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On Intuitionistic Fuzzy N_b Metric Space and Related Fixed Point Results with Application to Nonlinear Fractional Differential Equations

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Abstract: This manuscript contains several new notions including intuitionistic fuzzy N_b metric space, intuitionistic fuzzy quasi- S_b -metric space, intuitionistic fuzzy pseudo- S_b -metric space, intuitionistic fuzzy quasi- N -metric space and intuitionistic fuzzy pseudo N_b fuzzy metric space. We prove decomposition theorem and fixed-point results in the setting of intuitionistic fuzzy pseudo N_b fuzzy metric space. Further, we provide several non-trivial examples to show the validity of introduced notions and results. At the end, we solve an integral equation, system of linear equations and nonlinear fractional differential equations as applications.

Keywords: intuitionistic fuzzy metric space; fixed point; integral equations; system of linear equations; decomposition theorem

Citation: Ishtiaq, U.; Kattan, D.A.; Ahmad, K.; Lazăr, T.A.; Lazăr, V.L.; Guran, L. On Intuitionistic Fuzzy N_b Metric Space and Related Fixed Point Results with Application to Nonlinear Fractional Differential Equations. *Fractal Fract.* **2023**, *7*, 529. <https://doi.org/10.3390/fractalfract7070529>

Academic Editor: Ivanka Stamova

Received: 19 June 2023

Revised: 1 July 2023

Accepted: 2 July 2023

Published: 4 July 2023



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1. Introduction

There are numerous generalizations pertaining to metric space (MS) found in the literature of mathematical analysis. Gahlar [1] introduced the famous notion called 2-MS but while it is not continuous, MS is. Subsequently, Dhage [2] provided the concept of D-MS but Mustafa and Sims [3] proved that several topological properties were not true of D-MS and introduced the notion of G-MS. The majority of the fixed-point results in the setting of G-MS can be quickly determined by employing MS and quasi-MS, according to Jleli and Samet [4]. In 2012, Sedghi et al. [5] introduced the concept of S-MS and proved several fixed-point results in the framework of complete S-MS. In 1989, Bakhtin [6] defined b-MS by multiplying a real number $\tau \geq 1$ on the right side of triangle inequality. If we consider $\tau = 1$ in b-MS then it becomes MS. After that, Sedghi et al. [7] introduced the notion of S_b -MS by combining the definitions of b-MS and S-MS; however, problematically, S_b -MS is not continuous.

The fuzzy logic paradigm was established by Zadeh [8]. Contrary to traditional logic, some numbers not contained within the set are defined as an element within the interval $[0, 1]$ according to the parameters of fuzzy logic. Uncertainty, the mandatory factor of real difficulty, has helped Zadeh learn theories of fuzzy sets (FSs) that bear the difficulty of infinity. The theory is seen as a fixed point in the fuzzy metric space (FMS) for various

processes, one of them utilizing a fuzzy logic. Later on, building upon Zadeh's outcomes, Heilpern [9] established the fuzzy mapping notion and a theorem on an FP for fuzzy contraction mapping in linear MS, expressing a fuzzy general form of Banach's contraction theory. In the definition of FMSs provided by Kaleva and Seikkala [10], the imprecision is introduced when the distance between the elements is not a precise integer. After that first work by Kramosil and Michalek [11], and further work by George and Veeramani [12], the notation of an FMS was introduced. Nadaban [13] defined Fuzzy b-MS and proved several of its properties. Malviya [14] introduced the concepts of N-FMS, Pseudo N-FMS and proved several fixed-point results by using contraction mappings. Recently, Fernandez et al. [15] defined the concept of N_b -FMS and proved its topological properties with several fixed-point results for contraction mappings. In 2004, Park [16] introduced intuitionistic fuzzy metric spaces (IFMSs) as a generalization of FMSs using the concepts of intuitionistic FSs, continuous t-norm (CTN) and continuous t-conorm (CTCN). After that, several authors [17–22] worked on generalization of an IFMS and proved fixed point results for contraction mappings. Ionescu et al. [23] and Mehmood et al. [24] worked on intuitionistic fuzzy metric spaces and extended b-metric spaces and proved several fixed-point results for contraction mappings under several new conditions.

In this paper, we introduce the definitions of intuitionistic fuzzy N_b metric space (IFNBMS), intuitionistic fuzzy quasi- S_b -metric space (IFQSBMS), intuitionistic fuzzy pseudo- S_b -metric space (IFPSbMS), intuitionistic fuzzy quasi- N -metric space (IFQNMS) and intuitionistic fuzzy pseudo N_b metric space (IFPNbMS). We prove decomposition theorem and fixed-point results in new setting. At the end, we provide applications to integral equation, system of linear equations and nonlinear fractional differential equations.

2. Preliminaries

In this section, we recall several basic definitions from the existing literature.

Definition 1 [22]. A mapping $*$: $I^3 \rightarrow I$ ($I = [0,1]$) is said to be CTN if it verifies the below axioms:

- (i) $*(a, 1, 1) = a, *(0, 0, 0) = 0$;
- (ii) $*(a, b, c) = *(a, c, b) = *(b, c, a)$;
- (iii) $*$ is continuous;
- (iv) $*(a_1, b_1, c_1) \geq *(a_2, b_2, c_2)$ for $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$.

$a * b * c = abc$ and $a * b * c = \min\{a, b, c\}$ are called product CTN and minimum CTN respectively.

Definition 2 [22]. A mapping $*$: $I^3 \rightarrow I$ ($I = [0,1]$) is said to be CTCN if it verifies the below axioms:

- (i) $\diamond(a, 0, 0) = a, \diamond(0, 0, 0) = 0$;
- (ii) $\diamond(a, b, c) = \diamond(a, c, b) = \diamond(b, c, a)$;
- (iii) $\diamond$ is continuous;
- (iv) $\diamond(a_1, b_1, c_1) \geq \diamond(a_2, b_2, c_2)$ for $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$.

$a \diamond b \diamond c = \max\{a, b, c\}$ are called a maximum CTCN.

Definition 3 [14]. Suppose $\mathcal{E} \neq \emptyset$ is an arbitrary set. Then $(\mathcal{E}, N, *)$ is an NFMS, if $*$ is a CTN and N is a FS on $\mathcal{E}^3 \times (0, +\infty)$ verifying the below axioms for all $\kappa, \varpi, \omega, a \in \mathcal{E}$ and $\kappa, \tau, \sigma > 0$:

- (a) $N(\kappa, \varpi, \omega, \sigma) + \theta(\kappa, \varpi, \omega, \sigma) \leq 1$,

- (b) $N(\kappa, \varpi, \omega, \sigma) > 0$,
- (c) $N(\kappa, \varpi, \omega, \sigma) = 1$ if and only if $\kappa = \varpi = \omega$,
- (d) $N(\kappa, \varpi, \omega, \kappa + \tau + \sigma) \geq N(\kappa, \kappa, a, \kappa) * N(\varpi, \varpi, a, \tau) * N(\omega, \omega, a, \sigma)$,
- (e) $N(\kappa, \varpi, \omega, \cdot): (0, +\infty) \rightarrow (0, 1]$ is a continuous function.

Definition 4 [7]. Suppose $\mathcal{E} \neq \emptyset$ is an arbitrary set and $\mu \geq 1$ a real number. A function $S_b: \mathcal{E}^3 \rightarrow [0, +\infty)$ is called S_b -metric if and only if it verifies the below axioms for all $\kappa, \varpi, \omega, a \in \mathcal{E}$:

- (S1) $S_b(\kappa, \varpi, \omega) = 0$ if and only if $\kappa = \varpi = \omega$,
- (S2) $S_b(\kappa, \varpi, \omega) \leq \mu[S_b(\kappa, \kappa, a) + S_b(\varpi, \varpi, a) + S_b(\omega, \omega, a)]$.

Then, a pair $(\mathcal{E}, S_b)$ is known as S_b -MS.

Definition 5 [16]. A six tuple $(\mathcal{E}, N, \theta, *, \diamond)$ is an IFMS if $\mathcal{E} \neq \emptyset$ arbitrary set, $*$ is a CTN, $\diamond$ is a CTCN, N and θ are FSs on $\mathcal{E}^2 \times (0, +\infty)$ verifying the below axioms for all $\kappa, \varpi, \omega \in \mathcal{E}$ and $\tau, \sigma > 0$:

- (i) $N(\kappa, \varpi, \sigma) + \theta(\kappa, \varpi, \sigma) \leq 1$,
- (ii) $N(\kappa, \varpi, \sigma) > 0$,
- (iii) $N(\kappa, \varpi, \sigma) = 1$ if and only if $\kappa = \varpi$,
- (iv) $N(\kappa, \varpi, \tau + \sigma) \geq N(\kappa, \omega, \tau) * N(\omega, \varpi, \tau)$,
- (v) $N(\kappa, \varpi, \cdot): (0, +\infty) \rightarrow (0, 1]$ is a continuous function,
- (vi) $\theta(\kappa, \varpi, \sigma) > 0$,
- (vii) $\theta(\kappa, \varpi, \sigma) = 0$ if and only if $\kappa = \varpi$,
- (viii) $\theta(\kappa, \varpi, \tau + \sigma) \leq \theta(\kappa, \omega, \tau) \diamond \theta(\omega, \varpi, \tau)$,
- (ix) $\theta(\kappa, \varpi, \cdot): (0, +\infty) \rightarrow (0, 1]$ is a continuous function.

3. Intuitionistic Fuzzy N_b Metric Space

In this section, we introduce the notion of IFNbMS and provide several examples.

Definition 6. A six tuple $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is an IFNbMS if $\mathcal{E} \neq \emptyset$ is an arbitrary set, $*$ is a CTN, $\diamond$ is a CTCN, $\mu \geq 1$ is a real number, N and θ are FSs on $\mathcal{E}^3 \times (0, +\infty)$ verifying the below axioms for all $\kappa, \varpi, \omega, a \in \mathcal{E}$ and $\kappa, \tau, \sigma > 0$.

- (x) $N(\kappa, \varpi, \omega, \sigma) + \theta(\kappa, \varpi, \omega, \sigma) \leq 1$,
- (xi) $N(\kappa, \varpi, \omega, \sigma) > 0$,
- (xii) $N(\kappa, \varpi, \omega, \sigma) = 1$ if and only if $\kappa = \varpi = \omega$,
- (xiii) $N(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \geq N(\kappa, \kappa, a, \kappa) * N(\varpi, \varpi, a, \tau) * N(\omega, \omega, a, \sigma)$,
- (xiv) $N(\kappa, \varpi, \omega, \cdot): (0, +\infty) \rightarrow (0, 1]$ is a continuous function,
- (xv) $\theta(\kappa, \varpi, \omega, \sigma) > 0$,
- (xvi) $\theta(\kappa, \varpi, \omega, \sigma) = 0$ if and only if $\kappa = \varpi = \omega$,
- (xvii) $\theta(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \leq \theta(\kappa, \kappa, a, \kappa) \diamond \theta(\varpi, \varpi, a, \tau) \diamond \theta(\omega, \omega, a, \sigma)$,
- (xviii) $\theta(\kappa, \varpi, \omega, \cdot): (0, +\infty) \rightarrow (0, 1]$ is a continuous function.

Here, $N(\kappa, \varpi, \omega, \sigma)$ and $\theta(\kappa, \varpi, \omega, \sigma)$ are membership and non-membership functions of κ, ϖ and ω with respect to σ .

Remark 1. If we consider $\mu = 1$ in the above definition, then we get the definition of IFNMS.

Example 1. Let $\mathcal{E} = \mathbb{R}$ be the set of real numbers. Let N and Θ are the functions on $\mathcal{E}^3 \times (0, +\infty)$ defined by

$$N(\kappa, \varpi, \omega, \sigma) = \frac{\sigma}{\sigma + \mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2}$$

and

$$\Theta(\kappa, \varpi, \omega, \sigma) = \frac{\mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2}{\sigma + \mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2},$$

for all $\kappa, \varpi, \omega \in \mathcal{E}$ and $\sigma > 0$. Then $(\mathcal{E}, N, \Theta, *, \diamond, \mu)$ is an IFNbMS with CTN $a * b * c = abc$, CTCN $a \diamond b \diamond c = \max\{a, b, c\}$ and constant $\mu \geq 1$. The graphical behavior of N and Θ are shown in Figures 1 and 2 respectively.

Proof. Here, we prove only (iv) and (viii), others are obvious. Firstly, we investigate (iv),

$$\begin{aligned} & N(\kappa, \kappa, a, \kappa) * N(\varpi, \varpi, a, \tau) * N(\omega, \omega, a, \sigma) \\ &= \frac{\kappa}{\kappa + \mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2} \cdot \frac{\tau}{\tau + \mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2} \cdot \frac{\sigma}{\sigma + \mu[|\omega - a| + |\omega + a - 2\omega|]^2} \\ &= \frac{1}{1 + \frac{\mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}{\kappa}} \cdot \frac{1}{1 + \frac{\mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}{\tau}} \cdot \frac{1}{1 + \frac{\mu[|\omega - a| + |\omega + a - 2\omega|]^2}{\sigma}} \\ &\leq \frac{1}{1 + \frac{\mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}{(\kappa + \tau + \sigma)}} \cdot \frac{1}{1 + \frac{\mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}{(\kappa + \tau + \sigma)}} \cdot \frac{1}{1 + \frac{\mu[|\omega - a| + |\omega + a - 2\omega|]^2}{(\kappa + \tau + \sigma)}} \\ &\leq \frac{1}{1 + \frac{\mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2 + \mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2 + \mu[|\omega - a| + |\omega + a - 2\omega|]^2}{(\kappa + \tau + \sigma)}} \\ &\leq \frac{1}{1 + \frac{[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2}{\mu(\kappa + \tau + \sigma)}} \\ &\leq \frac{\mu(\kappa + \tau + \sigma)}{\mu(\kappa + \tau + \sigma) + [|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2} \\ &= N(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)). \end{aligned}$$

Now, let

$$\begin{aligned}
& \mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2 = \mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2 \\
& \max \left\{ \frac{\mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}{\mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}, \frac{\mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}{\mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}, \frac{\mu[|\omega - a| + |\omega + a - 2\omega|]^2}{\mu[|\omega - a| + |\omega + a - 2\omega|]^2} \right\} \\
& \leq [(\kappa + \tau + \sigma) + \mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2] \\
& \max \left\{ \frac{\mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}{\kappa + \mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}, \frac{\mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}{\tau + \mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}, \frac{\mu[|\omega - a| + |\omega + a - 2\omega|]^2}{\sigma + \mu[|\omega - a| + |\omega + a - 2\omega|]^2} \right\} \\
& \frac{\mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2}{(\kappa + \tau + \sigma) + \mu[|\kappa - \omega| + |\kappa + \omega - 2\varpi|]^2} \\
& \leq \max \left\{ \frac{\mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}{\kappa + \mu[|\kappa - a| + |\kappa + a - 2\kappa|]^2}, \frac{\mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}{\tau + \mu[|\varpi - a| + |\varpi + a - 2\varpi|]^2}, \frac{\mu[|\omega - a| + |\omega + a - 2\omega|]^2}{\sigma + \mu[|\omega - a| + |\omega + a - 2\omega|]^2} \right\} \\
& \theta(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \leq \theta(\kappa, \kappa, a, \kappa) \diamond \theta(\varpi, \varpi, a, \tau) \diamond \theta(\omega, \omega, a, \sigma).
\end{aligned}$$

Hence (iv) and (viii) are satisfied. $\square$

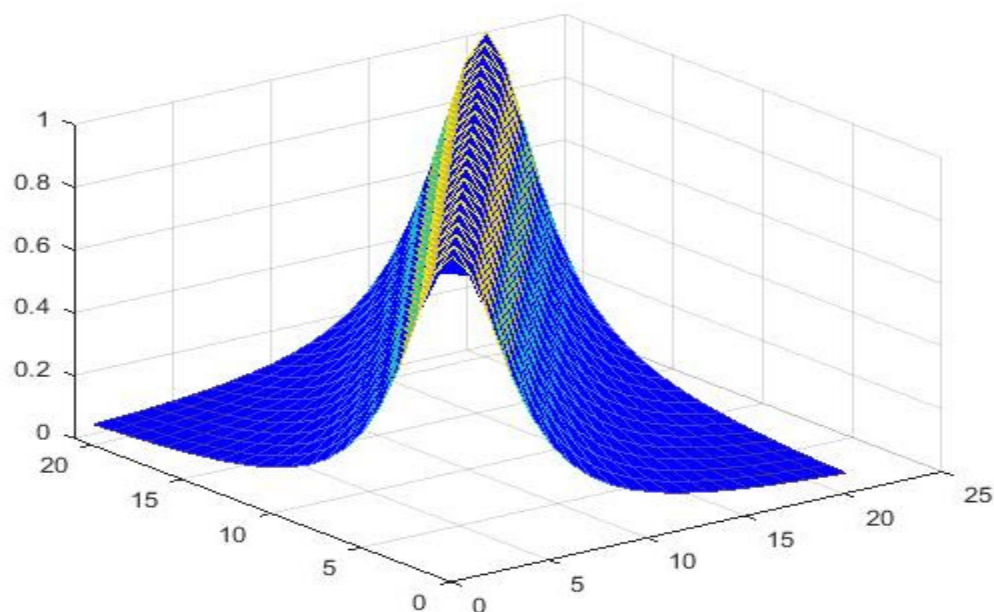


Figure 1. Depicts the behavior of N for $\mathcal{E} = [0,1]$, $\sigma = 1$ and $\mu = 4$.

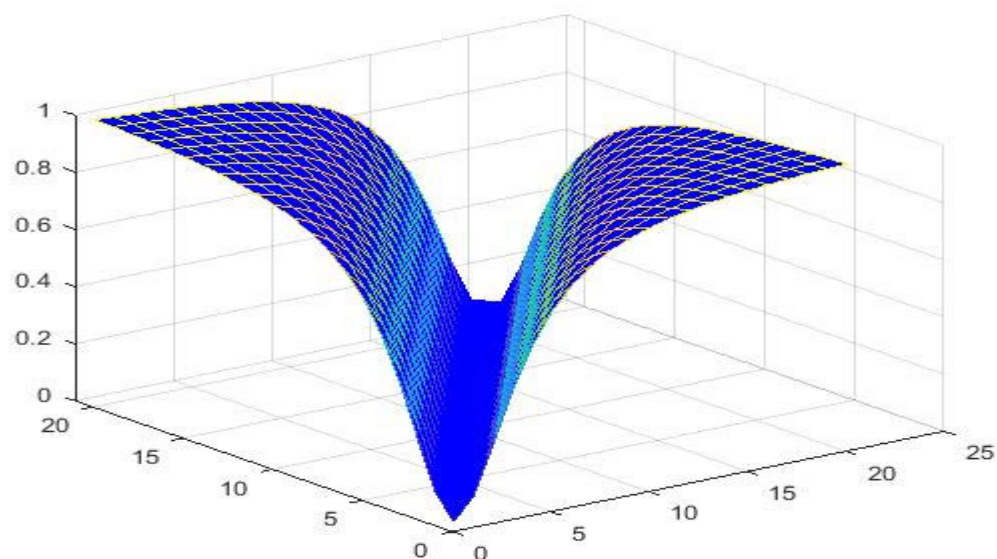


Figure 2. Depicts the behavior of θ for $\mathcal{E} = [0,1]$, $\sigma = 1$ and $\mu = 4$.

Example 2. Suppose $\mathcal{E} = \mathbb{R}$ is a set of real numbers. Let N and θ are the functions on $\mathcal{E}^3 \times (0, +\infty)$ defined by

$$N(\kappa, \varpi, \omega, \sigma) = e^{\frac{\mu[|\kappa-\omega|+|\kappa+\omega-2\varpi|]^2}{\sigma}}$$

and

$$\theta(\kappa, \varpi, \omega, \sigma) = 1 - e^{\frac{\mu[|\kappa-\omega|+|\kappa+\omega-2\varpi|]^2}{\sigma}},$$

for all $\kappa, \varpi, \omega \in \mathcal{E}$ and $\sigma > 0$. Then $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is an IFNbMS with CTN $a * b * c = abc$, CTCN $a \diamond b \diamond c = \max\{a, b, c\}$ and constant $\mu \geq 1$. The graphical behavior of N and θ are shown in Figures 3 and 4 respectively.

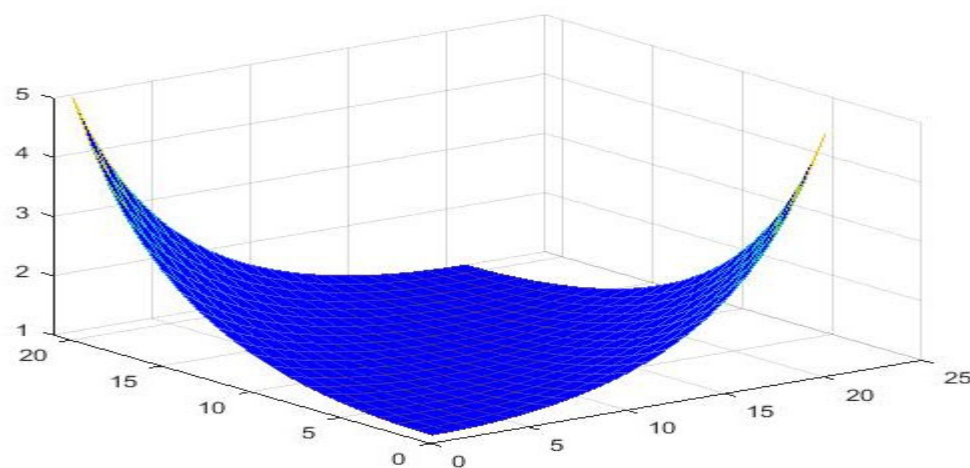


Figure 3. Depicts the behavior of N for $\mathcal{E} = [0,1]$, $\sigma = 10$ and $\mu = 4$.

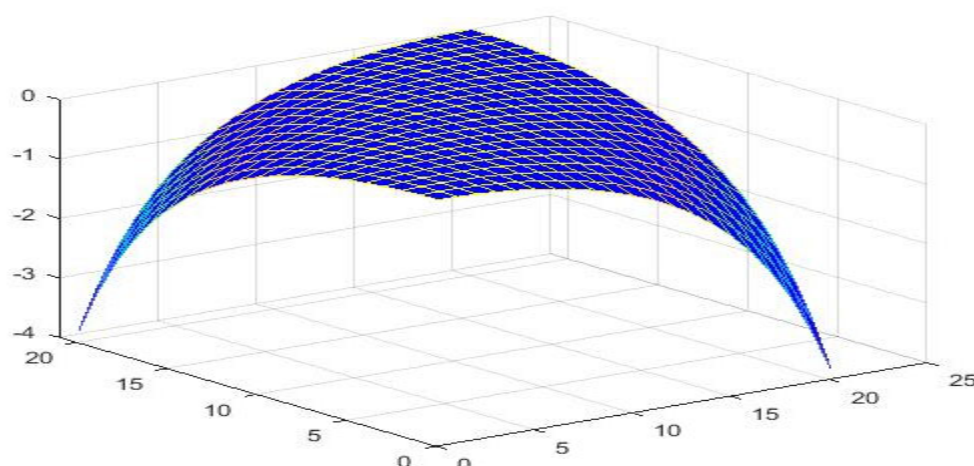


Figure 4. Depicts the behavior of θ for $\mathcal{E} = [0,1]$, $\sigma = 10$ and $\mu = 4$.

Definition 7. Suppose $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is an IFNbMS. Then $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is called symmetric if

$$N(\kappa, \kappa, \varpi, \sigma) = N(\varpi, \varpi, \kappa, \sigma) \quad (1)$$

and

$$\theta(\kappa, \kappa, \varpi, \sigma) = \theta(\varpi, \varpi, \kappa, \sigma), \quad (2)$$

for all $\kappa, \varpi \in \mathcal{E}$ and $\sigma > 0$.

Example 3. Let $\mathcal{E} = \mathbb{R}$ be the set of real numbers. Let N and θ are the functions on $\mathcal{E}^3 \times (0, +\infty)$ defined by

$$N(\kappa, \varpi, \omega, \sigma) = \frac{\sigma}{\sigma + [|\kappa - \varpi| + |\varpi - \omega| + |\omega - \kappa|]^p}$$

and

$$\theta(\kappa, \varpi, \omega, \sigma) = \frac{[|\kappa - \varpi| + |\varpi - \omega| + |\omega - \kappa|]^p}{\sigma + [|\kappa - \varpi| + |\varpi - \omega| + |\omega - \kappa|]^p},$$

for all $\kappa, \varpi, \omega \in \mathcal{E}$ and $\sigma > 0$. Then $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is a symmetric IFNbMS with CTN $a * b * c = abc$, CTCN $a \diamond b \diamond c = \max\{a, b, c\}$ and constant $\mu = 2^{2(p-1)}$.

4. Generalized Definitions

In this section, we provide several generalized definitions and examples.

Definition 8. A six tuple $(\mathcal{E}, N_{\tau}, \theta_{\tau}, *, \diamond, \mu)$ is an IFQSbMS iff $\mathcal{E} \neq \emptyset$ arbitrary set, $*$ is a CTN, $\diamond$ is a CTCN, $\mu \geq 1$ is a real number, N_{τ} and θ_{τ} are FSs on $\mathcal{E}^3 \times [0, +\infty)$ verifying the below axioms for all $\kappa, \varpi, \omega, a \in \mathcal{E}$ and $\kappa, \tau, \sigma > 0$.

- (a) $N_{\tau}(\kappa, \varpi, \omega, \sigma) + \theta_{\tau}(\kappa, \varpi, \omega, \sigma) \leq 1$,
- (b) $N_{\tau}(\kappa, \varpi, \omega, \sigma) \geq 0$,
- (c) $N_{\tau}(\kappa, \varpi, \omega, \sigma) = N_{\tau}(P\{\kappa, \varpi, \omega, \sigma\}) = 1$ if and only if $\kappa = \varpi = \omega$, where P is permutation,

- (d) $N_{\tau}(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \geq N_{\tau}(\kappa, \kappa, a, \kappa) * N_{\tau}(\varpi, \varpi, a, \tau) * N_{\tau}(\omega, \omega, a, \sigma),$
- (e) $N_{\tau}(\kappa, \varpi, \omega, \cdot) : (0, +\infty) \rightarrow (0, 1]$ is a continuous function,
- (f) $\theta_{\tau}(\kappa, \varpi, \omega, \sigma) \geq 0,$
- (g) $\theta_{\tau}(\kappa, \varpi, \omega, \sigma) = \theta_{\tau}(P\{\kappa, \varpi, \omega, \sigma\}) = 0$ if and only if $\kappa = \varpi = \omega$, where P is permutation,
- (h) $\theta_{\tau}(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \leq \theta_{\tau}(\kappa, \kappa, a, \kappa) \diamond \theta_{\tau}(\varpi, \varpi, a, \tau) \diamond \theta_{\tau}(\omega, \omega, a, \sigma),$
- (i) $\theta_{\tau}(\kappa, \varpi, \omega, \cdot) : (0, +\infty) \rightarrow (0, 1]$ is a continuous function.

Here, $N_{\tau}(\kappa, \varpi, \omega, \sigma)$ and $\theta_{\tau}(\kappa, \varpi, \omega, \sigma)$ are membership and non-membership functions of κ, ϖ and ω with respect to σ .

Remark 2. If we consider $\mu = 1$ in the above definition, then we get the definition of IFQSbMS.

Example 4. Suppose $\mathcal{E} = \mathbb{R}^+ \cup \{0\}$. Define N_{τ} and θ_{τ} by

$$N_{\tau}(\kappa, \varpi, \omega, \sigma) = \begin{cases} 1, & \text{if } \kappa = \varpi = \omega, \\ \frac{\sigma}{\sigma + \left[\left| \kappa - \frac{\omega}{2} \right| + \left| \varpi - \frac{\omega}{2} \right| \right]^2}, & \text{otherwise,} \end{cases}$$

and

$$\theta_{\tau}(\kappa, \varpi, \omega, \sigma) = \begin{cases} 0, & \text{if } \kappa = \varpi = \omega, \\ \frac{\left[\left| \kappa - \frac{\omega}{2} \right| + \left| \varpi - \frac{\omega}{2} \right| \right]^2}{\sigma + \left[\left| \kappa - \frac{\omega}{2} \right| + \left| \varpi - \frac{\omega}{2} \right| \right]^2}, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{E}, N_{\tau}, \theta_{\tau}, *, \diamond, \mu)$ is said to be an IFQSbMS with $\mu = 2$.

Remark 3. If $\kappa \neq \varpi \neq \omega$ then by definition of N_{τ} and θ_{τ} in Example 4

$$N_{\tau}(\kappa, \varpi, \omega, \sigma) \neq N_{\tau}(P\{\kappa, \varpi, \omega, \sigma\}) \text{ and } \theta_{\tau}(\kappa, \varpi, \omega, \sigma) \neq \theta_{\tau}(P\{\kappa, \varpi, \omega, \sigma\}).$$

Furthermore, $N_{\tau}(\kappa, \kappa, \varpi, \sigma) \neq N_{\tau}(\varpi, \varpi, \omega, \sigma)$. Hence, IFQSbMS is not symmetric in general.

Definition 9. A six tuple $(\mathcal{E}, N_{\tau b}, \theta_{\tau b}, *, \diamond, \mu)$ is an IFPSbMS if $\mathcal{E} \neq \emptyset$ is an arbitrary set, $*$ is a CTN, $\diamond$ is a CTCN, $\mu \geq 1$ is a real number, $N_{\tau b}$ and $\theta_{\tau b}$ are FSs on $\mathcal{E}^3 \times [0, +\infty)$ verifying the below axioms for all $\kappa, \varpi, \omega, a \in \mathcal{E}$ and $\kappa, \tau, \sigma > 0$.

- (I) $N_{\tau b}(\kappa, \varpi, \omega, \sigma) + \theta_{\tau b}(\kappa, \varpi, \omega, \sigma) \leq 1,$
- (II) $N_{\tau b}(\kappa, \varpi, \omega, \sigma) \geq 0,$
- (III) $N_{\tau b}(\kappa, \varpi, \omega, \sigma) = 1$ if $\kappa = \varpi = \omega,$
- (IV) $N_{\tau b}(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \geq N_{\tau b}(\kappa, \kappa, a, \kappa) * N_{\tau b}(\varpi, \varpi, a, \tau) * N_{\tau b}(\omega, \omega, a, \sigma),$
- (V) $N_{\tau b}(\kappa, \varpi, \omega, \cdot) : (0, +\infty) \rightarrow (0, 1]$ is a continuous function,
- (VI) $\theta_{\tau b}(\kappa, \varpi, \omega, \sigma) \geq 0,$
- (VII) $\theta_{\tau b}(\kappa, \varpi, \omega, \sigma) = 0$ if $\kappa = \varpi = \omega,$
- (VIII) $\theta_{\tau b}(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \leq \theta_{\tau b}(\kappa, \kappa, a, \kappa) \diamond \theta_{\tau b}(\varpi, \varpi, a, \tau) \diamond \theta_{\tau b}(\omega, \omega, a, \sigma),$
- (IX) $\theta_{\tau b}(\kappa, \varpi, \omega, \cdot) : (0, +\infty) \rightarrow (0, 1]$ is a continuous function.

Remark 4. If we consider $\mu = 1$ in the above definition, then we get the definition of IFPSbMS.

Example 5. Suppose $\mathcal{E} = \mathbb{R}^+$. Define N_{τ} and θ_{τ} by

$$N_{\tau b}(\kappa, \varpi, \omega, \sigma) = \begin{cases} 1, & \text{if } \kappa = \varpi = \omega, \\ \frac{\sigma}{\sigma + [|\kappa^2 - \omega^2| + |\varpi^2 - \omega^2|]^2}, & \text{otherwise,} \end{cases}$$

and

$$\theta_{\tau b}(\kappa, \varpi, \omega, \sigma) = \begin{cases} 0, & \text{if } \kappa = \varpi = \omega, \\ \frac{[|\kappa^2 - \omega^2| + |\varpi^2 - \omega^2|]^2}{\sigma + [|\kappa^2 - \omega^2| + |\varpi^2 - \omega^2|]^2}, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{E}, N_{\tau b}, \theta_{\tau b}, *, \diamond, \mu)$ is said to be an IFPSbMS.

Definition 10. A six tuple $(\mathcal{E}, N_N, \theta_N, *, \diamond)$ is an IFQNMS if $\mathcal{E} \neq \emptyset$ arbitrary set, $*$ is a CTN, $\diamond$ is a CTCN, N_N and θ_N are FSs on $\mathcal{E}^3 \times (0, +\infty)$ verifying the below axioms for all $\kappa, \varpi, \omega, a \in \mathcal{E}$ and $\kappa, \tau, \sigma > 0$.

- (a) $N_N(\kappa, \varpi, \omega, \sigma) + \theta_N(\kappa, \varpi, \omega, \sigma) \leq 1$,
- (b) $N_N(\kappa, \varpi, \omega, \sigma) > 0$,
- (c) $N_N(\kappa, \varpi, \omega, \sigma) = N_N(P\{\kappa, \varpi, \omega, \sigma\}) = 1$ if and only if $\kappa = \varpi = \omega$, where P is permutation,
- (d) $N_N(\kappa, \varpi, \omega, \kappa + \tau + \sigma) \geq N_N(\kappa, \kappa, a, \kappa) * N_N(\varpi, \varpi, a, \tau) * N_N(\omega, \omega, a, \sigma)$,
- (e) $N_N(\kappa, \varpi, \omega, \cdot). (0, +\infty) \rightarrow (0, 1]$ is a continuous function,
- (f) $\theta_N(\kappa, \varpi, \omega, \sigma) > 0$,
- (g) $\theta_N(\kappa, \varpi, \omega, \sigma) = \theta_N(P\{\kappa, \varpi, \omega, \sigma\}) = 0$ if and only if $\kappa = \varpi = \omega$, where P is permutation,
- (h) $\theta_N(\kappa, \varpi, \omega, \kappa + \tau + \sigma) \leq \theta_N(\kappa, \kappa, a, \kappa) \diamond \theta_N(\varpi, \varpi, a, \tau) \diamond \theta_N(\omega, \omega, a, \sigma)$,
- (i) $\theta_N(\kappa, \varpi, \omega, \cdot). (0, +\infty) \rightarrow (0, 1]$ is a continuous function.

Remark 5. An IFNMS is symmetric, but IFQNMS is not symmetric i.e., $N_N(\kappa, \kappa, \varpi, \sigma) \neq N_N(\varpi, \varpi, \kappa, \sigma)$ and $\theta_N(\kappa, \kappa, \varpi, \sigma) \neq \theta_N(\varpi, \varpi, \kappa, \sigma)$.

Definition 11. A six tuple $(\mathcal{E}, N_q, \theta_q, *, \diamond, \mu)$ is an IFPNbMS if $\mathcal{E} \neq \emptyset$ arbitrary set, $*$ is a CTN, $\diamond$ is a CTCN, $\mu \geq 1$ is a real number, N_q and θ_q are FSs on $\mathcal{E}^3 \times (0, +\infty)$ verifying the below axioms for all $\kappa, \varpi, \omega, a \in \mathcal{E}$ and $\kappa, \tau, \sigma > 0$.

- (i) $N_q(\kappa, \varpi, \omega, \sigma) + \theta_q(\kappa, \varpi, \omega, \sigma) \leq 1$,
- (ii) $N_q(\kappa, \varpi, \omega, \sigma) > 0$,
- (iii) $N_q(\kappa, \varpi, \omega, \sigma) = 1$ if $\kappa = \varpi = \omega$,
- (iv) $N_q(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \geq N_q(\kappa, \kappa, a, \kappa) * N_q(\varpi, \varpi, a, \tau) * N_q(\omega, \omega, a, \sigma)$,
- (v) $N_q(\kappa, \varpi, \omega, \cdot). (0, +\infty) \rightarrow (0, 1]$ is a continuous function,
- (vi) $\theta_q(\kappa, \varpi, \omega, \sigma) > 0$,
- (vii) $\theta_q(\kappa, \varpi, \omega, \sigma) = 0$ if $\kappa = \varpi = \omega$,
- (viii) $\theta_q(\kappa, \varpi, \omega, \mu(\kappa + \tau + \sigma)) \leq \theta_q(\kappa, \kappa, a, \kappa) \diamond \theta_q(\varpi, \varpi, a, \tau) \diamond \theta_q(\omega, \omega, a, \sigma)$,
- (ix) $\theta_q(\kappa, \varpi, \omega, \cdot). (0, +\infty) \rightarrow (0, 1]$ is a continuous function.

Example 6. Let $\mathbb{R}$ equipped with a usual metric and $\mathcal{E} = \{\{\mathcal{X}_n\} : \{\mathcal{X}_n\} \text{ is convergent in } \mathbb{R}\}$. Define CTN by $a * b * c = abc$ and CTCN by $a \diamond b \diamond c = \max\{a, b, c\}$ for all $a, b, c \in [0, 1]$ and

$$N_q(\mathcal{X}_n, \varpi_n, \mathcal{W}_n, \sigma) = \left[e^{\frac{(|\mathcal{X}_n - \varpi_n| + |\varpi_n - \mathcal{W}_n|)^2}{\sigma}} \right]^{-1}$$

and

$$\theta_q(\mathcal{X}_n, \varpi_n, \mathcal{W}_n, \sigma) = 1 - \left[e^{\frac{(|\mathcal{X}_n - \varpi_n| + |\varpi_n - \mathcal{W}_n|)^2}{\sigma}} \right]^{-1}.$$

Observe that $(\mathcal{E}, N_q, \theta_q, *, \diamond, \mu)$ is an IFPNbMS, but it is not an IFNbMS. For this, take $\{\mathcal{X}_n\} = \frac{2}{n}$, $\{\varpi_n\} = \frac{3}{n}$ and $\{\mathcal{W}_n\} = \frac{5}{n}$. Then, $\{\mathcal{X}_n\} \neq \{\varpi_n\} \neq \{\mathcal{W}_n\}$ for $\{\mathcal{X}_n\}$, $\{\varpi_n\}$ and $\{\mathcal{W}_n\}$ in $\mathcal{E}$ but $N_q(\mathcal{X}_n, \varpi_n, \mathcal{W}_n, \sigma) = 1$ and $\theta_q(\mathcal{X}_n, \varpi_n, \mathcal{W}_n, \sigma) = 0$.

Remark 6. Every IFNbMS is an IFPNbMS, but converse does not hold.

Definition 12. Suppose $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ be a symmetric IFNbMS. A sequence $\{\mathcal{X}_n\}$ in $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is called convergent, if $N(\mathcal{X}_n, \mathcal{X}_n, \mathcal{X}, \sigma) \rightarrow 1$ and $\theta(\mathcal{X}_n, \mathcal{X}_n, \mathcal{X}, \sigma) \rightarrow 0$ or $N(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \sigma) \rightarrow 1$ and $\theta(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \sigma) \rightarrow 0$ as $n \rightarrow +\infty$ for every $\sigma > 0$. That is, for $\kappa > 0$ and $\sigma > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $N(\mathcal{X}_n, \mathcal{X}_n, \mathcal{X}, \sigma) > 1 - \kappa$ and $\theta(\mathcal{X}_n, \mathcal{X}_n, \mathcal{X}, \sigma) < \kappa$ or $N(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \sigma) > 1 - \kappa$ and $\theta(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \sigma) < \kappa$.

Lemma 1. Suppose $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is a symmetric IFNbMS with CTN $a * b * c = abc$ and CTCN $a \diamond b \diamond c = \max\{a, b, c\}$. Suppose $\{\mathcal{X}_n\}$ be a sequence in $\mathcal{E}$. If $\{\mathcal{X}_n\}$ converges to $\mathcal{X}$ and $\{\mathcal{X}_n\}$ converges to ϖ then $\mathcal{X} = \varpi$. That is, the limit of $\{\mathcal{X}_n\}$ is unique if it exists.

Proof. Suppose $\{\mathcal{X}_n\}$ converges to $\mathcal{X}$ and ϖ . Then $N(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \tau) \rightarrow 1$ and $\theta(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \tau) \rightarrow 0$ as $n \rightarrow +\infty$ for every $\tau > 0$ and $N(\varpi, \varpi, \mathcal{X}_n, \sigma - 2\tau) \rightarrow 1$ and $\theta(\varpi, \varpi, \mathcal{X}_n, \sigma - 2\tau) \rightarrow 0$ as $n \rightarrow +\infty$ for every $\sigma - 2\tau > 0$.

$$N(\mathcal{X}, \mathcal{X}, \varpi, \sigma) \geq N(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \tau) * N(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \tau) * N\left(\varpi, \varpi, \mathcal{X}_n, \frac{\sigma}{\mu} - 2\tau\right) \rightarrow 1 * 1 * 1 = 1$$

and

$$\theta(\mathcal{X}, \mathcal{X}, \varpi, \sigma) \leq \theta(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \tau) \diamond \theta(\mathcal{X}, \mathcal{X}, \mathcal{X}_n, \tau) \diamond \theta\left(\varpi, \varpi, \mathcal{X}_n, \frac{\sigma}{\mu} - 2\tau\right) \rightarrow 0 \diamond 0 \diamond 0 = 0,$$

as $n \rightarrow +\infty$. $\square$

Definition 13. Let $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ be a symmetric IFNbMS. A sequence $\{\mathcal{X}_n\}$ be a sequence in $\mathcal{E}$ is termed a Cauchy, if for all $\kappa > 0$ and $\sigma > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$N(\mathcal{X}_n, \mathcal{X}_n, \mathcal{X}_m, \sigma) > 1 - \kappa \text{ and } \theta(\mathcal{X}_n, \mathcal{X}_n, \mathcal{X}_m, \sigma) < \kappa,$$

or

$$N(\mathcal{X}_m, \mathcal{X}_m, \mathcal{X}_n, \sigma) > 1 - \kappa \text{ and } \theta(\mathcal{X}_m, \mathcal{X}_m, \mathcal{X}_n, \sigma) < \kappa,$$

for each $n, m \geq n_0$.

Definition 14. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ be a symmetric IFNbMS. If every Cauchy in $\mathfrak{E}$ is convergent in $\mathfrak{E}$, then $\mathfrak{E}$ is said to be a complete symmetric IFNbMS.

Definition 15. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ be a symmetric IFNbMS. A subset A of $\mathfrak{E}$ is called F -bounded if $\sigma > 0$ and $0 < \kappa < 1$ such that

$$N(\kappa, \kappa, \varpi, \sigma) > 1 - \kappa \text{ and } \theta(\kappa, \kappa, \varpi, \sigma) < \kappa, \text{ for each } \kappa, \varpi \in A.$$

Definition 16. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ be an IFNbMS. A mapping $f: \mathfrak{E} \rightarrow \mathfrak{E}$ is called an intuitionistic fuzzy b -contraction if, for each $\kappa, \varpi, \omega \in \mathfrak{E}$ and for some $q \in (0, 1)$, we have

$$N(f(\kappa), f(\kappa), f(\varpi), q\sigma) \geq N(\kappa, \kappa, \varpi, \sigma) \text{ and } \theta(f(\kappa), f(\kappa), f(\varpi), q\sigma) \leq \theta(\kappa, \kappa, \varpi, \sigma).$$

Lemma 2. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ be an IFNbMS with CTN $a * b * c = abc$ and CTCN $a \diamond b \diamond c = \max\{a, b, c\}$. Let $\{\kappa_n\}$ is a sequence in $\mathfrak{E}$ and converges to κ , then $\{\kappa_n\}$ is a Cauchy sequence.

Proof. There is $p \in \mathbb{N}$ for every $\tau, \sigma > 0$, such that

$$N(\kappa_n, \kappa_n, \kappa, \tau) \rightarrow 1 \text{ and } \theta(\kappa_n, \kappa_n, \kappa, \tau) \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

and

$$N\left(\kappa_{n+p}, \kappa_{n+p}, \kappa, \frac{\sigma}{\mu} - 2\tau\right) \rightarrow 1 \text{ and } \theta\left(\kappa_{n+p}, \kappa_{n+p}, \kappa, \frac{\sigma}{\mu} - 2\tau\right) \rightarrow 0 \text{ as } n \rightarrow +\infty, \text{ for all } \frac{\sigma}{\mu} - 2\tau > 0.$$

$$N(\kappa_n, \kappa_n, \kappa_{n+p}, \sigma) \geq N(\kappa_n, \kappa_n, \kappa, \tau) * N(\kappa_n, \kappa_n, \kappa, \tau) * N\left(\kappa_{n+p}, \kappa_{n+p}, \kappa, \frac{\sigma}{\mu} - 2\tau\right)$$

$$\rightarrow 1 * 1 * 1 = 1 \text{ as } n \rightarrow +\infty,$$

and

$$\theta(\kappa_n, \kappa_n, \kappa_{n+p}, \sigma) \leq \theta(\kappa_n, \kappa_n, \kappa, \tau) \diamond \theta(\kappa_n, \kappa_n, \kappa, \tau) \diamond \theta\left(\kappa_{n+p}, \kappa_{n+p}, \kappa, \frac{\sigma}{\mu} - 2\tau\right)$$

$$\rightarrow 0 \diamond 0 \diamond 0 = 0 \text{ as } n \rightarrow +\infty.$$

Hence, $\{\kappa_n\}$ is a Cauchy sequence. $\square$

Definition 17. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ and $(\mathfrak{E}', N', \theta', *, \diamond', \mu')$ are symmetric IFNbMSs. Then a function $f: \mathfrak{E} \rightarrow \mathfrak{E}'$ is said to be continuous at a point $\kappa \in \mathfrak{E}$ if and only if it is sequentially continuous at κ , that is whenever $\{\kappa_n\}$ is convergent to κ we have $\{f\kappa_n\}$ converge to $f(\kappa)$.

Proposition 1. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ be symmetric IFNbMSs and f be a fuzzy q -contraction. If any fixed point κ of f satisfies

$$N(\kappa, \kappa, \kappa, \sigma) > 0,$$

$$\theta(\kappa, \kappa, \kappa, \sigma) < 1,$$

then

$$N(\kappa, \kappa, \kappa, \sigma) = 1,$$

$$\theta(\kappa, \kappa, \kappa, \sigma) = 0.$$

Proof. Let $\kappa \in \mathfrak{E}$ be a fixed point of f , as f is a fuzzy q -contraction, so

$$\begin{aligned} N(\kappa, \kappa, \kappa, \sigma) &= N(f(\kappa), f(\kappa), f(\kappa), \sigma) \\ &\geq N\left(\kappa, \kappa, \kappa, \frac{\sigma}{q}\right) \geq N\left(\kappa, \kappa, \kappa, \frac{\sigma}{q^2}\right) \\ &\geq \dots \geq N\left(\kappa, \kappa, \kappa, \frac{\sigma}{q^n}\right) \rightarrow 1 \end{aligned}$$

as $n \rightarrow +\infty$ and so

$$N(\kappa, \kappa, \kappa, \sigma) = 1.$$

Similarly,

$$\begin{aligned} \theta(\kappa, \kappa, \kappa, \sigma) &= \theta(f(\kappa), f(\kappa), f(\kappa), \sigma) \\ &\leq \theta\left(\kappa, \kappa, \kappa, \frac{\sigma}{q}\right) \leq \theta\left(\kappa, \kappa, \kappa, \frac{\sigma}{q^2}\right) \\ &\leq \dots \leq \theta\left(\kappa, \kappa, \kappa, \frac{\sigma}{q^n}\right) \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$ and so

$$\theta(\kappa, \kappa, \kappa, \sigma) = 0.$$

□

Lemma 3. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ be symmetric IFNbMSs. Let $\{\kappa_n\}$ and $\{\varpi_n\}$ be two sequences in $\mathfrak{E}$ and suppose $\kappa_n \rightarrow \kappa, \varpi_n \rightarrow \varpi$, as $n \rightarrow +\infty$, $N(\kappa, \kappa, \varpi, \sigma_n) \rightarrow N(\kappa, \kappa, \varpi, \sigma)$ and $\theta(\kappa, \kappa, \varpi, \sigma_n) \rightarrow \theta(\kappa, \kappa, \varpi, \sigma)$ as $n \rightarrow +\infty$. Then $N(\kappa_n, \kappa_n, \varpi_n, \sigma_n) \rightarrow N(\kappa, \kappa, \kappa, \sigma)$ and $\theta(\kappa_n, \kappa_n, \varpi_n, \sigma_n) \rightarrow \theta(\kappa, \kappa, \kappa, \sigma)$ as $n \rightarrow +\infty$.

Proof. Since $\lim_{n \rightarrow +\infty} \kappa_n = \kappa, \lim_{n \rightarrow +\infty} \varpi_n = \varpi, \lim_{n \rightarrow +\infty} N(\kappa, \kappa, \varpi, \sigma_n) = N(\kappa, \kappa, \varpi, \sigma)$ and $\lim_{n \rightarrow +\infty} \theta(\kappa, \kappa, \varpi, \sigma_n) = \theta(\kappa, \kappa, \varpi, \sigma)$ there is $n_0 \in \mathbb{N}$ such that $|\sigma - \sigma_n| < \delta$ for $n \geq n_0$ and $\delta < \frac{\sigma}{2}$. We know $N(\kappa, \kappa, \varpi, \sigma)$ is non-decreasing and $\theta(\kappa, \kappa, \varpi, \sigma)$ is non-increasing with respect to σ , so we have

$$\begin{aligned} N(\kappa_n, \kappa_n, \varpi_n, \sigma_n) &\geq N(\kappa_n, \kappa_n, \varpi_n, \sigma - \delta) \\ &\geq N\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) * N\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) * N\left(\varpi_n, \varpi_n, \kappa, \frac{\sigma}{\mu} - \frac{5\delta}{3\mu}\right) \\ &\geq N\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) * N\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) * N\left(\varpi_n, \varpi_n, \varpi, \frac{\delta}{6\mu^2}\right) \\ &\quad * N\left(\varpi_n, \varpi_n, \varpi, \frac{\delta}{6\mu^2}\right) * N\left(\varpi, \varpi, \kappa, \frac{\sigma}{\mu^2} - \frac{\delta}{6\mu^2}\right), \end{aligned}$$

and

$$\begin{aligned}
 N(\kappa, \kappa, \varpi, \sigma + 2\delta) &\geq N(\kappa, \kappa, \varpi, \sigma_n + 2\delta) \\
 &\geq N\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) * N\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) * N\left(\varpi, \varpi, \kappa_n, \frac{\sigma_n}{\mu} + \frac{\delta}{3\mu}\right) \\
 &\geq N\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) * N\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) * N\left(\varpi, \varpi, \kappa_n, \frac{\delta}{6\mu^2}\right) \\
 &\quad * N\left(\varpi, \varpi, \varpi_n, \frac{\delta}{6\mu^2}\right) * N\left(\kappa_n, \kappa_n, \varpi_n, \frac{\sigma_n}{\mu^2}\right).
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \theta(\kappa_n, \kappa_n, \varpi_n, \sigma_n) &\leq \theta(\kappa_n, \kappa_n, \varpi_n, \sigma - \delta) \\
 &\leq \theta\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) \diamond \theta\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) \diamond \theta\left(\varpi_n, \varpi_n, \kappa, \frac{\sigma}{\mu} - \frac{5\delta}{3\mu}\right) \\
 &\leq \theta\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) \diamond \theta\left(\kappa_n, \kappa_n, \kappa, \frac{\delta}{3\mu}\right) \diamond \theta\left(\varpi_n, \varpi_n, \varpi, \frac{\delta}{6\mu^2}\right) \\
 &\quad \diamond \theta\left(\varpi_n, \varpi_n, \varpi, \frac{\delta}{6\mu^2}\right) \diamond \theta\left(\varpi, \varpi, \kappa, \frac{\sigma}{\mu^2} - \frac{\delta}{6\mu^2}\right),
 \end{aligned}$$

and

$$\begin{aligned}
 \theta(\kappa, \kappa, \varpi, \sigma + 2\delta) &\leq \theta(\kappa, \kappa, \varpi, \sigma_n + 2\delta) \\
 &\leq \theta\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) \diamond \theta\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) \diamond \theta\left(\varpi, \varpi, \kappa_n, \frac{\sigma_n}{\mu} + \frac{\delta}{3\mu}\right) \\
 &\leq \theta\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) \diamond \theta\left(\kappa, \kappa, \kappa_n, \frac{\delta}{3\mu}\right) \diamond \theta\left(\varpi, \varpi, \kappa_n, \frac{\delta}{6\mu^2}\right) \\
 &\quad \diamond \theta\left(\varpi, \varpi, \varpi_n, \frac{\delta}{6\mu^2}\right) \diamond \theta\left(\kappa_n, \kappa_n, \varpi_n, \frac{\sigma_n}{\mu^2}\right).
 \end{aligned}$$

In the view of definition of 12 and combining the arbitrariness of δ and the continuity for $N(\kappa, \kappa, \varpi, \cdot)$ and $\theta(\kappa, \kappa, \varpi, \cdot)$ with respect to σ . For large enough n , we have

$$\begin{aligned}
 N(\kappa, \kappa, \varpi, \sigma) &\geq N(\kappa_n, \kappa_n, \varpi_n, \sigma_n) \geq N(\varpi, \varpi, \kappa, \sigma) \\
 N(\kappa, \kappa, \varpi, \sigma) &\geq N(\kappa_n, \kappa_n, \varpi_n, \sigma_n),
 \end{aligned}$$

by Definition 7

$$\geq N(\kappa, \kappa, \varpi, \sigma),$$

Consequently,

$$\lim_{n \rightarrow +\infty} N(\kappa_n, \kappa_n, \varpi_n, \sigma_n) = N(\kappa, \kappa, \varpi, \sigma).$$

and

$$\begin{aligned}
 \theta(\kappa, \kappa, \varpi, \sigma) &\leq \theta(\kappa_n, \kappa_n, \varpi_n, \sigma_n) \leq \theta(\varpi, \varpi, \kappa, \sigma) \\
 \theta(\kappa, \kappa, \varpi, \sigma) &\leq \theta(\kappa_n, \kappa_n, \varpi_n, \sigma_n),
 \end{aligned}$$

by Definition 7

$$\leq \theta(\kappa, \kappa, \varpi, \sigma).$$

Consequently,

$$\lim_{n \rightarrow +\infty} \theta(\kappa_n, \kappa_n, \varpi_n, \sigma_n) = \theta(\kappa, \kappa, \varpi, \sigma).$$

□

Lemma 4. Let $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ be symmetric IFNbMSs. If there exists $q \in (0, 1)$ such that $N(\kappa, \kappa, \varpi, \sigma) \geq N\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right)$ and $\theta(\kappa, \kappa, \varpi, \sigma) \leq \theta\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right)$ for all $\kappa, \varpi \in \mathfrak{E}, \sigma > 0$ and

$$\lim_{\sigma \rightarrow +\infty} N(\kappa, \varpi, \omega, \sigma) = 1,$$

$$\lim_{\sigma \rightarrow +\infty} \theta(\kappa, \varpi, \omega, \sigma) = 0.$$

Then $\kappa = \varpi$.

Proof. Suppose that there exists $q \in (0, 1)$ such that $N(\kappa, \kappa, \varpi, \sigma) \geq N\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right)$ and

$\theta(\kappa, \kappa, \varpi, \sigma) \leq \theta\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right)$ for all $\kappa, \varpi \in \mathfrak{E}$ and $\sigma > 0$. Then,

$$N(\kappa, \kappa, \varpi, \sigma) \geq N\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right) \geq N\left(\kappa, \kappa, \varpi, \frac{\sigma}{q^2}\right),$$

and so

$$N(\kappa, \kappa, \varpi, \sigma) \geq N\left(\kappa, \kappa, \varpi, \frac{\sigma}{q^n}\right),$$

For positive integer n . Taking limit as $n \rightarrow +\infty$, $N(\kappa, \kappa, \varpi, \sigma) \geq 1$,

$$\theta(\kappa, \kappa, \varpi, \sigma) \leq \theta\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right) \leq \theta\left(\kappa, \kappa, \varpi, \frac{\sigma}{q^2}\right),$$

and so

$$\theta(\kappa, \kappa, \varpi, \sigma) \leq \theta\left(\kappa, \kappa, \varpi, \frac{\sigma}{q^n}\right),$$

For positive integer n . Taking limit as $n \rightarrow +\infty$, $\theta(\kappa, \kappa, \varpi, \sigma) = 0$ and hence $\kappa = \varpi$.

□

5. Application in Fixed Point Theory

Now, we present a Banach contraction principle as an application via intuitionistic fuzzy q -contraction in symmetric IFNbMSs.

Theorem 1. Suppose $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ is a symmetric complete IFNbMSs with

$$\lim_{\sigma \rightarrow +\infty} N(\kappa, \varpi, \omega, \sigma) = 1, \quad \lim_{\sigma \rightarrow +\infty} \theta(\kappa, \varpi, \omega, \sigma) = 0. \quad (3)$$

and f be an intuitionistic fuzzy q -contraction. Then f has a unique fixed point.

Proof. Let $\kappa_0 \in \mathcal{E}$ and generate a sequence $\{\kappa_n\}$ by the iterative process $\kappa_n = f^n(\kappa_0), n \in \mathbb{N}$. Since $n, \sigma > 0$. We have

$$\begin{aligned} N(\kappa_n, \kappa_n, \kappa_{n+1}, q\sigma) &= N(f\kappa_{n-1}, f\kappa_{n-1}, f\kappa_n, q\sigma) \\ &\geq N(\kappa_{n-1}, \kappa_{n-1}, \kappa_n, \sigma) \\ &\geq N\left(\kappa_{n-2}, \kappa_{n-2}, \kappa_n, \frac{\sigma}{q}\right) \\ &\vdots \\ &\geq N\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n-1}}\right), \end{aligned}$$

and

$$\begin{aligned} \theta(\kappa_n, \kappa_n, \kappa_{n+1}, q\sigma) &= \theta(f\kappa_{n-1}, f\kappa_{n-1}, f\kappa_n, q\sigma) \\ &\leq \theta(\kappa_{n-1}, \kappa_{n-1}, \kappa_n, \sigma) \\ &\leq \theta\left(\kappa_{n-2}, \kappa_{n-2}, \kappa_n, \frac{\sigma}{q}\right) \\ &\vdots \\ &\leq \theta\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n-1}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} N(\kappa_n, \kappa_n, \kappa_{n+1}, q\sigma) &\geq N\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n-1}}\right) \\ N(\kappa_n, \kappa_n, \kappa_{n+p}, \sigma) &\geq N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_{n+p}, \kappa_{n+p}, \kappa_{n+1}, \frac{\sigma}{3\mu}\right). \end{aligned}$$

By using Definition 6, we have

$$= N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+p}, \frac{\sigma}{3\mu}\right)$$

by symmetric property, we obtain

$$\begin{aligned} &\geq N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right) \\ &\quad * N\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right) * N\left(\kappa_{n+p}, \kappa_{n+p}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right), \\ &= N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) * N\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right) \\ &\quad * N\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right) * N\left(\kappa_{n+2}, \kappa_{n+2}, \kappa_{n+p}, \frac{\sigma}{(3\mu)^2}\right), \\ &\geq N\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^n(3\mu)}\right) * N\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^n(3\mu)}\right) * N\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n+1}(3\mu)^2}\right) \\ &\quad * N\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n+1}(3\mu)^2}\right). \end{aligned}$$

By Equation (3), intuitionistic fuzzy q -contraction (i.e., $q < 1$) and taking $n \rightarrow +\infty$, we obtain

$$\lim_{n \rightarrow +\infty} N(\kappa_n, \kappa_n, \kappa_{n+1}, \sigma) = 1 * 1 * \dots * 1 = 1.$$

Similarly,

$$\theta(\kappa_n, \kappa_n, \kappa_{n+1}, q\sigma) \leq \theta\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n-1}}\right)$$

$$\theta(\kappa_n, \kappa_n, \kappa_{n+p}, \sigma) \leq \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_{n+p}, \kappa_{n+p}, \kappa_{n+1}, \frac{\sigma}{3\mu}\right).$$

By using Definition 6, we get

$$= \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+p}, \frac{\sigma}{3\mu}\right),$$

by symmetric property, we have

$$\leq \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right)$$

$$\diamond \theta\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right) \diamond \theta\left(\kappa_{n+p}, \kappa_{n+p}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right),$$

$$= \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_n, \kappa_n, \kappa_{n+1}, \frac{\sigma}{3\mu}\right) \diamond \theta\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right)$$

$$\diamond \theta\left(\kappa_{n+1}, \kappa_{n+1}, \kappa_{n+2}, \frac{\sigma}{(3\mu)^2}\right) \diamond \theta\left(\kappa_{n+2}, \kappa_{n+2}, \kappa_{n+p}, \frac{\sigma}{(3\mu)^2}\right)$$

$$\leq \theta\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^n(3\mu)}\right) \diamond \theta\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^n(3\mu)}\right) \diamond \theta\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n+1}(3\mu)^2}\right)$$

$$\diamond \theta\left(\kappa_0, \kappa_0, \kappa_1, \frac{\sigma}{q^{n+1}(3\mu)^2}\right).$$

By Equation (3), intuitionistic fuzzy q -contraction (i.e., $q < 1$) and taking $n \rightarrow +\infty$, we obtain

$$\lim_{n \rightarrow +\infty} N(\kappa_n, \kappa_n, \kappa_{n+1}, \sigma) = 0 \diamond 0 \diamond \dots \diamond 0 = 0.$$

Hence, $\{\kappa_n\}$ is Cauchy sequence. Therefore, $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is a symmetric complete IFNbMSs, there exist $\kappa \in \mathcal{E}$, we have

$$\lim_{n \rightarrow +\infty} \kappa_n = \kappa.$$

Now, we will show that κ is a fixed point of f .

$$N(f(\kappa), f(\kappa), \kappa, \sigma) \geq N\left(f(\kappa), f(\kappa), \kappa_n, \frac{\sigma}{3\mu}\right) * N\left(f(\kappa), f(\kappa), \kappa_n, \frac{\sigma}{3\mu}\right)$$

$$* N\left(\kappa, \kappa, f(\kappa_n), \frac{\sigma}{3\mu}\right)$$

$$\geq N\left(\kappa, \kappa, \kappa_n, \frac{\sigma}{3\mu q}\right) * N\left(\kappa, \kappa, \kappa_n, \frac{\sigma}{3\mu q}\right) * N\left(\kappa, \kappa, \kappa_{n+1}, \frac{\sigma}{3\mu}\right).$$

Since f is intuitionistic fuzzy q -contraction and $f(\kappa_n) = \kappa_{n+1}$ as $n \rightarrow +\infty$

$$\rightarrow 1 * 1 * 1 = 1.$$

On the other hand, we have

$$\theta(f(\kappa), f(\kappa), \kappa, \sigma) \leq \theta\left(f(\kappa), f(\kappa), \kappa_n, \frac{\sigma}{3\mu}\right) \diamond \theta\left(f(\kappa), f(\kappa), \kappa_n, \frac{\sigma}{3\mu}\right)$$

$$\diamond \theta\left(\kappa, \kappa, f(\kappa_n), \frac{\sigma}{3\mu}\right)$$

$$\leq \theta\left(\kappa, \kappa, \kappa_n, \frac{\sigma}{3\mu q}\right) \diamond \theta\left(\kappa, \kappa, \kappa_n, \frac{\sigma}{3\mu q}\right) \diamond \theta\left(\kappa, \kappa, \kappa_{n+1}, \frac{\sigma}{3\mu}\right),$$

Since f is intuitionistic fuzzy q -contraction and $f(\kappa_n) = \kappa_{n+1}$ as $n \rightarrow +\infty$

$$\rightarrow 0 \diamond 0 \diamond 0 = 0.$$

That is $f(\kappa) = \kappa$, hence, κ is a fixed point of f . Now, we examine the uniqueness, suppose $f(\varpi) = \varpi$ for some $\varpi \in \mathfrak{E}$, then

$$\begin{aligned} N(\varpi, \varpi, \kappa, \sigma) &= N(f(\varpi), f(\varpi), f(\kappa), \sigma) \\ &\geq N\left(\varpi, \varpi, \kappa, \frac{\sigma}{q}\right), \\ &= N\left(f(\varpi), f(\varpi), f(\kappa), \frac{\sigma}{q}\right) \geq N\left(\varpi, \varpi, \kappa, \frac{\sigma}{q^2}\right) \geq \dots \geq N\left(\varpi, \varpi, \kappa, \frac{\sigma}{q^n}\right) \rightarrow 1, \end{aligned}$$

as $n \rightarrow +\infty$. On the other hand, we have

$$\begin{aligned} \theta(\varpi, \varpi, \kappa, \sigma) &= \theta(f(\varpi), f(\varpi), f(\kappa), \sigma) \\ &\leq \theta\left(\varpi, \varpi, \kappa, \frac{\sigma}{q}\right), \\ &= \theta\left(f(\varpi), f(\varpi), f(\kappa), \frac{\sigma}{q}\right) \leq \theta\left(\varpi, \varpi, \kappa, \frac{\sigma}{q^2}\right) \leq \dots \leq \theta\left(\varpi, \varpi, \kappa, \frac{\sigma}{q^n}\right) \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$. That is $\kappa = \varpi$. $\square$

Example 7. Suppose $\mathfrak{E} = [0, 1]$ and $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ is symmetric complete IFNbMS where N, θ are defined by

$$N(\kappa, \varpi, \omega, \sigma) = e^{-\frac{[|\kappa-\omega|+|\varpi-\omega|]^2}{\sigma}},$$

$$\theta(\kappa, \varpi, \omega, \sigma) = 1 - e^{-\frac{[|\kappa-\omega|+|\varpi-\omega|]^2}{\sigma}} \quad \text{for all } \kappa, \varpi, \omega \in \mathfrak{E}, \sigma > 0.$$

Proof. Let $f(\kappa) = \lambda\kappa, \lambda < \frac{\sqrt{2}}{2}, \kappa \in \mathfrak{E}, \sigma > 0$. Then, for $\frac{1}{2} > q$

$$\begin{aligned} N(f(\kappa), f(\kappa), f(\varpi), \sigma) &= e^{-\frac{[|f(\kappa)-f(\varpi)|+|f(\kappa)-f(\varpi)|]^2}{\sigma}}, \\ &= e^{-\frac{[2|f(\kappa)-f(\varpi)|]^2}{\sigma}} = e^{-\frac{[2|\lambda\kappa-\lambda\varpi|]^2}{\sigma}} = e^{-\frac{4\lambda^2|\kappa-\varpi|^2}{\sigma}} \\ &= e^{-\frac{[|\kappa-\varpi|+|\kappa-\varpi|]^2}{\frac{\sigma}{\lambda^2}}} = e^{-\frac{[|\kappa-\varpi|+|\kappa-\varpi|]^2}{\frac{\sigma}{q}}} = N\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right), \end{aligned}$$

where, $\lambda^2 = q$. On the other hand

$$\begin{aligned} \theta(f(\kappa), f(\kappa), f(\varpi), \sigma) &= 1 - e^{-\frac{[|f(\kappa)-f(\varpi)|+|f(\kappa)-f(\varpi)|]^2}{\sigma}}, \\ &= 1 - e^{-\frac{[2|f(\kappa)-f(\varpi)|]^2}{\sigma}} = 1 - e^{-\frac{[2|\lambda\kappa-\lambda\varpi|]^2}{\sigma}} = 1 - e^{-\frac{4\lambda^2|\kappa-\varpi|^2}{\sigma}} \\ &= 1 - e^{-\frac{[|\kappa-\varpi|+|\kappa-\varpi|]^2}{\frac{\sigma}{\lambda^2}}} = 1 - e^{-\frac{[|\kappa-\varpi|+|\kappa-\varpi|]^2}{\frac{\sigma}{q}}} = \theta\left(\kappa, \kappa, \varpi, \frac{\sigma}{q}\right), \end{aligned}$$

where, $\lambda^2 = q$. That is, Theorem 1 hold and 0 is a unique fixed point of f in $\mathfrak{E}$.

Let $\theta: (0, +\infty) \rightarrow (0, +\infty)$ as

$$\theta(\sigma) = \int_0^\sigma \phi(\sigma) d\sigma, \quad \text{for all } \sigma > 0,$$

be a non-decreasing and continuous function. Furthermore, for every $\kappa > 0$, $\phi(\kappa) > 0$. This implies $\phi(\sigma) = 0$ if and only if $\sigma = 0$. $\square$

Theorem 2. Suppose $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ is a complete symmetric IFNbMSs and $f: \mathfrak{E} \rightarrow \mathfrak{E}$ is a mapping verifying

$$\int_0^{N(f(\kappa), f(\kappa), f(\varpi), q\sigma)} \phi(\sigma) d\sigma \geq \int_0^{N(\kappa, \kappa, \varpi, \sigma)} \phi(\sigma) d\sigma,$$

$$\int_0^{\theta(f(\kappa), f(\kappa), f(\varpi), q\sigma)} \phi(\sigma) d\sigma \leq \int_0^{\theta(\kappa, \kappa, \varpi, \sigma)} \phi(\sigma) d\sigma,$$

for all $\kappa, \varpi \in \mathfrak{E}$, $\phi \in \theta$ and $q \in (0, 1)$. Then there exists a unique fixed point of f .

Proof. Letting $\phi(1) = 1$ and by utilizing Theorem 1, it is immediate. $\square$

6. Application to Integral Equations

The most significant and engaging activities in mathematics are those involving solving equations of any form. There are numerous methods for dealing with various types of equations. Seeking solutions to the issue at hand and determining whether they are singular or multiple. Since fixed-point theory is an iterative process with a wide range of contexts, it is one of the key techniques that has achieved considerable success in the field of integral equations.

Fixed-point theory is crucial in the investigation of whether there is a solution to a differential or integral problem. In this part, we demonstrate how Theorem 1 applies to a specific nonlinear integral problem. The solution to the question “The solution for a specific nonlinear integral Equation (6) exists or not?” is provided by the following theorem.

Let $\mathfrak{E} = C[0, I]$ be the set of real valued continuous functions on a bounded interval $[0, I]$. Then $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ is complete symmetric IFNbMSs defined by $N, \theta: \mathfrak{E}^3 \times (0, +\infty) \rightarrow [0, 1]$ by

$$N(\kappa, \varpi, \omega, \sigma) = e^{-\frac{\sup_{\tau \in [0, I]} [|\kappa(\tau) - \omega(\tau)| + |\varpi(\tau) + \omega(\tau)|]^2}{\sigma}}, \quad (4)$$

$$\theta(\kappa, \varpi, \omega, \sigma) = 1 - e^{-\frac{\sup_{\tau \in [0, I]} [|\kappa(\tau) - \omega(\tau)| + |\varpi(\tau) + \omega(\tau)|]^2}{\sigma}}, \quad (5)$$

for $\sigma > 0$ and for all $\kappa, \varpi, \omega \in \mathfrak{E}$ and let

$$\kappa(\sigma) = g(\sigma) + \int_0^I A(\sigma, \tau) H(\sigma, \tau, \kappa(\tau)) d\tau, \quad (6)$$

where $I > 0$ and $g: [0, I] \rightarrow R$ and $H: [0, 1]^2 \times R \rightarrow R$ are continuous functions.

Theorem 3. Suppose $(\mathfrak{E}, N, \theta, *, \diamond, \mu)$ is a symmetric complete IFNbMSs provided in Equations (4) and (5). Define the integral operator $f: \mathfrak{E} \rightarrow \mathfrak{E}$ by

$$f(\kappa(\sigma)) = g(\sigma) + \int_0^I A(\sigma, \tau) H(\sigma, \tau, \kappa(\tau)) d\tau, \quad (7)$$

for all $\kappa \in \mathfrak{E}$ and $\sigma, \tau \in [0, I]$. Suppose that the following conditions are satisfied.

(a) For all $\sigma, \tau \in [0, I]$ and $\kappa, \varpi \in \mathfrak{E}$

$$|H(\sigma, \tau, \kappa(\tau)) - H(\sigma, \tau, \varpi(\tau))| \leq |\kappa(\tau) - \varpi(\tau)|. \quad (8)$$

(b) For all $\sigma, \tau \in [0, I]$,

$$\sup_{\tau \in [0, I]} \left| \int_0^I (A(\sigma, \tau)^2) d\tau \right| \leq q < 1. \quad (9)$$

Then $\kappa^* \in \mathcal{E}$ is a unique solution for Equation (6).

Proof. For each $\kappa, \varpi \in \mathcal{E}$, we get

$$\begin{aligned} N(f(\kappa), f(\kappa), f(\varpi), q\sigma) &= e^{-\frac{\sup_{\tau \in [0, I]} [|f(\kappa(\sigma)) - f(\varpi(\sigma))| + |f(\kappa(\sigma)) - f(\varpi(\sigma))|]^2}{q\sigma}}, \\ &= e^{-\frac{\sup_{\tau \in [0, I]} [2|f(\kappa(\sigma)) - f(\varpi(\sigma))|]^2}{q\sigma}}, = e^{-\frac{\sup_{\tau \in [0, I]} 4 \left| \int_0^I (A(\sigma, \tau) H(\sigma, \tau, \kappa(\tau)) - A(\sigma, \tau) H(\sigma, \tau, \varpi(\tau))) d\tau \right|^2}{q\sigma}}, \\ &\geq e^{-\frac{\sup_{\tau \in [0, I]} 4 \left| \int_0^I (A(\sigma, \tau))^2 d\tau \right| \int_0^I |H(\sigma, \tau, \kappa(\tau)) - H(\sigma, \tau, \varpi(\tau))| d\tau|^2}{q\sigma}} \geq e^{-\frac{4q \int_0^I |(\kappa(\tau) - \varpi(\tau)) d\tau|^2}{q\sigma}} \\ &\geq e^{-\frac{\sup_{\tau \in [0, I]} 4|\kappa(\tau) - \varpi(\tau)|^2}{\sigma}} = e^{-\frac{\sup_{\tau \in [0, I]} [|\kappa(\tau) - \varpi(\tau)| + |\kappa(\tau) - \varpi(\tau)|]^2}{\sigma}} = N(\kappa, \kappa, \varpi, \sigma). \end{aligned}$$

Similarly,

$$\begin{aligned} \theta(f(\kappa), f(\kappa), f(\varpi), q\sigma) &= 1 - e^{-\frac{\sup_{\tau \in [0, I]} [|f(\kappa(\sigma)) - f(\varpi(\sigma))| + |f(\kappa(\sigma)) - f(\varpi(\sigma))|]^2}{q\sigma}}, \\ &= 1 - e^{-\frac{\sup_{\tau \in [0, I]} [2|f(\kappa(\sigma)) - f(\varpi(\sigma))|]^2}{q\sigma}}, = 1 - e^{-\frac{\sup_{\tau \in [0, I]} 4 \left| \int_0^I (A(\sigma, \tau) H(\sigma, \tau, \kappa(\tau)) - A(\sigma, \tau) H(\sigma, \tau, \varpi(\tau))) d\tau \right|^2}{q\sigma}}, \\ &\leq 1 - e^{-\frac{\sup_{\tau \in [0, I]} 4 \left| \int_0^I (A(\sigma, \tau))^2 d\tau \right| \int_0^I |H(\sigma, \tau, \kappa(\tau)) - H(\sigma, \tau, \varpi(\tau))| d\tau|^2}{q\sigma}} \leq 1 - e^{-\frac{4q \int_0^I |(\kappa(\tau) - \varpi(\tau)) d\tau|^2}{q\sigma}} \\ &\leq 1 - e^{-\frac{\sup_{\tau \in [0, I]} 4|\kappa(\tau) - \varpi(\tau)|^2}{\sigma}} = 1 - e^{-\frac{\sup_{\tau \in [0, I]} [|\kappa(\tau) - \varpi(\tau)| + |\kappa(\tau) - \varpi(\tau)|]^2}{\sigma}} = \theta(\kappa, \kappa, \varpi, \sigma). \end{aligned}$$

Since all conditions of Theorem 1 are satisfied. Hence, the integral Equation (6) has a unique solution. $\square$

7. Application to Linear Equations

Suppose $\mathcal{E} = \mathbb{R}^n$ and define a complete symmetric IFNbMS on $\mathcal{E}^3 \times (0, +\infty)$ by

$$N(\kappa, \varpi, \omega, \sigma) = \frac{\sigma}{\sigma + [\sum_{i=1}^n |\kappa_i - \varpi_i| + \sum_{i=1}^n |\varpi_i - \omega_i|]^2}, \quad (10)$$

$$\theta(\kappa, \varpi, \omega, \sigma) = \frac{[\sum_{i=1}^n |\kappa_i - \varpi_i| + \sum_{i=1}^n |\varpi_i - \omega_i|]^2}{\sigma + [\sum_{i=1}^n |\kappa_i - \varpi_i| + \sum_{i=1}^n |\varpi_i - \omega_i|]^2}, \quad (11)$$

for all $\kappa, \varpi, \omega \in \mathbb{R}^n$ and $\mu = 2$, if

$$\left[\max_{1 \leq j \leq n} \sum_{i=1}^n |c_{ij}| \right]^2 \leq q < 1. \quad (12)$$

There is only one solution to the set of linear equations below.

$$\begin{cases} c_{11}\kappa_1 + c_{12}\kappa_2 + \cdots + c_{1n}\kappa_n = d_1, \\ c_{21}\kappa_1 + c_{22}\kappa_2 + \cdots + c_{2n}\kappa_n = d_2, \\ \vdots \\ c_{n1}\kappa_1 + c_{n2}\kappa_2 + \cdots + c_{nn}\kappa_n = d_n. \end{cases} \quad (13)$$

Proof. Let $f: \mathcal{E} \rightarrow \mathcal{E}$ be defined by $f(\kappa) = c\kappa + d$ where $\kappa, d \in \mathbb{R}^n$ and

$$c = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}.$$

For $\kappa, \varpi \in \mathbb{R}^n$, we get

$$\begin{aligned} N(f(\kappa), f(\kappa), f(\varpi), q\sigma) &= \frac{q\sigma}{q\sigma + 4\left[\sum_{i=1}^n \left|\sum_{j=1}^n c_{ij}(\kappa_j - \varpi_j)\right|\right]^2} \\ &\geq \frac{q\sigma}{q\sigma + 4\left[\sum_{i=1}^n \sum_{j=1}^n |c_{ij}| \left|\sum_{j=1}^n (\kappa_j - \varpi_j)\right|\right]^2} = \frac{q\sigma}{q\sigma + \left[\sum_{j=1}^n 2|\kappa_j - \varpi_j| \sum_{i=1}^n |c_{ij}|\right]^2} \\ &\geq \frac{q\sigma}{q\sigma + \left[\max_{1 \leq j \leq n} \sum_{i=1}^n |c_{ij}| \right]^2 \left[\sum_{j=1}^n 2|\kappa_j - \varpi_j|\right]^2}. \end{aligned}$$

Using the Equation (12), we have

$$\geq \frac{q\sigma}{q\sigma + q\left[\sum_{j=1}^n 2|\kappa_j - \varpi_j|\right]^2} = \frac{\sigma}{\sigma + \left[\sum_{j=1}^n |\kappa_j - \varpi_j| + |\kappa_j - \varpi_j|\right]^2} = N(\kappa, \kappa, \varpi, \sigma).$$

and

$$\begin{aligned} \theta(f(\kappa), f(\kappa), f(\varpi), q\sigma) &= \frac{4\left[\sum_{i=1}^n \left|\sum_{j=1}^n c_{ij}(\kappa_j - \varpi_j)\right|\right]^2}{q\sigma + 4\left[\sum_{i=1}^n \left|\sum_{j=1}^n c_{ij}(\kappa_j - \varpi_j)\right|\right]^2} \\ &\leq \frac{4\left[\sum_{i=1}^n \sum_{j=1}^n |c_{ij}| \left|\sum_{j=1}^n (\kappa_j - \varpi_j)\right|\right]^2}{q\sigma + 4\left[\sum_{i=1}^n \sum_{j=1}^n |c_{ij}| \left|\sum_{j=1}^n (\kappa_j - \varpi_j)\right|\right]^2} = \frac{\left[\sum_{j=1}^n 2|\kappa_j - \varpi_j| \sum_{i=1}^n |c_{ij}|\right]^2}{q\sigma + \left[\sum_{j=1}^n 2|\kappa_j - \varpi_j| \sum_{i=1}^n |c_{ij}|\right]^2} \\ &\leq \frac{\left[\max_{1 \leq j \leq n} \sum_{i=1}^n |c_{ij}| \right]^2 \left[\sum_{j=1}^n 2|\kappa_j - \varpi_j|\right]^2}{q\sigma + \left[\max_{1 \leq j \leq n} \sum_{i=1}^n |c_{ij}| \right]^2 \left[\sum_{j=1}^n 2|\kappa_j - \varpi_j|\right]^2}. \end{aligned}$$

Using the Equation (12)

$$\leq \frac{q\left[\sum_{j=1}^n 2|\kappa_j - \varpi_j|\right]^2}{q\sigma + q\left[\sum_{j=1}^n 2|\kappa_j - \varpi_j|\right]^2} = \frac{\left[\sum_{j=1}^n |\kappa_j - \varpi_j| + |\kappa_j - \varpi_j|\right]^2}{\sigma + \left[\sum_{j=1}^n |\kappa_j - \varpi_j| + |\kappa_j - \varpi_j|\right]^2} = \theta(\kappa, \kappa, \varpi, \sigma).$$

Hence, f is an intuitionistic fuzzy q -contraction and Theorem 1 hold. That is, the system of linear Equation (13) has a unique solution in $\mathcal{E}$. $\square$

8. Application to Nonlinear Fractional Differential Equation

In this section, we apply Theorem 1 is to determine the existence and uniqueness of a solution to nonlinear fractional differential equation given by

$$D_c^\alpha \kappa(\varrho) = \psi(\varrho, \kappa(\varrho)) \quad (\varrho \in (0,1), \alpha \in (1,2]),$$

with boundary conditions

$$\kappa(0) = 0, \kappa'(0) = I\kappa(\varrho) \quad \varrho \in (0,1),$$

where D_c^α means caputo fractional derivative of order α , defined by

$$D_c^\alpha \psi(\varrho) = \frac{1}{\Gamma(n-\alpha)} \int_0^\varrho (\varrho - \tau)^{n-\alpha-1} \psi^n(\tau) d\tau \quad (n-1 < \alpha < n, \quad n = [\alpha] + 1),$$

and $\psi: [0,1] \times \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function. We suppose that $\mathcal{E} = C([0,1], \mathbb{R})$, from $[0,1]$ into $\mathbb{R}$ with supremum $|\kappa| = \sup_{\varrho \in [0,1]} |\kappa(\varrho)|$.

The Riemann-Liouville fractional integral of order α is given by

$$I^\alpha \psi(\varrho) = \frac{1}{\Gamma(\alpha)} \int_0^\varrho (\varrho - \tau)^{\alpha-1} \psi(\tau) d\tau \quad (\alpha > 0).$$

Firstly, we provide an acceptable form for a nonlinear fractional differential equation before investigating the existence of a solution. Now, we suppose the following fractional differential equation

$$D_c^\alpha \kappa(\varrho) = \psi(\varrho, \kappa(\varrho)) \quad (\varrho \in (0,1), \alpha \in (1,2]), \quad (14)$$

with the boundary conditions

$$\kappa(0) = 0, \quad \kappa'(0) = I\kappa(\varrho) \quad (\varrho \in (0,1)),$$

where

i $\psi: [0,1] \times \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function,

ii $\kappa(\varrho): [0,1] \rightarrow \mathbb{R}$ is continuous,

and satisfying the following condition

$$|\psi(\varrho, \kappa) - \psi(\varrho, \varpi) - \psi(\varrho, \omega)| \leq \mathcal{L}|\kappa - \varpi - \omega|,$$

for all $\varrho \in [0,1]$ and L is a constant with $L\mathcal{L} < 1$, where

$$\mathcal{L} = \frac{1}{\Gamma(\alpha+1)} + \frac{2\varpi^{\alpha+1}\Gamma(\alpha)}{(2-\varpi^2)\Gamma(\alpha+1)}.$$

Then the Equation (14) has a unique solution.

Proof. Suppose that

$$N(\kappa, \varpi, \omega, \sigma) = \frac{\sigma}{\sigma + |\kappa - \varpi - \omega|^p}$$

$$\theta(\kappa, \varpi, \omega, \sigma) = \frac{|\kappa - \varpi - \omega|^p}{\sigma + |\kappa - \varpi - \omega|^p}, \text{ for all } \kappa, \varpi \in \mathcal{E} \text{ and } \sigma > 0,$$

defined by $a * b * c = abc$, and $a \diamond b \diamond c = \max\{a, b, c\}$. Let $|\kappa - \varpi - \omega| = \sup_{\varrho \in [0,1]} |\kappa - \varpi - \omega|^p$, for all $\kappa, \varpi \in \mathcal{E}$. Then $(\mathcal{E}, N, \theta, *, \diamond, \mu)$ is a complete IFNbMS. We define a mapping $f: \mathcal{E} \rightarrow \mathcal{E}$ by

$$f\kappa(\varrho) = \frac{1}{\Gamma(\alpha)} \int_0^\varrho (\varrho - \tau)^{\alpha-1} \psi(\tau, \kappa(\tau)) d\tau + \frac{2\varrho}{(2-\varpi^2)\Gamma(\alpha)} \int_0^\varpi \left(\int_0^\tau (\tau - m)^{\alpha-1} \psi(m, \kappa(m)) dm \right) d\tau \quad (15)$$

for all $\varrho \in [0,1]$. An Equation (14) has a solution $\kappa \in \mathcal{E}$ iff $\kappa(\varrho) = f\kappa(\varrho)$ for all $\varrho \in [0,1]$. Now

$$N(\kappa(\varrho), \varpi(\varrho), \omega(\varrho), \sigma) = \frac{\sigma}{\sigma + |\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p},$$

$$\theta(\kappa(\varrho), \varpi(\varrho), \omega(\varrho), \sigma) = \frac{|\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p}{\sigma + |\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p}. \quad (16)$$

$$\begin{aligned} & |f\kappa(\varrho) - f\varpi(\varrho) - f\omega(\varrho)| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} \psi(\tau, \kappa(\tau)) d\tau \right. \\ &\quad \left. + \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} \psi(m, \kappa(m)) dm \right) d\tau \right| \\ &\quad - \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} \psi(\tau, \varpi(\tau)) d\tau \right. \\ &\quad \left. + \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} \psi(m, \varpi(m)) dm \right) d\tau \right| \\ &\quad - \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} \psi(\tau, \omega(\tau)) d\tau \right. \\ &\quad \left. + \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} \psi(m, \omega(m)) dm \right) d\tau \right|. \end{aligned}$$

That is,

$$\begin{aligned}
& |f\kappa(\varrho) - f\varpi(\varrho) - f\omega(\varrho)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} \psi(\tau, \kappa(\tau)) d\tau + \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} \psi(m, \kappa(m)) dm \right) d\tau \right. \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} \psi(\tau, \varpi(\tau)) d\tau - \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} \psi(m, \varpi(m)) dm \right) d\tau \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} \psi(\tau, \omega(\tau)) d\tau - \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} \psi(m, \omega(m)) dm \right) d\tau \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} |\psi(\tau, \kappa(\tau)) - \psi(\tau, \varpi(\tau)) - \psi(\tau, \omega(\tau))| d\tau \\
&\quad + \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} |\psi(m, \kappa(m)) - \psi(m, \varpi(m)) - \psi(m, \omega(m))| dm \right) d\tau \\
&\quad + \frac{2\varrho}{(2 - \varpi^2)\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} |\psi(m, \kappa(m)) - \psi(m, \varpi(m)) - \psi(m, \omega(m))| dm \right) d\tau \\
&\leq \frac{L|\kappa - \varpi - \omega|}{\Gamma(\alpha)} \int_0^{\varrho} (\varrho - \tau)^{\alpha-1} d\tau + \frac{2L|\kappa - \varpi|}{\Gamma(\alpha)} \int_0^{\varpi} \left(\int_0^{\tau} (\tau - m)^{\alpha-1} dm \right) d\tau \\
&\leq \frac{L|\kappa - \varpi - \omega|}{\Gamma(\alpha + 1)} + \frac{2\varpi^{\alpha+1}L|\kappa - \varpi|\Gamma(\alpha)}{(2 - \varpi^2)\Gamma(\alpha + 2)} \\
&\leq L|\kappa - \varpi - \omega| \left(\frac{1}{\Gamma(\alpha + 1)} + \frac{2\varpi^{\alpha+1}\Gamma(\alpha)}{(2 - \varpi^2)\Gamma(\alpha + 2)} \right) = L\mathcal{L}|\kappa - \varpi - \omega|.
\end{aligned}$$

Utilizing the fact $L\mathcal{L} < 1$ and Equation (16), we have

$$\begin{aligned}
N(f\kappa(\varrho), f\varpi(\varrho), f\omega(\varrho), \eta\sigma) &= \frac{\eta\sigma}{\eta\sigma + |f\kappa(\varrho) - f\varpi(\varrho) - f\omega(\varrho)|^p} \geq \frac{\eta\sigma}{\eta\sigma + L\mathcal{L}|\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p} \\
&\geq \frac{\sigma}{\sigma + |\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p} = N(\kappa(\varrho), \varpi(\varrho), \omega(\varrho), \sigma), \\
\theta(f\kappa(\varrho), f\varpi(\varrho), f\omega(\varrho), \eta\sigma) &= \frac{|f\kappa(\varrho) - f\varpi(\varrho) - f\omega(\varrho)|^p}{\eta\sigma + |f\kappa(\varrho) - f\varpi(\varrho) - f\omega(\varrho)|^p} \leq \frac{L\mathcal{L}|\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p}{\eta\sigma + L\mathcal{L}|\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p} \\
&\leq \frac{|\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p}{\sigma + |\kappa(\varrho) - \varpi(\varrho) - \omega(\varrho)|^p} = \theta(\kappa(\varrho), \varpi(\varrho), \omega(\varrho), \sigma).
\end{aligned}$$

As a result, all conditions of Theorem 1 are satisfied. This shows that f has a unique solution. $\square$

9. Conclusions

In this paper, we introduced the notions of IFNbMS, IFQSBMS, IFPSbMS, IFQNMS and IFPNbMS. We proved decomposition theorem and fixed-point results in the setting of IFPNbMS. Further, we provided several non-trivial examples to show the validity of

introduced notions and results. At the end, we solved an integral equation, system of linear equations and nonlinear fractional differential equations as applications. This work is extendable by introducing the notions of neutrosophic N_b metric spaces, neutrosophic quasi N_b metric spaces and neutrosophic pseudo N_b metric spaces.

Author Contributions: Conceptualization, U.I. and K.A.; methodology, D.A.K. and T.A.L.; software, U.I. and L.G.; validation, U.I., D.A.K. and V.L.L.; formal analysis, T.A.L. and L.G.; investigation, U.I.; resources, K.A.; data curation, D.A.K.; writing—original draft preparation, U.I. and K.A.; writing—review and editing, U.I., D.A.K., T.A.L. and V.L.L.; visualization, U.I. and L.G.; supervision, U.I. and V.L.L.; project administration, D.A.K. and T.A.L.; funding acquisition, D.A.K. and T.A.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data will be available on demand from corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

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