Dynamical Behaviour, Control, and Boundedness of a Fractional-Order Chaotic System

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Abstract: In this paper, the fractional-order chaotic system form of a four-dimensional system with cross-product nonlinearities is introduced. The stability of the equilibrium points of the system and then the feedback control design to achieve this goal have been analyzed. Furthermore, further dynamical behaviors including, phase portraits, bifurcation diagrams, and the largest Lyapunov exponent are presented. Finally, the global Mittag–Leffler attractive sets (MLASs) and Mittag–Leffler positive invariant sets (MLPIs) of the considered fractional order system are presented. Numerical simulations are provided to show the effectiveness of the results.

Keywords: fractional-order hyperchaotic system; global Mittag–Leffler attractive sets (MLASs); Mittag–Leffler positive invariant sets (MLPIs); chaos control

1. Introduction

In recent years, fractional-order calculus has become a prominent part of applied research. Since applied sciences are naturally related to systems with memory, which dominates most physical and scientific systems, modeling with fractional order equations has been an undeniable necessity [1–5]. Fractional-order differential equations, which are actually generalizations of integer order calculus, deal with derivatives and integrals with non-integer orders. These equations have a unique feature which is the description of the memory effect [6–8]. Fractional derivative chaotic systems are now an important topic of study in nonlinear dynamics. The fractional-order chaotic system is considered a new alternative for which significant attention has been focused on developing techniques for modelling, synchronization, and control of the generalized dynamical systems [9–13]. Recently, fractional-order calculus has been applied to financial market changes. Investigations have indicated that economics and finance are extremely complex nonlinear systems involving many subjective factors, and there are many characteristics that cannot be modeled by integer-order calculus. Therefore, it is necessary to try to use the bifurcation and chaos of fractional nonlinear dynamics [14–16].

A chaotic system is a bounded nonlinear deterministic system with a long-term nature depending on the initial condition. The importance of the boundedness of a dynamical system can be found from the definition of chaos. Therefore, estimating the ultimate bound and positive invariant set of a chaotic system plays an important role in predicting its behavior, controlling chaos and preventing possible failures in physical and engineering systems. It can be clearly seen that the existence of the global attractor set for a system means the absence of hidden attractors outside this set [17–20].

By studying the ultimate bound of chaotic systems by Leonov [18], a new direction for research in this field was introduced. Many efforts have been made to develop methods
for estimating the ultimate bound and the global attractive set of chaotic systems [21–24]. Despite many problems in determining the ultimate bound of some dynamical systems, various methods were presented that generally used optimization and the method of Lagrange coefficients [25–27]. Among the interesting studies that have been carried out so far, we can mention the calculation of the ultimate bound for various classical and generalized Lorenz systems [28], autonomous high-dimensional systems [29], complex Lorenz system [30], financial risk system [14], etc.

The four-dimensional autonomous system that indicates the equations related to the laser is defined as follows:

\[
\begin{align*}
\dot{y}_1 &= -ky_1 + ky_2, \\
\dot{y}_2 &= ry_1 - y_2 - ey_3 - y_1y_4, \\
\dot{y}_3 &= ey_2 - y_3, \\
\dot{y}_4 &= -by_4 + y_1y_2,
\end{align*}
\]

where \(k, r, b, e\) are real system parameters. The dynamical behaviour of this system such as stability of equilibria, Hopf bifurcations and randomness of multistability regions were studied by Natiq et al. [31]. One of the subjects that is necessary to investigate for this system is the estimation of the bound set. Of course, the ultimate bound set for system (1) has not been taken into account neither in the integer-order nor fractional-order. In order to develop methods for fractional-order systems, in this article we introduce the four-dimensional fractional-order chaotic system.

In this paper, a four-dimensional fractional-order system with cross-product nonlinearities has been introduced and analyzed. Using the concept of stability in fractional-order systems, the stability of equilibrium points and chaos control are performed theoretically and numerically. Furthermore, the dynamical characteristics of the considered system, such as the existence or absence of chaos, bifurcation diagrams, and Lyapunov exponents are investigated. The calculation of the ultimate bound called Mittag–Leffler bound set for chaotic dynamical systems has been implemented for a small number of systems. The MLASs and MLPISs for the chaotic fractional systems have been estimated so far [32–34]. The main contribution of this research work is to estimate the MLASs and MLPISs for a fractional system that belongs to a special class of nonlinear systems with only cross-product nonlinearities. By constructing a suitable generalized Lyapunov function and using the extremum principle of function, we obtain the new 3D ellipsoid estimations of the bounds for the presented fractional system, which improve the earlier publications and can deduce some new estimations.

This article is organized in the following sections. Section 2 presents a fractional-order chaotic system and its dynamical behaviors are theoretically and numerically investigated. In Section 3, we apply the feedback control method to study chaos control. In Section 4, we will study the Mittag–Leffler attractive sets (MLASs) and Mittag–Leffler positive invariant sets (MLPISs) of the fractional-order chaotic system. Conclusions are drawn in Section 5.

2. Fractional-Order Chaotic System

In this section, we present a fractional-order system derived from system (1). This four-dimensional system has nonlinear terms in the product form, which we define in a special class of chaotic systems. The new fractional-order chaotic system can be expressed as follows:

\[
\begin{align*}
D_q^\alpha y_1(t) &= -ky_1 + ky_2, \\
D_q^\alpha y_2(t) &= ry_1 - y_2 - ey_3 - y_1y_4, \\
D_q^\alpha y_3(t) &= ey_2 - y_3, \\
D_q^\alpha y_4(t) &= -by_4 + y_1y_2,
\end{align*}
\]
where $D_q^t$ is $q$-order Caputo differential operator and $q$ is the derivative order:

$$
D_q^t y_i(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} y_i^{(n)}(s) ds, \quad i = 1, 2, 3.
$$

(3)

**Definition 1** ([32]). The Mittag–Leffler function $E_{\eta_1, \eta_2}(\cdot)$ with two parameters is defined as

$$
E_{\eta_1, \eta_2}(c) = \sum_{i=0}^{\infty} \frac{c^i}{\Gamma(i\eta_1 + \eta_2)},
$$

(4)

where $\eta_1 > 0, \eta_2 > 0,$ and $c$ is a complex number. Obviously,

$$
E_{\eta_1}(c) = E_{\eta_1, 1}(c), E_{0,1}(c) = \frac{1}{1-c}, \quad E_{1,1}(c) = e^c.
$$

2.1. Stability Analysis of Equilibrium Points

Here, we consider the stability of the fractional-order nonlinear system (2). The equilibrium points of fractional-order system (2) can be obtained by solving

$$
\begin{align*}
-ky_1 + ky_2 &= 0, \\
ry_1 - y_2 - ey_3 - y_1 y_4 &= 0, \\
ev_2 - y_3 &= 0, \\
-by_4 + y_1 y_2 &= 0.
\end{align*}
$$

(5)

If selecting the parameters as $k = 4, e = 0.5, r = 27,$ and $b = 1,$ there are three equilibrium points of system (2), which as follows:

$$
E_1 = (0, 0, 0, 0),
$$

$$
E_2 = \left(5.074445783, 5.074445783, 2.537222891, 25.75\right),
$$

$$
E_3 = \left(-5.074445783, -5.074445783, -2.537222891, 25.75\right).
$$

(6)

Then, we have the following conclusions for the stability of these equilibrium points.

**Theorem 1** ([4]). Consider the fractional-order nonlinear dynamical system

$$
D_q^t y(t) = f(y(t)), \quad 0 < q < 1,
$$

(7)

where $f = [f_1, f_2, \ldots, f_n]^T$, is a vector function and $f_i(i = 1, 2, \ldots, n)$ are continuous differential nonlinear functions. The equilibrium points $E^*$ of system (7) are asymptotically stable if all the eigenvalues $\lambda_i(i = 1, 2, \ldots, n)$ of the Jacobian matrix $J = \frac{\partial f}{\partial y}$, satisfy the following condition:

$$
|\arg(\lambda_i)| > \frac{q\pi}{2}(i = 1, 2, \ldots, n).
$$

The Jacobian matrix of the system (2) at the equilibrium point of $E^*$ is

$$
J(E^*) = \begin{bmatrix}
-k & k & 0 & 0 \\
-r - y_4^* & -1 & -e & -y_1^* \\
0 & e & -1 & 0 \\
y_2^* & y_1^* & 0 & -b
\end{bmatrix}.
$$
The characteristic equation of the Jacobian matrix \(J(E^*)\) is given by

\[
C(\lambda) = \begin{vmatrix}
\lambda + k & -k & 0 & 0 \\
y_4^* - r & \lambda + 1 & e & y_1^* \\
0 & -e & \lambda + 1 & 0 \\
y_2^* & -y_1^* & 0 & \lambda + b
\end{vmatrix}
\]

\[
= \lambda^4 + (b + k + 2)\lambda^3 + \left((b - r + y_4^* + 2)k + e^2 + y_1^* + 2b + 1\right)\lambda^2 \\
+ \left((-r + y_4^* + 2)b + e^2 + y_1^* + y_2^* - r + y_4^* + 1\right)k + e^2b + y_1^* + b)\lambda \\
+ k\left((e^2 - r + y_4^* + 1)b + y_1^*(y_1^* + y_2^*)\right) = 0.
\]

At the equilibrium point \(E_1\) and parameters \(k = 4, e = 0.5, r = 27,\) and \(b = 1\) the characteristic equation, becomes

\[
C(\lambda) = \lambda^4 + 7\lambda^3 + 36\lambda^2 + 236\lambda + 206 = 0,
\]

and eigenvalues of Equation (10) are

\[
\lambda_1 = 0.1842463692 - 5.684434590i, \quad \lambda_2 = -1, \\
\lambda_3 = 0.1842463692 + 5.684434590i, \quad \lambda_4 = -6.368492738.
\]

Since,

\[
|\arg(\lambda_1)| = 1.538395235, \quad |\arg(\lambda_2)| = \pi, \\
|\arg(\lambda_3)| = 1.538395235, \quad |\arg(\lambda_4)| = 1.538395235,
\]

by Theorem 1, equilibrium point \(E_1\) is asymptotically stable, when \(q < \frac{2|\arg(\lambda_{1,4})|}{\pi} = 0.9793.\)

2.2. Dynamical Behaviors

In this section, the basic dynamical properties such as phase portraits, bifurcation diagram, Lyapunov exponents and chaos diagram of the fractional-order chaotic system (2) are presented.

Selecting the parameters \(k = 4, e = 0.5, r = 27,\) and \(b = 1,\) Figure 1 depicts Lyapunov exponents. When \(q = 0.999,\) the values of Lyapunov exponents at 500th second are \(L_1 = 0.5291, L_2 = -0.0022, L_3 = -0.9789,\) and \(L_4 = -4.7122.\)

![Figure 1. Lyapunov exponent spectra for system (2) with \(k = 4, e = 0.5, r = 27,\) and \(b = 1.\)](image)

The Benettin–Wolf algorithm to determine all Lyapunov exponents for the fractional-order systems modeled by Caputo’s derivative is used [35]. The values of Lyapunov...
exponents at 500th second are obtained as $L_1 = -0.0783$, $L_2 = -0.0839$, $L_3 = -0.9501$, and $L_4 = -2.9103$. As it is clear from Figures 1 and 2, for some values of $q$, system (2) has positive Lyapunov exponent and therefore is in a chaotic state and for different values of $q$, system (2) converges to the equilibrium point.

The bifurcation diagram of system (2) is drawn based on the change of the derivative order parameter and by fixing the values of other parameters as $k = 4, e = 0.5, r = 27$, and $b = 1$. Figure 3, shows the effect of varying parameter $q$ in the interval $(0.954, 1)$. It is obvious that with different values of the derivative order of $q$, the system (2) is in stable, periodic and chaotic state and the bifurcation diagram is well consistent with the Lyapunov exponent spectrum. The system is stable in $q \in (0.954, 0.978)$ and the system exhibits chaotic behavior when $q \in (0.978, 1)$.

Figure 2. Lyapunov exponent spectra for system (2) with $k = 4, e = 0.5, r = 27$, and $b = 1$.

Figure 3. Bifurcation diagram with fixed values $k = 4, e = 0.5, r = 27, b = 1$, and $q \in (0.954, 1)$.

When $k = 4, e = 0.5, r = 27$, and $b = 1$, the phase portraits of system (2) are shown in Figure 4, that indicates for the value of $q = 0.97$, the trajectories of the system converge to the equilibrium point.

When $q = 0.99$, $k = 4, e = 0.5, r = 27$, and $b = 1$, the phase portraits of system (2) are shown in Figure 5. The system exhibits the chaotic behavior and the equilibrium point is not stable.
Figure 4. Phase portraits for (2) with $q = 0.97$, $k = 4$, $e = 0.5$, $r = 27$, and $b = 1$.

Figure 5. Cont.
3. Chaos Control

In this section, we apply the linear feedback control method to stabilize system (2) to its equilibrium points. Let consider the following controlled form of fractional-order system (2):

\[
\begin{align*}
D^q_t y_1(t) &= -ky_1 + ky_2 - w_1(y_1 - y^*_1), \\
D^q_t y_2(t) &= ry_1 - y_2 - ey_3 - y_1y_4 - w_2(y_2 - y^*_2), \\
D^q_t y_3(t) &= ey_2 - y_3 - w_3(y_3 - y^*_3), \\
D^q_t y_4(t) &= -by_4 + y_1y_2 - w_4(y_4 - y^*_4),
\end{align*}
\]

(9)

where \(w_1, w_2, w_3, w_4\) are control parameters and \(E^*\) is the equilibrium point of the system (2).

The Jacobian matrix at \(E^*\) for the system (9) is obtained as

\[
J(E^*) = \begin{bmatrix}
-k - w_1 & k & 0 & 0 \\
0 & -1 - w_2 & -e & -y^*_1 \\
0 & e & -1 - w_3 & 0 \\
y^*_2 & y^*_1 & 0 & -b - w_4
\end{bmatrix}.
\]

The characteristic equation of the Jacobian matrix \(J(E^*)\) is given by

\[
f(\lambda) = \begin{vmatrix}
\lambda + k + w_1 & -k & 0 & 0 \\
y^*_4 - r & \lambda + 1 + w_2 & e & y^*_1 \\
0 & -e & \lambda + 1 + w_3 & 0 \\
-y^*_2 & -y^*_1 & 0 & \lambda + b + w_4
\end{vmatrix} = 0.
\]
Selecting \( k_1 = 1, k_2 = 2, k_3 = 3 \) and \( k_4 = 4 \), the characteristic polynomial of for the equilibrium point \( E_1 \) with the parameters \( k = 4, e = 0.5, r = 27, \) and \( b = 1 \) is given as

\[
C(\lambda) = \lambda^4 + 17\lambda^3 + 128\lambda^2 + 587.25\lambda + 1133.25 = 0, \tag{10}
\]

and the characteristic roots of Equation (10) are calculated as

\[
\lambda_1 = -2.539703887 - 5.413874345i, \quad \lambda_2 = -7.918563161, \quad \lambda_3 = -4.002029066, \quad \lambda_4 = -2.539703887 + 5.413874345i.
\]

Since, all the eigenvalues of Equation (10) are negative real numbers and complex numbers with negative real parts, therefore the equilibrium point \( E_1 \) is asymptotically stable for all \( 0 < q < 1 \). When \( q = 0.99, k = 4, e = 0.5, r = 27, \) and \( b = 1 \), the phase portraits and time series of the controlled system (9) are shown in Figure 6.

Figure 6. Phase portraits and time series of the controlled system (9) with \( q = 0.99, k = 4, e = 0.5, \) \( r = 27, \) and \( b = 1 \).

4. Global Mittag–Leffler Attractive Sets

In this section, our main aim is to estimate the global Mittag–Leffler attractive sets (MLASs) and Mittag–Leffler positive invariant sets (MLPIs) for the fractional-order chaotic
system (2). Firstly, we introduce some definitions for calculating the MLAS and MLPIS of fractional-order systems with only cross-product nonlinearities.

**Lemma 1** ([32]). Let \( y(t) \in \mathbb{R} \), is the continuous and differentiable functions, then

\[
\frac{1}{2} D^\alpha (y^2(t)) \leq y(t) D^\alpha (y(t)),
\]

(11)

**Lemma 2** ([32]). For \( q \in (0, 1) \) and constant \( \gamma \in \mathbb{R} \), if a continuous function \( y(t) \) meets

\[
D^\alpha (y(t)) \leq \gamma y(t), \quad t \geq 0,
\]

(12)

then

\[
y(t) \leq y(0) E_q(\gamma t^\alpha), \quad t \geq 0.
\]

(13)

**Definition 2** ([32]). Let us consider the following fractional-order system

\[
D^\alpha_i y_i = f(y_i), \quad y_i(t_0) = y_{i,0}, \quad i = 1, 2, \cdots, n,
\]

(14)

where \( y_i \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n \) is sufficiently smooth, and \( y_i(t, t_0, y_{i,0}) \) is the solution.

For a given Lyapunov function \( \mathcal{L}(Y(t)) \), if there exist constants \( K_\gamma > 0 \) and \( \gamma > 0 \) for all \( Y_0 \in \mathbb{R}^n \) such that

\[
\mathcal{L}(Y(t)) - K_\gamma \leq (\mathcal{L}(Y(t_0)) - K_\gamma) E_q(-\gamma(t - t_0)^\alpha), \quad t \geq t_0,
\]

(15)

for \( \mathcal{L}(Y(t)) > K_\gamma \), then \( \mathcal{G} = \{ (y_1, y_2, \cdots, y_n) | \mathcal{L}(Y(t)) \leq K_\gamma \} \) is said to be the global Mittag–Leffler attractive sets (MLASs) of system (14). If for any \( Y_0 \in \mathcal{G} \) and any \( t > t_0 \), \( Y(t, t_0, Y_0) \in \mathcal{G} \), then \( \mathcal{G} \) is said to be a Mittag–Leffler positive invariant sets (MLPISs).

### 4.1. Main Results

Here we will discuss the ultimate bound of a class of the fractional-order chaotic systems with only cross-product nonlinearities:

\[
D^\alpha_1 y_1(t) = \sum_{j=1}^{4} a_1 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{1jk} y_j y_k + r_1,
\]

\[
D^\alpha_2 y_2(t) = \sum_{j=1}^{4} a_2 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{2jk} y_j y_k + r_2,
\]

\[
D^\alpha_3 y_3(t) = \sum_{j=1}^{4} a_3 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{3jk} y_j y_k + r_3,
\]

(16)

\[
D^\alpha_4 y_4(t) = \sum_{j=1}^{4} a_4 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{4jk} y_j y_k + r_4.
\]

(17)

The candidate Lyapunov function for this problem is defined as follows

\[
\mathcal{L}(Y(t)) = \frac{1}{2} (\omega_1 y_1 - s_1)^2 + \frac{1}{2} (\omega_2 y_2 - s_2)^2 + \frac{1}{2} (\omega_3 y_3 - s_3)^2 + \frac{1}{2} (\omega_4 y_4 - s_4)^2,
\]

(18)

where \( \omega_i, s_i, i = 1, 2, 3, 4 \) are known parameters. The function \( h(Y, \delta) \) is considered as follows
\[ h(Y, \delta) = \omega_1(\omega_1y_1 - s_1) \left( \sum_{j=1}^{4} \alpha_1 y_j + \frac{\delta_1}{\omega_1} (\omega_1y_1 - s_1) + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{1jk} y_j y_k + r_1 \right) + \omega_2(\omega_2 y_2 - s_2) \left( \sum_{j=1}^{4} \alpha_2 y_j + \frac{\delta_2}{\omega_2} (\omega_2 y_2 - s_2) + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{2jk} y_j y_k + r_2 \right) + \omega_3(\omega_3 y_3 - s_3) \left( \sum_{j=1}^{4} \alpha_3 y_j + \frac{\delta_3}{\omega_3} (\omega_3 y_3 - s_3) + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{3jk} y_j y_k + r_3 \right) + \omega_4(\omega_4 y_4 - s_4) \left( \sum_{j=1}^{4} \alpha_4 y_j + \frac{\delta_4}{\omega_4} (\omega_4 y_4 - s_4) + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{4jk} y_j y_k + r_4 \right), \] (19)

where \( \delta = (\delta_1, \delta_2, \delta_3, \delta_4). \)

**Theorem 2.** Let \( h(Y_0, \delta) > 0 \) is the supremum of function \( h(Y, \delta) \) on \( Y \in \mathbb{R}^4 \) defined in (19), \( \gamma = \min \{\delta_1, \delta_2, \delta_3, \delta_4\} \). Then the following inequality is fulfilled:

\[ \mathcal{L}(Y(t)) - \frac{1}{2\gamma} h(Y_0, \delta) \leq \left( \mathcal{L}(Y(t_0)) - \frac{1}{2\gamma} h(Y_0, \delta) \right) \mathcal{E}_4(-2\gamma(t - t_0)^\eta), \quad t \geq t_0. \] (20)

Furthermore,

\[ \mathcal{G} = \{ Y | \mathcal{L}(Y(t)) \leq \frac{1}{2\gamma} h(Y_0, \delta) \} = \{ Y | (\omega_1 y_1 - s_1)^2 + (\omega_2 y_2 - s_2)^2 + (\omega_3 y_3 - s_3)^2 + (\omega_4 y_4 - s_4)^2 \leq \frac{1}{2\gamma} h(Y_0, \delta) \}, \] (21)

is the Mittag–Leffler attractive sets (MLASs) and Mittag–Leffler positive invariant sets (MLPISs) of system \( (2) \).

**Proof.** Computing the fractional derivative of \( \mathcal{L}(Y(t)) \) along the trajectory of system (16) and using Lemma 1, we have

\[
D^\eta \mathcal{L}(Y(t)) \leq (\omega_1 y_1 - s_1) D^\eta \omega_1 y_1 + (\omega_2 y_2 - s_2) D^\eta \omega_2 y_2 + (\omega_3 y_3 - s_3) D^\eta \omega_3 y_3 + (\omega_4 y_4 - s_4) D^\eta \omega_4 y_4
\]
\[
= \omega_1(\omega_1 y_1 - s_1) \left( \sum_{j=1}^{4} \alpha_1 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{1jk} y_j y_k + r_1 \right) + \omega_2(\omega_2 y_2 - s_2) \left( \sum_{j=1}^{4} \alpha_2 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{2jk} y_j y_k + r_2 \right) + \omega_3(\omega_3 y_3 - s_3) \left( \sum_{j=1}^{4} \alpha_3 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{3jk} y_j y_k + r_3 \right) + \omega_4(\omega_4 y_4 - s_4) \left( \sum_{j=1}^{4} \alpha_4 y_j + \sum_{j=1}^{4} \sum_{k=1}^{4} \beta_{4jk} y_j y_k + r_4 \right)
\]
\[
= -\delta_1 (\omega_1 y_1 - s_1)^2 - \delta_2 (\omega_2 y_2 - s_2)^2 - \delta_3 (\omega_3 y_3 - s_3)^2 - \delta_4 (\omega_4 y_4 - s_4)^2 + h(Y, \delta) \leq -\gamma \sum_{j=1}^{4} (\omega_j y_j - s_j)^2 + h(Y, \delta)
\]
\[
\leq -\gamma \sum_{j=1}^{4} (\omega_j y_j - s_j)^2 + h(Y_0, \delta).
\] (22)
Theorem 3. Let consider (MLASs) and Mittag–Leffler positive invariant sets (MLPISs) of the system (2).

Proof. is the Mittag–Leffler attractive sets (MLASs) and Mittag–Leffler positive invariant sets (MLPISs) matrix on \( E \) conditions for estimating the MLAS and MLPIS for the system (2). The following theorem determines the necessary conditions for estimating the MLAS and MLPIS for the system (2).

4.2. MLAS and MLPIS for the Fractional-Order System (2)

To investigate the correctness of Theorem 4.1, we consider the calculation of the bound set for the fractional-order system (2). The following theorem determines the necessary conditions for estimating the MLAS and MLPIS for the system (2).

Theorem 3. Let consider \( \delta_1 < k, \delta_2 < 1, \delta_3 < 1, \delta_4 < b, \) and \( E_0 = (y_1^0, y_2^0, y_3^0, y_4^0) \) be the stable point of \( h(Y, \delta) \) defined by (19). Moreover, the Hessian Matrix of \( h(Y, \delta) \) is a negative definite matrix on \( E_0 \). Then,

\[
\mathcal{G} = \left\{ Y(t) \in \mathbb{R}^4 \mid \omega_1 y_1^2 + \omega_2 y_2^2 + \omega_3 y_3^2 + \omega_2 (y_4 - \frac{kw_1 + rw_2}{\omega_2})^2 \leq \frac{b^2(kw_1 + rw_2)^2}{4(b - \delta_4)} \right\},
\]

is the Mittag–Leffler attractive sets (MLASs) and Mittag–Leffler positive invariant sets (MLPISs) of the system (2).

Proof. Let \( a_{11} = -k, a_{12} = k, a_{21} = r, a_{22} = -1, a_{32} = -e, a_{33} = e, a_{33} = -1, a_{44} = -b, b_{214} = -1, b_{32} = 1, b_1 = t_2 = t_3 = t_4 = 0 \), else \( a_{ij} = 0, b_{ijk} = 0 \), and \( s_1 = s_2 = s_3 = 0, s_4 = kw_1 + rw_2 \).

Thus, \( \mathcal{L}(Y(t)) \) can be rewritten as:

\[
\mathcal{L}(Y(t)) = \frac{1}{2} \omega_1 y_1^2 + \frac{1}{2} \omega_2 y_2^2 + \frac{1}{2} \omega_3 y_3^2 + \frac{1}{2} \omega_2 (y_4 - \frac{kw_1 + rw_2}{\omega_2})^2,
\]

and

\[
h(Y, \delta) = \omega_1^2 y_1 (-ky_1 + ky_2 + \delta_1 y_1) + \omega_2^2 y_2 (ry_1 - y_2 - ey_3 - y_1 y_4 + \delta_2 y_2) + \omega_3^2 y_3 (ey_2 - y_3 + \delta_3 y_3) + \omega_2 (\omega_2 y_4 - (kw_1 + rw_2)) \left( -by_4 + y_1 y_2 + \frac{\delta_4}{\omega_2} (\omega_2 y_4 - (kw_1 + rw_2)) \right).
\]

Since \( E_0 \) is the stable point of the \( h(Y, \delta) \), we have

\[
\nabla h|_{E_0} = (h_{y_1}, h_{y_2}, h_{y_3}, h_{y_4}) = 0.
\]

and the Hessian Matrix \( H \) of the \( h(Y, \delta) \) is a negative definite matrix. Therefore,
\[ h'_{y_1} = \omega_1^2(2 - ky_1 + ky_2 + \delta_1 y_1) + \omega_1^2 y_1(-k + \delta_1) + \omega_2^2 y_2(r - y_4) + \omega_3^2(-k\omega_1 - r\omega_2 + \omega_2 y_4)y_2, \]

\[ h'_{y_2} = \omega_1^2 y_1 k + \omega_2^2(-ey_3 + ry_1 + \delta_2 y_2 - y_1 y_4 - y_2) + \omega_2^2 y_2(\delta_2 - 1) + \omega_2^2 y_3 e + \omega_3^2(-k\omega_1 - r\omega_2 + \omega_2 y_4)y_1, \]

\[ h'_{y_3} = -\omega_2^2 y_2 e + \omega_2^2(e y_2 + \delta_3 y_3 - y_3) + \omega_2^2 y_3(\delta_3 - 1), \]

\[ h'_{y_4} = -\omega_2^2 y_2 y_1 + \omega_2^2(-by_4 + y_1 y_2 + \delta_4(-k\omega_1 - r\omega_2 + \omega_2 y_4)/\omega_2) + \omega_3^2(-k\omega_1 - r\omega_2 + \omega_2 y_4)(-b + \delta_4), \]

\[ h''_{y_1 y_1} = 2\omega_1^2(-k + \delta_1), \]

\[ h''_{y_1 y_2} = 2\omega_2^2(\delta_2 - 1), \]

\[ h''_{y_3 y_3} = 2\omega_2^2(\delta_3 - 1), \]

\[ h''_{y_4 y_4} = 2\omega_2^2(-b + \delta_4), \]

\[ h''_{y_1 y_2} = h''_{y_2 y_1} = k\omega_1^2 - k\omega_1\omega_2, \]

\[ h''_{y_1 y_3} = h''_{y_3 y_1} = 0, \]

\[ h''_{y_1 y_4} = h''_{y_4 y_1} = 0, \]

\[ h''_{y_2 y_3} = h''_{y_3 y_2} = -e\omega_2^2 + e\omega_3^2, \]

\[ h''_{y_2 y_4} = h''_{y_4 y_2} = 0, \]

\[ h''_{y_3 y_3} = h''_{y_4 y_4} = 0, \]

\[ \begin{vmatrix} h''_{y_1 y_1} & h''_{y_1 y_2} \\ h''_{y_2 y_1} & h''_{y_2 y_2} \end{vmatrix} = \begin{vmatrix} 2\omega_1^2(-k + \delta_1) & k\omega_1^2 - k\omega_1\omega_2 \\ k\omega_1^2 - k\omega_1\omega_2 & 2\omega_2^2(\delta_2 - 1) \end{vmatrix} > 0, \]

\[ \begin{vmatrix} h''_{y_1 y_1} & h''_{y_1 y_2} & h''_{y_1 y_3} \\ h''_{y_2 y_1} & h''_{y_2 y_2} & h''_{y_2 y_3} \\ h''_{y_3 y_1} & h''_{y_3 y_2} & h''_{y_3 y_3} \end{vmatrix} = \begin{vmatrix} 2\omega_1^2(-k + \delta_1) & k\omega_1^2 - k\omega_1\omega_2 & 0 \\ k\omega_1^2 - k\omega_1\omega_2 & 2\omega_2^2(\delta_2 - 1) & -e\omega_2^2 + e\omega_3^2 \\ 0 & -e\omega_2^2 + e\omega_3^2 & 2\omega_2^2(\delta_3 - 1) \end{vmatrix} < 0. \]

And

\[ \begin{vmatrix} h''_{y_1 y_1} & h''_{y_1 y_2} & h''_{y_1 y_3} & h''_{y_1 y_4} \\ h''_{y_2 y_1} & h''_{y_2 y_2} & h''_{y_2 y_3} & h''_{y_2 y_4} \\ h''_{y_3 y_1} & h''_{y_3 y_2} & h''_{y_3 y_3} & h''_{y_3 y_4} \\ h''_{y_4 y_1} & h''_{y_4 y_2} & h''_{y_4 y_3} & h''_{y_4 y_4} \end{vmatrix} = \begin{vmatrix} 2\omega_1^2(-k + \delta_1) & k\omega_1^2 - k\omega_1\omega_2 & 0 & 0 \\ k\omega_1^2 - k\omega_1\omega_2 & 2\omega_2^2(\delta_2 - 1) & -e\omega_2^2 + e\omega_3^2 & 0 \\ 0 & -e\omega_2^2 + e\omega_3^2 & 2\omega_2^2(\delta_3 - 1) & 0 \\ 0 & 0 & 0 & 2\omega_2^2(-b + \delta_4) \end{vmatrix} > 0. \]

By solving the following system we will have

\[
\begin{cases}
  h'_{y_1} = 0, \\
  h'_{y_2} = 0, \\
  h'_{y_3} = 0, \\
  h'_{y_4} = 0.
\end{cases}
\]
\[ y_1 = y_2 = y_3 = 0, \quad y_4 = \frac{(b-2\delta_4)(k\omega_1 + r\omega_2)}{2\omega_2(b-\delta_4)}, \quad \text{and} \quad h(Y, \delta) \leq \sup_{Y \in \mathbb{R}^4} h(Y, \delta) = \frac{b^2(k\omega_1 + r\omega_2)^2}{4(b-\delta_4)}.\]

Using Lemma 2, gives the following result
\[
\mathcal{L}(Y(t)) - \frac{b^2(k\omega_1 + r\omega_2)^2}{8\gamma(b-\delta_4)} \leq \left[ \mathcal{L}(Y(0)) - \frac{b^2(k\omega_1 + r\omega_2)^2}{8\gamma(b-\delta_4)} \right] E_q(-2\gamma t^q), \quad t \geq 0.
\]

and the Mittag–Leffler attractive sets (MLASs) and Mittag–Leffler positive invariant sets (MLPISs) of the system (2) is
\[
\mathcal{G} = \left\{ Y(t) \in \mathbb{R}^4 \mid \omega_1 y_1^2 + \omega_2 y_2^2 + \omega_3 y_3^2 + \omega_2 \left( y_4 - \frac{k\omega_1 + r\omega_2}{\omega_2} \right)^2 \leq \frac{b^2(k\omega_1 + r\omega_2)^2}{4\gamma(b-\delta_4)} \right\}, \quad (30)
\]

\[ \square \]

Remark 1. It is obvious that by changing the parameters \( \omega_1, \omega_2, \omega_3, \delta_1, \delta_2, \delta_3, \text{and} \delta_4, \) different estimates of the global attractive set and the positive reliable set for system (2) can be obtained.

Let us take \( \omega_1 = \omega_2 = \omega_3 = 1, \delta_1 = 1, \delta_2 = \delta_3 = \delta_4 = 0.999, \quad q = 0.99, k = 4, \quad e = 0.5, r = 27, \quad \text{and} \quad b = 1. \) Then
\[
\mathcal{G} = \left\{ (y_1, y_2, y_3, y_4) \mid y_1^2 + y_2^2 + y_3^2 + (y_4 - 31)^2 \leq (31)^2 \right\}.
\]

is the Mittag–Leffler MLAS and Mittag–Leffler MLPIS of system (2). Figure 7 shows the Mittag–Leffler MLAS and phase portrait of system (2).

When \( q = 0.9, k = 4, e = 0.5, r = 27, \quad \text{and} \quad b = 1, \) Figure 8 shows the phase portrait and Mittag–Leffler MLAS of system (2).

![Figure 7](image_url)

Figure 7. The estimated Mittag–Leffler MLAS of system (2), where \( q = 0.99, k = 4, e = 0.5, r = 27, \quad \text{and} \quad b = 1. \)
5. Conclusions

In this paper, we present a four-dimensional fractional-order system with cross-product nonlinearities. Using the concept of stability in fractional-order systems, the stability of equilibrium points and chaos control were performed theoretically and numerically. Furthermore, the dynamical characteristics of the considered system, such as the existence or absence of chaos, bifurcation diagrams, and Lyapunov exponents are investigated. Finally, the estimation of the MLASs and MLPISs for a fractional system that belongs to a special class of nonlinear systems is presented.

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