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Dynamic of Some Relapse in a Giving Up Smoking Model Described by Fractional Derivative

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Abstract: Smoking is associated with various detrimental health conditions, including cancer, heart disease, stroke, lung illnesses, diabetes, and fatal diseases. Motivated by the application of fractional calculus in epidemiological modeling and the exploration of memory and nonlocal effects, this paper introduces a mathematical model that captures the dynamics of relapse in a smoking cessation context and presents the dynamic behavior of the proposed model utilizing Caputo fractional derivatives. The model incorporates four compartments representing potential, persistent (heavy), temporally recovered, and permanently recovered smokers. The basic reproduction number R_0 is computed, and the local and global dynamic behaviors of the free equilibrium smoking point ($\mathscr{Y}_0$) and the smoking-present equilibrium point ($\mathscr{Y}^*$) are analyzed. It is demonstrated that the free equilibrium smoking point ($\mathscr{Y}_0$) exhibits global asymptotic stability when $R_0 \leq 1$, while the smoking-present equilibrium point ($\mathscr{Y}^*$) is globally asymptotically stable when $R_0 > 1$. Additionally, analytical results are validated through a numerical simulation using the predictor–corrector PECE method for fractional differential equations in Matlab software.

Keywords: smoking model; Caputo fractional derivatives; existence and uniqueness; smoking-free equilibrium; basic reproduction number; numerical simulation

MSC: 34A08; 37C75; 37N25; 65L07

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1. Introduction

Smoking addiction is a major global cause of respiratory and cancer diseases, leading to premature deaths. It is estimated that over five million people die each year due to tobacco consumption. According to statistics from the World Health Organization, this number could exceed eight million by 2030 if effective control systems are not established [1]. Smokers face a 70% higher risk of heart attack than nonsmokers, and their life expectancy is typically 10–13 years shorter [2]. The harmful effects of smoking extend beyond the smokers themselves. Secondhand smoke, which comprises both exhaled smoke and the smoke directly emitted from burning tobacco, contains harmful substances. Nonsmokers frequently exposed to secondhand tobacco smoke are at increased risk of developing similar diseases as smokers, including lung cancer and cardiovascular disease [3]. Carcinogens are the primary agents responsible for causing cancer, and studies have identified over 60 different carcinogens present in tobacco smoke [4].

Researchers, physicians, and mathematicians desire to reduce cigarette use to extend human life expectancy. Mathematicians have developed various models to depict the smoking phenomenon accurately, and several researchers have contributed to these smoking models. Castillo-Garsow et al. [5] presented an initial smoking model that focused on studying smoking dynamics in society, particularly the behavior of individuals trying to quit smoking. Zaman et al. [6] concentrated on smoking control by identifying optimal control factors. Recognizing that some smokers experience relapses while others continue smoking due to constant interactions, Rahman et al. [7] developed a smoking model that included terms for quit smoker relapse. Huo and Zhu [8] derived and analyzed a model taking into account light smokers compartment, recovery compartment, and two relapses in the giving up smoking model based on ordinary differential equations. Since smoking can have harmful effects after some time, many mathematical models of evolutionary systems with a memory effect on dynamics have used fractional calculus (see [9] and references therein). There have been many mathematical models for the smoking epidemic using fractional derivatives. Erturk et al. [10] investigated a model for smoking cessation linked to the Caputo fractional derivative. Zaman explored the optimal campaign in a smoking dynamical system [6]. Numerous researchers have also examined the effects of smoking. Lubin and Caporaso [11] discussed the association between cigarettes and lung cancer. Garsow et al. analyzed the mathematical description of tobacco use, cessation, and relapse in [5]. Alkhudhari et al. [12] studied the global dynamics of mathematical equations describing smoking. Khalid et al. [13] explained a fractional mathematical model for smoking cessation. Singh et al. [14] analyzed a fractional smoking cessation model in relation to a new fractional derivative with a nonsingular kernel. Ahmad et al. [15] created and studied a smoking epidemic model using Atangana–Baleanu (AB) with the Mittag–Leffler kernel and Atangana–Toufik method (ATM) fractional derivative. Addai et al. [16] presented a nonlinear fractional mathematical model for the smoke epidemic that includes two age groups using the Atangana–Baleanu–Caputo fractional derivative. More recently, Addai et al. [17] studied the dynamics of the age-structure smoking model under fractal-fractional (F-F) derivatives with government intervention coverage in the Caputo–Fabrizio framework. Zeb et al. [18] presented a four-class mathematical model—S (potential smokers), C (chain smokers), R (temporary quitters that can be movable to the relapse habit), and Q (permanent quitters of smoking)—in the form of fractional order. However, they did not assume that a fraction of the heavy smokers can enrich the permanently recovered smokers class and that temporally recovered smokers can relapse into the heavy smokers class. Therefore, we construct a new mathematical model that incorporates this phenomenon. Moreover, motivated by applying fractional calculus in the epidemiology model and examining the memory and the nonlocal you effect, the considering model's dynamic is presented in terms of Caputo fractional derivatives of order $\alpha \in (0, 1]$.

The control strategy is implemented by considering tobacco as an epidemic that causes several deaths. Its spread is mainly linked to the human factor, including factors such as the living environment, curiosity, and contact between people (smokers), which facilitate tobacco use. Mathematical models, described by differential equations, are utilized to interpret the spread of an infectious agent (smokers) within a population. The numbers of healthy and sick individuals evolve over time based on the contacts during which this agent passes from an infected individual to a healthy, immunized individual, subsequently infecting them in turn. The dynamics of the propagation in the population are determined through the resolution of these equations. Hence, we study the dynamics of the smoking epidemic problem. The model is based on four compartment classes: potential, persistent or heavy, temporally recovered, and permanently recovered smokers.

The paper's organization is as follows: Section 2 provides the essential introductory concepts required throughout the article. Section 3 describes the construction and development of our model. The mathematical analysis of the model is presented in Section 4. In Section 5, numerical simulations of the proposed model are provided. Ultimately, the conclusions are shown in the final section.

2. Preliminary

In this section, important definitions and preliminaries for fractional calculus are given, and for more details, see [19].

- The Riemann–Liouville fractional integral of order α is defined by

$$I^{\alpha}_{0,t}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\varsigma)^{\alpha-1}f(\varsigma)d\varsigma, \tag{1}$$

  where $\alpha \in \mathbb{R}_+$ is the order of integration, and $\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}dt$ is the gamma function.
- The Caputo fractional derivatives of order $\alpha \in \mathbb{R}_+$ are defined as

$${}^{C}\mathcal{D}^{\alpha}_{0,t}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\varsigma)^{n-\alpha-1}f^{(n)}(\varsigma)d\varsigma, \tag{2}$$

  where $n = [\alpha] + 1$, with $[\alpha]$ being the integer part of $\alpha \in \mathbb{R}_+$.
- The Caputo derivative and the Riemann–Liouville integral satisfy the following properties:
  (a) ${}^{C}\mathcal{D}^{\alpha}_{0,t}(I^{\alpha}_{0,t}f(t)) = f(t)$.
  (b) ${}^{C}\mathcal{D}^{\alpha}_{0,t}(C) = 0$, where $C \in \mathbb{R}$.
  (c) $I^{\alpha}_{0,t}({}^{C}\mathcal{D}^{\alpha}_{0,t}f(t)) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k$.
  (d) If α is such that $0 < \alpha < 1$, then $I^{\alpha}_{0,t}({}^{C}\mathcal{D}^{\alpha}_{0,t}f(t)) = f(t) - f(0)$.

3. Mathematical Modeling of a Giving Up Smoking Model

In this section, we develop a mathematical model of the dynamic of some relapse in a giving up smoking described by fractional derivatives. The model is based on four compartments: potential, persistent or heavy, temporally recovered, and permanently recovered smokers. As shown in Figure 1 (the flow diagram), the total host population $N(t)$ is partitioned into four classes, namely, the potential smokers $P(t)$, persistent smokers $S(t)$, temporally recovered smokers $X(t)$, and permanently recovered smokers $Q(t)$. The number of recruits per unit of time is denoted by ω. The smoking-related death rates are denoted by ϖ_i, $i = 1,2,3,4$. We assume that $\varpi_1 < \varpi_4 < \varpi_3 < \varpi_2$, which is biologically relevant since the death rate is higher if the smokers do not quit smoking.

A person joins the potential smoker's compartment, denoted as $P(t)$, at a constant recruitment rate ω. It is assumed that the potential smokers start smoking as a result of contact with the persistent smokers class, denoted as $S(t)$, at a rate $\wp_1 P(t)S(t)$, where $\wp_1$ represents the contact rate between potential smokers and the persistent smokers class (see Table 1). Upon leaving the persistent smokers class $S(t)$, a fraction $(1-\theta)$ enters the temporary quit smokers class, while the remaining fraction θ enters the permanently quit smokers class $Q(t)$, both at a rate ϑ. Temporary quit smokers have the possibility of relapsing into the persistent smokers class at the rate $\wp_2 X(t)S(t)$, where $\wp_2$ represents the contact rate between temporary quit smokers and the persistent smokers class. Finally, individuals in each compartment will vacate the compartment at a constant natural death $\mho$ and the smoking-related death rate ϖ_i, $i = 1,\ldots,4$.

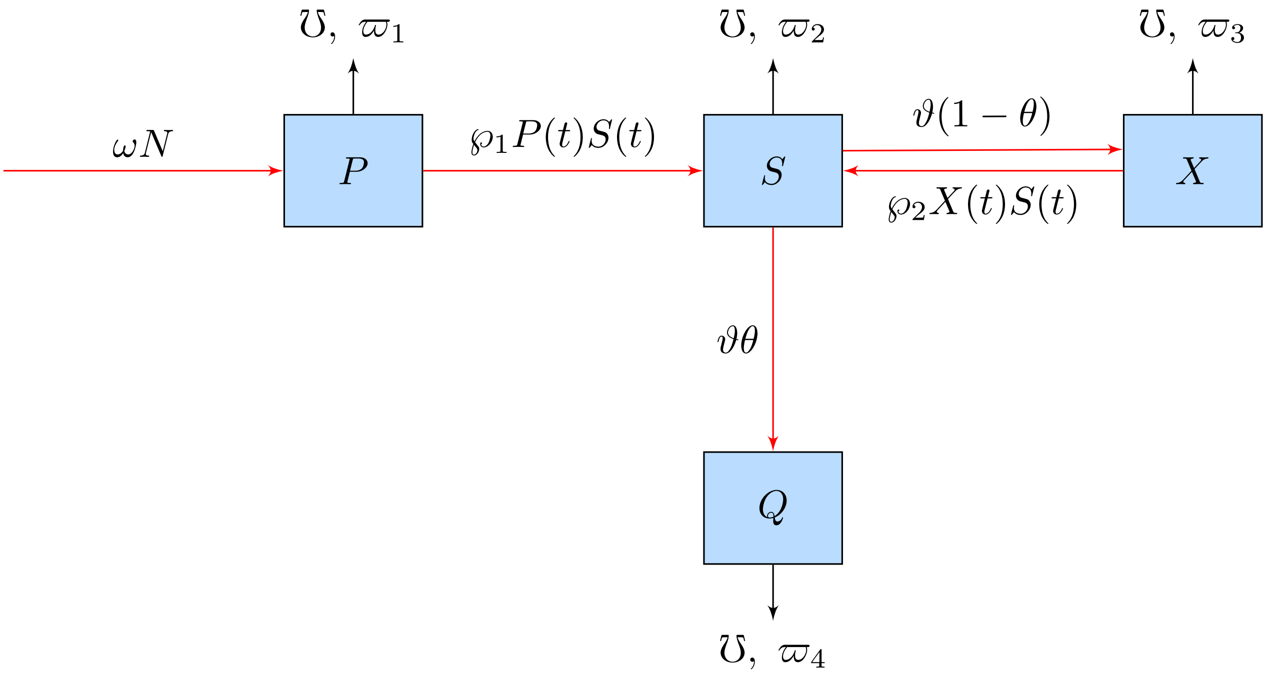

Figure 1. Diagram of the smoking model.

Table 1. Model parameter description (3).

Parameter	Description
ω	The overall recruits number into the considered homogeneously mixed population.
$\wp_1$	The rate of spread from potential to persistent smokers.
$\wp_2$	The relapse rate of temporally recovered smokers who contact persistent smokers.
$\vartheta(1-\theta),\ 0<\theta<1$	The rate of smokers who temporarily stop.
$\vartheta\theta$	The rate of people who have successfully stopped smoking.
$\mho$	The natural rate of mortality.
$\varpi_i,\ i=1,\ldots,4$	The mortality rate from smoking.

In view of the transfer diagram shown in Figure 1, we can derive the following system of ordinary differential equations:

$$\begin{cases} \frac{dP(t)}{dt} = \omega - \wp_1 P(t)S(t) - (\varpi_1+\mho)P(t), \\ \frac{dS(t)}{dt} = \wp_1 P(t)S(t) + \wp_2 X(t)S(t) - (\vartheta+\varpi_2+\mho)S(t), \\ \frac{dX(t)}{dt} = \vartheta(1-\theta)S(t) - \wp_2 X(t)S(t) - (\varpi_3+\mho)X(t), \\ \frac{dQ(t)}{dt} = \vartheta\theta S(t) - (\varpi_4+\mho)Q(t). \end{cases} \tag{3}$$

where $N(t) = P(t)+S(t)+X(t)+Q(t)$.

Remark 1. *Adding up the equations given in System (3), we have*

$$\begin{aligned} \frac{dN(t)}{dt} &= \omega - (\varpi_1+\mho)P(t) - (\varpi_2+\mho)S(t) - (\varpi_3+\mho)X(t) - (\varpi_4+\mho)Q(t) \\ &= \omega - \mho N(t) - (\varpi_1 P(t) + \varpi_2 S(t) + \varpi_3 X(t) + \varpi_4 Q(t)) \\ &\le \omega - \mho N(t). \end{aligned}$$

If, in addition, we know the size of the total population $N(0)$, it follows that

$$0 \le N(t) \le \frac{\omega}{\mho} + N(0)e^{-\mho t}.$$

Thus, $0 \leq N(t) \leq \frac{\omega}{\mho}$ *as* $t \to +\infty$*.*

Lemma 1. *Thesolutions of System (3) remain bounded and enter the region*

$$\Gamma = \{(P, S, X, Q) \in \mathbb{R}^4_+,\ P + S + X + Q \leq \frac{\omega}{\mho}\}.$$

We wish to investigate System (3) for fractional orders. Fractional order differential equations have been widely utilized in the literature to model real-life phenomena, as they provide a more accurate representation compared with classical order differential equations. It is important to note that fractional order differential equations generalize the classical order counterparts. Fractional calculus is recognized for its various advantages in diverse applications, allowing for the modeling of complex phenomena beyond the limitations of classical derivatives. These advantages include capturing the memory effect by incorporating past information, enabling long-term (nonlocal) dynamics without focusing on local aspects of derivation, and facilitating the study of stability through control of the derivation order. Moreover, fractional calculus is commonly employed in epidemiological modeling, and as smoking is regarded as an epidemic, we have incorporated fractional calculus into our model.

In our study, we specifically focus on the Caputo fractional derivative ${}^C\mathcal{D}^{\alpha}_{0,t}$ with an order of α, where $0 < \alpha < 1$. Subsequently, we analyze the following system:

$$\begin{cases} {}^C\mathcal{D}^{\alpha}_{0,t}(P(t)) = \omega - \wp_1 P(t)S(t) - (\varpi_1 + \mho)P(t), \\ {}^C\mathcal{D}^{\alpha}_{0,t}(S(t)) = \wp_1 P(t)S(t) + \wp_2 X(t)S(t) - (\vartheta + \varpi_2 + \mho)S(t), \\ {}^C\mathcal{D}^{\alpha}_{0,t}(X(t)) = \vartheta(1-\theta)S(t) - \wp_2 X(t)S(t) - (\varpi_3 + \mho)X(t), \\ {}^C\mathcal{D}^{\alpha}_{0,t}(Q(t)) = \vartheta\theta S(t) - (\varpi_4 + \mho)Q(t), \end{cases} \tag{4}$$

with initial conditions

$$P(0) = P_0,\ S(0) = S_0,\ X(0) = X_0,\ Q(0) = Q_0. \tag{5}$$

Remark 2. *Adding up equations given in System (4), taking into account the linearity of the Caputo fractional derivative operator, we have*

$${}^C\mathcal{D}^{\alpha}_{0,t}(N(t)) = \omega - \mho N(t) - (\varpi_1 P(t) + \varpi_2 S(t) + \varpi_3 X(t) + \varpi_4 Q(t)).$$

4. Mathematical Analysis

In this section, System (4) is found to have a unique solution that is positive whenever the initial condition is positive. Moreover, System (4) is found to have two equilibria, the smoking-free equilibrium and the smoking-present equilibrium. Finally, the equilibria's local and global stability results are also obtained.

4.1. Existence and Uniqueness

For the existence and uniqueness of the solution to System (4), we proceed in several steps. Applying the fractional integral (1), taking into count the property (d), we obtain another version of System (4) in the following manner:

$$\begin{cases} P(t) - P_0 = I^{\alpha}_{0,t}(\omega - \wp_1 P(t)S(t) - (\varpi_1 + \mho)P(t)), \\ S(t) - S_0 = I^{\alpha}_{0,t}(\wp_1 P(t)S(t) + \wp_2 X(t)S(t) - (\vartheta + \varpi_2 + \mho)S(t)), \\ X(t) - X_0 = I^{\alpha}_{0,t}(\vartheta(1-\theta)S(t) - \wp_2 X(t)S(t) - (\varpi_3 + \mho)X(t)), \\ Q(t) - Q_0 = I^{\alpha}_{0,t}(\vartheta\theta S(t) - (\varpi_4 + \mho)Q(t)), \end{cases} \tag{6}$$

or

$$\begin{cases} P(t) - P_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1}((\omega - \wp_1 P(\varsigma)S(\varsigma) - (\varpi_1 + \mho)P(\varsigma))d\varsigma, \\ S(t) - S_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1}(\wp_1 P(\varsigma)S(\varsigma) + \wp_2 X(\varsigma)S(\varsigma) - (\vartheta + \varpi_2 + \mho)S(\varsigma))d\varsigma, \\ X(t) - X_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1}(\vartheta(1-\theta)S(\varsigma) - \wp_2 X(\varsigma)S(\varsigma) - (\varpi_3 + \mho)X(\varsigma))d\varsigma, \\ Q(t) - Q_0 = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1}(\vartheta\theta S(\varsigma) - (\varpi_4 + \mho)Q(\varsigma))d\varsigma. \end{cases} \tag{7}$$

Definition 1. *The space of continuous valued functions f on an interval I with the norm $\| f \| = \sup_{t \in I} |f(t)|$ is denoted by $C^0(I)$.*

Remark 3. *We define the following kernels:*

$$K_1(t, P) = \omega - \wp_1 P(t)S(t) - (\varpi_1 + \mho)P(t), \tag{8}$$
$$K_2(t, S) = \wp_1 P(t)S(t) + \wp_2 X(t)S(t) - (\vartheta + \varpi_2 + \mho)S(t), \tag{9}$$
$$K_3(t, X) = \vartheta(1-\theta)S(t) - \wp_2 X(t)S(t) - (\varpi_3 + \mho)X(t), \tag{10}$$
$$K_4(t, Q) = \vartheta\theta S(t) - (\varpi_4 + \mho)Q(t). \tag{11}$$

Proposition 1. *If $P(t)$, $S(t)$, $X(t)$, and $Q(t)$ are bounded functions in $C^0(I)$, and let*

$$\lambda = \max\{\wp_1 M_2 + \varpi_1 + \mho,\ \wp_1 M_1 + \wp_2 M_3 + \vartheta + \varpi_2 + \mho,\ (\vartheta(1-\theta) + \wp_2)M_2 + \varpi_3 + \mho,\ \vartheta\theta M_2 + \varpi_4 + \mho\},$$

where $M_1 = \sup_{t \in I} |P(t)|$, $M_2 = \sup_{t \in I} |S(t)|$, $M_3 = \sup_{t \in I} |X(t)|$, $M_4 = \sup_{t \in I} |Q(t)|$, then the kernels $K_1(t, P)$, $K_2(t, S)$, $K_3(t, X)$, $K_4(t, Q)$ satisfy the Lipschitz condition and contraction if $0 \leq \lambda < 1$.

Proof. We proof the proposition for $K_1(t, P)$; the statement for $K_2(t, S)$, $K_3(t, X)$, $K_4(t, Q)$ can be proved using similar arguments. Let P_1 and P_2 be two functions, so we have

$$\begin{aligned} \| K_1(t, P_1) - K_1(t, P_2) \| &= \| \omega - \wp_1 P_1(t)S(t) - (\varpi_1 + \mho)P_1(t) - \omega + \wp_1 P_2(t)S(t) + (\varpi_1 + \mho)P_2(t) \| \\ &= \| -(\wp_1 S(t))(P_1(t) - P_2(t)) - ((\varpi_1 + \mho)(P_1(t) - P_2(t)) \| \\ &\leq (\wp_1 M_2 + \varpi_1 + \mho) \| P_1(t) - P_2(t) \| . \end{aligned}$$

Hence, the Lipschitz condition is satisfied for K_1, and since $\lambda < 1$, K_1 is also a contraction. □

In the following, we adopt the approach in [20]. First, we define

$$\begin{aligned} \zeta_1 &= \wp_1 M_2 + \varpi_1 + \mho, \\ \zeta_2 &= \wp_1 M_1 + \wp_2 M_3 + \vartheta + \varpi_2 + \mho, \\ \zeta_3 &= (\vartheta(1-\theta) + \wp_2)M_2 + \varpi_3 + \mho, \\ \zeta_4 &= \vartheta\theta M_2 + \varpi_4 + \mho. \end{aligned}$$

Using the definition of kernels in (8), then the equations in (7) become

$$\begin{cases} P(t) = P(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_1(\varsigma, P)d\varsigma, \\ S(t) = S(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_2(\varsigma, S)d\varsigma, \\ X(t) = X(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_3(\varsigma, X)d\varsigma, \\ Q(t) = Q(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_4(\varsigma, Q)d\varsigma. \end{cases} \tag{12}$$

Additionally, we give the following recursive formula:

$$\begin{cases} P_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_1(\varsigma, P_{n-1}) d\varsigma, \\ S_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_2(\varsigma, S_{n-1}) d\varsigma, \\ X_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_3(\varsigma, X_{n-1}) d\varsigma, \\ Q_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} K_4(\varsigma, Q_{n-1}) d\varsigma. \end{cases} \tag{13}$$

where the initial conditions are defined by

$$\begin{aligned} P_0(t) =& P(0), \\ S_0(t) =& S(0), \\ X_0(t) =& X(0), \\ Q_0(t) =& Q(0). \end{aligned}$$

Definition 2. *We define the difference between successive terms by*

$$\Upsilon_n(t) = P_n(t) - P_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, P_{n-1}) - K_1(\varsigma, P_{n-2})) d\varsigma, \tag{14}$$

$$\Phi_n(t) = S_n(t) - S_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_2(\varsigma, S_{n-1}) - K_2(\varsigma, S_{n-2})) d\varsigma, \tag{15}$$

$$\Psi_n(t) = X_n(t) - X_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_3(\varsigma, X_{n-1}) - K_3(\varsigma, X_{n-2})) d\varsigma, \tag{16}$$

$$\Omega_n(t) = Q_n(t) - Q_{n-1}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_4(\varsigma, Q_{n-1}) - K_4(\varsigma, Q_{n-2})) d\varsigma. \tag{17}$$

Proposition 2. *With the notations in Definition 2, we have*

$$\begin{aligned} P_n(t) &= \sum_{k=0}^{n} \Upsilon_k(t), \\ S_n(t) &= \sum_{k=0}^{n} \Phi_k(t), \\ X_n(t) &= \sum_{k=0}^{n} \Psi_k(t), \\ Q_n(t) &= \sum_{k=0}^{n} \Omega_k(t), \end{aligned}$$

and

$$\begin{aligned} \| \Upsilon_n(t) \| &\leq \frac{\zeta_1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| \Upsilon_{n-1}(t) \| d\varsigma, \\ \| \Phi_n(t) \| &\leq \frac{\zeta_2}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| \Phi_{n-1}(t) \| d\varsigma, \\ \| \Psi_n(t) \| &\leq \frac{\zeta_3}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| \Psi_{n-1}(t) \| d\varsigma, \\ \| \Omega_n(t) \| &\leq \frac{\zeta_4}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| \Omega_{n-1}(t) \| d\varsigma. \end{aligned}$$

Proof. The first statement is the telescoping sums. Performing the norm to both sides of Equation (14), we obtain

$$\begin{aligned}\| \Upsilon_n(t) \| = \| P_n(t) - P_{n-1}(t) \| &= \| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, P_{n-1}) - K_1(\varsigma, P_{n-2})) d\varsigma \| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| (K_1(\varsigma, P_{n-1}) - K_1(\varsigma, P_{n-2})) \| d\varsigma.\end{aligned}$$

Since the kernel K_1 satisfies the Lipschitz condition (Proposition 1), we find

$$\| \Upsilon_n(t) \| \leq \frac{\zeta_1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| \Upsilon_{n-1}(t) \| d\varsigma.$$

Similarly, one can obtained the results for $\Phi_n(t),\ \Psi_n(t),\ \Omega_n(t)$. □

Proposition 3. *If $P(t),\ S(t),\ X(t)$, and $Q(t)$ are bounded functions in $C^0(I)$, then we have*

$$\begin{aligned}\| \Upsilon_n(t) \| &\leq \| P_0(t) \| \left(\frac{\zeta_1\, t^\alpha}{\alpha\, \Gamma(\alpha)} \right)^n, \\ \| \Phi_n(t) \| &\leq \| S_0(t) \| \left(\frac{\zeta_2\, t^\alpha}{\alpha\, \Gamma(\alpha)} \right)^n, \\ \| \Psi_n(t) \| &\leq \| X_0(t) \| \left(\frac{\zeta_3\, t^\alpha}{\alpha\, \Gamma(\alpha)} \right)^n, \\ \| \Omega_n(t) \| &\leq \| Q_0(t) \| \left(\frac{\zeta_4\, t^\alpha}{\alpha\, \Gamma(\alpha)} \right)^n.\end{aligned}$$

Proof. For the proof, we use Proposition 2, the Lipschitz condition, and employ the recursive method. □

Theorem 1. *Let $t_0 = \min(t_1, t_2, t_3, t_4)$, where*

$$\begin{aligned}t_1 &< \sqrt[\alpha]{\frac{\alpha\, \Gamma(\alpha)}{\zeta_1}}, \\ t_2 &< \sqrt[\alpha]{\frac{\alpha\, \Gamma(\alpha)}{\zeta_2}}, \\ t_3 &< \sqrt[\alpha]{\frac{\alpha\, \Gamma(\alpha)}{\zeta_3}}, \\ t_4 &< \sqrt[\alpha]{\frac{\alpha\, \Gamma(\alpha)}{\zeta_4}}.\end{aligned}$$

Then $P_n(t),\ S_n(t),\ X_n(t)$, and $Q_n(t)$ defined by Proposition 2 exist and are smooth.

Proof. For $t_0 = \min(t_1, t_2, t_3, t_4)$, $\frac{\zeta_1\, t_0^\alpha}{\alpha\, \Gamma(\alpha)} < 1$; then the series $\sum_{k=0}^{n} \Upsilon_k(t)$ converges and $P_n(t)$ exists. □

We can write

$$\begin{aligned}
P(t) - P(0) =& P_n(t) - \Theta_n^P(t), \\
S(t) - S(0) =& S_n(t) - \Theta_n^S(t), \\
X(t) - X(0) =& X_n(t) - \Theta_n^X(t), \\
Q(t) - Q(0) =& Q_n(t) - \Theta_n^Q(t),
\end{aligned}$$

where

$$\begin{aligned}
\Theta_n^P(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, P_{n-1}) - K_1(\varsigma, P)) d\varsigma, \\
\Theta_n^S(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, S_{n-1}) - K_1(\varsigma, S)) d\varsigma, \\
\Theta_n^X(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, X_{n-1}) - K_1(\varsigma, X)) d\varsigma, \\
\Theta_n^Q(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, Q_{n-1}) - K_1(\varsigma, Q)) d\varsigma.
\end{aligned}$$

For $\Theta_n^P(t)$, we have

$$\begin{aligned}
\| \Theta_n^P(t) \| =& \| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, P_{n-1}) - K_1(\varsigma, P)) d\varsigma \| \\
\leq& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| K_1(\varsigma, P_{n-1}) - K_1(\varsigma, P) \| d\varsigma \\
\leq& \frac{t^\alpha}{\alpha\, \Gamma(\alpha)} \zeta_1 \| P - P_{n-1} \| .
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\| P - P_{n-1} \| =& \| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, P_{n-2}) - K_1(\varsigma, P)) d\varsigma \| \\
\leq& \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \| K_1(\varsigma, P_{n-2}) - K_1(\varsigma, P) \| d\varsigma \\
\leq& \frac{t^\alpha}{\alpha\, \Gamma(\alpha)} \zeta_1 \| P - P_{n-2} \| .
\end{aligned}$$

Therefore, repeating this process recursively, we obtain

$$\begin{aligned}
\| \Theta_n^P(t) \| \leq& \left(\frac{t^\alpha \zeta_1}{\alpha\, \Gamma(\alpha)} \right)^{n+1} \| P_0 \| \\
\leq& \left(\frac{t^\alpha \zeta_1}{\alpha\, \Gamma(\alpha)} \right)^{n+1} M_1.
\end{aligned}$$

Thus, we also have

$$\begin{aligned}
\| \Theta_n^S(t) \| \leq& \left(\frac{t^\alpha \zeta_2}{\alpha\, \Gamma(\alpha)} \right)^{n+1} M_2, \\
\| \Theta_n^X(t) \| \leq& \left(\frac{t^\alpha \zeta_3}{\alpha\, \Gamma(\alpha)} \right)^{n+1} M_3, \\
\| \Theta_n^Q(t) \| \leq& \left(\frac{t^\alpha \zeta_4}{\alpha\, \Gamma(\alpha)} \right)^{n+1} M_4.
\end{aligned}$$

Lemma 2. *For $t_0 = \min(t_1, t_2, t_3, t_4)$, we have*

$$\lim_{n\to+\infty} \parallel \Theta_n^P(t) \parallel = 0,$$
$$\lim_{n\to+\infty} \parallel \Theta_n^S(t) \parallel = 0,$$
$$\lim_{n\to+\infty} \parallel \Theta_n^X(t) \parallel = 0,$$
$$\lim_{n\to+\infty} \parallel \Theta_n^Q(t) \parallel = 0.$$

Proof. For $t_0 = \min(t_1, t_2, t_3, t_4)$, we have $\frac{t^\alpha \zeta_i}{\alpha\,\Gamma(\alpha)} < 1 \quad \text{for all } i = 1, 2, 3, 4.$ □

Combining the above results, we have established the following theorem, which ensures that System (4) has a solution.

Theorem 2 (Existence)**.** *For $t_0 = \min(t_1, t_2, t_3, t_4)$, the giving up smoking model (4) has a solution defined by*

$$P(t) = P(0) + \lim_{n\to+\infty} P_n(t). \tag{18}$$

The following theorem ensures that Model (4) has a unique solution, which is an important condition for Model (4) to be well posed.

Theorem 3 (Uniqueness)**.** *For $t_0 = \min(t_1, t_2, t_3, t_4)$, the giving up smoking model (4) has a unique solution.*

Proof. For the uniqueness of the solution of System (4), let $P_1(t),\ S_1(t),\ X_1(t),\ Q_1(t)$ be another solution to System (4). We have

$$P(t) - P_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} (K_1(\varsigma, P) - K_1(\varsigma, P_1)) d\varsigma,$$

and

$$\begin{aligned} \parallel P(t) - P_1(t) \parallel &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\varsigma)^{\alpha-1} \parallel (K_1(\varsigma, P) - K_1(\varsigma, P_1)) \parallel d\varsigma \\ &\leq \frac{t^\alpha}{\alpha\,\Gamma(\alpha)} \zeta_1 \parallel P(t) - P_1(t) \parallel . \end{aligned}$$

Then, we have

$$\left(1 - \frac{t^\alpha}{\alpha\,\Gamma(\alpha)} \zeta_1\right) \parallel P(t) - P_1(t) \parallel \leq 0.$$

Therefore, also

$$\left(1 - \frac{t^\alpha}{\alpha\,\Gamma(\alpha)} \zeta_2\right) \parallel S(t) - S_1(t) \parallel \leq 0,$$
$$\left(1 - \frac{t^\alpha}{\alpha\,\Gamma(\alpha)} \zeta_3\right) \parallel X(t) - X_1(t) \parallel \leq 0,$$
$$\left(1 - \frac{t^\alpha}{\alpha\,\Gamma(\alpha)} \zeta_4\right) \parallel Q(t) - Q_1(t) \parallel \leq 0.$$

For $t_0 = \min(t_1, t_2, t_3, t_4)$, we have

$$\left(1 - \frac{t_0^\alpha}{\alpha\,\Gamma(\alpha)} \zeta_2\right) > 0,$$

and

$$\left(1 - \frac{t^{\alpha}}{\alpha\,\Gamma(\alpha)}\zeta_1\right) \parallel P(t) - P_1(t) \parallel \leq 0,$$

so

$$\parallel P(t) - P_1(t) \parallel = 0\,;\text{ then } P(t) = P_1(t).$$

The statement for $S(t),\ X(t),\ Q(t)$ can be shown in a similar way. □

4.2. Non-Negative Solutions

Let $\mathbb{R}^4_+ = \{Y \in \mathbb{R}^4,\ Y \geq 0\}$ and $Y(t) = (P(t), S(t), X(t), Q(t))^T$; we investigate the non-negative solution of System (4). To proceed, we need the following lemmas:

Lemma 3. *[21] Let $f(x) \in \mathcal{C}([a,b])$ and ${}^{C}\mathcal{D}^{\alpha}_{a,t}f(x) \in \mathcal{C}((a,b])$ for $0 < \alpha \leq 1$; then we have*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha)}\,{}^{C}\mathcal{D}^{\alpha}_{a,t}f(x)(x-\xi)^{\alpha},$$

with $a \leq \xi \leq x$, $\forall x \in (a,b]$.

Remark 4. *Notice, for $a = 0$ in Lemma 3, we have*

$$f(x) = f(0) + \frac{1}{\Gamma(\alpha)}\,{}^{C}\mathcal{D}^{\alpha}_{0,t}f(x)(x)^{\alpha}.$$

Then,

1. *If ${}^{C}\mathcal{D}^{\alpha}_{0,t}f(x) \geq 0$, then the function f is nondecreasing for all $x \in (0,b]$.*
2. *If ${}^{C}\mathcal{D}^{\alpha}_{0,t}f(x) \leq 0$, then the function f is nonincreasing for all $x \in (0,b]$.*

Lemma 4. *[22] Let $0 < \alpha \leq 1$ and consider the two fractional differential equations*

$$ {}^{C}\mathcal{D}^{\alpha}_{0,t}(S(t)) = F(t,S) + \frac{1}{k} \quad \text{and} \quad {}^{C}\mathcal{D}^{\alpha}_{0,t}(S(t)) = F(t,S), \tag{19}$$

with the same initial condition $S(0) = S_0$, where $k \in \mathbb{N}^*$ and $F : [0,b] \times \mathbb{R} \to \mathbb{R}$ are a continuous function and Lipschitz with respect to the second component; that is, there exists a constant L such that

$$|F(t,S_1) - F(t,S_2)| \leq L\|S_1 - S_2\|.$$

If S^*_k and S^* are the solution of (19), respectively, then

$$S^*_k \xrightarrow[k \to +\infty]{} S^*,$$

for all $t \in [0,b]$.

Theorem 4. *The solution to the initial value problem given by (4) and (5), if it exists, belongs to*

$$\mathbb{R}^4_+ = \{Y = (P(t), S(t), X(t), Q(t))^T \in \mathbb{R}^4,\ Y \geq 0\}.$$

Proof. First, we prove that $S(t) > 0$. In view of Lemma 19, we consider the following alternative equation of the fractional differential equation:

$${}^{C}\mathcal{D}^{\alpha}_{0,t}(S^*_k(t)) = \underbrace{\wp_1 P(t)S(t) + \wp_2 X(t)S(t) - (\vartheta + \varpi_2 + \mho)S(t) + \frac{1}{k}}_{=F(t,S)}.$$

Obviously, $F(t, S)$ is Lipschitz with respect to the second variable with a Lipschitz constant $L = \frac{\omega}{\mho}(\wp_1 + \wp_2) + (\vartheta + \varpi_2 + \mho)$. We use the contradiction argument. Let us assume that there exists t_0 such that $(S_k^*(t_0)) = 0$. Since

$$
\begin{aligned}
{}^C\mathcal{D}_{0,t}^{\alpha}(S_k^*(t_0)) &= \wp_1 P(t_0)S(t_0) + \wp_2 X(t_0)S(t_0) - (\vartheta + \varpi_2 + \mho)S(t_0) + \frac{1}{k} \\
&= \frac{1}{k} > 0.
\end{aligned}
$$

By Lemma 3, we obtain that $(S_k^*(t_0)) > 0$ since t_0 is arbitrary, so $S_k^* > 0$, obtaining a contradiction. By Lemma 4, as $k \to +\infty$, we obtain that $S^* > 0$.
On the other hand, since

$$
\begin{aligned}
&{}^C\mathcal{D}_{0,t}^{\alpha}(P(t))|_{P=0} = \omega > 0, \\
&{}^C\mathcal{D}_{0,t}^{\alpha}(X(t))|_{X=0} = \vartheta(1-\theta)S(t) > 0, \\
&{}^C\mathcal{D}_{0,t}^{\alpha}(Q(t))|_{X=0} = \vartheta\theta S(t) > 0,
\end{aligned}
$$

we conclude by Lemma 3 that $P(t) > 0,\ X(t) > 0,\ Q(t) > 0$; that is, the domain $\mathbb{R}_+^4$ is positively invariant with respect to System (4). □

4.3. Equilibrium and Smokers Generation Number

In this subsection, we analyze the existence of the smoking-free equilibrium (SFE) point of Model (4). In view of [23], the smoker compartment is S, which gives $m = 1$, and using the next-generation matrix method as formulated in [23], with $Y = (S, X, Q, P)^T$, Model (4) can be written as

$$
{}^C\mathcal{D}_{0,t}^{\alpha}(Y(t)) = \mathcal{F}(Y) - \mathcal{V}(Y),
$$

where

$$
\mathcal{F}(Y) = \begin{pmatrix} \wp_1 PS \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{20}
$$

and

$$
\mathcal{V}(Y) = \begin{pmatrix} (\vartheta + \varpi_2 + \mho)S(t) - \wp_2 XS \\ \wp_2 X(t)S(t) + (\varpi_3 + \mho)X(t) - \vartheta(1-\theta)S(t) \\ (\varpi_4 + \mho)Q(t) - \vartheta\theta S(t) \\ \wp_1 P(t)S(t) + (\varpi_1 + \mho)P(t) - \omega \end{pmatrix}. \tag{21}
$$

Proposition 4. *For the giving up smoking model (4), there exists the smoking-free equilibrium* $\mathscr{Y}_0 = (\frac{\omega}{\varpi_1 + \mho}, 0, 0, 0)$.

Proof. Thanks to (20) and (21), an equilibrium solution with $S = 0$ has the form $\mathscr{Y}_0 = (\frac{\omega}{\varpi_1 + \mho}, 0, 0, 0)$. □

Proposition 5. *The basic reproduction number, denoted by R_0, is given by*

$$
R_0 = \rho(FV^{-1}) = \frac{\wp_1 b}{(\varpi_1 + \mho)(\vartheta + \varpi_2 + \mho)},
$$

where the matrix F and V are such that

$$D\mathcal{F}(\mathscr{Y}_0) = \left(\begin{array}{c:c} F_{3\times 3} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hdashline \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{array}\right), \quad D\mathcal{V}(\mathscr{Y}_0) = \left(\begin{array}{c:c} V_{3\times 3} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hdashline J_{1\times 3} & c \end{array}\right),$$

where $D\mathcal{F}(\mathscr{Y}_0)$ and $D\mathcal{V}(\mathscr{Y}_0)$ are the Jacobian matrix of $\mathcal{F}(Y)$ and $\mathcal{V}(Y)$ at the smoking-free equilibrium $\mathscr{Y}_0$, respectively.

Proof. The definitions of $D\mathcal{F}(\mathscr{Y}_0)$ *and* $D\mathcal{V}(\mathscr{Y}_0)$ are given in [23]. Furthermore, a simple calculation gives

$$D\mathcal{F}(\mathscr{Y}_0) = \left(\begin{array}{ccc:c} \frac{\wp_1 b}{\varpi_1+\mho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right), \quad D\mathcal{V}(\mathscr{Y}_0) = \left(\begin{array}{ccc:c} \vartheta+\varpi_2+\mho & 0 & 0 & 0 \\ -\vartheta(1-\theta) & \varpi_3+\mho & 0 & 0 \\ -\vartheta\theta & 0 & \varpi_4+\mho & 0 \\ \hdashline \frac{\wp_1 b}{\varpi_1+\mho} & 0 & 0 & \varpi_1+\mho \end{array}\right).$$

Thus,

$$F = \begin{pmatrix} \frac{\wp_1 b}{\varpi_1+\mho} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} \vartheta+\varpi_2+\mho & 0 & 0 \\ -\vartheta(1-\theta) & \varpi_3+\mho & 0 \\ -\vartheta\theta & 0 & \varpi_4+\mho \end{pmatrix}.$$

Next, we have

$$V^{-1} = \begin{pmatrix} \frac{1}{\vartheta+\varpi_2+\mho} & 0 & 0 \\ \frac{\vartheta(1-\theta)}{(\vartheta+\varpi_2+\mho)(\varpi_3+\mho)} & \frac{1}{(\varpi_3+\mho)} & 0 \\ \frac{\vartheta\theta}{(\vartheta+\varpi_2+\mho)(\varpi_4+\mho)} & 0 & \frac{1}{d4+\mho} \end{pmatrix},$$

and

$$FV^{-1} = \begin{pmatrix} \frac{\wp_1 b}{(\varpi_1+\mho)(\vartheta+\varpi_2+\mho)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$\rho(FV^{-1}) = \frac{\wp_1 b}{(\varpi_1+\mho)(\vartheta+\varpi_2+\mho)}.$$

□

Proposition 6. *For the giving up smoking model (4), there exists the smoking-present equilibrium $\mathscr{Y}^* = (P^*, S^*, X^*, Q^*)$, where*

$$P^* = \frac{\omega}{\wp_1 S^* + (\varpi_1+\mho)},$$
$$X^* = \frac{\vartheta(1-\theta)S^*}{\wp_2 S^* + (\varpi_3+\mho)},$$
$$Q^* = \frac{\vartheta\theta S^*}{\varpi_4+\mho},$$

and S^* satisfies the equation

$$S^{*2} + AS^* + B = 0,$$

where

$$A=\frac{\wp_1(\mho+\varpi_3)(\vartheta+\mho+\varpi_2)+\wp_2(\mho+\varpi_1)(\vartheta\theta+\mho+\varpi_2)-\wp_1\wp_2 b}{\wp_1\wp_2(\vartheta\theta+\mho+\varpi_2)},$$
$$B=\frac{(\vartheta+\mho+\varpi_2)(\mho+\varpi_1)(\mho+\varpi_3)-\wp_1\omega(\mho+\varpi_3)}{\wp_1\wp_2(\vartheta\theta+\mho+\varpi_2)}.$$

Proof. To evaluate the existence of the positive smoking-present equilibrium $\mathscr{Y}^*$ of System (4), let $S^*>0$ and

$$\begin{aligned}{}^{C}\mathcal{D}^{\alpha}_{0,t}(P^*)&=0,\\ {}^{C}\mathcal{D}^{\alpha}_{0,t}(S^*)&=0,\\ {}^{C}\mathcal{D}^{\alpha}_{0,t}(X^*)&=0,\\ {}^{C}\mathcal{D}^{\alpha}_{0,t}(Q^*)&=0.\end{aligned}$$

This gives

$$\omega-\wp_1P^*S^*-(\varpi_1+\mho)P^*=0, \tag{22}$$
$$\wp_1P^*S^*+\wp_2X^*S^*-(\vartheta+\varpi_2+\mho)S^*=0, \tag{23}$$
$$\vartheta(1-\theta)S^*-\wp_2X^*S^*-(\varpi_3+\mho)X^*=0, \tag{24}$$
$$\vartheta\theta S^*-(\varpi_4+\mho)Q^*=0. \tag{25}$$

From Equation (25), we obtain

$$Q^*=\frac{\vartheta\theta S^*}{\varpi_4+\mho}. \tag{26}$$

From Equation (24), we have

$$X^*=\frac{\vartheta(1-\theta)S^*}{\wp_2S^*+(\varpi_3+\mho)}. \tag{27}$$

Then it follows from Equation (22)

$$P^*=\frac{\omega}{\wp_1S^*+(\varpi_1+\mho)}. \tag{28}$$

Finally, (23) gives

$$S^*(\wp_1P^*+\wp_2X^*-(\vartheta+\varpi_2+\mho))=0.$$

Since $S^*\neq 0$, we obtain

$$\wp_1P^*+\wp_2X^*-(\vartheta+\varpi_2+\mho)=0. \tag{29}$$

Substituting (28) and (27) in Equation (29), we obtain

$$S^{*2}+AS^*+B=0, \tag{30}$$

with

$$A=\frac{\wp_1(\mho+\varpi_3)(\vartheta+\mho+\varpi_2)+\wp_2(\mho+\varpi_1)(\vartheta\theta+\mho+\varpi_2)-\wp_1\wp_2 b}{\wp_1\wp_2(\vartheta\theta+\mho+\varpi_2)},$$
$$B=\frac{(\vartheta+\mho+\varpi_2)(\mho+\varpi_1)(\mho+\varpi_3)-\wp_1\omega(\mho+\varpi_3)}{\wp_1\wp_2(\vartheta\theta+\mho+\varpi_2)}.$$

□

Theorem 5. *For the existence of a smoking-present equilibrium point $\mathscr{Y}^* = (P^*, S^*, X^*, Q^*)$, we have*

(i) *If $R_0 = 1$, there is no positive present equilibrium point $\mathscr{Y}^*$.*
(ii) *If $R_0 < 1$, and $\frac{\mho}{\wp_1} > \frac{\omega}{\mho}$, then there is no positive present equilibrium point $\mathscr{Y}^*$.*
(iii) *If $R_0 > 1$, there exists one positive present equilibrium point $\mathscr{Y}^*$ given by (26), (27), (28), and (30).*

Proof. Using the quadratic formula, notice that Equation (30) can be solved as follows:

$$S_1^* = -\frac{A}{2} + \frac{1}{2}\sqrt{A^2 - 4B},$$
$$S_2^* = -\frac{A}{2} - \frac{1}{2}\sqrt{A^2 - 4B},$$

with

$$A = \frac{\wp_1(\mho + \varpi_3)(\vartheta + \mho + \varpi_2) + \wp_2(\mho + \varpi_1)(\vartheta\theta + \mho + \varpi_2) - \wp_1\wp_2 b}{\wp_1\wp_2(\vartheta\theta + \mho + \varpi_2)},$$
$$B = \frac{(\vartheta + \mho + \varpi_2)(\mho + \varpi_1)(\mho + \varpi_3) - \wp_1\omega(\mho + \varpi_3)}{\wp_1\wp_2(\vartheta\theta + \mho + \varpi_2)}$$
$$= \frac{(\vartheta + \mho + \varpi_2)(\mho + \varpi_1)(\mho + \varpi_3)(1 - R_0)}{\wp_1\wp_2(\vartheta\theta + \mho + \varpi_2)}.$$

Then, we have

1. If $R_0 = 1$, then $B = 0$, which gives $S_1^* = 0$ and $S_2^* = -A < 0$; then there is no positive solution.
2. If $R_0 > 1$, then $B < 0$. Consequently, $\sqrt{A^2 - 4B} > A$, so we obtain one positive solution

$$S^* = -\frac{A}{2} + \frac{1}{2}\sqrt{A^2 - 4B}.$$

3. If $R_0 < 1$, then $B > 0$; if, in addition, $A > 0$, then there is no positive solution. However, we can write A as

$$A = \frac{\mho(\wp_1(\vartheta + \mho + \varpi_2 + \varpi_3) + \wp_2(\vartheta\theta + \varpi_2) + \wp_2(\mho - \wp_1\frac{\omega}{\mho})) + \wp_1\varpi_3(\vartheta + \varpi_2) + \wp_2\varpi_1(\vartheta\theta + \mho + \varpi_2)}{\wp_1\wp_2(\vartheta\theta + \mho + \varpi_2)}.$$

Then, to ensure that A remains positive, we take $\mho > \wp_1\frac{\omega}{\mho}$; that is, $\frac{\mho}{\wp_1} > \frac{\omega}{\mho}$.

□

4.4. Local Stability for the Free Smoker Equilibrium Point

We use the result proven in [24,25]. The local stability of the equilibrium point $\mathscr{Y}_0 = (\frac{\omega}{\varpi_1+\mho}, 0, 0, 0)$ is studied in the following theorem.

Theorem 6. *The smoking-free equilibrium point $\mathscr{Y}_0 = (\frac{\omega}{\varpi_1+\mho}, 0, 0, 0)$ is locally asymptotically stable if $R_0 < 1$, locally stable if $R_0 = 1$, and unstable if $R_0 > 1$.*

Proof. By using the Jacobian matrix of System (4) evaluated at $\mathscr{Y}_0 = (\frac{\omega}{\varpi_1+\mho}, 0, 0, 0)$, we obtain

$$J(\mathscr{Y}_0) = \begin{pmatrix} -(\varpi_1+\mho) & -\frac{\wp_1 b}{(\varpi_1+\mho)} & 0 & 0 \\ 0 & \frac{\wp_1\omega-(\vartheta+\varpi_2+\mho)}{(\varpi_1+\mho)} & 0 & 0 \\ 0 & \vartheta(1-\theta) & -(\varpi_3+\mho) & 0 \\ 0 & \vartheta\theta & 0 & -(\varpi_4+\mho) \end{pmatrix}$$
$$= \begin{pmatrix} -(\varpi_1+\mho) & (\vartheta+\varpi_2+\mho)R_0 & 0 & 0 \\ 0 & -(\vartheta+\varpi_2+\mho)(1-R_0) & 0 & 0 \\ 0 & \vartheta(1-\theta) & -(\varpi_3+\mho) & 0 \\ 0 & \vartheta\theta & 0 & -(\varpi_4+\mho) \end{pmatrix}.$$

Thus, the characteristic polynomial is given by

$$P(\lambda) = (\lambda + (\varpi_1 + u))(\lambda + (\varpi_3 + \mho))(\lambda + (\varpi_4 + \mho))(\lambda + (\varpi_2 + \mho + \vartheta)(1 - R_0)).$$

Therefore, the eigenvalues of $J(\mathscr{Y}_0)$ are

$$\lambda_1 = -(\varpi_1 + u),\ \lambda_2 = -(\varpi_3 + u),\ \lambda_3 = -(\varpi_4 + u),\ \lambda_4 = -(\varpi_2 + \mho + \vartheta)(1 - R_0)).$$

Clearly, we have $\lambda_4 < 0$ if $R_0 < 1$, so all eigenvalues of $J(\mathscr{Y}_0)$ are negative and verify the condition $|arg(\lambda_i)| > \alpha\frac{\pi}{2}$, $i = 1, \ldots, 4$; then $\mathscr{Y}_0$ is locally asymptotically stable (refer to [24,25]). On the other hand, when $R_0 = 1$, then $\lambda_4 = 0$, then $\mathscr{Y}_0$ is locally stable, and when $R_0 > 1$, then $\lambda_4 > 0$ and $|arg(\lambda_4)| = 0 < \alpha\frac{\pi}{2}$. Then, $\mathscr{Y}_0$ is unstable.

4.5. Local Stability for the Present Equilibrium Point

Here, we investigate the local stability of the present equilibrium point. By using the Jacobian matrix of System (4) evaluated at $\mathscr{Y}^* = (P^*, S^*, X^*, Q^*)$, we obtain

$$J(\mathscr{Y}^*) = \begin{pmatrix} -(\wp_1 S^* + (\varpi_1+\mho)) & -\wp_1 P^* & 0 & 0 \\ \wp_1 S^* & \wp_1 P^* + \wp_2 X^* - (\vartheta+\varpi_2+\mho) & \wp_2 S^* & 0 \\ 0 & \vartheta(1-\theta) - \wp_2 X^* & -(\wp_2 S^* + (\varpi_3+\mho)) & 0 \\ 0 & \vartheta\theta & 0 & -(\varpi_4+\mho) \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{P^*}{\omega} & -\wp_1 P^* & 0 & 0 \\ \wp_1 S^* & 0 & \wp_2 S^* & 0 \\ 0 & \vartheta(1-\theta) - \wp_2 X^* & -\frac{X^*}{\vartheta(1-\theta)S^*} & 0 \\ 0 & \vartheta\theta & 0 & -(\varpi_4+\mho) \end{pmatrix}.$$

Thus, the characteristic polynomial is given by

$$(\lambda + (\varpi_4 + \mho))(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3),$$

where

$$A_1 = \frac{\vartheta(1-\theta)P^*S^* + \omega X^*}{\omega\vartheta(1-\theta)S^*}, \tag{31}$$

$$A_2 = \frac{P^*X^* + v\vartheta(1-\theta)S^{*2}(\wp_1^2 P^* + \wp_2 X^*) - \omega\wp_2\vartheta^2(1-\theta)^2 S^{*2}}{\omega\vartheta(1-\theta)S^*}, \tag{32}$$

$$A_3 = \frac{P^*S^*X^*(\omega\wp_1^2 + \vartheta(1-\theta)\wp_2^2 S^*) - \wp_2\vartheta^2(1-\theta)^2 P^* S^{*2}}{\omega\vartheta(1-\theta)S^*}. \tag{33}$$

The eigenvalues of the characteristic polynomial are $\lambda_1 = -(\varpi_4 + \mho)$ and the solutions of the cubic equation

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0. \tag{34}$$

Using the Routh–Hurwitz criterion [26], all the eigenvalue of the characteristic Equation (34) has a negative real part if and only if

$$A_1 > 0,\ A_2 > 0,\ A_3 > 0,\ A_1A_2 > A_3;$$

then we have the following result. □

Theorem 7. *Let A_1, A_2, and A_3 be given by (31), (32), and (33), respectively; then the present equilibrium point of System (3) is locally asymptotically stable if*

$$A_1 > 0,\ A_2 > 0,\ A_3 > 0,\ A_1A_2 > A_3.$$

4.6. Global Stability

In this section, we establish results on global stability for the free smoker equilibrium and the present equilibrium. To do so, we introduce the following lemma:

Lemma 5. *[27,28] Suppose $\phi(t) \in \mathbb{R}_+$ is a continuous and differentiable function. Then, for any $t \geq 0$, we have the following inequalities:*

$${}^C\mathcal{D}^{\alpha}_{0,t}(\phi(t) - \phi^* - \phi^* \ln \frac{\phi(t)}{\phi^*}) \leq (1 - \frac{\phi^*}{\phi(t)})\ {}^C\mathcal{D}^{\alpha}_{0,t}(\phi(t)),$$

and

$$\frac{1}{2}\ {}^C\mathcal{D}^{\alpha}_{0,t}(\phi^2(t)) \leq \phi(t)\ {}^C\mathcal{D}^{\alpha}_{0,t}(\phi(t)).$$

The following theorem presents the global stability result for the free smoker equilibrium.

Theorem 8. *The smoker free equilibrium point $\mathscr{Y}_0 = (\frac{\omega}{\varpi_1+\mho)}, 0, 0, 0)$ of System (4) is globally asymptotically stable when $R_0 \leq 1$.*

Proof. Let $L(t)$ be the Lyapunov candidate function, such that

$$L(t) = (P(t) - P_0 - P_0 \ln \frac{P(t)}{P_0}) + S(t) + X(t) + Q(t), \tag{35}$$

$L(t)$ is defined, continuous and positive for all $t \leq 0$. For this function, we have

$${}^C\mathcal{D}^{\alpha}_{0,t}(L(t)) = {}^C\mathcal{D}^{\alpha}_{0,t}((P(t) - P_0 - P_0 \ln \frac{P(t)}{P_0}) + S(t) + X(t) + Q(t)).$$

Using the linearity propriety of the Caputo derivative and Lemma 5, we obtain

$$\begin{aligned} {}^C\mathcal{D}^{\alpha}_{0,t}(L(t)) &= {}^C\mathcal{D}^{\alpha}_{0,t}((P(t) - P_0 - P_0 \ln \frac{P(t)}{P_0}) + S(t) + X(t) + Q(t)) \\ &= {}^C\mathcal{D}^{\alpha}_{0,t}((P(t) - P_0 - P_0 \ln \frac{P(t)}{P_0})) + {}^C\mathcal{D}^{\alpha}_{0,t}(S(t)) + {}^C\mathcal{D}^{\alpha}_{0,t}(X(t)) + {}^C\mathcal{D}^{\alpha}_{0,t}(Q(t)) \\ &\leq (1 - \frac{P_0}{P(t)})\ {}^C\mathcal{D}^{\alpha}_{0,t}(P(t)) + {}^C\mathcal{D}^{\alpha}_{0,t}(S(t)) + {}^C\mathcal{D}^{\alpha}_{0,t}(X(t)) + {}^C\mathcal{D}^{\alpha}_{0,t}(Q(t)). \end{aligned}$$

From System (4), and with direct calculation, we obtain

$$
\begin{aligned}
{}^{C}\mathcal{D}^{\alpha}_{0,t}(L(t)) &\leq (1-\frac{P_0}{P(t)})(\omega-\wp_1 P(t)S(t)-(\varpi_1+\mho)P(t))+(\wp_1 P(t)S(t)+\wp_2 X(t)S(t)-(\vartheta+\varpi_2+\mho)S(t)) \\
&\quad +(\vartheta(1-\theta)S(t)-\wp_2 X(t)S(t)-(\varpi_3+\mho)X(t))+(\vartheta\theta S(t)-(\varpi_4+\mho)Q(t)) \\
&\leq \omega(2-\frac{P(t)}{P_0}-\frac{P_0}{P(t)})+(\wp_1 P_0-(\varpi_2+\mho))S(t)-(\varpi_3+\mho)X(t)-(\varpi_4+\mho)Q(t) \\
&\leq \omega(2-\frac{P(t)}{P_0}-\frac{P_0}{P(t)})+\frac{1}{\varpi_1+\mho}(\wp_1\omega-(\varpi_1+\mho)(\varpi_2+\mho))S(t)-(\varpi_3+\mho)X(t)-(\varpi_4+\mho)Q(t).
\end{aligned}
$$

We know by the arithmetic–geometric means that

$$(2-\frac{P(t)}{P_0}-\frac{P_0}{P(t)})\leq 0.$$

Then, if we have $(\wp_1\omega-(\varpi_1+\mho)(\varpi_2+\mho))\leq 0$, we obtain ${}^{C}\mathcal{D}^{\alpha}_{0,t}(L(t))\leq 0$. In addition, we have ${}^{C}\mathcal{D}^{\alpha}_{0,t}(L(t))=0$ if and only if $(P(t),S(t),X(t),Q(t))=\mathscr{Y}_0$; then the maximum invariant set for

$$\{(P(t),S(t),X(t),Q(t))\in\mathbb{R}^4_+,\ {}^{C}\mathcal{D}^{\alpha}_{0,t}(L(t))=0\}$$

is the set $\{\mathscr{Y}_0\}$, and according to LaSalle's invariance principle, the free equilibrium point $\mathscr{Y}_0$ is globally asymptotically stable. □

The following theorem presents the global stability result for the present equilibrium.

Theorem 9. *Let $R_0>1$ and suppose we have*

$$(\frac{Q^*}{Q}>\frac{X^*}{X}>1 \text{ and } \frac{S^*}{S}<1); \qquad or \qquad (\frac{Q}{Q^*}>\frac{X}{X^*}>1 \text{ and } \frac{S^*}{S}>1).$$

Then, the present smoker equilibrium $\mathscr{Y}^*$ given in Proposition 6 is globally asymptotically stable.

Proof. Let $L^*(t)$ be the Lyapunov function defined by

$$
\begin{aligned}
L^*(t) &= (P(t)-P^*-P^*\ln\frac{P(t)}{P^*})+(S(t)-S^*-S^*\ln\frac{S(t)}{S^*})+(X(t)-X^*-X^*\ln\frac{X(t)}{X^*}) \\
&\quad +(Q(t)-Q^*-Q^*\ln\frac{Q(t)}{Q^*}).
\end{aligned}
$$

The function $L^*(t)$ is defined as positive and continuous for all $t\geq 0$. Now using the linearity propriety of the Caputo derivative and Lemma 5, we obtain

$$
\begin{aligned}
{}^{C}\mathcal{D}^{\alpha}_{0,t}(L^*(t)) &\leq (1-\frac{P^*}{P(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(P(t))+(1-\frac{S^*}{S(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(S(t))+(1-\frac{X^*}{X(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(X(t)) \\
&\quad +(1-\frac{Q^*}{Q(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(Q(t)).
\end{aligned}
$$

From System (4), we have

$$(1-\frac{P^*}{P(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(P(t)) = (1-\frac{P^*}{P(t)})(\omega - \wp_1 P(t)S(t) - (\varpi_1 + \mho)P(t)), \tag{36}$$

$$(1-\frac{S^*}{S(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(S(t)) = (1-\frac{S^*}{S(t)})(\wp_1 P(t)S(t) + \wp_2 X(t)S(t) - (\vartheta + \varpi_2 + \mho)S(t)), \tag{37}$$

$$(1-\frac{X^*}{X(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(X(t)) = (1-\frac{X^*}{X(t)})(\vartheta(1-\theta)S(t) - \wp_2 X(t)S(t) - (\varpi_3 + \mho)X(t)), \tag{38}$$

$$(1-\frac{Q^*}{Q(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(Q(t)) = (1-\frac{Q^*}{Q(t)})(\vartheta\theta S(t) - (\varpi_4 + \mho)Q(t)). \tag{39}$$

On the other hand, Equations (22) to (25) give

$$(\varpi_1 + \mho) = \frac{\omega}{P^*} - \wp_1 S^*, \tag{40}$$

$$(\vartheta + \varpi_2 + \mho) = \wp_1 P^* + \wp_2 X^*, \tag{41}$$

$$(\varpi_3 + \mho) = \vartheta(1-\theta)\frac{S^*}{X^*} - \wp_2 S^*, \tag{42}$$

$$(\varpi_4 + \mho) = \vartheta\theta\frac{S^*}{Q^*}. \tag{43}$$

Substituting Equations (40) to (43) in Equations (36) to (39), we obtain

$$(1-\frac{P^*}{P(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(P(t)) = (1-\frac{P^*}{P(t)})(\omega - \wp_1 P(t)S(t) - (\frac{\omega}{P^*} - \wp_1 S^*)P(t)), \tag{44}$$

$$(1-\frac{S^*}{S(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(S(t)) = (1-\frac{S^*}{S(t)})(\wp_1 P(t)S(t) + \wp_2 X(t)S(t) - (\wp_1 P^* + \wp_2 X^*)S(t)), \tag{45}$$

$$(1-\frac{X^*}{X(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(X(t)) = (1-\frac{X^*}{X(t)})(\vartheta(1-\theta)S(t) - \wp_2 X(t)S(t) - (\vartheta(1-\theta)\frac{S^*}{X^*} - \wp_2 S^*)X(t)), \tag{46}$$

$$(1-\frac{Q^*}{Q(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(Q(t)) = (1-\frac{Q^*}{Q(t)})(\vartheta\theta S(t) - (\vartheta\theta\frac{S^*}{Q^*})Q(t)). \tag{47}$$

Direct calculation gives

$$\begin{aligned}
(1-\frac{P^*}{P(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(P(t)) &= \omega(2-\frac{P^*}{P}-\frac{P}{P^*}) - \wp_1(PS + P^*S^* - PS^* - P^*S),\\
(1-\frac{S^*}{S(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(S(t)) &= \wp_1(PS + P^*S^* - PS^* - P^*S) + \wp_2(XS + X^*S^* - X^*S - XS^*),\\
(1-\frac{X^*}{X(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(X(t)) &= \vartheta S(1-\frac{X^*}{X}) + \vartheta S^*(1-\frac{X}{X^*}) + \vartheta\theta S^*\frac{X}{X^*} + \vartheta\theta S\frac{X^*}{X} - \vartheta\theta(S+S^*)\\
&\quad - \wp_2(XS + X^*S^* - X^*S - XS^*),\\
(1-\frac{Q^*}{Q(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(Q(t)) &= \vartheta\theta(S+S^*) - \vartheta\theta S^*\frac{Q}{Q^*} - \vartheta\theta S\frac{Q^*}{Q}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
{}^{C}\mathcal{D}^{\alpha}_{0,t}(L^*(t)) \leq\ & (1-\frac{P^*}{P(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(P(t)) + (1-\frac{S^*}{S(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(S(t)) + (1-\frac{X^*}{X(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(X(t))\\
&+ (1-\frac{Q^*}{Q(t)})\,{}^{C}\mathcal{D}^{\alpha}_{0,t}(Q(t))\\
&= \omega(2-\frac{P^*}{P}-\frac{P}{P^*}) + \vartheta S(1-\frac{X^*}{X}) + \vartheta S^*(1-\frac{X}{X^*}) + \vartheta\theta S(\frac{X^*}{X} - \frac{Q^*}{Q(t)}) + \vartheta\theta S^*(\frac{X}{X^*} - \frac{Q}{Q^*}).
\end{aligned}$$

We know that

$$(2-\frac{P^*}{P}-\frac{P}{P^*}) \leq 0,$$

and if

$$(\frac{Q^*}{Q} > \frac{X^*}{X} > 1 \text{ and } \frac{S^*}{S} < 1), \qquad \text{or} \qquad (\frac{Q}{Q^*} > \frac{X}{X^*} > 1 \text{ and } \frac{S^*}{S} > 1).$$

We obtain

$$^C\mathcal{D}_{0,t}^{\alpha}(L^*(t)) \leq 0.$$

In addition, we have $^C\mathcal{D}_{0,t}^{\alpha}(L^*(t)) = 0$ if and only if $(P(t), S(t), X(t), Q(t)) = \mathscr{Y}^*$; then the maximum invariant set for $\{(P(t), S(t), X(t), Q(t)) \in \mathbb{R}_+,\ ^C\mathcal{D}_{0,t}^{\alpha}(L^*(t)) = 0\}$ is the set $\{\mathscr{Y}^*\}$, and according to LaSalle's invariance principle, the present smoker equilibrium point $\mathscr{Y}^*$ is globally asymptotically stable. □

5. Numerical Simulations and Discussions

We present here some numerical implementation of System (4); the data are taken from [8] and are presented in Table 2 for $R_0 < 1$ and Table 3 for $R_0 > 1$.

Additionally, we use the following initial condition:

$$(P(0), S(0), X(0), Q(0)) = (0.75,\ 0.15,\ 0.1,\ 0),$$

such that $P + S + X + Q = 1$. For the data from Table 2, we obtain $R_0 = \frac{\wp_1 b}{(\varpi_1+\mho)(\vartheta+\varpi_2+\mho)} = 0.59 < 1$, so the smoking-free equilibrium $\mathscr{U}_0$ of System (4) is $(3.63,\ 0,\ 0,\ 0)$, while for the data from Table 3, we obtain $R_0 = 2.5123 > 1$. According to the data in Tables 2 and 3 and with the help of the Matlab program for fractional differential equations, we obtain the following graphical representation for the smoking dynamic progression of System (4).

Table 2. The parameter values for $R_0 < 1$.

Parameter	Estimation	Source
ω	0.2	[8]
$\wp_1$	0.009	Estimated
$\wp_2$	0.003	Estimated
ϑ	0.0013	Estimated
θ	0.35	Estimated
$\mho$	0.05	Estimated
ϖ_1	0.005	Estimated
ϖ_2	0.0021	[8]
ϖ_3	0.0037	[8]
ϖ_4	0.0012	[8]

Table 3. The parameter values for $R_0 > 1$.

Parameter	Estimation	Source
ω	0.2	[8]
$\wp_1$	0.038	[8]
$\wp_2$	0.0411	[8]
ϑ	0.0013	Estimated
θ	0.35	Estimated
$\mho$	0.05	Estimated
ϖ_1	0.005	Estimated
ϖ_2	0.0021	[8]
ϖ_3	0.0037	[8]
ϖ_4	0.0012	[8]

Discussions

Figure 2 demonstrates that each compartment class converges to its free equilibrium point $\mathscr{U}_0$ as time progresses. The simulation confirms the asymptotic stability of the smoking-free equilibrium point, consistent with the expected theoretical result when $R_0 < 1$.

Additionally, we observe the influence of the fractional derivative order on the numerical solutions for values close to $\alpha = 1$, such as $\alpha = 0.9$ and $\alpha = 0.8$, and notice that the graphs of the numerical solutions approach the solutions obtained from ordinary derivatives at $\alpha = 1$. On the other hand, Figure 3 represents the scenario where $R_0 = 2.5123 > 1$. We observe the existence and stability of the smoking-present equilibrium $\mathscr{Y}^*$ as predicted by the theoretical result.

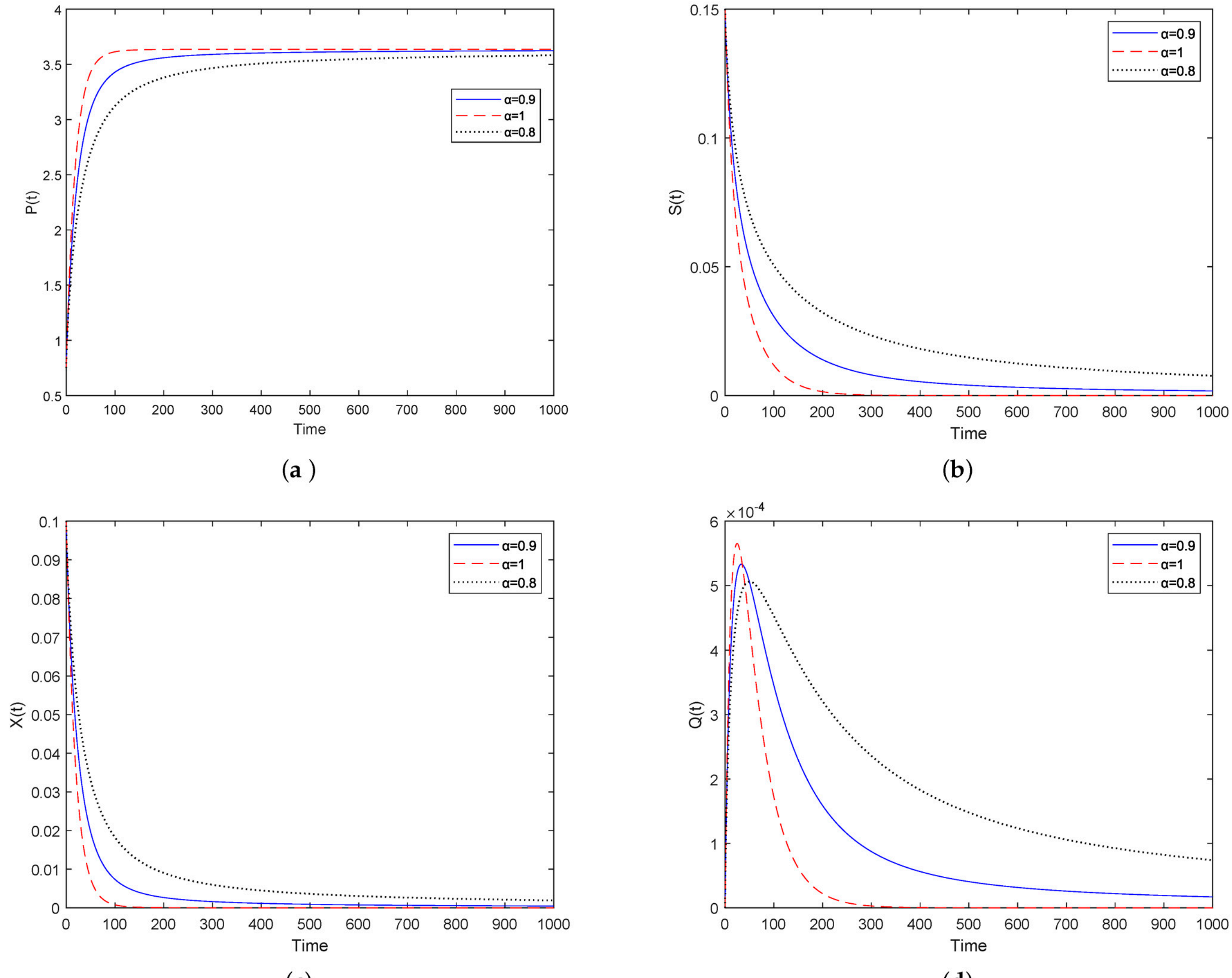

Figure 2. Time series plots of potential, persistent, temporally recovered, and permanently recovered smokers for $R_0 = 0.5950 < 1$. (**a**) Potential smokers with a different value of α. (**b**) Persistent smokers with a different value of α. (**c**) Temporally recovered smokers with a different value of α. (**d**) Permanently recovered smokers with a different value of α.

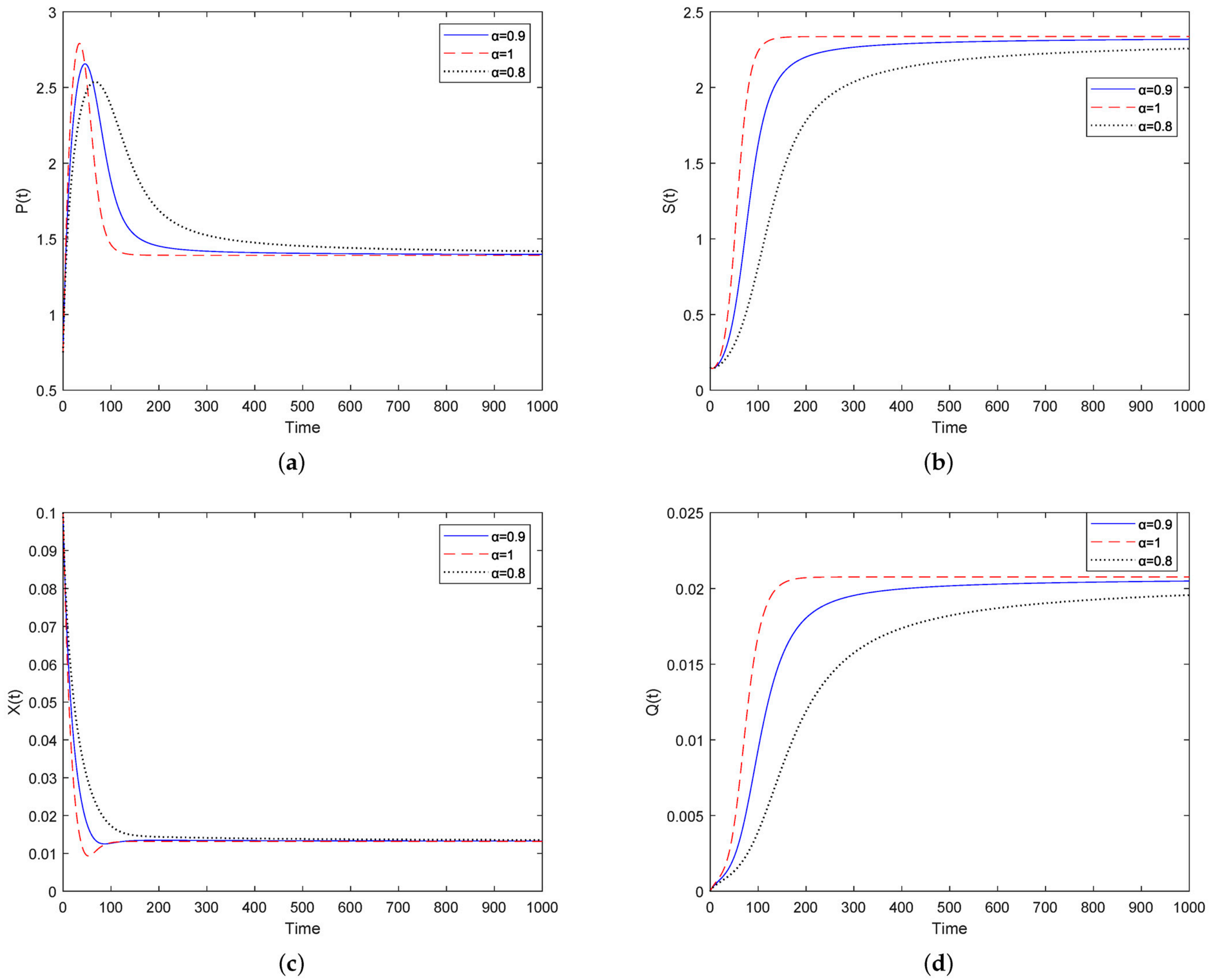

Figure 3. Timeseries plots of potential, persistent, temporally recovered, and permanently recovered smokers for $R_0 = 2.5123 > 1$. (**a**) Potential smokers with a different value of α. (**b**) Persistent smokers with a different value of α. (**c**) Temporally recovered smokers with a different value of α. (**d**) Permanently recovered smokers with a different value of α.

6. Conclusions

In this study, we introduced a mathematical model that describes the dynamics of relapse in the context of smoking cessation and presents the dynamic behavior of the proposed model using Caputo fractional derivatives (see Figure 1). The model considers four categories of individuals: potential smokers, persistent smokers, temporarily recovered smokers, and permanently recovered smokers. Each class was described by a Caputo fractional differential equation of order α. Although specific parameters were introduced and estimated in this work [8], the model could accurately represent the dynamics of a smoking epidemic in real life if the actual parameter values were obtained. The model exhibited a smoking-free equilibrium point, representing a state without smoking, and a present smoking equilibrium point. We derived the reproduction number R_0 by employing the next-generation matrix method. The analysis revealed that if $R_0 < 1$, the smoking-free equilibrium point is asymptotically stable. Conversely, if $R_0 > 1$, the present smoking equilibrium point is asymptotically stable. We performed numerical simulations using the predictor–corrector PECE method for fractional differential equations to validate our findings. The simulation results confirmed the conclusions drawn from the analytical investigation.

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