have made great efforts to the dynamics of neural networks with both discrete and distributed delays, and there have been some excellent results. Notice that these works were mainly concerned with integer-order neural networks. Research on fractional-order neural networks with discrete and distributed time delays has received little attention.

**5. Conclusions**

The synchronization of discrete-time fractional-order complex-valued neural networks with distributed delays is examined in this research. By building suitable Lyapunov functions, sufficient conditions are attained. The resulting results are fresh and add to the global Mittag–Leffler synchronization findings for fractional networks that already exist. Some adequate requirements are derived from the theory of discrete fractional calculus, the discrete Laplace transform, the theory of complex functions, and discrete Mittag–Leffler functions in order to guarantee the global stability and synchronization of the Mittag–Leffler function for the suggested networks. In future research, we will further study the dynamical behaviors, such as projective synchronization and finite time stability, of more
sophisticated neural networks, including fractional-order coupled discontinuous neural networks with time-varying delays.

**Author Contributions:** Methodology, R.P. and M.H.; Software, T.F.I.; Validation, M.S.A.; Formal analysis, B.A.A.M.; Investigation, W.M.O. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by the Deanship of Scientific Research at King Khalid University under grant number RGP/2/141/44.

**Data Availability Statement:** There is no data associated with this study.

**Acknowledgments:** The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large groups (project under grant number RGP/2/141/44).

**Conflicts of Interest:** The authors declare no conflict of interest.

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