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Article

The Existence of Mild Solutions for Hilfer Fractional Stochastic Evolution Equations with Order $\mu \in (1, 2)$

Qien Li [1] and Yong Zhou [1,2,*]

[1] Faculty of Innovation Engineering, Macau University of Science and Technology, Macau 999078, China; 2109853mii30001@student.must.edu.mo
[2] School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China
* Correspondence: yzhou@xtu.edu.cn

Abstract: In this study, we investigate the existence of mild solutions for a class of Hilfer fractional stochastic evolution equations with order $\mu \in (1, 2)$ and type $\nu \in [0, 1]$. We prove the existence of mild solutions of Hilfer fractional stochastic evolution equations when the semigroup is compact as well as noncompact. Our approach is based on the Schauder fixed point theorem, the Ascoli–Arzelà theorem and the Kuratowski measure of noncompactness. An example is also provided, to demonstrate the efficacy of this method.

Keywords: stochastic evolution equation; Hilfer fractional derivative; mild solution

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1. Introduction

We discuss the Hilfer fractional stochastic systems:

$$\begin{cases} {}^{H}D_{0^+}^{\mu,\nu} y(t) = Ay(t) + f(t, y(t)) + h(t, y(t))\frac{d\mathbb{W}(t)}{dt}, \quad t \in (0, b], \\ (I_{0^+}^{2-\alpha} y)(0) = y_0, \quad (I_{0^+}^{2-\alpha} y)'(0) = y_1. \end{cases} \tag{1}$$

In the above equation, ${}^{H}D_{0^+}^{\mu,\nu}$ represents the Hilfer fractional derivative with order $\mu \in (1, 2)$ and type $\nu \in [0, 1]$, while $I_{0^+}^{2-\alpha}$ is the Riemann–Liouville integral operator with order $(2 - \alpha)$, where $\alpha = \mu + \nu(2 - \mu)$. We assume that $\mathbb{H}$ is a separable Hilbert space, and that A: $D(A) \subset \mathbb{H} \to \mathbb{H}$, being the infinitesimal generator of a cosine family $\{C(t)\}_{t \geq 0}$, consisting of strongly continuous and uniformly bounded linear operators. Moreover, we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and that $\mathbb{K}$ is a separable Hilbert space. The $\mathbb{K}$-value Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is denoted by $\{\mathbb{W}(t)\}_{t\geq 0}$, and it has a finite trace nuclear covariance operator $\mathcal{Q} \geq 0$. Finally, we assume that f: $(0, b] \times \mathbb{H} \to \mathbb{H}$ and h: $(0, b] \times \mathbb{H} \to L(\mathbb{K}, \mathbb{H})$ are given functions that satisfy appropriate assumptions, and that $y_0, y_1 \in L_0^2(\Omega, \mathbb{H})$.

Fractional differential equations can effectively describe complex systems that many integer differential equations cannot. As a result, fractional differential equations are widely used in the field of science, and many scholars around the world have taken an interest in studying them [1–3]. Moreover, Hilfer fractional derivatives encompass Caputo and Riemann–Liouville fractional derivatives as special cases, which has led some scholars to concentrate on Hilfer fractional differential equations in their research [4–8].

Random disturbances are commonplace in many real-life scenarios. Stochastic differential equations can solve option pricing problems in the field of economics, and can be used for escape and jump problems of Brownian particles in the field of physics, among other applications. As a result, stochastic differential equations are widely employed in diverse scientific fields, including economics, chemistry, physics and social sciences. The existence, uniqueness, stability and controllability of solutions for fractional stochastic differential equations are very important. To this end, researchers have established mild solutions for various classes of stochastic evolution equations. Specifically, Refs. [9–11] demonstrated the existence of mild solutions for a class of Caputo fractional stochastic evolution equations with order $\mu \in (0,1)$. Refs. [12,13] established the existence of mild solutions for a class of stochastic evolution equations with Riemann–Liouville fractional derivatives. Additionally, refs. [14,15] proved the existence of mild solutions for a class of stochastic evolution equations with Hilfer fractional derivatives ${}^{H}D_{0^+}^{\mu,\nu}$, where $\mu \in (0,1)$, $\nu \in [0,1]$.

While numerous research results have been conducted on fractional stochastic differential equations with order $\mu \in (1,2)$ and type $\nu \in [0,1]$, no research has been conducted on Hilfer fractional stochastic differential equations with order $\mu \in (1,2)$ and type $\nu \in [0,1]$. To address this gap, this paper introduces a novel concept of mild solutions for Hilfer fractional evolution equations with order $\mu \in (1,2)$ and type $\nu \in [0,1]$, which is based on the cosine family.

To provide a clear structure for this article, we have divided it into several sections. In Section 2, we introduce some basic facts that are needed for our analysis. In Section 3, we prove some lemmas, and make certain assumptions. In Section 4, we verify the existence problem of Equation (1) based on the proved lemma and some new methods. In Section 5, we provide an example, to verify the validity of our results. Finally, a summary of this thesis is provided in Section 6.

2. Preliminaries

Denoting $L^2(\Omega,\mathbb{H})$ as a Banach space of all strongly measurable, square-integrable and $\mathbb{H}$-valued random variables, the norm $\|y(\cdot)\|_{L^2(\Omega,\mathbb{H})} = (E\|y(\cdot,\mathbb{W})\|^2)^{\frac{1}{2}}$, where the expectation E is defined by $E(y) = \int_\Omega y(\mathbb{W})d\mathbb{P}$. An important subspace of $L^2(\Omega,\mathbb{H})$ is $L_0^2(\Omega,\mathbb{H}) = \{y \in L^2(\Omega,\mathbb{H});\ y \text{ is } \mathcal{F}_0 - measurable\}$. Let $C([0,b],L^2(\Omega,\mathbb{H}))$ be the Banach space of all continuous maps from $[0,b]$ into $L^2(\Omega,\mathbb{H})$, with the norm $\|y(\cdot)\|_C = (\sup_{t\in[0,b]} E\|y(t)\|^2)^{\frac{1}{2}} < \infty$.

$L(\mathbb{K},\mathbb{H})$ denote the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$, with the norm $\|\cdot\|$. We assume that there exists a complete orthonormal basis $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$. Denote $Tr(\mathcal{Q}) = \sum_{k=1}^{\infty}\lambda_k < \infty$, that satisfies $\mathcal{Q}e_k = \lambda_k e_k, k \in \mathbb{N}$. By Proposition 2.9 in [16], provided that $\psi(t) \in L(\mathbb{K},\mathbb{H})$, and that $\psi(t)$ is measurable with respect to $\mathcal{F}_t$ for $t \in [0,b]$, and satisfies

$$\int_0^t E\|\psi(s)\|^2 ds < \infty,$$

then we have the following property:

$$E\Big\|\int_0^t \psi(s)d\mathbb{W}(s)\Big\|^2 \leq Tr(\mathcal{Q})\int_0^t E\|\psi(s)\|^2 ds. \tag{2}$$

For convenience of calculation in this paper, we introduce the function $g_\alpha(\cdot)$, which is defined as

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\alpha > 0$, and gamma function $\Gamma(\cdot)$ satisfies $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$. In case $\alpha = 0$, we denote $g_0(t) = \delta(t)$; the Dirac measure is concentrated at the origin.

Definition 1 (see [2])**.** *The Riemann–Liouville fractional integral is defined as follows:*

$$I_{0^+}^{\mu} y(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}y(s)ds = (g_\mu * y)(t),\ t > 0,\ \mu > 0,$$

where $*$ is the convolution.

Definition 2 (see [2])**.** *The Riemann–Liouville fractional derivative is defined as follows:*

$$^{RL}D^{\mu}_{0^+}y(t) = \frac{1}{\Gamma(n-\mu)}\frac{d^n}{dt^n}(\int_0^t (t-s)^{n-\mu-1}y(s)ds) = \frac{d^n}{dt^n}(g_{n-\mu} * y)(t),\ t>0,\ n-1<\mu<n;$$

in particular, its Laplace transform is as follows:

$$\mathfrak{L}\big(^{RL}D^{\mu}_{0^+}y(t)\big) = \lambda^{\mu-1}\mathfrak{L}\big(y(t)\big)(\lambda) - \sum_{k=0}^{n}(g_{n-\mu} * u)(0)\lambda^{n-1-k}. \tag{3}$$

Definition 3 (see [2]). *The Caputo fractional derivative is defined as follows:*

$$^{C}D^{\mu}_{0^+}y(t) = \frac{1}{\Gamma(n-\mu)}\int_0^t (t-s)^{n-\mu-1}y^n(s)ds := g_{n-\mu} * (\frac{d^n}{dt^n}y)(t),\quad t>0,\ n-1<\mu<n,$$

where the function $y(t)$ is $n-1$ times continuously differentiable, and is absolutely continuous.

Definition 4 (see [1]). *The Hilfer fractional derivative is defined as follows:*

$$^{H}D^{\mu,\nu}_{0^+}y(t) = I^{\nu(n-\mu)}_{0^+}\frac{d^n}{dt^n}I^{(1-\nu)(n-\mu)}_{0^+}y(t),\ t>0,$$

where $n-1<\mu<n, 0\le\nu\le 1$.

Remark 1. *(i) In particular, if $\nu=0, n-1<\mu<n$, then*

$$^{H}D^{\mu,0}_{0^+}y(t) = \frac{d^n}{dt^n}I^{(n-\mu)}_{0^+}y(t) = {}^{RL}D^{\mu}_{0^+}y(t);$$

(ii) If $\nu=1, n-1<\mu<n$, then

$$^{H}D^{\mu,1}_{0^+}y(t) = I^{(n-\mu)}_{0^+}\frac{d^n}{dt^n}y(t) = {}^{C}D^{\mu}_{0^+}y(t).$$

Remark 2. *The focus of this article is on the case where $0<\nu<1$; however, we note that when $\nu=1$ or $\nu=0$, the conclusions in this article still apply.*

Definition 5 (see [3]). *If $(I^{2-\alpha}_{0^+}y)(t)$ is continuous and $(I^{2-\alpha}_{0^+}y)'(t)$ is absolutely continuous, then*

$$I^{\alpha}_{0^+}(\,^{RL}D^{\alpha}_{0^+}y(t)) = y(t) - \frac{(I^{2-\alpha}_{0^+}y)(0)}{\Gamma(\alpha-1)}t^{\alpha-2} - \frac{(I^{2-\alpha}_{0^+}y)'(0)}{\Gamma(\alpha)}t^{\alpha-1},$$

where $1<\alpha<2$.

Let $\mathbb{U}$ be the bounded subset of Banach space X with the norm $\|\cdot\|_X$. The Kuratowski measure of noncompactness χ is defined as follows:

$$\chi(\mathbb{U}) = \inf\{r>0 : \mathbb{U}\subset\bigcup_{j=1}^{n}\mathbb{V}_j \ and\ diam(\mathbb{V}_j)\le r\},$$

where $diam(\mathbb{V}_j) = sup\{||x-y||_X : x,y\in\mathbb{V}_j\}, j=1,2,\cdots,n$.

Lemma 1 (see [17]). *Let $\{y_n(t)\}_{n=1}^{\infty} : [0,b]\to X$ be a sequence of Bochner integrable function, if there exists $\phi\in L([0,b],\mathbb{R}^+)$, such that*

$$\|y_n(t)\|_X \le \phi(t),\quad t\in[0,b].$$

Then, $\chi(\{y_n(t)\}_{n=1}^{\infty})$ belongs to $L([0,b],\mathbb{R}^+)$, and satisfies

$$\chi(\{\int_0^t y_n(s)ds : n=1,2,\cdots\}) \leq 2\int_0^t \chi(\{y_n(s) : n=1,2,\cdots\})ds.$$

Definition 6 (see [18]). *The Wright function M_ν is defined as follows:*

$$M_\nu(\vartheta) = \sum_{m=1}^{\infty} \frac{(-\vartheta)^{m-1}}{(m-1)!\Gamma(1-\nu m)},\ 0<\nu<1,\ \vartheta \in \mathbb{C},$$

which satisfies

$$\int_0^\infty \vartheta^\delta M_\nu(\vartheta)d\vartheta = \frac{\Gamma(1+\delta)}{\Gamma(1+\nu\delta)},\ for\ \delta \geq 0.$$

Definition 7 (see [19]). *If X is a Banach space, then bounded linear operators mapping $\{\mathcal{C}(t)\}_{t\in\mathbb{R}} : X \to X$ are called a strongly continuous cosine family if and only if*

(i) $\mathcal{C}(t''+t') + \mathcal{C}(t''-t') = 2\mathcal{C}(t'')\mathcal{C}(t')$ *for all* $t'', t' \in \mathbb{R}$,
(ii) $\mathcal{C}(0)$ *is the identity operator I, and*
(iii) $\mathcal{C}(t)y$ *is continuous for $t \in \mathbb{R}$ and $y \in X$.*

One parameter family, $\{\mathcal{S}(t)\}_{t\in\mathbb{R}}$, is defined by

$$\mathcal{S}(t)y = \int_0^t \mathcal{C}(s)yds,\ t \in \mathbb{R},\ y \in X,$$

where $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family in X.

The infinitesimal generator of a strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ is the operator A: $X \to X$, defined by

$$Ay = \left(\frac{d^2}{dt^2}\mathcal{C}(t)y\right)_{t=0},\ y \in D(A),$$

where $D(A) = \{y \in X : \mathcal{C}(t)y$ is a twice continuously differentiable function with respect to t$\}$.

Lemma 2 (see [19]). *Strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$ satisfying $||\mathcal{C}(t)||_X \leq M_0 e^{\omega|t|}$ in X, for all $t \geq 0$ and some $\omega \geq 0$, $M_0 \geq 1$, and A being the infinitesimal generator of $\{\mathcal{C}(t)\}_{t\in\mathbb{R}}$. Then, for $Re\lambda > \omega$, $\lambda^2 \in \rho(A)$ and*

$$\lambda R(\lambda^2; A)y = \int_0^\infty e^{-\lambda t}\mathcal{C}(t)ydt, \quad R(\lambda^2; A)y = \int_0^\infty e^{-\lambda t}\mathcal{S}(t)ydt, \quad for\ y \in X.$$

In this paper, because A is the infinitesimal generator of a strongly continuous cosine family of uniformly bounded linear operators $\{\mathcal{C}(t)\}_{t\geq 0}$ in $\mathbb{H}$, there exists a constant $M \geq 1$, such that $\|\mathcal{C}(t)\|_{L(\mathbb{H})} \leq M$ for $t \geq 0$.

Lemma 3. *The problem Equation (1) is considered in the corresponding integral form, as follows:*

$$\begin{aligned} y(t) = {} & g_{\alpha-1}(t)y_0 + g_\alpha(t)y_1 + \int_0^t g_\mu(t-s)[Ay(s) + f(s,y(s))]ds \\ & + \int_0^t g_\mu(t-s)h(s,y(s))d\mathbb{W}(s),\ t \in (0,b], \end{aligned} \tag{4}$$

where $\alpha = \mu + \nu(2-\mu)$.

Proof. When $0 < t \leq b$, it follows from Definitions 2 and 4 that

$$\begin{aligned} I^{\mu}_{0^+}({}^{H}D^{\mu,\nu}_{0^+}y)(t) &= I^{\mu}_{0^+}I^{\nu(2-\mu)}_{0^+}(\frac{d^2}{dt^2}I^{(1-\nu)(2-\mu)}_{0^+}y)(t) \\ &= I^{\mu+\nu(2-\mu)}_{0^+}({}^{RL}D^{\mu+\nu(2-\mu)}_{0^+}y)(t) \\ &= I^{\alpha}_{0^+}({}^{RL}D^{\alpha}_{0^+}y)(t). \end{aligned} \tag{5}$$

We can deduce, based on Definition 5, that

$$I^{\alpha}_{0^+}({}^{RL}D^{\alpha}_{0^+}y)(t) = y(t) - \frac{(I^{2-\alpha}_{0^+}y)(0)}{\Gamma(\alpha-1)}t^{\alpha-2} - \frac{(I^{2-\alpha}_{0^+}y)'(0)}{\Gamma(\alpha)}t^{\alpha-1}. \tag{6}$$

Thus, the operators $I^{\mu}_{0^+}$ act simultaneously on both sides of Equation (1), by using (5) and (6), and we can deduce that

$$\begin{aligned} y(t) &= \frac{y_0}{\Gamma(\alpha-1)}t^{\alpha-2} + \frac{y_1}{\Gamma(\alpha)}t^{\alpha-1} + I^{\mu}_{0^+}(Ay(t)) + I^{\mu}_{0^+}(f(t,y(t))) + I^{\mu}_{0^+}[h(t,y(t))\frac{d\mathbb{W}(t)}{dt}] \\ &= g_{\alpha-1}(t)y_0 + g_{\alpha}(t)y_1 + \int_0^t g_{\mu}(t-s)[Ay(s)+f(s,y(s))]ds + \int_0^t g_{\mu}(t-s)h(s,y(s))\frac{d\mathbb{W}(s)}{ds}ds \\ &= g_{\alpha-1}(t)y_0 + g_{\alpha}(t)y_1 + \int_0^t g_{\mu}(t-s)[Ay(s)+f(s,y(s))]ds + \int_0^t g_{\mu}(t-s)h(s,y(s))d\mathbb{W}(s). \end{aligned}$$

The proof is complete. □

Lemma 4. *If y(t) satisfies integral Equation (4), then*

$$\begin{aligned} y(t) = {}^{RL}D^{1-\zeta}_{0^+}(t^{p-1}Q_p(t)y_0) + I^{\zeta}_{0^+}(t^{p-1}Q_p(t)y_1) + \int_0^t (t-s)^{p-1}Q_p(t-s)f(s,y(s))ds \\ + \int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y(s))d\mathbb{W}(s), \end{aligned} \tag{7}$$

for $t \in (0,b]$, $\zeta = \nu(2-\mu) \in (0,1)$, $\mu = 2p$, *where*

$$Q_p(t) = \int_0^{\infty} p\vartheta M_p(\vartheta)\mathcal{S}(t^p\vartheta)d\vartheta.$$

Proof. From Lemma 3, and using convolution calculation, we can obtain

$$y(t) = g_{\alpha-1}(t)y_0 + g_{\alpha}(t)y_1 + g_{\mu} * Ay(t) + g_{\mu} * f(t,y(t)) + g_{\mu} * [h(t,y(t))\frac{d\mathbb{W}(t)}{dt}]. \tag{8}$$

Assuming $Re\lambda > 0$, we denote the Laplace transform by $\mathfrak{L}$, and then

$$\begin{aligned} a(\lambda) &:= \mathfrak{L}(y(t))(\lambda) = \int_0^{\infty} e^{-\lambda s}y(s)ds, \\ b(\lambda) &:= \mathfrak{L}(f(t,y(t)))(\lambda) = \int_0^{\infty} e^{-\lambda s}f(s,y(s))ds, \\ c(\lambda) &:= \mathfrak{L}\big(h(t,y(t))\frac{d\mathbb{W}(t)}{dt}\big)(\lambda) = \int_0^{\infty} e^{-\lambda s}[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]ds. \end{aligned}$$

By using the Laplace transform on Equation (8), we obtain

$$\begin{aligned} a(\lambda) &= \frac{1}{\lambda^{\alpha-1}}y_0 + \frac{1}{\lambda^{\alpha}}y_1 + \frac{1}{\lambda^{\mu}}Aa(\lambda) + \frac{1}{\lambda^{\mu}}b(\lambda) + \frac{1}{\lambda^{\mu}}c(\lambda) \\ &= \lambda^{\mu-\alpha+1}(\lambda^{\mu}I-A)^{-1}y_0 + \lambda^{\mu-\alpha}(\lambda^{\mu}I-A)^{-1}y_1 + (\lambda^{\mu}I-A)^{-1}b(\lambda) + (\lambda^{\mu}I-A)^{-1}c(\lambda). \end{aligned} \tag{9}$$

By Lemma 2 and $\mu = 2p$, we obtain

$$a(\lambda) = \lambda^{\mu-\alpha+1}\int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)y_0 dt + \lambda^{\mu-\alpha}\int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)y_1 dt \\ + \int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)b(\lambda)dt + \int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)c(\lambda)dt.$$

In Appendix B of [18], the one-sided stable probability is denoted as follows:

$$\psi_p(\vartheta) = \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m-1}\vartheta^{-pm-1}\frac{\Gamma(pm+1)}{m!}sin(pm\pi),\ \vartheta \in (0,\infty),$$

and its Laplace transform is

$$\int_0^\infty e^{-\lambda\vartheta}\psi_p(\vartheta)d\vartheta = e^{-\lambda^p},\ p \in (\frac{1}{2},1).$$

Meanwhile, we can obtain

$$M_p(\vartheta) = \frac{1}{p}\vartheta^{-\frac{1}{p}-1}\psi_p(\vartheta^{-\frac{1}{p}}),\ \vartheta \in (0,\infty),$$

where $M_p(\vartheta)$ refers to the Wright function defined in Definition 6.

Thus, we have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)c(\lambda)dt \\
&= \int_0^\infty p t^{p-1}e^{-(\lambda t)^p}\mathcal{S}(t^p)c(\lambda)dt \\
&= \int_0^\infty\int_0^\infty p t^{p-1}e^{-(\lambda t)^p}\mathcal{S}(t^p)e^{-\lambda s}[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]dsdt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty p t^{p-1}e^{-\lambda t\vartheta}\psi_p(\vartheta)\mathcal{S}(t^p)e^{-\lambda s}[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]d\vartheta dsdt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty p\frac{t^{p-1}}{\vartheta^p}e^{-\lambda(t+s)}\psi_p(\vartheta)\mathcal{S}(\frac{t^p}{\vartheta^p})[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]d\vartheta dsdt \\
&= \int_0^\infty\int_0^t\int_0^\infty p\frac{(t-s)^{p-1}}{\vartheta^p}e^{-\lambda t}\psi_p(\vartheta)\mathcal{S}(\frac{(t-s)^p}{\vartheta^p})[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]d\vartheta dsdt \\
&= \int_0^\infty e^{-\lambda t}\left\{\int_0^t\int_0^\infty p\vartheta(t-s)^{p-1}\frac{1}{p}\vartheta^{-\frac{1}{p}-1}\psi_p(\vartheta^{-\frac{1}{p}})\mathcal{S}((t-s)^p\vartheta)[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]d\vartheta ds\right\}dt \\
&= \int_0^\infty e^{-\lambda t}\left\{\int_0^t\int_0^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}((t-s)^p\vartheta)[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]d\vartheta ds\right\}dt \\
&= \int_0^\infty e^{-\lambda t}\left\{\int_0^t (t-s)^{p-1}Q_p(t-s)[h(s,y(s))\frac{d\mathbb{W}(s)}{ds}]ds\right\}dt \\
&= \mathfrak{L}\left\{\int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y(s))d\mathbb{W}(s)\right\}(\lambda).
\end{aligned}$$

Similarly, we can derive

$$\int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)b(\lambda)dt = \mathfrak{L}\left\{\int_0^t (t-s)^{p-1}Q_p(t-s)f(s,y(s))ds\right\}(\lambda).$$

As $\mu - \alpha = -v(2-\mu) = -\zeta \in (-1,0)$, and as the Laplace inverse transform $\mathfrak{L}^{-1}(\lambda^{-\zeta}) = \frac{t^{\zeta-1}}{\Gamma(\zeta)}$, we can derive

$$\begin{aligned}
\lambda^{\mu-\alpha}\int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)y_1 dt &= \lambda^{-\zeta}\int_0^\infty pt^{p-1}e^{-(\lambda t)^p}\mathcal{S}(t^p)y_1 dt \\
&= \lambda^{-\zeta}\int_0^\infty\int_0^\infty pt^{p-1}e^{-\lambda t\vartheta}\psi_p(\vartheta)\mathcal{S}(t^p)y_1 d\vartheta dt \\
&= \lambda^{-\zeta}\int_0^\infty\int_0^\infty p\frac{t^{p-1}}{\vartheta^p}e^{-\lambda t}\psi_p(\vartheta)\mathcal{S}(\frac{t^p}{\vartheta^p})y_1 d\vartheta dt \\
&= \lambda^{-\zeta}\int_0^\infty e^{-\lambda t}\left\{\int_0^\infty p\vartheta t^{p-1}\frac{1}{p}\vartheta^{-\frac{1}{p}-1}\psi_p(\vartheta^{-\frac{1}{p}})\mathcal{S}(t^p\vartheta)y_1 d\vartheta\right\}dt \\
&= \lambda^{-\zeta}\int_0^\infty e^{-\lambda t}\left\{\int_0^\infty p\vartheta t^{p-1}M_p(\vartheta)\mathcal{S}(t^p\vartheta)y_1 d\vartheta\right\}dt \\
&= \lambda^{-\zeta}\mathfrak{L}(t^{p-1}Q_p(t)y_1)(\lambda) \\
&= \mathfrak{L}(\frac{t^{\zeta-1}}{\Gamma(\zeta)})(\lambda)\cdot\mathfrak{L}(t^{p-1}Q_p(t)y_1)(\lambda) \\
&= \mathfrak{L}(g_\zeta * (t^{p-1}Q_p(t)y_1))(\lambda).
\end{aligned}$$

As $\mu - \alpha + 1 = 1 - v(2-\mu) = 1 - \zeta \in (0,1)$, according to Equation (3) and $\lim_{t\to 0} I_{0^+}^{\zeta} t^{p-1}Q_p(t) = 0$, we can obtain

$$\begin{aligned}
\lambda^{\mu-\alpha+1}\int_0^\infty e^{-\lambda^p t}\mathcal{S}(t)y_0 dt &= \lambda^{1-\zeta}\mathfrak{L}(t^{p-1}Q_p(t)y_0)(\lambda) \\
&= \lambda^{1-\zeta}\mathfrak{L}(t^{p-1}Q_p(t)y_0)(\lambda) - \lim_{t\to 0} I_{0^+}^{\zeta} t^{p-1}Q_p(t) \\
&= \mathfrak{L}({}^{RL}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y_0))(\lambda).
\end{aligned}$$

Then, by combining the aforementioned arguments with the uniqueness theorem of Laplace transform, Equation (7) can be derived. The proof is completed. □

Definition 8. *y: $(0,b] \to \mathbb{H}$ is an $\mathcal{F}_t$-adapted stochastic process; it is said to be the mild solution of the Cauchy problem (1), if $y \in C((0,b]; L^2(\Omega,\mathbb{H}))$, $y_0, y_1 \in L_0^2(\Omega,\mathbb{H})$ and*

$$\begin{aligned}
y(t) = {}^{RL}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y_0) + I_{0^+}^{\zeta}(t^{p-1}Q_p(t)y_1) + \int_0^t (t-s)^{p-1}Q_p(t-s)f(s,y(s))ds \\
+ \int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y(s))d\mathbb{W}(s),\ t \in (0,b].
\end{aligned} \tag{10}$$

Lemma 5. *For any $y \in \mathbb{H}$, the following inequality is true:*

$$||Q_p(t)y|| \le \frac{Mt^p}{\Gamma(2p)}\|y\|,\ t \ge 0.$$

In addition, $Q_p(t)$ is uniformly continuous: that is, for any $t_2, t_1 \ge 0$,

$$||Q_p(t_2) - Q_p(t_1)|| \to 0,\ as\ t_2 \to t_1.$$

Proof. As $\|\mathcal{C}(t)\| \le M$ for any $t \ge 0$, we can obtain

$$\|\mathcal{S}(t)y\| = \|\int_0^t \mathcal{C}(s)yds\| \le Mt\|y\|,\ t \ge 0,\ y \in \mathbb{H}.$$

Thus, by Definition 6 and Lemma 4, we can derive

$$\begin{aligned}\|Q_p(t)y\| &\leq \int_0^\infty p\vartheta M_p(\vartheta)\|\mathcal{S}(t^p\vartheta)y\|d\vartheta \\ &\leq \int_0^\infty p\vartheta^2 M_p(\vartheta)Mt^p\|y\|d\vartheta \\ &= pMt^p\|y\|\int_0^\infty \vartheta^2 M_p(\vartheta)d\vartheta \\ &= \frac{Mt^p}{\Gamma(2p)}\|y\|.\end{aligned}$$

We will now demonstrate the uniform continuity of operator $Q_p(t)$ for $t_2 > t_1 \geq 0$:

$$\begin{aligned}\|Q_p(t_2) - Q_p(t_1)\| &\leq \int_0^\infty p\vartheta M_p(\vartheta)\|\mathcal{S}((t_2)^p\vartheta) - \mathcal{S}((t_1)^p\vartheta)\|d\vartheta \\ &\leq \frac{M|t_2^p - t_1^p|}{\Gamma(2p)} \to 0, \ as\ t_2 \to t_1.\end{aligned}$$

Thus, the proof is concluded. □

Lemma 6. *If Equation (4) holds for any $t > 0$ and $y \in \mathbb{H}$, the following formula is true:*

$$\frac{d}{dt}(t^{p-1}Q_p(t)y) = (p-1)t^{p-2}Q_p(t)y + t^{2p-2}\int_0^\infty p^2\vartheta^2 M_p(\vartheta)\mathcal{C}(t^p\vartheta)yd\vartheta.$$

Moreover,

$$\|\frac{d}{dt}(t^{p-1}Q_p(t)y)\| \leq \frac{Mt^{2p-2}}{\Gamma(2p)}\|y\|, \ \ t > 0.$$

Proof. As $\frac{d}{dt}\mathcal{S}(t^p\vartheta)y = pt^{p-1}\vartheta\mathcal{C}(t^p\vartheta)y$ for $t > 0$ and $y \in \mathbb{H}$, it can be calculated that

$$\begin{aligned}\frac{d}{dt}Q_p(t)y &= \frac{d}{dt}\int_0^\infty p\vartheta M_p(\vartheta)\mathcal{S}(t^p\vartheta)yd\vartheta \\ &= \int_0^\infty p\vartheta M_p(\vartheta)\frac{d}{dt}\mathcal{S}(t^p\vartheta)yd\vartheta \\ &= t^{p-1}\int_0^\infty p^2\vartheta^2 M_p(\vartheta)\mathcal{C}(t^p\vartheta)yd\vartheta.\end{aligned}$$

Thus, we obtain

$$\frac{d}{dt}(t^{p-1}Q_p(t)y) = (p-1)t^{p-2}Q_p(t)y + t^{2p-2}\int_0^\infty p^2\theta^2 M_p(\vartheta)\mathcal{C}(t^p\vartheta)yd\vartheta, \ t > 0, \ y \in \mathbb{H}.$$

By Definition 6 and Lemma 5, we can obtain

$$\begin{aligned}\|\frac{d}{dt}(t^{p-1}Q_p(t)y)\| &\leq \|(p-1)t^{p-2}Q_p(t)y\| + \|t^{2p-2}\int_0^\infty p^2\vartheta^2 M_p(\vartheta)\mathcal{C}(t^p\vartheta)y\| \\ &\leq \frac{M(1-p)}{\Gamma(2p)}t^{2p-2}\|y\| + \|p^2 t^{2p-2}M\int_0^\infty \vartheta^2 M_p(\vartheta)yd\vartheta\| \\ &\leq \frac{M(1-p)}{\Gamma(2p)}t^{2p-2}\|y\| + \frac{Mp}{\Gamma(2p)}t^{2p-2}\|y\| \\ &= \frac{Mt^{2p-2}}{\Gamma(2p)}\|y\|, \ t > 0, \ y \in \mathbb{H}.\end{aligned}$$

In conclusion, the proof is finished. □

Lemma 7 (A generalized Gronwall inequality; see [20])**.** *Suppose that (i) $\gamma > 0$, $0 < T \leq \infty$, (ii) non-negative function $a(t)$ and $u(t)$ are locally integrable on $0 \leq t < T$, and (iii) continuous function $g(t)$ is a non-negative, non-decreasing and bounded, $0 \leq t < T$. If*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\gamma-1} u(s)ds,$$

then

$$u(t) \leq a(t) + \int_0^t [\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{\gamma-1} a(s)]ds, \quad for\ 0 \leq t < T;$$

in particular, if $a(t) = 0$, then $u(t) = 0$ for all $0 \leq t < T$.

Lemma 8 (Ascoli–Arzelà theorem)**.** *The set $U \subset C([0,b], L^2(\Omega, \mathbb{H}))$ is relatively compact if and only if the following conditions hold:*

(i) the set U is equicontinuous on [0,b];
(ii) for any $t^* \in [0,b]$, $U(t^*)$ is relatively compact in $L^2(\Omega, \mathbb{H})$.

Lemma 9 (Schauder fixed point theorem; see [3])**.** *Let B be a closed, covex and nonempty subset of a Banach space X. Let $\Psi : B \to B$ be a continuous mapping, such that ΨB is a relatively compact subset of X. Then, Ψ has at least one fixed point in B.*

3. Some Lemmas

To demonstrate the main outcome of this paper, the following assumptions are necessary:
(A_1): For any $0 < t \leq b$, we assume that $f(t,\cdot)$ is Lebesgue measurable; for each $y \in \mathbb{H}$, we assume that $f(\cdot, y)$ is continuous;
(A_2): For any $0 < t \leq b$, we assume that $h(t,\cdot)$ is $\mathcal{F}_t$-measurable, and $\int_0^t E\|h(s,\cdot)\|^2 ds < \infty$; for each $y \in \mathbb{H}$, we assume that $h(\cdot, y)$ is continuous;
(A_3): There exists $m \in L^1([0,b]; \mathbb{R}^+)$, which satisfies

$$E\|f(t,y)\|^2 \vee E\|h(t,y)\|^2 \leq m(t), \quad for\ all\ y \in \mathbb{H}, \quad t \in [0,b];$$

(A_4): There exists a constant, $l > 0$, and a bounded set, $D \subset \mathbb{H}$, such that

$$\chi\big(f(t,D)\big) \vee \chi\big(h(t,D)\big) \leq l t^{2-\alpha} \chi(D), \quad for\ a.e.\ t \in [0,b],$$

where $\vee$ means the maximum of the two—for example, if $a > b$, then $a \vee b = a$.

Now, we introduce another space:

$$C_1\big((0,b], L^2(\Omega,\mathbb{H})\big) := \Big\{ y \in C\big((0,b], L^2(\Omega,\mathbb{H})\big) : \lim_{t \to 0^+} t^{2-\alpha} y(t)\ exists\ and\ finite \Big\};$$

with the norm $\|y(\cdot)\|_{C_1} = \big(\sup_{t\in(0,b]} E\|t^{2-\alpha} y(t)\|^2 \big)^{\frac{1}{2}}$, it is clear that the space is a Banach space.

Define mapping Φ :

$$(\Phi y)(t) = (\Phi_1 y)(t) + (\Phi_2 y)(t),\ y \in C_1\big((0,b], L^2(\Omega,\mathbb{H})\big),$$

where

$$\begin{aligned}
(\Phi_1 y)(t) =& {}^{RL}D_{0^+}^{1-\zeta} \big(t^{p-1} Q_p(t) y_0\big) + I_{0^+}^{\zeta} \big(t^{p-1} Q_p(t) y_1\big),\ for\ t \in (0,b], \\
(\Phi_2 y)(t) =& \int_0^t (t-s)^{p-1} Q_p(t-s) f(s, y(s)) ds \\
&+ \int_0^t (t-s)^{p-1} Q_p(t-s) h(s, y(s)) d\mathbb{W}(s),\ for\ t \in (0,b].
\end{aligned}$$

Clearly, if Φ has a fixed point $y^* \in C_1\big((0,b], L^2(\Omega,\mathbb{H})\big)$, then problem (1) has a mild solution, $y \in C_1\big((0,b], L^2(\Omega,\mathbb{H})\big)$.

For $\forall z \in C\big([0,b], L^2(\Omega,\mathbb{H})\big)$, set

$$y(t) = t^{\alpha-2}z(t),\ t \in (0,b].$$

Evidently, $y \in C_1\big((0,b], L^2(\Omega,\mathbb{H})\big)$.

Define operator Ψ :

$$(\Psi z)(t) = (\Psi_1 z)(t) + (\Psi_2 z)(t),\ for\ t \in [0,b],$$

where

$$(\Psi_1 z)(t) = \begin{cases} t^{2-\alpha}(\Phi_1 y)(t), & for\ t \in (0,b], \\ \frac{y_0}{\Gamma(\zeta+2p-1)}, & for\ t = 0, \end{cases}$$

$$(\Psi_2 z)(t) = \begin{cases} t^{2-\alpha}(\Phi_2 y)(t), & for\ t \in (0,b], \\ 0, & for\ t = 0. \end{cases}$$

According to (A_3) above, there is a positive constant R, such that

$$\begin{aligned} \sup_{t\in[0,b]} \Big\{ & 4\Big(\frac{M}{(2p-1)\Gamma(\zeta+2p-1)}\Big)^2 E\|y_0\|^2 + 4\Big(\frac{t}{\Gamma(\zeta+2p)}\Big)^2 E\|y_1\|^2 \\ & + 4\Big(\frac{Mt^{(2-\alpha+p)}}{\Gamma(2p)}\Big)^2 \frac{1}{2p} \int_0^t (t-s)^{2p-1} m(s) ds \\ & + 4\Big(\frac{Mt^{(2-\alpha)}}{\Gamma(2p)}\Big)^2 Tr(\mathcal{Q}) \int_0^t (t-s)^{2(2p-1)} m(s) ds \Big\} \le R. \end{aligned} \tag{11}$$

Let

$$B_R = \Big\{ z : z \in C\big([0,b], L^2(\Omega,\mathbb{H})\big), \|z\|_C \le R \Big\},\ \tilde{B}_R = \Big\{ y : y \in C_1\big((0,b], L^2(\Omega,\mathbb{H})\big), \|y\|_{C_1} \le R \Big\}.$$

Evidently, B_R and $\tilde{B}_R$ are convex, nonempty and closed subsets of $C\big([0,b], L^2(\Omega,\mathbb{H})\big)$ and $C_1\big((0,b], L^2(\Omega,\mathbb{H})\big)$, respectively.

Let

$$\mathbb{U} := \Big\{ u : u(t) = (\Psi z)(t), z \in B_R \Big\}.$$

Next, we prove several lemmas that are relevant to our main result.

Lemma 10. *If (A_1)–(A_3) hold, then the set $\mathbb{U}$ is equicontinuous.*

Proof. As part of our analysis, we aim to prove that $\mathbb{U}$ is equicontinuous; therefore, we need to show that $\lim\limits_{t_2\to t_1} E\big\|(\Psi z)(t_2) - (\Psi z)(t_1)\big\|^2 \to 0$ for $t_1, t_2 \in [0,b]$. This proof can be divided into the following two steps:

Step I: $\Big\{ u : u(t) = (\Psi_1 z)(t), z \in B_R \Big\}$ is equicontinuous.

Because $\mathcal{C}(0) = I$ and Definition 6, we can obtain

$$
\begin{aligned}
&\lim_{t\to 0} t^{2-\alpha} t^{\zeta+p-2}(p-1)\int_0^1\int_0^\infty g_\zeta(1-s)s^{p-2}p\vartheta M_p(\vartheta)\mathcal{S}((ts)^p\vartheta)y_0 d\vartheta ds \\
&= (p-1)\int_0^1\int_0^\infty g_\zeta(1-s)s^{p-2}p\vartheta M_p(\vartheta)\lim_{t\to 0}\frac{\mathcal{S}((ts)^p\vartheta)}{t^p}y_0 d\vartheta ds \\
&= (p-1)p\int_0^1 g_\zeta(1-s)s^{2p-2}ds\Big(\int_0^\infty \vartheta^2 M_p(\vartheta)y_0 d\vartheta\Big) \\
&= \frac{(p-1)y_0}{(2p-1)\Gamma(\zeta+2p-1)}.
\end{aligned}
\tag{12}
$$

Similarly, we obtain

$$
\lim_{t\to 0} t^{2-\alpha} t^{\zeta+p-1}\int_0^1\int_0^\infty g_\zeta(1-s)s^{p-2}p\vartheta M_p(\vartheta)\mathcal{S}((ts)^p\vartheta)y_1 d\vartheta ds = 0. \tag{13}
$$

$$
\lim_{t\to 0} t^{2-\alpha} t^{\zeta+2p-2}\int_0^1\int_0^\infty g_\zeta(1-s)s^{2p-2}p^2\vartheta^2 M_p(\vartheta)\mathcal{C}((ts)^p\vartheta)y_0 d\vartheta ds = \frac{py_0}{(2p-1)\Gamma(\zeta+2p-1)}. \tag{14}
$$

Due to the known relations between the Caputo fractional derivative and the Riemann–Liouville fractional derivative, we obtain

$$
{}^{C}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y) = {}^{RL}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y), \;\; \zeta\in(0,1), \;\; y\in\mathbb{H}. \tag{15}
$$

By using Equations (12)–(15), as well as Lemmas 4 and 6, we can establish the following result:

$$
\begin{aligned}
&\lim_{t\to 0} t^{2-\alpha}(\Phi_1 y)(t) \\
&= \lim_{t\to 0} t^{2-\alpha}\Big({}^{RL}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y_0) + I_{0^+}^{\zeta}(t^{p-1}Q_p(t)y_1)\Big) \\
&= \lim_{t\to 0} t^{2-\alpha}\Big({}^{C}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y_0) + I_{0^+}^{\zeta}(t^{p-1}Q_p(t)y_1)\Big) \\
&= \lim_{t\to 0} t^{2-\alpha}\Big(I_{0^+}^{\zeta}\big(\frac{d}{dt}(t^{p-1}Q_p(t)y_0)\big) + I_{0^+}^{\zeta}(t^{p-1}Q_p(t)y_1)\Big) \\
&= \lim_{t\to 0} t^{2-\alpha} I_{0^+}^{\zeta}\Big((p-1)t^{p-2}Q_p(t)y_0 + t^{2p-2}\int_0^\infty p^2\vartheta^2 M_p(\vartheta)\mathcal{C}(t^p\vartheta)y_0 d\vartheta + t^{p-1}Q_p(t)y_1\Big) \\
&= \lim_{t\to 0} t^{2-\alpha}(p-1)\int_0^t\int_0^\infty g_\zeta(t-s)s^{p-2}p\vartheta M_p(\vartheta)\mathcal{S}(s^p\vartheta)y_0 d\vartheta ds \\
&\quad + \lim_{t\to 0} t^{2-\alpha}\int_0^t\int_0^\infty g_\zeta(t-s)s^{2p-2}p^2\vartheta^2 M_p(\vartheta)\mathcal{C}(s^p\vartheta)y_0 d\vartheta ds \\
&\quad + \lim_{t\to 0} t^{2-\alpha}\int_0^t\int_0^\infty g_\zeta(t-s)s^{p-1}p\vartheta M_p(\vartheta)\mathcal{S}(s^p\vartheta)y_1 d\vartheta ds \\
&= \lim_{t\to 0} t^{2-\alpha} t^{\zeta+p-2}(p-1)\int_0^1\int_0^\infty g_\zeta(1-s)s^{p-2}p\vartheta M_p(\vartheta)\mathcal{S}((ts)^p\vartheta)y_0 d\vartheta ds \\
&\quad + \lim_{t\to 0} t^{2-\alpha} t^{\zeta+2p-2}\int_0^1\int_0^\infty g_\zeta(1-s)s^{2p-2}p^2\vartheta^2 M_p(\vartheta)\mathcal{C}((ts)^p\vartheta)y_0 d\vartheta ds \\
&\quad + \lim_{t\to 0} t^{2-\alpha} t^{\zeta+p-1}\int_0^1\int_0^\infty g_\zeta(1-s)s^{p-1}p\vartheta M_p(\vartheta)\mathcal{S}((ts)^p\vartheta)y_1 d\vartheta ds \\
&= \frac{y_0}{\Gamma(\zeta+2p-1)}.
\end{aligned}
$$

By using the equality mentioned above, and the C_r inequality, we can obtain the following result, when $t_1 = 0$ and $t_2 \in (0, b]$:

$$\begin{aligned}
&E\left\|(\Psi_1 z)(t_2) - (\Psi_1 z)(0)\right\|^2 \\
&= E\left\|t_2^{2-\alpha}(\Phi_1 y)(t_2) - \frac{y_0}{\Gamma(\zeta+2p-1)}\right\|^2 \\
&\leq 2E\left\|t_2^{2-\alpha} t_2^{\zeta+p-2}(p-1)\int_0^1\int_0^\infty g_\zeta(1-s)s^{p-2}p\vartheta M_p(\vartheta)\mathcal{S}((t_2 s)^p\vartheta)y_0 d\vartheta ds \right. \\
&\quad \left. + t_2^{2-\alpha} t_2^{\zeta+2p-2}\int_0^1\int_0^\infty g_\zeta(1-s)s^{2p-2}p^2\vartheta^2 M_p(\vartheta)\mathcal{C}((t_2 s)^p\vartheta)y_0 d\vartheta ds - \frac{y_0}{\Gamma(\zeta+2p-1)}\right\|^2 \\
&\quad + 2E\left\|t_2^{2-\alpha} t_2^{\zeta+p-1}\int_0^1\int_0^\infty g_\zeta(1-s)s^{p-1}p\vartheta M_p(\vartheta)\mathcal{S}((t_2 s)^p\vartheta)y_1 d\vartheta ds\right\|^2 \\
&\to 0,\ as\ t_2 \to 0.
\end{aligned}$$

For any $0 < t_1 < t_2 \leq b$, we can apply the C_r inequality, to obtain the following result:

$$\begin{aligned}
&E\left\|(\Psi_1 z)(t_2) - (\Psi_1 z)(t_1)\right\|^2 \\
&= E\left\|t_2^{2-\alpha}(\Phi_1 y)(t_2) - t_1^{2-\alpha}(\Phi_1 y)(t_1)\right\|^2 \\
&= E\left\|t_2^{2-\alpha}\left[{}^{RL}D_{0^+}^{1-\zeta}\left(t_2^{p-1}Q_p(t_2)y_0\right) + I_{0^+}^{\zeta}\left(t_2^{p-1}Q_p(t_2)y_1\right)\right]\right. \\
&\quad \left. - t_1^{2-\alpha}\left[{}^{RL}D_{0^+}^{1-\zeta}\left(t_1^{p-1}Q_p(t_1)y_0\right) + I_{0^+}^{\zeta}\left(t_1^{p-1}Q_p(t_1)y_1\right)\right]\right\|^2 \\
&\leq 2E\left\|t_2^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}\left(t_2^{p-1}Q_p(t_2)y_0\right)\right) - t_1^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}\left(t_1^{p-1}Q_p(t_1)y_0\right)\right)\right\|^2 \\
&\quad + 2E\left\|t_2^{2-\alpha}I_{0^+}^{\zeta}\left(t_2^{p-1}Q_p(t_2)y_1\right) - t_1^{2-\alpha}I_{0^+}^{\zeta}\left(t_1^{p-1}Q_p(t_1)y_1\right)\right\|^2 \\
&=: 2I_1 + 2I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= E\left\|t_2^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}\left(t_2^{p-1}Q_p(t_2)y_0\right)\right) - t_1^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}\left(t_1^{p-1}Q_p(t_1)y_0\right)\right)\right\|^2, \\
I_2 &= E\left\|t_2^{2-\alpha}I_{0^+}^{\zeta}\left(t_2^{p-1}Q_p(t_2)y_1\right) - t_1^{2-\alpha}I_{0^+}^{\zeta}\left(t_1^{p-1}Q_p(t_1)y_1\right)\right\|^2.
\end{aligned}$$

By employing the C_r inequality, we obtain

$$
\begin{aligned}
I_1 &\le 2E\left\| t_2^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0)\right) - t_1^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0)\right)\right\|^2 \\
&\quad + 2E\left\| t_1^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0)\right) - t_1^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}(t_1^{p-1}Q_p(t_1)y_0)\right)\right\|^2 \\
&= 2\left| t_2^{2-\alpha} - t_1^{2-\alpha}\right|^2 E\left\| {}^{RL}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0)\right\|^2 \\
&\quad + 2t_1^{2(2-\alpha)}E\left\| {}^{RL}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0) - {}^{RL}D_{0^+}^{1-\zeta}(t_1^{p-1}Q_p(t_1)y_0)\right\|^2 \\
&=: I_{11} + 2t_1^{2(2-\alpha)}I_{12}.
\end{aligned}
$$

We can observe that $I_{11} \to 0$ as $t_2 \to t_1$. According to Equation (15), we can derive

$$
\begin{aligned}
& E\left\| {}^{RL}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0) - {}^{RL}D_{0^+}^{1-\zeta}(t_1^{p-1}Q_p(t_1)y_0)\right\|^2 \\
&= E\left\| {}^{C}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0) - {}^{C}D_{0^+}^{1-\zeta}(t_1^{p-1}Q_p(t_1)y_0)\right\|^2 \\
&= E\left\| I_{0^+}^{\zeta}\left(\frac{d}{dt_2}(t_2^{p-1}Q_p(t_2)y_0)\right) - I_{0^+}^{\zeta}\left(\frac{d}{dt_1}(t_1^{p-1}Q_p(t_1)y_0)\right)\right\|^2 \\
&= E\left\| \int_0^{t_2} g_\zeta(t_2-s)\frac{d}{ds}(s^{p-1}Q_p(s)y_0)ds - \int_0^{t_1} g_\zeta(t_1-s)\frac{d}{ds}(s^{p-1}Q_p(s)y_0)ds\right\|^2 \\
&\le 2E\left\| \int_{t_1}^{t_2} g_\zeta(t_2-s)\frac{d}{ds}(s^{p-1}Q_p(s)y_0)ds\right\|^2 + 2E\left\| \int_0^{t_1} (g_\zeta(t_2-s)-g_\zeta(t_1-s))\frac{d}{ds}(s^{p-1}Q_p(s)y_0)ds\right\|^2.
\end{aligned}
$$

By using Lemma 6, we can derive the following result:

$$
\begin{aligned}
E\left\| \int_{t_1}^{t_2} g_\zeta(t_2-s)\frac{d}{ds}(s^{p-1}Q_p(s)y_0)ds\right\|^2 &\le \left(\frac{M}{\Gamma(2p)}\right)^2 E\left\| \int_{t_1}^{t_2} g_\zeta(t_2-s)s^{2p-2}y_0ds\right\|^2 \\
&\le \left(\frac{Mt_1^{p-2}}{\Gamma(2p)\Gamma(\zeta)}\right)^2\left(\int_{t_1}^{t_2}(t_2-s)^{\zeta-1}ds\right)^2 E\|y_0\|^2 \\
&= \left(\frac{Mt_1^{p-2}}{\Gamma(2p)\Gamma(\zeta+1)}\right)^2 (t_2-t_1)^{2\zeta}E\|y_0\|^2 \\
&\to 0,\ as\ t_2 \to t_1.
\end{aligned} \tag{16}
$$

Noting that

$$
((t_2-s)^{\zeta-1} - (t_1-s)^{\zeta-1})s^{2p-2} \le (t_2-s)^{\zeta-1}s^{2p-2},\ for\ a.e.\ s \in [0,t_1),
$$

then, by Lebesgue's dominated convergence theorem and Lemma 6, we derive

$$
\begin{aligned}
& E\left\| \int_0^{t_1} (g_\zeta(t_2-s) - g_\zeta(t_1-s))\frac{d}{ds}(s^{p-1}Q_p(s)y_0)ds\right\|^2 \\
&\le \left(\frac{M}{\Gamma(\zeta)\Gamma(2p)}\right)^2 E\left\| \int_0^{t_1} ((t_2-s)^{\zeta-1} - (t_1-s)^{\zeta-1})s^{2p-2}y_0ds\right\|^2 \\
&\to 0,\ as\ t_2 \to t_1;
\end{aligned}
$$

This implies that

$$I_{12}=E\left\|{}^{RL}D_{0^+}^{1-\zeta}\left(t_2^{p-1}Q_p(t_2)y_0\right)-{}^{RL}D_{0^+}^{1-\zeta}\left(t_1^{p-1}Q_p(t_1)y_0\right)\right\|^2\to 0,\ as\ t_2\to t_1.$$

Thus, by $I_{11}\to 0$ and $I_{12}\to 0$ as $t_2\to t_1$, we derive

$$I_1=E\left\|t_2^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}(t_2^{p-1}Q_p(t_2)y_0)\right)-t_1^{2-\alpha}\left({}^{RL}D_{0^+}^{1-\zeta}(t_1^{p-1}Q_p(t_1)y_0)\right)\right\|^2\to 0,\ as\ t_2\to t_1.$$

Using similar methods for $I_1\to 0$ as $t_2\to t_1$, we can obtain the following result:

$$I_2=E\left\|t_2^{2-\alpha}I_{0^+}^{\zeta}\left(t_2^{p-1}Q_p(t_2)y_1\right)-t_1^{2-\alpha}I_{0^+}^{\zeta}\left(t_1^{p-1}Q_p(t_1)y_1\right)\right\|^2\to 0,\ as\ t_2\to t_1.$$

Thus, we obtain

$$E\left\|(\Psi_1 z)(t_2)-(\Psi_1 z)(t_1)\right\|^2\to 0,\ as\ t_2\to t_1.$$

After conducting the aforementioned analysis, we can conclude that the set $\{u:u(t)=(\Psi_1 z)(t),z\in B_R\}$ is equicontinuous.

Step II: $\{u:u(t)=(\Psi_2 z)(t),z\in B_R\}$ is equicontinuous.

When $t_1=0,0<t_2\le b$, according to Lemma 5, Equation (2), (A_3) and Hölder's inequality, we obtain

$$\begin{aligned}
&E\|(\Psi_2 z)(t_2)-(\Psi_2 z)(0)\|^2\\
&\le 2E\left\|t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{p-1}Q_p(t_2-s)f(s,y(s))ds\right\|^2+2E\left\|t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{p-1}Q_p(t_2-s)h(s,y(s))d\mathbb{W}(s)\right\|^2\\
&\le 2\left(\frac{Mt_2^{(2-\alpha+p)}}{\Gamma(2p)}\right)^2\frac{1}{2p}\int_0^{t_2}(t_2-s)^{2p-1}m(s)ds+2\left(\frac{Mt_2^{2-\alpha}}{\Gamma(2p)}\right)^2 Tr(\mathcal{Q})\int_0^{t_2}(t_2-s)^{2(2p-1)}m(s)ds\\
&\to 0,\ as\ t_2\to 0.
\end{aligned}$$

When $0<t_1<t_2\le b$, we obtain

$$\begin{aligned}
&E\left\|(\Psi_2 z)(t_2)-(\Psi_2 z)(t_1)\right\|^2\\
&\le 2E\left\|t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{p-1}Q_p(t_2-s)f(s,y(s))ds-t_1^{2-\alpha}\int_0^{t_1}(t_1-s)^{p-1}Q_p(t_1-s)f(s,y(s))ds\right\|^2\\
&\quad+2E\left\|t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{p-1}Q_p(t_2-s)h(s,y(s))d\mathbb{W}(s)-t_1^{2-\alpha}\int_0^{t_1}(t_1-s)^{p-1}Q_p(t_1-s)h(s,y(s))d\mathbb{W}(s)\right\|^2\\
&=:2J_1+2J_2,
\end{aligned}$$

where

$$\begin{aligned}
J_1&=E\left\|t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{p-1}Q_p(t_2-s)f(s,y(s))ds-t_1^{2-\alpha}\int_0^{t_1}(t_1-s)^{p-1}Q_p(t_1-s)f(s,y(s))ds\right\|^2,\\
J_2&=E\left\|t_2^{2-\alpha}\int_0^{t_2}(t_2-s)^{p-1}Q_p(t_2-s)h(s,y(s))d\mathbb{W}(s)-t_1^{2-\alpha}\int_0^{t_1}(t_1-s)^{p-1}Q_p(t_1-s)h(s,y(s))d\mathbb{W}(s)\right\|^2.
\end{aligned}$$

Next, we prove $J_2 \to 0$ as $t_2 \to t_1$, according to C_r inequality and Lemma 5, obtaining

$$\begin{aligned} J_2 &\leq 3E\left\| t_1^{2-\alpha} \int_{t_1}^{t_2} (t_2-s)^{p-1} Q_p(t_2-s) h(s,y(s)) d\mathbb{W}(s) \right\|^2 \\ &\quad + 3E\left\| t_1^{2-\alpha} \int_0^{t_1} \Big((t_2-s)^{p-1} Q_p(t_2-s) - (t_1-s)^{p-1} Q_p(t_1-s) \Big) h(s,y(s)) d\mathbb{W}(s) \right\|^2 \\ &\quad + 3\left(t_2^{2-\alpha} - t_1^{2-\alpha} \right)^2 E\left\| \int_0^{t_2} (t_2-s)^{p-1} Q_p(t_2-s) h(s,y(s)) d\mathbb{W}(s) \right\|^2 \\ &\leq 3\sum_{i=1}^{3} J_{2i}, \end{aligned}$$

where

$$\begin{aligned} J_{21} &= \left(\frac{M t_1^{(2-\alpha)}}{\Gamma(2p)} \right)^2 Tr(\mathcal{Q}) \left(\int_0^{t_2} (t_2-s)^{2(2p-1)} m(s) ds - \int_0^{t_1} (t_2-s)^{2(2p-1)} m(s) ds \right), \\ J_{22} &= t_1^{2(2-\alpha)} E\left\| \int_0^{t_1} \Big((t_2-s)^{p-1} Q_p(t_2-s) - (t_1-s)^{p-1} Q_p(t_1-s) \Big) h(s,y(s)) d\mathbb{W}(s) \right\|^2, \\ J_{23} &= \left(t_2^{2-\alpha} - t_1^{2-\alpha} \right)^2 E\left\| \int_0^{t_2} (t_2-s)^{p-1} Q_p(t_2-s) h(s,y(s)) d\mathbb{W}(s) \right\|^2. \end{aligned}$$

We can deduce that $\lim\limits_{t_1 \to t_2} J_{21} = 0$ and $\lim\limits_{t_1 \to t_2} J_{23} = 0$.

Then,

$$J_{22} = t_1^{2(2-\alpha)} E\left\| \int_0^{t_1} \int_{t_1-s}^{t_2-s} \frac{d}{dt}\left\{ t^{p-1} Q_p(t) h(s,y(s)) \right\} dt d\mathbb{W}(s) \right\|^2.$$

Moreover, Lemma 6 and Equation (2) imply that

$$\begin{aligned} J_{22} &\leq t_1^{2(2-\alpha)} E\left\| \frac{M}{\Gamma(2p)} \int_0^{t_1} \int_{t_1-s}^{t_2-s} t^{2p-2} h(s,y(s)) dt d\mathbb{W}(s) \right\|^2 \\ &\leq \left(\frac{M t_1^{(2-\alpha)}}{(2p-1)\Gamma(2p)} \right)^2 Tr(\mathcal{Q}) \int_0^{t_1} \left((t_2-s)^{2p-1} - (t_1-s)^{2p-1} \right)^2 m(s) ds \\ &\to 0, \ as \ t_2 \to t_1; \end{aligned}$$

Hence, $J_2 \to 0$ as $t_2 \to t_1$. Thus, we can prove $J_1 \to 0$ as $t_2 \to t_1$ in a way similar to $J_2 \to 0$.

Consequently,

$$E\left\| (\Psi_2 z)(t_2) - (\Psi_2 z)(t_1) \right\|^2 \to 0, \ as \ t_2 \to t_1.$$

According to the above analysis, $\lim\limits_{t_2 \to t_1} \left\| (\Psi_2 z)(t_2) - (\Psi_2 z)(t_1) \right\|_c \to 0$ for $t_1, t_2 \in [0,b]$; therefore, $\mathbb{U} = \{u : u(t) = (\Psi_2 z)(t), z \in B_R\}$ is equicontinuous. □

Lemma 11. *If $(A_1) - (A_3)$ hold, then Ψ is continuous.*

Proof. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence which is convergent to z in B_R. Then,

$$\lim_{n \to \infty} z_n(t) = z(t) \ and \ \lim_{n \to \infty} t^{\alpha-2} z_n(t) = t^{\alpha-2} z(t), \ for \ t \in (0,b].$$

As $y(t) = t^{\alpha-2}z(t)$, $t \in (0,b]$, according to (A_1) and (A_2), we obtain

$$\lim_{n\to\infty} E\big\|f(t,y_n(t))\big\|^2 = \lim_{n\to\infty} E\big\|f(t,t^{\alpha-2}z_n(t))\big\|^2 = E\big\|f(t,t^{\alpha-2}z(t))\big\|^2 = E\big\|f(t,y(t))\big\|^2,$$
$$\lim_{n\to\infty} E\big\|h(t,y_n(t))\big\|^2 = \lim_{n\to\infty} E\big\|h(t,t^{\alpha-2}z_n(t))\big\|^2 = E\big\|h(t,t^{\alpha-2}z(t))\big\|^2 = E\big\|h(t,y(t))\big\|^2.$$

By employing (A_3), we can obtain

$$(t-s)^{2p-1}E\big\|f(s,y_n(s)) - f(s,y(s))\big\|^2 \le 4(t-s)^{2p-1}m(s),\ t \in (0,b].$$

As $s \to 4(t-s)^{2p-1}m(s)$ is integrable for $s \in [0,t]$, we can use the Lebesgue dominated convergence theorem to derive

$$E\bigg\|\int_0^t (t-s)^{2p-1}\big[f(s,y_n(s)) - f(s,y(s))\big]ds\bigg\|^2 \to 0,\ as\ n \to \infty.$$

Similarly, we obtain

$$E\bigg\|\int_0^t (t-s)^{2p-1}\big[h(s,y_n(s)) - h(s,y(s))\big]d\mathbb{W}(s)\bigg\|^2 \to 0,\ as\ n \to \infty.$$

So, for each $t \in [0,b]$, we obtain

$$\begin{aligned}
&E\bigg\|(\Psi_2 z_n)(t) - (\Psi_2 z)(t)\bigg\|^2 \\
&\le 2t^{2(2-\alpha)}E\bigg\|\int_0^t (t-s)^{p-1}Q_p(t-s)(f(s,y_n(s)) - f(s,y(s)))ds\bigg\|^2 \\
&\quad + 2t^{2(2-\alpha)}E\bigg\|\int_0^t (t-s)^{p-1}Q_p(t-s)(h(s,y_n(s)) - h(s,y(s)))d\mathbb{W}(s)\bigg\|^2 \\
&\le 2(\frac{Mt^{(2-\alpha)}}{\Gamma(2p)})^2 E\bigg\|\int_0^t (t-s)^{2p-1}(f(s,y_n(s)) - f(s,y(s)))ds\bigg\|^2 \\
&\quad + 2(\frac{Mt^{(2-\alpha)}}{\Gamma(2p)})^2 E\bigg\|\int_0^t (t-s)^{2p-1}(h(s,y_n(s)) - h(s,y(s)))d\mathbb{W}(s)\bigg\|^2 \\
&\to 0,\ as\ n \to \infty.
\end{aligned}$$

Hence, Ψ is continuous. □

Lemma 12. *If $(A_1) - (A_3)$ hold, then $\Psi(B_R) \subset B_R$.*

Proof. When $t > 0$, by using Equation (15) and Lemma 6, we can derive

$$\begin{aligned}
\bigg\|{}^{RL}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y)\bigg\| &= \bigg\|{}^{C}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y)\bigg\| \\
&= \bigg\|I_{0^+}^{\zeta}(\frac{d}{dt}t^{p-1}Q_p(t)y)\bigg\| \\
&\le \int_0^t g_\zeta(t-s)\bigg\|\frac{d}{ds}s^{p-1}Q_p(s)y\bigg\|ds \\
&\le \frac{M}{\Gamma(2p)}\int_0^t g_\zeta(t-s)s^{2p-2}\|y\|ds \\
&= \frac{Mt^{2p+\zeta-2}}{(2p-1)\Gamma(\zeta+2p-1)}\|y\|,
\end{aligned}$$

which implies that

$$\left\| {}^{RL}D_{0^+}^{1-\zeta}\left(t^{p-1}Q_p(t)y\right)\right\| \le \frac{Mt^{2p+\zeta-2}}{(2p-1)\Gamma(\zeta+2p-1)}\|y\|,\ t>0, y\in\mathbb{H}. \tag{17}$$

Similarly, by using Lemma 5, we can obtain

$$\left\| I_{0^+}^{\zeta}\left(t^{p-1}Q_p(t)y\right)\right\| \le \frac{Mt^{2p+\zeta-1}}{\Gamma(\zeta+2p)}\|y\|,\ t>0, y\in\mathbb{H}. \tag{18}$$

For $t\in(0,b]$, according to (A_3), Lemmas 5 and 6 and Equations (11), (17) and (18), we obtain

$$\begin{aligned}
E\left\|(\Psi z)(t)\right\|^2 &= E\left\|t^{2-\alpha}(\Phi y)(t)\right\|^2 \\
&= t^{2(2-\alpha)}E\Big\| {}^{RL}D_{0^+}^{1-\zeta}\left(t^{p-1}Q_p(t)y_0\right) \\
&\quad + I_{0^+}^{\zeta}\left(t^{p-1}Q_p(t)y_1\right) + \int_0^t (t-s)^{p-1}Q_p(t-s)f(s,y(s))ds \\
&\quad + \int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y(s))d\mathbb{W}(s)\Big\|^2 \\
&\le 4t^{2(2-\alpha)}E\left\| {}^{RL}D_{0^+}^{1-\zeta}\left(t^{p-1}Q_p(t)y_0\right)\right\|^2 + 4t^{2(2-\alpha)}E\left\|I_{0^+}^{\zeta}\left(t^{p-1}Q_p(t)y_1\right)\right\|^2 \\
&\quad + 4t^{2(2-\alpha)}E\left\|\int_0^t (t-s)^{p-1}Q_p(t-s)f(s,y(s))ds\right\|^2 \\
&\quad + 4t^{2(2-\alpha)}E\left\|\int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y(s))d\mathbb{W}(s)\right\|^2 \\
&\le 4\left(\frac{M}{(2p-1)\Gamma(\zeta+2p-1)}\right)^2 E\|y_0\|^2 + 4\left(\frac{t}{\Gamma(\zeta+2p)}\right)^2 E\|y_1\|^2 \\
&\quad + 4\left(\frac{Mt^{(2-\alpha+p)}}{\Gamma(2p)}\right)^2 \frac{1}{2p}\int_0^t (t-s)^{2p-1}E\|f(s,y(s))\|^2 ds \\
&\quad + 4\left(\frac{Mt^{(2-\alpha)}}{\Gamma(2p)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}E\|h(s,y(s))\|^2 ds \\
&\le \sup_{t\in(0,b]}\Big\{4\left(\frac{M}{(2p-1)\Gamma(\zeta+2p-1)}\right)^2 E\|y_0\|^2 + 4\left(\frac{t}{\Gamma(\zeta+2p)}\right)^2 E\|y_1\|^2 \\
&\quad + 4\left(\frac{Mt^{(2-\alpha+p)}}{\Gamma(2p)}\right)^2 \frac{1}{2p}\int_0^t (t-s)^{2p-1}m(s)ds \\
&\quad + 4\left(\frac{Mt^{2-\alpha}}{\Gamma(2p)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}m(s)ds\Big\} \\
&\le R.
\end{aligned}$$

For $t=0$, as $M>1$, we obtain

$$E\left\|(\Psi z)(0)\right\|^2 = E\left\|\frac{y_0}{\Gamma(\zeta+2p-1)}\right\|^2 \le \left(\frac{M}{(2p-1)\Gamma(\zeta+2p-1)}\right)^2 E\|y_0\|^2 < R;$$

therefore, for any $z\in B_R$, we obtain $\Psi(B_R)\subset B_R$. □

4. Main Results

Theorem 1. *Suppose that* (A_1)–(A_3) *are satisfied, and that* $\{\mathcal{S}(t)\}_{t>0}$ *is compact, then there exists at least one mild solution to problem* (1) *in* $\tilde{B}_R$.

Proof. Evidently, if the operator Ψ has a fixed point, $z \in B_R$ is equivalent to Equation (1), and there exists a mild solution $y \in \tilde{B}_R$, where $z(t) = t^{2-\alpha}y(t)$. Hence, we only need to prove that the operator Ψ has a fixed point in B_R. According to Lemmas 11 and 12, we know that Ψ is continuous, and that $\Psi B_R \subset B_R$. In order to prove that Ψ is a completely continuous operator, we need to prove that $\Psi(B_R)$ is a relatively compact set. From Lemma 10, the set $\mathbb{U} = \{u : u(t) = (\Psi z)(t)), z \in B_R\}$ is equicontinuous. According to the Ascoli–Arzelà theorem, we only need to prove that $\mathbb{U}(t) = \{u(t) : u(t) = (\Psi z)(t)), z \in B_R\}$ is relatively compact in $L^2(\Omega, \mathbb{H})$. If $\mathbb{U}(0)$ is relatively compact in $L^2(\Omega, \mathbb{H})$, we only need to prove that the set $\mathbb{U}(t)$ is relatively compact in $L^2(\Omega, \mathbb{H})$ for $t > 0$.

For $\forall \epsilon \in (0, t)$ and $\gamma > 0$, we define $\Psi_{\epsilon,\gamma}$ on B_R as follows:

$$
\begin{aligned}
(\Psi_{\epsilon,\gamma}z)(t) := {} & t^{2-\alpha}(\Phi_{\epsilon,\gamma}y)(t) \\
= {} & t^{2-\alpha}\, {}^{RL}D_{0^+}^{1-\zeta}\big(t^{p-1}Q_p(t)y_0\big) + t^{2-\alpha} I_{0^+}^{\zeta}\big(t^{p-1}Q_p(t)y_1\big) \\
& + t^{2-\alpha}\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\int_0^{t-\epsilon}\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta - \epsilon^p\gamma\big)f(s, y(s))d\vartheta ds \\
& + t^{2-\alpha}\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\int_0^{t-\epsilon}\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta - \epsilon^p\gamma\big)h(s, y(s))d\vartheta d\mathbb{W}(s).
\end{aligned}
$$

Because $\{\mathcal{S}(t)\}_{t>0}$ is compact, $\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}$ is also compact. Then, for any $\epsilon \in (0, t)$ and any $\gamma > 0$, we can deduce that $\{(\Psi_{\epsilon,\gamma}z)(t), z \in B_R\}$ is relatively compact in $L^2(\Omega, \mathbb{H})$. In addition, for any $z \in B_R$, we obtain

$$
\begin{aligned}
& E\big\|(\Psi z)(t) - (\Psi_{\epsilon,\gamma}z)(t)\big\|^2 \\
& = t^{2(2-\alpha)}2E\bigg\|\int_0^t\int_0^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta\big)f(s, y(s))d\vartheta ds \\
& \quad - \frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\int_0^{t-\epsilon}\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta - \epsilon^p\gamma\big)f(s, y(s))d\vartheta ds\bigg\|^2 \\
& \quad + t^{2(2-\alpha)}2E\bigg\|\int_0^t\int_0^\infty p\theta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta\big)h(s, y(s))d\vartheta d\mathbb{W}(s) \\
& \quad - \frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\int_0^{t-\epsilon}\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta - \epsilon^p\gamma\big)h(s, y(s))d\vartheta d\mathbb{W}(s)\bigg\|^2 \\
& =: d_1 + d_2.
\end{aligned}
$$

In order to prove $E\|(\Psi z)(t)-(\Psi_{\epsilon,\gamma}z)(t)\|^2\to 0$, we first prove $d_2\to 0$:

$$\begin{aligned}
d_2 &= t^{2(2-\alpha)}2E\Big\|\int_0^t\int_0^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta\big)h(s,y(s))d\vartheta d\mathbb{W}(s)\\
&\quad-\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\int_0^{t-\epsilon}\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta-\epsilon^p\gamma\big)h(s,y(s))d\vartheta d\mathbb{W}(s)\Big\|^2\\
&\le t^{2(2-\alpha)}\Big(6E\Big\|\int_0^t\int_0^\gamma p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta\big)h(s,y(s))d\vartheta d\mathbb{W}(s)\Big\|^2\\
&\quad+6E\Big\|\int_{t-\epsilon}^t\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}\big((t-s)^p\vartheta\big)h(s,y(s))d\vartheta d\mathbb{W}(s)\Big\|^2\\
&\quad+6E\Big\|\int_0^{t-\epsilon}\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\Big(\mathcal{S}((t-s)^p\vartheta)-\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\mathcal{S}((t-s)^p\vartheta-\epsilon^p\gamma)\Big)h(s,y(s))d\vartheta d\mathbb{W}(s)\Big\|^2\Big)\\
&=: d_{21}+d_{22}+d_{23}.
\end{aligned}$$

As $\|\mathcal{S}(t)\|\le Mt$ for any $t\ge 0$ and Equation (11), we obtain

$$\begin{aligned}
d_{21} &= t^{2(2-\alpha)}6E\Big\|\int_0^t\int_0^\gamma p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}((t-s)^p\vartheta)h(s,y(s))d\vartheta d\mathbb{W}(s)\Big\|^2\\
&\le 6M^2Tr(\mathcal{Q})t^{2(2-\alpha)}\int_0^t(t-s)^{2(2p-1)}m(s)ds\Big(\int_0^\gamma p\vartheta^2M_p(\vartheta)d\vartheta\Big)^2\\
&\le \frac{3}{2}R(\Gamma(2p))^2\Big(\int_0^\gamma p\vartheta^2M_p(\vartheta)d\vartheta\Big)^2\\
&\to 0,\ as\ \gamma\to 0,
\end{aligned}$$

and

$$\begin{aligned}
d_{22} &= t^{2(2-\alpha)}6E\Big\|\int_{t-\epsilon}^t\int_\gamma^\infty p\vartheta(t-s)^{p-1}M_p(\vartheta)\mathcal{S}((t-s)^p\vartheta)h(s,y(s))d\vartheta d\mathbb{W}(s)\Big\|^2\\
&\le 6M^2Tr(\mathcal{Q})t^{2(2-\alpha)}\int_{t-\epsilon}^t(t-s)^{2(2p-1)}m(s)ds\Big(\int_0^\infty p\vartheta^2M_p(\vartheta)d\vartheta\Big)^2\\
&\le 6M^2Tr(\mathcal{Q})\Big(\frac{t^{(2-\alpha)}}{\Gamma(2p)}\Big)^2\int_{t-\epsilon}^t(t-s)^{2(2p-1)}m(s)ds\\
&\to 0,\ as\ \epsilon\to 0.
\end{aligned}$$

Regarding d_{23}, using $\|\mathcal{S}(t)-\mathcal{S}(k)\|\le M|t-k|$ and the $\lim_{t\to 0}\|\frac{\mathcal{S}(t)y}{t}-y\|=0$, for any $y\in\mathbb{H}$, we can conclude that

$$\begin{aligned}
&\Big\|\mathcal{S}\Big((t-s)^p\vartheta\Big)y-\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\mathcal{S}\Big((t-s)^p\vartheta-\epsilon^p\gamma\Big)y\Big\|\\
&\le\Big\|\Big(\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}-I\Big)\mathcal{S}\Big((t-s)^p\vartheta\Big)y\Big\|+\Big\|\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\Big(\mathcal{S}((t-s)^p\vartheta-\epsilon^p\gamma)-\mathcal{S}((t-s)^p\vartheta)\Big)y\Big\|\\
&\le M(t-s)^p\vartheta\Big\|\Big(\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}-I\Big)y\Big\|+M\Big\|\Big(\mathcal{S}\big((t-s)^p\vartheta-\epsilon^p\gamma\big)-\mathcal{S}\big((t-s)^p\vartheta\big)\Big)y\Big\|\\
&\to 0,\ as\ \epsilon,\ \gamma\to 0.
\end{aligned}$$

Thus, we can obtain

$$t^{2(2-\alpha)}Tr(\mathcal{Q})\bigg(\int_{\gamma}^{\infty} p\vartheta(t-s)^{p-1}M_p(\vartheta)\bigg(\mathcal{S}((t-s)^p\vartheta)-\frac{\mathcal{S}(\epsilon^p\gamma)}{\epsilon^p\gamma}\mathcal{S}\big((t-s)^p\vartheta-\epsilon^p\gamma\big)d\vartheta\bigg)\bigg)^2 E\|h(s,y(s))\|^2$$
$$\leq t^{2(2-\alpha)}Tr(\mathcal{Q})\bigg(\int_{\gamma}^{\infty} p\vartheta(t-s)^{p-1}M_p(\vartheta)M(t-s)^p\vartheta+M^2(t-s)^p\vartheta d\vartheta\bigg)^2 m(s)$$
$$\leq t^{2(2-\alpha)}Tr(\mathcal{Q})(t-s)^{2(2p-1)}M^2(M+1)^2 m(s)\bigg(\int_0^{\infty} p\vartheta^2 M_p(\vartheta)d\vartheta\bigg)^2$$
$$=(M+1)^2(\frac{Mt^{2-\alpha}}{\Gamma(2p)})^2 Tr(\mathcal{Q})(t-s)^{2(2p-1)}m(s).$$

Furthermore, as the above inequality is integrable for $s\in[0,t]$, by Lebesgue's dominated convergence theorem, we derive $d_{23}\to 0$ as $\epsilon\to 0$ or $\gamma\to 0$. Thus, $d_2\to 0$.

Similarly, we can obtain $d_1\to 0$. Thus, $\mathbb{U}(t)$ is also relatively compact in $L^2(\Omega,\mathbb{H})$; therefore, by employing the Schauder fixed point theorem, we can deduce that Ψ has at least one fixed point $z^*\in B_R$. Let $y^*=t^{\alpha-2}z^*$ for $t\in(0,b]$. Thus,

$$y^*={}^{RL}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y_0)+I_{0^+}^{\zeta}(t^{p-1}Q_p(t)y_1)+\int_0^t(t-s)^{p-1}Q_p(t-s)f(s,y^*(s))ds$$
$$+\int_0^t(t-s)^{p-1}Q_p(t-s)h(s,y^*(s))d\mathbb{W}(s),\ t\in(0,b].$$

The proof is completed. □

Theorem 2. *Suppose that (A_1)–(A_4) are satisfied, and that $\{\mathcal{S}(t)\}_{t>0}$ is noncompact, then there exists at least one mild solution to the problem of Equation (1) in $\tilde{B}_R$.*

Proof. Let $z_0(t)=t^{2-\alpha}\,{}^{RL}D_{0^+}^{1-\zeta}\big(t^{p-1}Q_p(t)y_0\big)+t^{2-\alpha}I_{0^+}^{\zeta}\big(t^{p-1}Q_p(t)y_1\big)$ for all $t\in[0,b]$, and $z_{m+1}=\Psi z_m, m\in N$. According to Lemmas 11 and 12, we can deduce that $\Psi z_m: B_R\to B_R$ is continuous. We will prove set $\mathbb{V}=\{v_m: v_m(t)=(\Psi z_m)(t), z_m\in B_R\}_{m=0}^{\infty}$ is relatively compact. By Lemma 10, we can deduce that the set $\mathbb{V}$ is equicontinuous. According to the Ascoli–Arzelà theorem, we only need to prove that $\mathbb{V}(t)=\big\{v_m(t): v_m(t)=(\Psi z_m)(t), z_m\in B_R\big\}_{m=0}^{\infty}$ is relatively compact in $L^2(\Omega,\mathbb{H})$.

According to Lemmas 1 and 5 and (A_5), we obtain

$$\chi\bigg(\bigg\{t^{2-\alpha}\int_0^t(t-s)^{p-1}Q_p(t-s)f(s,y_m(s))ds\bigg\}_{m=0}^{\infty}\bigg)$$
$$\leq t^{2-\alpha}\frac{2M}{\Gamma(2p)}\int_0^t(t-s)^{2p-1}\chi\big(f(s,\{s^{\alpha-2}z_m(s)\}_{m=0}^{\infty})\big)ds$$
$$\leq t^{2-\alpha}\frac{2Ml}{\Gamma(2p)}\int_0^t(t-s)^{2p-1}s^{2-\alpha}\chi\big(\{s^{\alpha-2}z_m(s)\}_{m=0}^{\infty}\big)ds$$
$$\leq t^{2-\alpha}\frac{2Ml}{\Gamma(2p)}\int_0^t(t-s)^{2p-1}\chi\big(\{z_m(s)\}_{m=0}^{\infty}\big)ds.$$

For any $y_1,y_2\in\mathbb{H}$, by employing Lemma 5 and Equation (2), we can derive

$$\bigg(E\bigg\|\int_0^t(t-s)^{p-1}Q_p(t-s)h(s,y_1(s))d\mathbb{W}(s)-\int_0^t(t-s)^{p-1}Q_p(t-s)h(s,y_2(s))d\mathbb{W}(s)\bigg\|^2\bigg)^{\frac{1}{2}}$$
$$\leq\frac{M}{\Gamma(2p)}\bigg(E\bigg\|\int_0^t(t-s)^{2p-1}\big(h(s,y_1(s))-h(s,y_2(s))\big)d\mathbb{W}(s)\bigg\|^2\bigg)^{\frac{1}{2}} \tag{19}$$
$$\leq\frac{M}{\Gamma(2p)}\bigg(Tr(\mathcal{Q})\int_0^t(t-s)^{2(2p-1)}E\big\|h(s,y_1(s))-h(s,y_2(s))\big\|^2 ds\bigg)^{\frac{1}{2}}.$$

Thus, according to Equation (19), (A_4) and [14], we obtain

$$\begin{aligned}
&\chi\left(\left\{t^{2-\alpha}\int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y_m(s))d\mathbb{W}(s)\right\}_{m=0}^{\infty}\right)\\
&\leq t^{2-\alpha}\frac{M}{\Gamma(2p)}\left(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}\left[\chi(h(s,\{s^{\alpha-2}z_m(s)\}_{m=0}^{\infty}))\right]^2 ds\right)^{\frac{1}{2}}\\
&\leq t^{2-\alpha}\frac{Ml}{\Gamma(2p)}\left(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}s^{2(2-\alpha)}\left[\chi(\{s^{\alpha-2}z_m(s)\}_{m=0}^{\infty})\right]^2 ds\right)^{\frac{1}{2}}\\
&\leq t^{2-\alpha}\frac{Ml}{\Gamma(2p)}\left(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}[\chi(\{z_m(s)\}_{m=0}^{\infty})]^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

The above estimates yield

$$\begin{aligned}
\chi\Big(\mathbb{V}(t)\Big) &= \chi\left(\left\{(\Psi z_m)(t)\right\}_{m=0}^{\infty}\right)\\
&= \chi\Bigg(\Bigg\{\,{}^{RL}D_{0^+}^{1-\zeta}(t^{p-1}Q_p(t)y_0) + I_{0^+}^{\zeta}(t^{p-1}Q_p(t)y_1)\\
&\quad + \int_0^t (t-s)^{p-1}Q_p(t-s)f(s,y_m(s))ds + \int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y_m(s))d\mathbb{W}(s)\Bigg\}_{m=0}^{\infty}\Bigg)\\
&= \chi\left(\left\{\int_0^t (t-s)^{p-1}Q_p(t-s)f(s,y_m(s))ds + \int_0^t (t-s)^{p-1}Q_p(t-s)h(s,y_m(s))d\mathbb{W}(s)\right\}_{m=0}^{\infty}\right)\\
&\leq t^{2-\alpha}\frac{2Ml}{\Gamma(2p)}\int_0^t (t-s)^{2p-1}\chi\left(\left\{z_m(s)\right\}_{m=0}^{\infty}\right)ds\\
&\quad + t^{2-\alpha}\frac{Ml}{\Gamma(2p)}\left(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}\left[\chi\left(\left\{z_m(s)\right\}_{m=0}^{\infty}\right)\right]^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

In addition, we obtain

$$\chi\Big(\{z_m(t)\}_{m=0}^{\infty}\Big) = \chi\Big(z_0(t)\cup\{z_m(t)\}_{m=1}^{\infty}\Big) = \chi\Big(\{z_m(t)\}_{m=1}^{\infty}\Big) = \chi\Big(\mathbb{V}(t)\Big),\ \ t\in[0,b].$$

Thus,

$$\begin{aligned}
\chi(\mathbb{V}(t)) &\leq \frac{2Mlb^{2-\alpha}}{\Gamma(2p)}\int_0^t (t-s)^{2p-1}\chi(\mathbb{V}(s))ds\\
&\quad + \frac{Mlb^{2-\alpha}}{\Gamma(2p)}\left(2Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}[\chi(\mathbb{V}(s))]^2 ds\right)^{\frac{1}{2}}\\
&:= L_1 + L_2.
\end{aligned} \tag{20}$$

If $L_1 > L_2$, according to Equation (20), we know that

$$\chi(\mathbb{V}(t)) \leq \frac{4Mlb^{2-\alpha}}{\Gamma(2p)}\int_0^t (t-s)^{2p-1}\chi(\mathbb{V}(s))ds;$$

therefore, according to Lemma 7, we obtain $\chi(\mathbb{V}(t)) = 0$.

If $L_2 > L_1$, according to Equation (20), we know that

$$\Big(\chi(\mathbb{V}(t))\Big)^2 \leq \left(\frac{4Mlb^{2-\alpha}}{\Gamma(2p)}\right)^2 Tr(\mathcal{Q})\int_0^t (t-s)^{2(2p-1)}\big[\chi(\mathbb{V}(s))\big]^2 ds.$$

According to Lemma 7, we can also obtain $\chi(\mathbb{V}(t)) = 0$; therefore, $\mathbb{V}(t)$ is relatively compact. According to the Ascoli–Arzelà theorem, $\mathbb{V}$ is relatively compact. Thus, there exists $y^* \in B_R$, such that $\lim\limits_{m\to\infty} z_m = z^*$.

As Ψ is continuous, we can obtain

$$z^* = \lim_{m\to\infty} z_m = \lim_{m\to\infty} \Psi z_{m-1} = \Psi(\lim_{m\to\infty} z_{m-1}) = \Psi z^*.$$

Let $y^*(t) = t^{2-\alpha} z^*(t)$. Thus, $y^* \in \tilde{B}_R$ is a mild solution to Equation (1). □

5. An Application

Example 1. *Consider the following equation:*

$$\begin{cases} \partial_t^{\mu,\nu} v(t,z) = \partial_z^2 v(t,z) + f_1(t,v(t,z)) + g_1(t,v(t,z))\frac{dw(t)}{dt}, & t \in (0,b],\ z \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, & t \in (0,b], \\ (I_{0^+}^{(2-\gamma)} v)(0,z) = v_0(z), (I_{0^+}^{2-\gamma} v)'(0,z) = v_1(z), & z \in [0,\pi], \end{cases} \tag{21}$$

where $\partial_t^{\mu,\nu}$ is the Hilfer fractional partial derivative of order $1 < \mu < 2$, $0 \le \nu \le 1$, $\alpha = \mu + \nu(2-\mu)$, $f_1(t,v(t,z)), g_1(t,v(t,z))$ satisfy (A_1) and (A_2), respectively, and there exists $\varphi \in L^1([0,b];\mathbb{R}^+)$, such that $E\|f_1(t,v(t,z))\|^2 \vee E\|g_1(t,v(t,z))\|^2 \le \varphi(t)$.

Let $\mathbb{H} = L^2([0,\pi])$ and defined A satisfies $Av = \frac{d^2}{dt^2}v$, $D(A) = \{v \in \mathbb{H} : v(0) = v(\pi)) = 0; v'' \in \mathbb{H}; v', v''$ are absolutely continuous$\}$. Then, A is the infinitesimal generator of a uniformly bounded strongly continuous cosine family $\{\mathcal{C}(t)\}_{t\ge 0}$. Let $\phi_m(z) = \sqrt{\frac{2}{\pi}}\sin(m\pi z)$, implying that $\{-m^2, m \in N\}$ are eigenvalues of A, and that $\{\phi_m\}_{m=1}^{\infty}$ is an orthonormal basis of $\mathbb{H}$. Then,

$$Av = -\sum_{m=1}^{\infty} m^2 < v, \phi_m > \phi_m, v \in D(A);$$

$< \cdot,\cdot >$ is the inner product in $\mathbb{H}$. According to [19], we can obtain

$$\mathcal{C}(t)v = \sum_{m=1}^{\infty} \cos(m\pi t) < v, \phi_m > \phi_m, \quad \mathcal{S}(t)v = \sum_{m=1}^{\infty} \frac{1}{m} \sin(m\pi t) < v, \phi_m > \phi_m,\ v \in \mathbb{H}.$$

According to [4], we obtain

$$Q_p(t)v = \sum_{n=1}^{\infty} t^{\frac{\mu}{2}} E_{\mu,\mu}(-m^2 t^{\mu}) < v, \phi_m > \phi_m,\ p = \frac{\mu}{2},$$

where $E_{\mu,\mu}(z) = \sum_{m=0}^{\infty} \frac{z^n}{\Gamma(\mu(m+1))}$ is the Mittag-Leffler function. Let $x(t)z = v(t,z)$. Then, problem (21) can be reformulated as problem (1) in $\mathbb{H}$; therefore, Theorem 1 implies that problem (21) has at least a mild solution.

6. Conclusions

By employing the Ascoli–Arzelà theorem and novel techniques, this paper has explored the existence of mild solutions for Hilfer fractional stochastic evolution equations with order $1 < \mu < 2$ and type $0 \le \nu \le 1$. Our proof demonstrates the theorem of the existence of mild solutions in both compact and noncompact cases: specifically, the satisfaction of the Lipschitz condition is not required for $f(t,\cdot)$ and $h(t,\cdot)$. The techniques presented in this paper are suitable for investigating the existence of solutions for non-autonomous evolution equations, fractional evolution equations with instantaneous/non-instantaneous impulses and fractional neutral functional evolution equations. We refer readers to the relevant papers [21,22].

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