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The Hölder Regularity for Abstract Fractional Differential Equation with Applications to Rayleigh–Stokes Problems

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Abstract: In this paper, we studied the Hölder regularities of solutions to an abstract fractional differential equation, which is regarded as an abstract version of fractional Rayleigh–Stokes problems, rising up to describing a non-Newtonian fluid with a Riemann–Liouville fractional derivative. The purpose of this article was to establish the Hölder regularities of mild solutions, classical solutions, and strict solutions. We introduced an interpolation space in terms of an analytic resolvent to lower the spatial regularity of initial value data. By virtue of the properties of analytic resolvent and the interpolation space, the Hölder regularities were obtained. As applications, the main conclusions were applied to the regularities of fractional Rayleigh–Stokes problems.

Keywords: fractional derivative; Hölder regularity; interpolation space; Rayleigh–Stokes problem

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1. Introduction

Abstract fractional differential equations have been applied to many fields in science and engineering, such as in viscoelastic mechanics, anomalous diffusion phenomena, materials science, electrochemistry etc.; for more details we refer to the books and the papers [1–7]. It is known that the abstract fractional differential equations can be used to study some partial differential equations with fractional derivatives in an appropriate work space using an operator-theoretic approach. When considering a nonlinear constitutive relationship between shear stress and shear strain rate in fluids, non-Newtonian fluids appear in human blood, oil, and mud-rock flow etc. that cannot be described in a single model, contrasted to the Newtonian fluids. As mentioned in [8], fractional calculus has proved an effective tool for describing viscoelastic fluids; a fractional Rayleigh–Stokes problem in non-Newtonian fluids is more suitable for describing its qualitative properties and behaviors. It is reasonable to analyze the properties and structures of solutions using the operator -theoretic approach.

The exact solutions of fractional Rayleigh–Stokes equations in second grade fluid [4,9,10], Maxwell fluid [3,11], and Oldroyed fluid [12] were obtained by virtue of the Fourier sine transform and fractional Laplace transform. Under the conditions of a non-local integral term, Luc et al. [13] obtained the existence and uniqueness of solutions for nonlinear equations; by using the Fourier truncation method, they constructed a regularization solution to tackle the ill-posedness of solutions. Wang et al. [14] obtained the well-posedness for nonlinear Rayleigh–Stokes equations in view of the fixed point arguments and they also showed the blow-up results. Nguyen et al. [15] obtained some regularity properties of the solutions to the backward problem of determining initial conditions. Lan [16] analyzed some sufficient conditions to ensure the global regularity of solutions and, if the nonlinearity is Lipschizian, then the mild solution of the given problem becomes a classical one. Wang et al. [17] obtained the existence, uniqueness, and regularity of a weak solution in $L^\infty(0,b;L^2(\Omega))\cap L^2(0,b;H_0^1(\Omega))$ by using the Galerkin method and they also proved an improved regularity result of a weak solution in the case of non-homogeneous term $f\in L^2(0,b;L^2(\Omega))$ and initial value $h\in H^2(\Omega)$. Bao et al. [18] studied an inverse problem with a nonlinear source and obtained some results on the existence and regularity of mild solutions. By using an operator-theoretic approach, Bazhlekova et al. [19] obtained the well-posedness and Sobolev regularity of the homogenous Rayleigh–Stokes problem and Bazhlekova [20] showed a well-posed result associated with the bounded C_0 a-semigroup by means of the subordination principle. Pham et al. [21] studied a final-value problem involving weak-valued nonlinearities and obtained the existence and Hölder regularity by using the regularity of the resolvent operators. Tran and Nguyen [22] obtained the solvability and Hölder regularity on the embeddings of fractional Sobolev spaces.

In this paper, we considered the following abstract fractional differential equations:

$$u'(t) = Au(t) + \gamma D_t^{\mu} Au(t) + f(t), \quad t > 0; \quad u(0) = u_0, \tag{1}$$

where D_t^{μ} is the Riemann–Liouville fractional derivative of order $\mu \in (0,1)$, γ is a positive parameter, and operator A generates a bounded analytic semigroup on a Banach X within some sectors $\Sigma(0,\vartheta)$ and $\vartheta \in (0,\pi/2]$, in which $\Sigma(\omega,\vartheta) = \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \vartheta\}$, u_0 is an initial value and f is a continuous function. A prototype example is given by the Rayleigh–Stokes problem on $\mathbb{R}^N$

$$\partial_t u - (1 + \partial_t^{\mu})\Delta u = f(t), \quad t > 0; \quad u(0,x) = u_0(x), \ \ x \in \mathbb{R}^N,$$

for ∂_t^{μ}, a Riemann–Liouville fractional partial derivative. Replacing $f(t)$ with a semilinear function $f(u)$, the global well-posed result with a small initial value and a local well-posed result on $C([0,T];L^p(\mathbb{R}^N))$ for some positive parameters $p > 1, N \geq 1, T \in (0,\infty]$ were considered by He et al. [23].

We list several highlights in the following. Firstly, we note that there are few works concerned with the classical solutions of abstract evolution problem (1), even with the fractional Rayleigh–Stokes problem on $\mathbb{R}^N$ or bounded domain Ω with smooth boundary $\partial\Omega$. The Hölder regularity of solutions is also still a considerable problem because the Hölder regularity of solutions plays an important role in the structure of solutions. Li [24] studied the Hölder regularities of mild solutions for a class of fractional evolution equations with an order of $\alpha \in (1,2)$ and the author showed that a mild solution is the classical one for $f \in C^{\rho}([0,T];X)$ ($\rho \in (0,1)$) especially. In [25], Li and Li also considered the case of the order of $\alpha \in (0,1)$ for the Hölder regularities of mild solutions. Alam et al. [26] established the Hölder regularity of mild/strict solutions of fractional abstract differential equations of the order of $\alpha \in (0,1)$; the obtained results improved the existing results presented in [25]. Allen et al. [27] established a Hölder regularity theorem of De Giorgi–Nash–Moser type for a fractional diffusion equation, see e.g., [28–30] . Secondly, when operator A acts on an analytical semigroup e^{tA}, it appears that $\|tAe^{tA}x\|$ is bounded near $t = 0$ for x in a Banach space X (and it goes to 0 as $t \to 0$ if $x \in \overline{D(A)}$), and $\|Ae^{tA}x\|$ is bounded in $(0,1)$ for $x \in D(A) \subset X$. This means that, for studying the properties of classical solutions, the concept of an intermediate space is naturally introduced in order to reduce the requirement for $x \in (X, D(A))_{\theta,p}$ ($\theta \in (0,1), p \geq 1$). For the consideration (1), we showed that $\|t^{\mu-1}AS(t)x\|$ is bounded in (0,1) for $x \in X$ ($S(t)$ is defined in Equation (2)); it is not suitable for discussing the requirements of $(X, D(A))_{\theta,p}$ since $S(t)$ is no longer an analytic semigroup essentially, and should construct a new interpolation space to lower the spatial regularity on initial condition for the classical solutions. Thirdly, we also proved that a mild solution to problem (1) is also a classical solution if $u_0 \in X$ and $f \in C^1([0,T];X)$, even if it is a strict solution with zero initial value data. In particular, the results we obtained reflect the relevant properties and the structure of solutions of problem (1).

For these targets, we established the existence and Hölder regularity of solutions to problem (1) under an analytic resolvent $S(t)$ determined by A as follows:

$$S(t) = \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt} G(z)dz, \tag{2}$$

where

$$G(z) = \frac{g(z)}{z} R(g(z), A), \quad g(z) = \frac{z}{1 + \gamma z^{\mu}},$$

and for $\vartheta \in (0, \pi/2)$ and $\delta > 0$, the contour $\Gamma_{\delta,\vartheta}$ is defined by

$$\Gamma_{\delta,\vartheta} = \{re^{-i\vartheta}, \, r \geq \delta\} \cup \{\delta e^{i\psi}, \, |\psi| \leq \vartheta\} \cup \{re^{i\vartheta}, \, r \geq \delta\};$$

the circular arc is oriented counterclockwise. We showed that the mild solution is Hölder-continuous for $f \in L^p(0, T; X)$, $p > 1$. Additionally, the solution shall be singular at $t = 0$ for considering the Hölder continuous, in order to lower the regularity of the initial value data. By using the K-method, we introduced a new interpolation space ${}^S D_A(\theta, p)$ compared to the classical one $D_A(\theta, p)$ driven by the analytic semigroup, and we proved that these two spaces are isometric isomorphic. In particular, if $f \in C^1([0, T]; X)$ and $u_0 \in X$, the mild solution is indeed a classical solution. If $u_0 \equiv 0$, the solution will still be a strict solution. Especially, it possesses a Hölder regularity with an exponent of $\mu \wedge (1 - \mu - \frac{1}{q})$ for $\frac{1}{1-\mu} < q < \frac{1}{(1-\mu)(1-\theta)}$ and $u_0 \in {}^S D_A(\theta, p)$. Our proofs of the main results are based on the analytic properties of $S(t)$ and the operator approach.

The present paper is constructed as follows. In Section 2, in view of the Hardy type inequality, we showed several main properties of the analytic resolvent $S(t)$. In Section 3, we constructed a new interpolation space in terms of the analytic resolvent and we analyzed its properties. In Section 4, we proved the existence and uniqueness of the solutions of the problem (1), and we established the Hölder regularity of solutions. Finally, some examples are presented to check the main results.

2. Preliminaries

Let X and X_1 be two Banach spaces; the notation $\mathcal{B}(X, X_1)$ denotes the space of all bounded linear operators mapping from X into X_1 with the norm $\| \cdot \|_{\mathcal{B}(X,X_1)}$—for short, $\| \cdot \|_{\mathcal{B}}$ by $x \mapsto \mathcal{B}(X)$. We denote by $C(J, X)$ the space of continuous functions that from an interval $J \subseteq \mathbb{R}_+$ to X. Let A be a linear closed operator; we set $\rho(A)$ and $\sigma(A)$ by the resolvent set and spectral set of A, respectively, and the resolvent operator of A is given by $R(z; A) = (zI - A)^{-1}$. I is an identity operator. The notation $\wedge$ denotes $a \wedge b = \min\{a, b\}$ for any constant $a, b \in \mathbb{R}$. For convenience, the notation C will denote a positive constant.

For $\gamma \in (0, 1)$, the Hölder continuous function space $C^{\gamma}(J; X)$ is defined by

$$C^{\gamma}(J; X) := \left\{ f \in C(J; X) : [f]_{\gamma} = \sup_{\tau,\sigma \in J, \tau \neq \sigma} \frac{\|f(\tau) - f(\sigma)\|}{|\tau - \sigma|^{\gamma}} < \infty \right\},$$

equipped with the norm $\|f\|_{C^{\gamma}(J;X)} = \sup_{\sigma \in J} \|f(\sigma)\| + [f]_{\gamma}$.

For $1 \leq p < \infty$, denote a space by $L^p_*(J) := L^p(J, dt/t)$, equipped with norm

$$\|h\|_{L^p_*(J)} = \left(\int_J |h(\tau)|^p \frac{d\tau}{\tau} \right)^{\frac{1}{p}}$$

and $\|h\|_{L^{\infty}_*(J)} = \operatorname*{ess\,sup}_{s \in J} |h(s)|$.

It is known that the K-method is a classical method for producing real interpolation spaces; for every $t > 0$ and $y \in X$, let

$$K(t, y; X, X_1) := \inf\{\|y_1\|_X + t\|y_2\|_{X_1} : y = y_1 + y_2, y_1 \in X, y_2 \in X_1\}.$$

For any $p \in [1, \infty]$, $\theta \in (0, 1)$, denote the following space

$$(X, X_1)_{\theta,p} = \{y \in X : t \mapsto \phi(t) = t^{-\theta} K(t, y; X, X_1) \in L^p_*(0, \infty)\},$$

with its norm $\|y\|_{(X,X_1)_{\theta,p}} = \|\phi\|_{L^p_*(0,\infty)}$. Then, the real interpolation $(X, X_1)_{\theta,p}$ is a Banach space.

Note that $K(t, y; X, X_1) \leq \|y\|_X$ as $X_1 \hookrightarrow X$; in order to check $y \in (X, X_1)_{\theta,p}$, it is sufficient to show that $t \mapsto \phi(t) \in L^p_*(0, c)$ for some fixed $c > 0$. In terms of the analytic semigroup $T(\cdot)$ generated by A, $(X, D(A))_{\theta,p}$ also has the following expression:

$$(X, D(A))_{\theta,p} = \{y \in X : t \mapsto \varphi(t) = t^{1-\theta}\|AT(t)y\|_X \in L^p_*(0,1)\} := D_A(\theta, p),$$

with norm $\|y\|_{\theta,p} = \|y\|_X + \|\varphi\|_{L^p_*(0,1)}$, see e.g., [31].

Consider the weak singular kernel in Riemann–Liouville fractional integral $g_\mu(t) = t^{\mu-1}/\Gamma(\mu)$, $t > 0$, $\mu > 0$, where $\Gamma(\cdot)$ is the Gamma function. And let $*$ denote the convolution of functions $a, b \in L^1(0, T; X)$ by

$$(a * b)(t) = \int_0^t a(t-\sigma)b(\sigma)d\sigma, \quad t > 0.$$

Definition 1. *Let $g \in L^1(0, T; X)$. The Riemann–Liouville fractional integral $I^\mu g(t)$ of order $\mu \geq 0$ is defined by*

$$I^\mu g(t) = \int_0^t \frac{1}{\Gamma(1-\mu)}(t-\sigma)^{\mu-1}g(\sigma)d\sigma, \ \ t \in [0, T].$$

Definition 2. *Let $h \in L^1(0, T; X)$. The Riemann–Liouville fractional derivative $D^\mu_t h(t)$ of order $\mu > 0$ is defined by*

$$D^\mu_t h(t) = \frac{d}{dt} I^{1-\mu} h(t).$$

Recall that the following definition of analytic resolvent $Q(t)$ is introduced by Prüss [32].

Definition 3. *A resolvent $Q(t)$ is called analytic if the function $Q(\cdot) : \mathbb{R}_+ \to \mathcal{B}(X)$ admits an analytic extension to a sector $\Sigma(0, \vartheta_0)$ for some $0 < \vartheta_0 \leq \pi/2$. An analytic resolvent $Q(t)$ is said to be of analyticity type (ω_0, ϑ_0) if, for each $\omega > \omega_0$ and $\vartheta < \vartheta_0$, there is $M = M(\omega, \vartheta)$ such that $\|Q(z)\|_{\mathcal{B}} \leq Me^{\omega Re(z)}$ for $z \in \Sigma(0, \vartheta_0)$.*

Suppose that (1) admits a solution, then the problem can be rewritten as the following integral equation:

$$u(t) = u_0 + \int_0^t (1 + \gamma g_{1-\mu}(t-\sigma))Au(\sigma)d\sigma + \int_0^t f(\sigma)d\sigma.$$

By the variation of the parameters formula—for example, see [20,23]—the solution is given by

$$u(t) = S(t)u_0 + (S * f)(t), \quad t \geq 0, \tag{3}$$

where $S(\cdot)$ is defined in (2). Note that, since $\hat{a}(z) = z^{-1} + \gamma z^{\mu-1} \neq 0$ admits meromorphic extension to $\Sigma(0, \vartheta + \pi/2)$, $1/\hat{a}(z) \in \rho(A)$, and $\|G(z)\|_{\mathcal{B}} \leq M|z|^{-1}$ for $z \in \Sigma(0, \vartheta + \pi/2)$ from [23], it yields that the analytic resolvent $S(\cdot)$ is an analyticity type $\Sigma(0, \vartheta)$ by Prüss ([32], Theorem 2.1).

We observe that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$, we know that $\hat{S}(\lambda) = G(\lambda)$, where $\hat{S}(\cdot)$ is the Laplace transform of $S(\cdot)$. From the identity of the Laplace transform,

$$R(z, A) = \int_0^\infty e^{-zt}T(t)dt,$$

it follows that

$$G(s) = \frac{g(s)}{s}\int_0^\infty e^{-tg(s)}T(t)dt = \int_0^\infty \left(\int_0^\infty e^{-st}\varphi(t,\sigma)dt\right)T(\sigma)d\sigma,$$

where the probability density function $\varphi(t,\sigma)$ satisfies the inverse Laplace integral,

$$\varphi(t,\sigma)=\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty}e^{st-\frac{\sigma s}{1+\gamma s^{\mu}}}\frac{1}{1+\gamma s^{\mu}}ds,\quad \tau>0,\ \ t,\sigma>0.$$

Therefore—for example, see [20]—by the uniqueness of the Laplace transform, it also yields

$$S(t)=\int_0^{\infty}\varphi(t,\sigma)T(\sigma)d\sigma,\ \ t\geq 0. \tag{4}$$

Lemma 1. *Let $S(\cdot)$ be defined in Equation (2). Then,*

(i) *For every $t>0$, $S(t)\in\mathcal{B}(X)$ and $\|S(t)\|_{\mathcal{B}}\leq C$;*
(ii) *For each $x\in X$, $\lim\limits_{t\to 0}\|S(t)x-x\|=0$;*
(iii) *For any $x\in X$, $S(\cdot)x\in C([0,\infty);X)$ and $AS(\cdot)x\in C((0,\infty);X)$;*
(iv) *For every $t>0$, $\|AS(t)\|_{\mathcal{B}}\leq Ct^{\mu-1}$.*

Proof. The proof is similar to ([23], Lemma 2.2), so we omit it. □

Let $\mathcal{C}$ be an arbitrary piecewise smooth simple curve in $\Sigma(0,\vartheta+\pi/2)$ running from $\infty e^{-i(\vartheta+\pi/2)}$ to $\infty e^{i(\vartheta+\pi/2)}$ and $\vartheta\in(0,\pi/2)$; we have the following Lemma.

Lemma 2 ([33], Lemma 4.1.1)**.** *Assume that the map $F:\Sigma(0,\vartheta+\pi/2)\times X\times\mathbb{R}^+\to X$, satisfying:*

(i) *For $(x,t)\in X\times\mathbb{R}^+$, $F(\cdot,x,t):\Sigma(0,\vartheta+\pi/2)\to X$ is holomorphic;*
(ii) *For $z\in\Sigma(0,\vartheta+\pi/2)$, $F(z,\cdot,\cdot)\in C(X\times\mathbb{R}^+,X)$;*
(iii) *For $(z,x,t)\in\Sigma(0,\vartheta+\pi/2)\times X\times\mathbb{R}^+$, there exists constant $\varsigma\in\mathbb{R}$ such that*

$$\|F(z,x,t)\|\leq C|z|^{\varsigma-1}e^{tRe(z)}.$$

Then,

$$(x,t)\mapsto\int_{\mathcal{C}}F(z,x,t)dz\in C(X\times\mathbb{R}^+,X)$$

and

$$\left\|\int_{\mathcal{C}}F(z,x,t)dz\right\|\leq Ct^{-\varsigma},\quad (x,t)\in X\times\mathbb{R}^+.$$

Lemma 3. *Let $S(\cdot)$ be defined in Equation (2). Then, the following results hold:*

(i) *For each $t>0$, $AS(t)x=S(t)Ax$ for $x\in D(A)$;*
(ii) *The mapping $t\mapsto S(t)\in C^n((0,\infty);\mathcal{B}(X))$ and for $n\in\mathbb{N}$, there holds*

$$\|S^{(n)}(t)\|_{\mathcal{B}}\leq Ct^{-n},\quad \text{and}\quad \|AS'(t)\|_{\mathcal{B}}\leq Ct^{\mu-2},\quad t>0;$$

(iii) *$\forall x\in X$ yields*

$$(g_1*S)(t)x+\gamma(g_{1-\mu}*S)(t)x\in D(A),$$

and

$$(g_1*AS)(t)x+\gamma(g_{1-\mu}*AS)(t)x=S(t)x-x, \tag{5}$$

uniformly in a compact interval.

Proof. Since $T(t)$ is an analytic semigroup, and for $x\in D(A)$, $AT(t)x=T(t)Ax$, it can readily be seen that $AS(t)x=S(t)Ax$ is true by (4) .

Let us check (ii). From ([23], Lemma 2.2), we know that $\|(g(z)I+A)^{-1}\|_{\mathcal{B}}\leq C|g(z)|^{-1}$ and $g(z)\in\Sigma(0,\vartheta+\pi/2)$ for $z\in\Sigma(0,\vartheta+\pi/2)$, also $|g(z)|\leq C|z|^{1-\mu}$. Thus, we get $\|G(z)\|_{\mathcal{B}}\leq C|z|^{-1}$. Function $F(z,x,t)=e^{zt}z^nG(z)x$ is continuous for $z\in\Sigma(0,\vartheta+\pi/2)$, $x\in X$, it yields from Lemma 2 that

$$S^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^n G(z)dz, \;\; t > 0,$$

belongs to $C((0,\infty);\mathcal{B}(X))$. Hence, it follows that $S(\cdot) \in C^n((0,\infty);\mathcal{B}(X))$. Moreover, by the analyticity of $G(z)$ in $\Gamma_{\delta,\vartheta}$, we get that

$$\begin{aligned}\|S^{(n)}(t)\|_{\mathcal{B}} \le& \int_{\Gamma_{1/t,\vartheta}} e^{Re(z)t}\|z^n G(z)\|_{\mathcal{B}}\,|dz|\\ \le& C\left(\int_{1/t}^{\infty} r^{n-1}e^{-rt\cos(\vartheta)}dr + \int_{-\vartheta}^{\vartheta} e^{\cos(\psi)}t^{-n}d\psi\right)\\ \le& Ct^{-n}.\end{aligned}$$

Therefore, for $z \in \Sigma(0,\vartheta+\pi/2)$, from the identity of the inequality $\|AG(z)\|_{\mathcal{B}} \le C|z|^{-\mu}$, we have

$$\begin{aligned}\|AS'(t)\|_{\mathcal{B}} \le& \int_{\Gamma_{1/t,\vartheta 1}} e^{Re(z)t}\|zAG(z)\|_{\mathcal{B}}\,|dz|\\ \le& 2C\int_{1/t}^{\infty} r^{1-\mu}e^{-rt\cos(\vartheta)}dr + C\int_{-\vartheta}^{\vartheta} e^{\cos(\psi)}t^{\mu-2}d\psi\\ \le& Ct^{\mu-2}\int_{cos(\vartheta)}^{\infty} e^{-u}u^{1-\mu}du + Ct^{\mu-2}\int_{-\vartheta}^{\vartheta} e^{cos(\psi)}d\psi\\ \le& Ct^{\mu-2}, \;\; t>0.\end{aligned}$$

For (iii), by Lemma 1, we note that

$$\|A(g_1 * S)(t)x\| \le Ct^{\mu}\|x\|, \quad \|A(g_{1-\mu} * S)(t)x\| \le C\|x\|, \;\; x \in X. \tag{6}$$

Therefore, we also obtain $(g_1 * S)(t)x + \gamma(g_{1-\mu} * S)(t)x \in D(A)$ uniformly in a compact interval.

By the Laplace transform and its uniqueness, obverse that the integrals

$$(g_1 * AS)(t)x = \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^{-1}AG(z)xdz$$

and

$$(g_{1-\mu} * AS)(t)x = \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^{\mu-1}AG(z)xdz$$

are uniformly bounded on a compact interval. For $x \in X$, by $AG(z) = (G(z) - z^{-1}I)g(z)$, we have

$$\begin{aligned}&(g_1 * AS)(t)x + \gamma(g_{1-\mu} * AS)(t)x\\ &\quad= \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^{-1}G(z)g(z)xdz - \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^{-2}g(z)xdz\\ &\quad\quad+ \frac{\gamma}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^{\mu-1}G(z)g(z)xdz - \frac{\gamma}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^{\mu-2}g(z)xdz.\end{aligned}$$

From the identity $z^{-1}g(z) + \gamma z^{\mu-1}g(z) = 1$ for $z \in \Gamma_{\delta,\vartheta}$, it follows that:

$$\begin{aligned}(g_1 * AS)(t)x + \gamma(g_{1-\mu} * AS)(t)x =& \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}G(z)xdz - \frac{1}{2\pi i}\int_{\Gamma_{\delta,\vartheta}} e^{zt}z^{-1}xdz\\ =& S(t)x - x,\end{aligned}$$

which shows Equation (5). The proof is completed. □

Corollary 1. *Let $S(\cdot)$ be defined in Equation (2). Then,*

$$\left\| D_t^{\mu} S(t)x \right\| \leq Ct^{-\mu} \|x\|, \quad \text{for } t > 0, \;\; x \in X,$$

and $D_t^{\mu} S(t) \in C((0,\infty); \mathcal{B}(X))$. In particular, it follows that:

$$\|D_t^{\mu} S(t)x - D_s^{\mu} S(s)x\| \leq C(t-s)^{-\mu} \|x\|, \quad \text{for } t > s > 0, \;\; x \in X.$$

Proof. Obverse that the integral,

$$(g_{1-\mu} * S)(t)x = \frac{1}{2\pi i} \int_{\Gamma_{\delta,\vartheta}} e^{zt} z^{\mu-1} G(z) x dz, \; t > 0,$$

is uniform in a compact interval, and then

$$\frac{d}{dt}(g_{1-\mu} * S)(t)x = \frac{1}{2\pi i} \int_{\Gamma_{\delta,\vartheta}} e^{zt} z^{\mu} G(z) x dz, \; t > 0.$$

By using Lemma 2 and applying the similar proof in Lemma 3 and the definition of operator D_t^{μ}, we get $\|D_t^{\mu} S(t)x\| \leq Ct^{-\mu}\|x\|$ for any $x \in X$. Another conclusion follows the same approach. We thus obtain this corollary. □

Remark 1. *By the similar proof in Corollary 1, from the analytic resolvent of $S(t)$, then for $t > 0$, $x \in X$, it also yields that $\|D_t^{\mu} AS(t)x\| \leq Ct^{-1}\|x\|$.*

Remark 2. *Since A generates a bounded analytic semigroup on a Banach X within some sector $\Sigma(0,\theta)$ and $\theta \in (0, \pi/2]$, the equation $(g_1 * AS)(t)x + \gamma(g_{1-\mu} * AS)(t)x = S(t)x - x$ is valid on X for every $t \geq 0$, which derives that analytic resolvent $S(t)$ is a solution operator of the homogeneous equation to problem (1) and also $S'(t)x = AS(t)x + \gamma D_t^{\mu} AS(t)x$ for $t > 0$, $x \in X$.*

Next, we introduce the concepts of solutions as follows:

Definition 4. *A function $u : [0,T] \to X$ is a mild solution of problem (1) on $[0,T]$ if the function u defined in Equation (3) belongs to $C([0,T];X)$.*

Definition 5. *A function $u : [0,T] \to X$ is called a classical solution of problem (1) on $[0,T]$ if u is continuous on $[0,T]$, continuously differentiable on $(0,T]$, $u(t)$ and $D_t^{\mu} u(t) \in D(A)$ for $0 < t \leq T$,and (1) is satisfied on $[0,T]$.*

Definition 6. *A function $u : [0,T] \to X$ is called a strict solution of problem (1) on $[0,T]$ if u is continuous on $D(A)$ and continuously differentiable on $[0,T]$, $D_t^{\mu} u(t) \in D(A)$ for $0 \leq t \leq T$ and (1) is satisfied on $[0,T]$.*

Clearly, the relations satisfy: the strict solution ⇒, the classical solution ⇒, and the mild solution. Next, we will use the Hardy-type inequalities involving the Riemann–Liouville fractional integral.

Lemma 4 ([34])**.** *Let $1 < p \leq q < \infty$ and $\frac{1}{p'} = 1 - \frac{1}{p}$. Then, for non-negative weight functions u and v, it yields*

$$I^{\mu} : L^p((0,\infty); v(s)ds) \to L^q((0,\infty); u(s)ds),$$

iff for $\forall\, R > 0$,

$$\left(\int_0^R (v(s))^{1-p'} ds \right)^{\frac{1}{p'}} \left(\int_{2R}^{\infty} s^{(\mu-1)q} u(s) ds \right)^{\frac{1}{q}} \leq C. \tag{7}$$

Lemma 5 ([35])**.** *Let u, v be non-negative weight functions. If there is constant $0 \leq \beta \leq 1$, such that*

$$\left(\int_r^\infty (s-r)^{(\mu-1)\beta}u(s)ds\right)\left(\operatorname*{ess\,sup}_{s\in(0,r)}(r-s)^{(\mu-1)(1-\beta)}[v(s)]^{-1}\right)\le C,\quad \forall\, r>0. \tag{8}$$

Then,

$$\int_0^\infty |(I^\mu f)(s)|u(s)ds \le C\int_0^\infty |f(s)|v(s)ds.$$

Let us recall the Hardy–Young inequality.

Lemma 6 ([36])**.** *Let $p\ge 1$ and $0<a\le\infty$. Then, for any $\beta>0$ and measurable function $g:(0,a)\to\mathbb{R}^+$, it yields*

$$\int_0^a s^{-\beta p}\left(\int_0^s g(\tau)\frac{d\tau}{\tau}\right)^p\frac{ds}{s}\le\frac{1}{\beta^p}\int_0^a \tau^{-\beta p}g^p(\tau)\frac{d\tau}{\tau}.$$

3. A New Interpolation Space

Let $1\le p\le\infty$, $0<\theta<1$; we now introduce an interpolation space with $\mu\in(0,1)$ by

$${}^S D_A(\theta,p):=\{y\in X: t\mapsto \omega(t)=t^{(1-\mu)(1-\theta)}\|AS(t)y\|\in L^p_*(0,1)\},$$

which is a Banach space endowed with norm ${}^S\|y\|_{\theta,p}=\|y\|+\|\omega\|_{L^p_*(0,1)}$.

Theorem 1. *Let $1\le p\le\infty$, $0<\theta<\frac{\mu}{1-\mu}\wedge 1$. Then,*

$$D_A(\theta,p)={}^S D_A(\theta,p);$$

the respective norms are equivalent.

Proof. Let $y\in D_A(\theta,p):=(X,D(A))_{\theta,p}$ and $y=y_1+y_2$, where $y_1\in X$ and $y_2\in D(A)$. Then, for $t>0$, by using Lemma 1 (*iv*), we have

$$\|AS(t)y\|\le\|AS(t)y_1\|+\|AS(t)y_2\|\le C(t^{\mu-1}\|y_1\|+\|Ay_2\|).$$

By the definition of $(X,D(A))_{\theta,p}$, we know that

$$t\mapsto\phi(t^{1-\mu})=t^{-\theta(1-\mu)}K(t^{1-\mu},y;X,D(A))\in L^p_*(0,\infty).$$

Therefore, the mapping follows

$$t\mapsto t^{-(1-\mu)\theta}\inf\{\|y_1\|+t^{1-\mu}\|y_2\|_{D(A)}:\ y=y_1+y_2,\ y_1\in X,\ y_2\in D(A)\}\in L^p_*(0,\infty).$$

Let $\omega(t)=t^{(1-\mu)(1-\theta)}\|AS(t)y\|$, since $D(A)\hookrightarrow X$, the map $t\mapsto\omega(t)\in L^p_*(0,1)$ and ${}^S\|y\|_{\theta,p}\le C\|y\|_{\theta,p}$. Thus, we get

$$D_A(\theta,p)\hookrightarrow{}^S D_A(\theta,p). \tag{9}$$

Conversely, let $y\in{}^S D_A(\theta,p)$. From Lemma 3 (*iii*), for $t\in(0,1)$, we have

$$\begin{aligned} y&=-\,(S(t)y-y)+S(t)y\\ &=-\,\big((g_1*AS)(t)y+\gamma(g_{1-\mu}*AS)(t)y\big)+S(t)y. \end{aligned}$$

From inequalities (6) and $\omega\in L^p_*(0,1)$, we know that $A(g_1*S)(t)y$ and $\gamma A(g_{1-\mu}*S)(t)y$ belong to X for every $t\in(0,1)$. Moreover, $S(\cdot)y\in D(A)$ from Lemma 1. Therefore, we derive that:

$$
\begin{aligned}
K(t^{1-\mu}, y; X, D(A)) \leq & \|S(t)y - y\| + t^{1-\mu}\|S(t)y\| \\
& \leq \int_0^t \|AS(\tau)y\| d\tau + \frac{\gamma}{\Gamma(1-\mu)} \int_0^t (t-\tau)^{-\mu} \|AS(\tau)y\| d\tau \\
& \quad + t^{1-\mu}\|AS(t)y\|.
\end{aligned}
$$

For the case of $p = \infty$, we first see that:

$$
\begin{aligned}
K(t^{1-\mu}, y; X, D(A)) \leq & \int_0^t \tau^{-(1-\mu)(1-\theta)} \omega(\tau) d\tau + \frac{\gamma}{\Gamma(1-\mu)} \int_0^t (t-\tau)^{-\mu} \tau^{-(1-\mu)(1-\theta)} \omega(\tau) d\tau \\
& + Ct^{1-\mu}\|y\| \\
\leq & Ct^{\mu+\theta(1-\mu)} \|\omega\|_{L_*^\infty(0,1)} + Ct^{\theta(1-\mu)} \|\omega\|_{L_*^\infty(0,1)} + Ct^{1-\mu}\|y\|,
\end{aligned}
$$

implying $t^{-(1-\mu)\theta} K(t^{1-\mu}, y; X, D(A)) \leq C^S \|y\|_{\theta,\infty}$. By the change of variable $t \mapsto t^{1-\mu}$, we derive that:

$$
\|y\|_{D_A(\theta,\infty)} \leq C^S \|y\|_{\theta,\infty}.
$$

For the case of $1 \leq p < \infty$, it also yields

$$
K(t^{1-\mu}, y; X, D(A)) \leq \|(g_1 * AS)(t)y + \gamma(g_{1-\mu} * AS)(t)y\| + t^{1-\mu}\|AS(t)y\|.
$$

Therefore, it follows from the elementary inequality $(a_1 + a_2)^p \leq 2^p(a_1^p + a_2^p)$ for $a_1, a_2 \geq 0$, $p \geq 1$, and $y \in {}^S D_A(\theta, p)$ that:

$$
\begin{aligned}
& \int_0^1 t^{-(1-\mu)\theta p} K(t^{1-\mu}, y; X, D(A))^p \frac{dt}{t} \\
& \leq \int_0^1 t^{-(1-\mu)\theta p} (\|(g_1 * AS)(t)y + \gamma(g_{1-\mu} * AS)(t)y\| + t^{1-\mu}\|AS(t)y\|)^p \frac{dt}{t} \\
& \leq 2^p \int_0^1 t^{-(1-\mu)\theta p} (\|(g_1 * AS)(t)y + \gamma(g_{1-\mu} * AS)(t)y\|^p + t^{(1-\mu)p}\|AS(t)y\|^p) \frac{dt}{t} \\
& \leq C \int_0^1 t^{-(1-\mu)\theta p} \|(g_1 * AS)(t)y + \gamma(g_{1-\mu} * AS)(t)y\|^p \frac{dt}{t} + C\|\omega\|^p_{L_*^p(0,1)}.
\end{aligned}
$$

The Hardy-type and Hardy–Young inequalities show that:

$$
\int_0^1 t^{-(1-\mu)\theta p} \|(g_1 * AS)(t)y + \gamma(g_{1-\mu} * AS)(t)y\|^p \frac{dt}{t} \leq C^S \|y\|^p_{\theta,p}. \tag{10}
$$

In fact, for $1 < p < \infty$, by $0 < \theta < \frac{\mu}{1-\mu} \wedge 1$, Lemmas 1 and 6 show that:

$$
\begin{aligned}
\int_0^1 t^{-(1-\mu)\theta p} \|(g_1 * AS)(t)y\|^p \frac{dt}{t} = & \int_0^1 t^{-(1-\mu)\theta p} \left\| \int_0^t AS(s) y ds \right\|^p \frac{dt}{t} \\
\leq & \int_0^1 t^{-(1-\mu)\theta p} \left(\int_0^t \|AS(s)y\| ds \right)^p \frac{dt}{t} \\
\leq & C \int_0^1 t^{-(1-\mu)\theta p} \left(\int_0^t s^{\mu-1} \|y\| ds \right)^p \frac{dt}{t} \\
\leq & C\|y\|^p \int_0^1 s^{-(1-\mu)\theta p} s^{\mu p} \frac{ds}{s} \\
\leq & C\|y\|^p.
\end{aligned} \tag{11}
$$

Therefore, we have

$$
\int_0^1 t^{-(1-\mu)\theta p} \|(g_1 * AS)(t)y\|^p \frac{dt}{t} \leq C^S \|y\|^p_{\theta,p}.
$$

It is easy to get that the inequality (7) is true for $u(t) = t^{-(1-\mu)\theta p-1}$ and $v(t) = t^{(1-\mu)(1-\theta)p-1}$. Hence, by Lemmas 1 and 4, we get

$$I^{1-\mu}: L^p((0,\infty); v(t)dt) \to L^p((0,\infty); u(t)dt);$$

this means that

$$\begin{aligned}
\int_0^1 t^{-(1-\mu)\theta p}\|(g_{1-\mu} * AS)(t)y\|^p \frac{dt}{t} &= \int_0^1 t^{-(1-\mu)\theta p}\|(I^{1-\mu} AS)(t)y\|^p \frac{dt}{t} \\
&\leq \int_0^\infty t^{-(1-\mu)\theta p}\|(I^{1-\mu} AS)(t)y\|^p \frac{dt}{t} \\
&\leq C \int_0^\infty t^{(1-\mu)(1-\theta)p}\|AS(t)y\|^p \frac{dt}{t} \\
&\leq C\Big(\int_0^1 t^{(1-\mu)(1-\theta)p}\|AS(t)y\|^p \frac{dt}{t} + \|y\|^p\Big),
\end{aligned} \tag{12}$$

which shows that

$$\int_0^1 t^{-(1-\mu)\theta p}\|(g_{1-\mu} * AS)(t)y\|^p \frac{dt}{t} \leq C(\|\omega\|^p_{L^p_*(0,1)} + \|y\|^p) \leq C^S\|y\|^p_{\theta,p}.$$

This deduces that (10) holds.

On the other hand, for the case of $p = 1,$ by using the similar procedure of (11), we obtain that

$$\int_0^t t^{-(1-\mu)\theta}\|(g_1 * AS)(t)y\| \frac{dt}{t} \leq C\|y\|. \tag{13}$$

Moreover, set $u(t) = t^{-(1-\mu)\theta-1}$, $v(t) = t^{(1-\mu)(1-\theta)-1}$, and $\beta = 1$, we see that the inequality (8) holds. Thus, by Lemma 5 and the similar procedure of (12) , we get that:

$$\int_0^t t^{-(1-\mu)\theta}\|(g_{1-\mu} * AS)(t)y\|^p \frac{dt}{t} \leq C(\|y\| + \|\omega\|_{L^1_*(0,1)}). \tag{14}$$

Consequently, combined (13) with (14), it follows that:

$$\int_0^1 t^{-(1-\mu)\theta}\|(g_1 * AS)(t)y + \gamma(g_{1-\mu} * AS)(t)y\| \frac{dt}{t} \leq C^S\|y\|_{\theta,1}.$$

Hence, it follows that (10) and $\|y\|_{D_A(\theta,p)} \leq C^S\|y\|_{D_A(\theta,p)}$ for all $1 \leq p \leq \infty$. Consequently, we get the embedding

$${}^S D_A(\theta,p) \hookrightarrow D_A(\theta,p), \tag{15}$$

The conclusion follows (9) and (15). $\square$

Remark 3 (Alam et al. [26], Theorem 3.2). *introduced two new interpolation spaces in terms of solution operators for considering the fractional differential equation of order $\mu \in (0,1)$ by*

$${}^{S_\beta} D_A(\theta,p) := \{x \in X : t \mapsto t^{\beta-\mu\theta}\|AS_\beta(t)x\| \in L^p_*(0,1)\},$$

where $\beta = 1$, or $\beta = \mu \in (0,1)$ and

$$S_\beta(t) = t^{\beta-1}\int_0^\infty (\mu s)^{[\beta]} M_\mu(s) T(st^\mu) ds.$$

The $M_\mu(\cdot)$ is a probability density function, see e.g., [6], and $[\beta]$ is the integer part of β. It is noted that the characterization of these interpolation spaces has the connection

$${}^{S_\mu} D_A(\theta,p) = {}^{S_1} D_A(\theta,p) = D_A(\theta,p), \quad \theta \in (0,1), \quad p \in [1,\infty].$$

Combining this result and Theorem 1, it follows that ${}^{S_1} D_A(\theta,p) = {}^S D_A(\theta,p)$ for $1 \leq p \leq \infty$ and the restriction $\theta \in (0, \frac{\mu}{1-\mu} \wedge 1)$. This means that we can construct different interpolation spaces under the classical interpolation space $(X, D(A))_{\theta,p}$, which are equivalent with respect to *the corresponding norms. We can thus construct an appropriate interpolation space for different considerable problems.*

Another construction of ${}^{S}D_A(\theta,p)$ is obtained below.

Theorem 2. *Let* $1 \le p \le \infty$ *and* $0 < \theta < \frac{\mu}{1-\mu} \wedge 1$. *Then,*

$$ {}^{S}D_A(\theta,p) = \{y \in X : t \mapsto \phi_\mu(t) = t^{-(1-\mu)\theta}\|S(t)y - y\| \in L_*^p(0,1)\}; $$

the norms $\|y\| + \|\phi_\mu\|_{L_*^p(0,1)}$ *and* ${}^{S}\|y\|_{\theta,p}$ *are equivalent.*

Proof. Let $y \in {}^{S}D_A(\theta,p)$ for $p=\infty$ and $t \in (0,1)$; by Lemma 3 (*iii*), we get

$$ \begin{aligned} \|S(t)y - y\| &= \|(g_1 * AS)(t)y + \gamma(g_{1-\mu} * AS)(t)y\| \\ &\le \int_0^t s^{-(1-\mu)(1-\theta)}\omega(s)ds + \frac{\gamma}{\Gamma(1-\mu)}\int_0^t (t-s)^{-\mu}s^{-(1-\mu)(1-\theta)}\omega(s)ds \\ &\le Ct^{(1-\mu)\theta}\|\omega\|_{L_*^\infty(0,1)}. \end{aligned} $$

This implies that $\phi_\mu(t) \in L_*^p(0,1)$ and $\|\phi_\mu\|_{L_*^\infty(0,1)} \le C\|\omega\|_{L_*^\infty(0,1)}$. Therefore, we have

$$ \|y\| + \|\phi_\mu\|_{L_*^\infty(0,1)} \le C^S\|y\|_{\theta,\infty}. $$

For $1 \le p < \infty$, by the same procedure of (10), we obtain

$$ \int_0^1 t^{-(1-\mu)\theta p}\|(g_1 * AS)y + \gamma(g_{1-\mu} * AS)(t)y\|^p \frac{dt}{t} \le C^S\|y\|_{\theta,p}^p. $$

Hence, for all $1 \le p \le \infty$, it yields

$$ \|y\| + \|\phi_\mu\|_{L_*^p(0,1)} \le C^S\|y\|_{\theta,p}. $$

Conversely, let $y \in X$ satisfy $t \mapsto \phi_\mu(t) = t^{-(1-\mu)\theta}\|S(t)y - y\| \in L_*^p(0,1)$. For every $t > 0$, we have the following identity:

$$ \begin{aligned} y =& t^{\mu-1}\Gamma(2-\mu)(g_{1-\mu} * (y - S(\cdot)y))(t) \\ &+ t^{\mu-1}\Gamma(2-\mu)\big((g_{1-\mu} * S)(t)y + \gamma^{-1}(g_1 * S)(t)y\big) - \gamma^{-1}t^{\mu-1}\Gamma(2-\mu)(g_1 * S)(t)y. \end{aligned} $$

Therefore, from Lemma 3 (*i*) and (*iii*), we obtain

$$ \begin{aligned} AS(t)y =& AS(t)t^{\mu-1}\Gamma(2-\mu)(g_{1-\mu} * (y - S(\cdot)y))(t) \\ &+ S(t)\Gamma(2-\mu)\gamma^{-1}t^{\mu-1}(S(t)y - y) - AS(t)\gamma^{-1}t^{\mu-1}\Gamma(2-\mu)(g_1 * S)(t)y. \end{aligned} $$

For $p = \infty$, by using Lemma 1, we have

$$ \begin{aligned} \|AS(t)y\| &\le Ct^{2(\mu-1)}\int_0^t (t-s)^{-\mu}s^{(1-\mu)\theta}\phi_\mu(s)ds + Ct^{-(1-\mu)(1-\theta)}\phi_\mu(t) + Ct^{2\mu-1}\|y\| \\ &\le Ct^{-(1-\mu)(1-\theta)}\|\phi_\mu\|_{L_*^\infty(0,1)} + Ct^{2\mu-1}\|y\|. \end{aligned} $$

Thus, we derive that:

$$ \omega(t) = t^{(1-\mu)(1-\theta)}\|AS(t)y\| \le C\|\phi_\mu\|_{L_*^\infty(0,1)} + Ct^{\mu-\theta+\mu\theta}\|y\|. $$

We also have $\omega \in L_*^\infty(0,1)$ for $\theta \in (0, \frac{\mu}{1-\mu} \wedge 1)$ and ${}^{S}\|y\|_{\theta,\infty} \le C(\|y\| + \|\phi_\mu\|_{L_*^\infty(0,1)})$. For $1 \le p < \infty$, we see that

$$ t^{(1-\mu)(1-\theta)}\|AS(t)y\| \le Ct^{-(1-\mu)(1+\theta)}\|(I^{1-\mu}(y - S(\cdot)y))(t)\| + C\phi_\mu(t) + Ct^{\mu-\theta+\mu\theta}\|y\|. $$

Set $u(t) = t^{-1-(1-\mu)(1+\theta)}, v(t) = t^{-1-(\mu+(1-\mu)\theta)p}, p = q$ and $\beta = 1$, then by Lemmas 4 and 5 and Minkowski inequality, we get

$$\|\omega\|_{L_*^p(0,1)} \le C(\|y\| + \|\phi_\mu\|_{L_*^p(0,1)}).$$

Therefore, for all $1 \le p \le \infty$, we deduce that

$$^{S}\|y\|_{\theta,p} \le C(\|y\| + \|\phi_\mu\|_{L_*^p(0,1)}).$$

The proof is completed. □

4. The Existence and Hölder Regularity

For convenience, we set

$$\psi(t) := \int_0^t S(t-\delta)f(\delta)d\delta, \quad t \ge 0.$$

Theorem 3. *Let $f \in L^p(0,T;X)$ for $p \in (1,\infty)$ and $u_0 \in X$. Then, problem (1) admits a unique mild solution.*

Proof. It is clear from Lemma 1 and Hölder's inequality that

$$\begin{aligned}\|u(t)\| \le & \|S(t)u_0\| + \left\|\int_0^t S(t-\delta)f(\delta)d\delta\right\| \\ \le & C\|u_0\| + C\int_0^t \|f(\delta)\|d\delta \le C\|u_0\| + Ct^{1-\frac{1}{p}}\|f\|_{L^p(0,T;X)}.\end{aligned} \tag{16}$$

From Lemma 1, we know that $S(\cdot)u_0 \in C([0,T];X)$. For $0 \le t < t+h \le T$, we have

$$\|S(t+h)x - S(t)x\| \le 2C\|x\|, \quad x \in X,\ t \in [0,T].$$

Moreover, Lemma 3 implies that

$$\|S(t+h)x - S(t)x\| = \left\|\int_t^{t+h} S'(\delta)x d\delta\right\| \le \frac{Ch}{t}\|x\|, \quad t \in (0,T].$$

Therefore, we get

$$\|S(t+h) - S(t)\|_{\mathcal{B}} \le \chi(h,t) := C\left(1 \wedge \frac{h}{t}\right). \tag{17}$$

Note from (17) that

$$\int_0^t (\chi(h,t-s))^{\frac{p}{p-1}}ds = \int_0^t (\chi(h,\tau))^{\frac{p}{p-1}}d\tau \le \int_0^\infty (\chi(h,\tau))^{\frac{p}{p-1}}d\tau = ph^{\frac{p}{p-1}}.$$

By Lemma 3, for $0 \le t < t+h \le T$ with small $h > 0$, we have

$$\begin{aligned}&\left\|\int_0^{t+h} S(t+h-\delta)f(\delta)d\delta - \int_0^t S(t-\delta)f(\delta)d\delta\right\| \\ \le & \left\|\int_0^t (S(t+h-\delta) - S(t-\delta))f(\delta)d\delta\right\| + \left\|\int_t^{t+h} S(t-\delta)f(\delta)d\delta\right\| \\ \le & \int_0^t \chi(h,t-\delta)\|f(\delta)\|d\delta + C\int_t^{t+h}\|f(\delta)\|d\delta \\ \le & \left(\int_0^t (\chi(h,t-\delta))^{\frac{p}{p-1}}d\delta\right)^{1-\frac{1}{p}}\|f\|_{L^p(0,T;X)} + C\|f\|_{L^p(0,T;X)}h^{1-\frac{1}{p}} \\ \le & C\|f\|_{L^p(0,T;X)}h^{1-\frac{1}{p}} \to 0, \quad \text{as } h \to 0.\end{aligned}$$

We also show that $u(t+h)-u(t) \to 0$ for $0 \le t < t+h \le T$ with small $h > 0$. Similarly, we can obtain $u(t)-u(t-h) \to 0$ for $0 \le t-h < t \le T$ with small $h > 0$. Therefore, we obtain the continuity of u and the uniqueness follows (16). This means that u is a unique mild solution to problem (1). □

A basic computation shows that the Riemann–Liouville fractional integral has the following property:

Lemma 7. *For $\mu \in (0, \frac{1}{2})$, $1 < p < \frac{1}{2\mu}$, let $k \in L^p(0,a)$ with $0 < a < \infty$, then $I^{1-\mu}k \in C^{\frac{p-1}{p} \wedge \mu}[0,a]$.*

Proof. We show the Hölder continuity of $I^{1-\mu}k$ for $k \in L^p(0,a)$. In fact, for any $0 \le t < t+h \le a$, it yields

$$|I^{1-\mu}k(t+h) - I^{1-\mu}k(t)| \le \int_t^{t+h} |g_{1-\mu}(t+h-\sigma)k(\sigma)|d\sigma + \int_0^t |(g_{1-\mu}(t+h-\sigma) - g_{1-\mu}(t-\sigma))k(\sigma)|d\sigma.$$

Since $g_{1-\mu}(\cdot) \in L^p(0,T;X)$ with $1 < p < \frac{1}{\mu}$, then

$$\int_t^{t+h} |g_{1-\mu}(t+h-\sigma)k(\sigma)|d\sigma \le C|k(\cdot)|_{L^p(0,a)} h^{1-\frac{1}{p}}.$$

By the inequality $a_2^\gamma - a_1^\gamma \le (a_2 - a_1)^\gamma$ for $0 \le a_1 < a_2 < \infty$ and $\gamma \in [0,1]$, we obtain

$$|(g_{1-\mu}(t+h-\sigma) - g_{1-\mu}(t-\sigma))k(\sigma)| \le g_{1-2\mu}(t-\sigma)|k(\sigma)|h^\mu,$$

integrating the above inequality a.e. $[0,a]$, and for $1 < p < \frac{1}{2\mu}$, we get

$$\int_0^t |(g_\mu(t+h-\sigma) - g_\mu(t-\sigma))k(\sigma)|d\sigma \le C|k(\cdot)|_{L^p(0,a)} h^\mu.$$

Therefore, we derive the conclusion as follows:

$$|I^{1-\mu}k(t+h) - I^{1-\mu}k(t)| \le C|k(\cdot)|_{L^p(0,a)} h^{1-\frac{1}{p}} + C|k(\cdot)|_{L^p(0,a)} h^\mu.$$

The proof is completed. □

Remark 4. *Note that in Lemma 4.1 in [24], for $\mu \in (0,1)$, $0 < \mu - \frac{1}{p} < 1$, let $k \in L^p(0,a)$ with $0 < a < \infty$, then $I^\mu k \in C^{\mu - \frac{1}{p}}[0,a]$.*

In the sequel, we show the Hölder regularity of the mild solution.

Theorem 4. *Let $u_0 \in X$, $f \in L^p(0,T;X)$ for $1 < p < \infty$. Then, for every $\varepsilon > 0$, $u \in C^{1-\frac{1}{p}}([\varepsilon,T];X)$ for $1 < p < \infty$. If, moreover, $u_0 \in X$ for $1 < p < \frac{1}{1-\mu} \wedge \frac{1}{2\mu}$, $0 < \mu < \frac{1}{2}$, then $u \in C^{\mu \wedge (1-\frac{1}{p})}([0,T];X)$, and if $u_0 \in {}^S D_A(\theta,p)$ for $p > \frac{1}{1-\mu}$, $\frac{1}{1-\mu} < q \le \frac{1}{(1-\mu)(1-\theta)} \wedge p$, $0 < \theta < \frac{\mu}{1-\mu} \wedge 1$, $0 < \mu < 1$, then $u \in C^{\mu \wedge (1-\mu-\frac{1}{q})}([0,T];X)$. Especially, if $u_0 \in D(A)$, then $u \in C^{1-\frac{1}{p}}([0,T];X)$ for $1 < p < \frac{1}{\mu}$, $0 < \mu < 1$. If $u_0 \in X$, $p > \frac{1}{\mu}$, then for any $\varepsilon > 0$, $u \in C^{\mu-\frac{1}{p}}([\varepsilon,T];X)$.*

Proof. The existence of the mild solution u to problem (1) follows Theorem 3 and it satisfies $u = S(\cdot)u_0 + \psi$. By Lemma 3, we know that $\|S'(t)\|_{\mathcal{B}} \le Ct^{-1}$ for all $t > 0$. Hence, for every

$\varepsilon > 0$, $u_0 \in X$, by the mean value theorem, $S(t)u_0$ is Lipschitz-continuous on $[\varepsilon, T]$. For $0 \le s < t \le T$, from Lemma 1, we obtain

$$\begin{aligned}\|\psi(t)-\psi(s)\| \le& \int_s^t \|S(t-\delta)f(\delta)\|d\delta + \int_0^s \|(S(t-\delta)-S(s-\delta))f(\delta)\|d\delta \\ \le& C(t-s)^{1-\frac{1}{p}}\|f\|_{L^p(0,T;X)} + \int_0^s \chi(t-s, s-\delta)\|f(\delta)\|d\delta \\ \le& C\|f\|_{L^p(0,T;X)}(t-s)^{1-\frac{1}{p}};\end{aligned}$$

this implies that $\psi \in C^{\frac{p-1}{p}}([0,T];X)$. Consequently, $u \in C^{\frac{p-1}{p}}([\varepsilon,T];X)$.

For $u_0 \in X$, by Remark 2, we know that

$$S(t)u_0 = (g_1 * AS)(t)u_0 + \gamma(g_{1-\mu} * AS)(t)u_0 + u_0.$$

Obviously, $A(g_1 * S)(t)u_0$ is Hölder-continuous with exponent μ on $[0,T]$. Let $k(t) = AS(t)u_0$. Clearly, $k(\cdot) \in L^p(0,T;X)$ for $p < \frac{1}{1-\mu}$. Lemma 7 shows that $(g_{1-\mu} * AS)(t)u_0 \in C^{\frac{p-1}{p}\wedge\mu}([0,T];X)$. Consequently, $S(\cdot)u_0 \in C^{\frac{p-1}{p}\wedge\mu}([0,T];X)$. Based on $\psi \in C^{\frac{p-1}{p}}([0,T];X)$, the second result is shown.

Due to $u_0 \in {}^S D_A(\theta,p)$, it yields that $k(\cdot) \in L^q(0,T;X)$ for $\frac{1}{1-\mu} < q \le \frac{1}{(1-\mu)(1-\theta)} \wedge p$. In fact, by Hölder inequality, let $r = (1-\mu)(1-\theta)$; we derive that

$$\begin{aligned}\int_0^t |k(s)|^q ds =& \int_0^t s^{-rq}\omega(s)^q ds \\ \le& \left(\int_0^t (s^{-rq+\frac{q}{p}})^{\frac{p}{p-q}} ds\right)^{\frac{p-q}{p}} \left(\int_0^t \omega(s)^p s^{-1} ds\right)^{\frac{q}{p}} \\ \le& C\|u_0\|_{\theta,p}\left(\int_0^t s^{\frac{q-rpq}{p-q}} ds\right)^{\frac{p-q}{p}},\end{aligned}$$

which means that $k(\cdot) \in L^q(0,T;X)$. By Remark 4, one can check that $(g_{1-\mu} * k)(t) \in C^{1-\mu-\frac{1}{q}}([0,T];X)$, thus $u \in C^{\mu\wedge(1-\mu-\frac{1}{q})}([0,T];X)$.

In particular, for $u_0 \in D(A)$, it suffices to check that $(g_{1-\mu} * S)(t)Au_0 \in C^{\frac{p-1}{p}}([0,T];X)$ for $1 < p < \frac{1}{\mu}$. In fact, for $0 \le t < t+h \le T$, we have

$$\begin{aligned}(g_{1-\mu} * S)(t+h)Au_0 - (g_{1-\mu} * S)(t)Au_0 =& \int_t^{t+h} S(t+h-\sigma)g_{1-\mu}(\sigma)Au_0 d\sigma \\ &+ \int_0^t (S(t+h-\sigma) - S(t-\sigma))g_{1-\mu}(\sigma)Au_0 d\sigma \\ =& I_1 + I_2.\end{aligned}$$

Since $g_{1-\mu}(\cdot) \in L^p(0,T)$ for $1 < p < \frac{1}{\mu}$. By Lemma 1 and Hölder inequality, we have

$$\|I_1\| \le C\|Au_0\| \int_t^{t+h} g_{1-\mu}(\sigma)d\sigma \le C\|Au_0\|\|g_{1-\mu}(\cdot)\|_{L^p(0,T)} h^{1-\frac{1}{p}}.$$

As for I_2, by Hölder inequality, we get

$$\begin{aligned}\|I_2\| \le& C\|Au_0\| \int_0^t \chi(h, t-\sigma)g_{1-\mu}(\sigma)d\sigma \\ \le& C\|Au_0\|\|g_{1-\mu}(\cdot)\|_{L^p(0,T)}\left(\int_0^t (\chi(h,\tau))^{\frac{p}{p-1}} d\tau\right)^{\frac{p-1}{p}} \\ \le& C\|Au_0\|\|g_{1-\mu}(\cdot)\|_{L^p(0,T)} h^{1-\frac{1}{p}}.\end{aligned}$$

Combined with the above arguments, the fourth conclusion follows.

For $u_0 \in X$, $p > \frac{1}{\mu}$, from the proof of the first result, we know that $S(t)u_0$ is Lipschitz-continuous on $[\varepsilon, T]$. Similar to the proof of Theorem 3, we obtain $\psi(\cdot) \in C^{\mu-\frac{1}{p}}([\varepsilon, T]; X)$. The proof is completed. □

The following means that the mild solution is a classical solution.

Theorem 5. *Let $f \in C^1([0, T]; X)$, $u_0 \in X$. Then, the following descriptions hold:*

(i) $\psi(\cdot) \in D(A)$, $0 \le t \le T$, *and* $\psi \in C([0, T]; D(A))$;

(ii) u *is a classical solution of (1).*

Proof. By Lemma 1, it follows that $\psi(\cdot) \in D(A)$ for all $t \in [0, T]$. In fact, one can derive that

$$\|A\psi(t)\| \le \int_0^t \|AS(t-\delta)f(\delta)\|d\delta \le C\|f\|_{C^1([0,T];X)}.$$

Since the integral

$$\mu(1-\mu)\int_0^s \int_{s-\delta}^{t-\delta} \tau^{\mu-2} d\tau d\delta = s^\mu - t^\mu + (t-s)^\mu$$

holds for all $0 \le s < t \le T$, we see that

$$\begin{aligned}\|A\psi(t) - A\psi(s)\| \le &\int_s^t \|AS(t-\delta)f(\delta)\|d\delta + \int_0^s \|(AS(t-\delta) - AS(s-\delta))f(\delta)\|d\delta \\ \le &C\int_s^t (t-\delta)^{\mu-1}\|f(\delta)\|d\delta + C\int_0^s \int_{s-\delta}^{t-\delta} \tau^{\mu-2}\|f(\delta)\|d\tau d\delta \\ \le &C\|f\|_{C^1([0,T];X)}(t-s)^\mu,\end{aligned}$$

which shows that $\psi \in C([0, T]; D(A))$.

From the assumptions and Theorem 3, we know that u is the mild solution of problem (1) and obviously $u \in C([0, T]; D(A))$ due to $S(t)u_0 \in D(A)$ for $u_0 \in X, 0 \le t \le T$, and $S(\cdot)u_0 \in C^1((0, T]; X)$ from Lemma 3, and $u(0) = u_0$. Corollary 1 and Remark 1 show that $D_t^\mu S(t)u_0 \in C((0, T]; X)$ and $D_t^\mu S(t)u_0 \in D(A)$ for $u_0 \in X, 0 < t \le T$. This means that $S(t)u_0$ is the classical solution of the homogeneous equation by Remark 2. Consequently, it suffices to check that the remaining is a classical solution of the nonhomogeneous equation.

From $f \in C^1([0, T]; X)$, we have $(S * f')(t) \in C([0, T]; X)$. In fact, it yields

$$\begin{aligned}\|(S*f')(t) - (S*f')(s)\| \le &\int_s^t \|S(t-\delta)f'(\delta)\|d\delta + \int_0^s \|(S(t-\delta) - S(s-\delta))f'(\delta)\|d\delta \\ \le &C\|f'\|_{C([0,T];X)}(t-s) + C\int_0^s \int_{s-\delta}^{t-\delta} \tau^{-1} d\tau d\delta \|f'\|_{C([0,T];X)} \\ \le &C\|f'\|_{C([0,T];X)}(t-s) + C\|f'\|_{C([0,T];X)}\sqrt{t-s},\end{aligned}$$

where we have used

$$\int_0^s \int_{s-\delta}^{t-\delta} \tau^{-1} d\tau d\delta = \int_0^s (\log(t-\delta) - \log(s-\delta))d\delta \le \frac{\sqrt{(t-s)s}}{2}, \ \ 0 < s < t,$$

and the inequality

$$\log a_2 - \log a_1 < \frac{a_2 - a_1}{\sqrt{a_1 a_2}}, \ \forall\, a_2 > a_1 > 0.$$

Hence, $\psi'(t) = (S * f')(t) + S(t)f(0) \in C([0, T]; X)$. Since $A\psi \in C([0, T]; X)$, by Remark 2, we also have $(g_{1-\mu} * AS * f)(t) = \gamma^{-1}[-(g_1 * AS * f)(t) + (S * f)(t) - 1 * f(t)]$ and

$$\frac{d}{dt}(g_{1-\mu} * AS * f)(t) = \gamma^{-1}(-A\psi(t) + \psi'(t) - f(t)).$$

The definition of the fractional derivative shows that

$$D_t^{\mu} A\psi(t) = \gamma^{-1}(-A\psi(t) + \psi'(t) - f(t)) \in C([0,T];X).$$

Therefore, we have

$$\psi'(t) = A\psi(t) + \gamma D_t^{\mu} A\psi(t) + f(t),$$

combined with $\psi(0) = 0$. Thus, u is a classical solution of (1). □

The following corollary is immediate.

Corollary 2. *Let $f \in C^1([0,T];X)$, $u_0 \equiv 0$. Then, u is a strict solution of (1).*

Lemma 8. *Let $f \in C([0,T];X)$. Then $\psi(\cdot) \in C^{\mu}([0,T];D(A))$.*

Proof. By Lemma 1, we have $\|A\psi\| \le C\|f\|_{C([0,T];X)}$ for $0 \le t \le T$. For $0 \le s < t \le T$, from Lemma 3, we see that

$$\begin{aligned}
\|A\psi(t) - A\psi(s)\| &\le \int_0^s \|[AS(t-\delta) - AS(s-\delta)]f(\delta)\|d\delta + \int_s^t \|AS(t-\delta)f(\delta)\|d\delta \\
&\le \int_0^s \int_{s-\delta}^{t-\delta} \|AS'(\tau)f(\delta)\|d\tau d\delta + C\int_s^t (t-\delta)^{\mu-1}\|f(\delta)\|d\delta \\
&\le C\|f\|_{C([0,T];X)}\left(\int_0^s \int_{s-\delta}^{t-\delta} \tau^{\mu-2} d\tau d\delta + (t-s)^{\mu}\right) \\
&\le C\|f\|_{C([0,T];X)}(t-s)^{\mu}.
\end{aligned}$$

Thus, $\psi \in C^{\mu}([0,T];D(A))$. □

In the sequel, we obtain the Hölder regularity of the classical solution.

Theorem 6. *Let $u_0 \in D(A)$, $f \in C^{\vartheta}([0,T];D(A))$ for $\vartheta \in (0,1)$. Then, there exists a classical solution u of (1) satisfying $u \in C^{\mu\wedge(1-\mu)}([0,T];X)$ for $\mu \in (0,1)$. Moreover, there holds*

$$\|u\|_{C^{\mu\wedge(1-\mu)}([0,T];X)} \le C(\|f\|_{C^{\vartheta}([0,T];D(A))} + \|Au_0\|).$$

Proof. Obviously, Theorem 3 implies that there is a unique mild solution u. Lemma 1 and Corollary 1 show that $S(\cdot)v \in C([0,\infty);X)$ and $D_t^{\mu} S(t)v \in C((0,\infty);X)$ for any $v \in X$. Hence, it suffices to check that $\psi(\cdot) \in C^1((0,T];X)$. Note that

$$\begin{aligned}
\int_s^t S'(t-\tau)f(\tau)d\tau &= \int_s^t S'(t-\tau)(f(\tau) - f(t))d\tau + \int_s^t S'(t-\tau)f(t)d\tau \\
&= \int_s^t S'(t-\tau)(f(\tau) - f(t))d\tau + \int_0^{t-s} S'(\tau)f(t)d\tau \\
&= \int_s^t S'(t-\tau)(f(\tau) - f(t))d\tau + \int_0^{t-s} AS(\tau)f(t)d\tau \\
&\quad + \int_0^{t-s} AD_\tau^{\mu} S(\tau)f(t)d\tau;
\end{aligned}$$

for any $\varepsilon > 0$, $\varepsilon \le s < t \le T$, it follows from Remark 2 that

$$
\begin{aligned}
&\|(S' * f)(t) - (S' * f)(s)\| \\
\le& \int_s^t \|S'(t-\tau) f(\tau)\| d\tau + \int_0^s \|(S'(t-\tau) - S'(s-\tau)) f(\tau)\| d\tau \\
\le& \int_s^t \|S'(t-\tau)(f(\tau) - f(t))\| d\tau + \int_0^{t-s} \|AS(\tau) f(t)\| d\tau \\
&+ \int_0^{t-s} \|D_\tau^\mu S(\tau) A f(t)\| d\tau + \int_0^s \|(S'(t-\tau) - S'(s-\tau)) f(\tau)\| d\tau \\
\le& C\|f\|_{C^\vartheta([0,T];X)} (t-s)^\vartheta + C(t-s)^\mu \|f\|_{C^\vartheta([0,T];D(A))} \\
&+ C(t-s)^{1-\mu} \|f\|_{C^\vartheta([0,T];D(A))} + \int_0^s \|(S'(t-\tau) - S'(s-\tau)) f(\tau)\| d\tau.
\end{aligned}
$$

From $S'(t)x = AS(t)x + AD_t^\mu S(t)x$, we have

$$
\begin{aligned}
&\int_0^s \|(S'(t-\tau) - S'(s-\tau)) f(\tau)\| d\tau \\
\le& \int_0^s \|(AS(t-\tau) - AS(s-\tau)) f(\tau)\| d\tau + \int_0^s \|(D_t^\mu S(t-\tau) - D_s^\mu S(s-\tau)) A f(\tau)\| d\tau \\
\le& \int_0^s \int_{s-\tau}^{t-\tau} \|AS'(\sigma) f(\tau)\| d\sigma d\tau + C(t-s)^{-\mu} \int_0^s \|A f(\tau)\| d\tau \\
\le& C \int_0^s \int_{s-\tau}^{t-\tau} \sigma^{\mu-2} d\sigma d\tau \|f\|_{C^\vartheta([0,T];D(A))} + C\|f\|_{C^\vartheta([0,T];D(A))} (t-s)^{-\mu}.
\end{aligned}
$$

Hence, $S' * f \in C([\varepsilon, T]; X)$ for any $f \in C^\vartheta([0,T]; D(A))$. Therefore, by $\psi'(t) = (S' * f)(t) + f(t)$, we obtain $\psi'(\cdot) \in C([\varepsilon, T]; X)$ for any $\varepsilon > 0$. Thus, u is the classical solution of (1).

Additionally, by Lemma 8, we get that $\psi(\cdot) \in C^\mu([0,T]; D(A))$. From Lemma 3 (*iii*), we next check $S(t)u_0 \in C^{\mu \wedge (1-\mu)}([0,T]; X)$. In fact, for any $0 \le s < t \le T$, by Remark 2 and Corollary 1, we see that

$$
\begin{aligned}
\|S(t)u_0 - S(s)u_0\| \le& \int_s^t \|S'(\tau) u_0\| d\tau \\
\le& \int_s^t \|AS(\tau) u_0\| d\tau + C \int_s^t \|D_\tau^\mu S(\tau) A u_0\| d\tau \\
\le& C(t-s)^{\mu \wedge (1-\mu)} \|A u_0\|.
\end{aligned}
$$

Thus, $S(\cdot)u_0 \in C^{\mu \wedge (1-\mu)}([0,T]; X)$ for $u_0 \in D(A)$. Then, $u \in C^{\mu \wedge (1-\mu)}([0,T]; X)$. The inequality easily follows the above arguments. The proof is completed. □

Theorem 7. *Let $f \in C^1([0,T]; X)$ and let $\frac{1}{1-\mu} < p \le \infty$, $0 < \mu < 1$, $0 < \theta < \frac{\mu}{1-\mu} \wedge 1$. If $u_0 \in {}^S D_A(\theta, p)$, for $\frac{1}{1-\mu} < q \le \frac{1}{(1-\mu)(1-\theta)} \wedge p$, then the classical solution $u \in C^{\mu \wedge (1-\mu-\frac{1}{q})}([0,T]; X)$. If, moreover, $u_0 \in D(A)$, then $Au \in C^{\mu \wedge (1-\mu-\frac{1}{q})}([0,T]; X)$. In particular, if $u_0 \equiv 0$, $f(0) \in {}^S D_A(\theta, p)$, then the strict solution $u, Au \in C^{\mu \wedge (1-\mu)}([0,T]; X)$ and $u' \in C^{\frac{1}{2} \wedge \mu \wedge (1-\mu-\frac{1}{q})}([0,T]; X)$.*

Proof. From Theorem 5, we know that the mild solution u is a classical one for $f \in C^1([0,T]; X)$. Using the proof of Theorem 4, we get that $S(\cdot)u_0 \in C^{\mu \wedge (1-\mu-\frac{1}{q})}([0,T]; X)$. Moreover, by the proof of Theorem 3, we see from Hölder inequality that

$$
\begin{aligned}
\|\psi(t+h) - \psi(t)\| =& \|(S * f)(t+h) - (S * f)(t)\| \\
\le& C\left(\int_0^t (\chi(h, t-\delta))^{\frac{p}{p-1}} d\delta\right)^{1-\frac{1}{p}} \|f\|_{C^1(0,T;X)} + C\|f\|_{C^1(0,T;X)} h \\
\le& C\|f\|_{C^1(0,T;X)} h,
\end{aligned}
$$

for $h \in (0,T)$. Consequently, $u \in C^{\mu\wedge(1-\mu-\frac{1}{q})}([0,T];X)$.

By using the proof of Lemma 8, it follows that $A\psi \in C^{\mu}([0,T];X)$. By $u_0 \in D(A)$ and Lemma 3, we have

$$AS(t)u_0 = (g_1 * AS)(t)Au_0 + \gamma(g_{1-\mu} * AS)(t)Au_0 + Au_0.$$

From the proof of the third conclusion of Theorem 4, we know that $AS(t)Au_0 \in L^q(0,T;X)$ for $\frac{1}{1-\mu} < q \leq \frac{1}{(1-\mu)(1-\theta)} \wedge p$. Thus, by Remark 4, it follows that $(g_{1-\mu} * AS)(t)Au_0 \in C^{1-\mu-\frac{1}{q}}([0,T];X)$. Furthermore, $(g_1 * AS)(t)Au_0$ is Hölder-continuous with exponent μ. Hence, $AS(\cdot)u_0 \in C^{\mu\wedge(1-\mu-\frac{1}{q})}([0,T];X)$. Therefore, $Au \in C^{\mu\wedge(1-\mu-\frac{1}{q})}([0,T];X)$.

Let us check the case $u_0 \equiv 0$. In fact, Theorem 6 and the mentioned arguments in the current proof imply that $u, Au \in C^{\mu\wedge(1-\mu)}([0,T];X)$. Since $u'(t) = (f' * S)(t) + S(t)f(0)$, from the proof of Theorem 5, we obtain $(f' * S)(t) \in C^{1/2}([0,T];X)$. For $f(0) \in {}^{S}D_A(\theta,p)$, by Remark 2 and the proof of Theorem 4, we get $S(\cdot)f(0) \in C^{\mu\wedge(1-\mu-\frac{1}{q})}([0,T];X)$. Hence, $u' \in C^{\frac{1}{2}\wedge\mu\wedge(1-\mu-\frac{1}{q})}([0,T];X)$. The proof is completed. □

5. Applications

Let $N \geq 2$. We consider the following fractional partial differential equation:

$$\begin{cases} \partial_t u - (1+\partial_t^{\mu})\Delta u = f(t,x), \quad t > 0, \ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{cases} \tag{18}$$

where ∂_t^{μ} is the Caputo fractional partial derivative of order $\mu \in (0,1)$, Δ is the Laplace operator, and f takes the $L^p(\mathbb{R}^N)$ data.

Note from ([37], Theorem 2.3.2) that the Laplace operator Δ with maximal domain $D(\Delta) = \{u \in L^p(\mathbb{R}^N) : \ \Delta u \in L^p(\mathbb{R}^N)\}$ generates a bounded analytic semigroup of the spectral angle that is less than or equal to $\pi/2$ on $L^p(\mathbb{R}^N)$ with $1 < p < +\infty$. We set $A = \Delta$. By He et al. [23], the problem (18) can be reformulated as problem (1). It follows that the analytic resolvent $S(t)$ generated by A is defined in (2). For the $L^p(\mathbb{R}^N)$ data of u_0, we know that there exists a unique mild solution of (18) from Theorem 3. Due to the interpolation

$$(W^{2,p}(\mathbb{R}^N), L^p(\mathbb{R}^N))_{\theta,q} = B^{2\theta}_{p,q}(\mathbb{R}^N), \ \ 1 \leq p,q \leq \infty, \ 0 < \theta < 1,$$

and if further $u_0 \in B^{2\theta}_{p,q}(\mathbb{R}^N)$ for $\frac{1}{1-\mu} < p \leq \frac{1}{(1-\mu)(1-\theta)}$, $1 \leq q \leq \infty$, $0 < \theta < \frac{\mu}{1-\mu} \wedge 1$, $0 < \mu < 1$, then $u \in C^{(1-\mu-\frac{1}{p})\wedge\mu}([0,T];L^p(\mathbb{R}^N))$ from Theorem 4. In addition, by Theorem 7, if $f \in C^1([0,T];L^p(\mathbb{R}^N))$, then the mild solution will be a classical solution which possesses the Hölder continuity with the same exponent $(1-\mu-\frac{1}{p}) \wedge \mu$ and the estimate

$$\begin{aligned} \|Au\|_{C^{\mu\wedge(1-\mu)}([0,T];L^p(\mathbb{R}^N))} + \|u\|_{C^{(1-\mu-\frac{1}{p})\wedge\mu}([0,T];L^p(\mathbb{R}^N))} \\ \leq C(\|f\|_{C^1([0,T];L^p(\mathbb{R}^N))} + \|u_0\|_{B^{2\theta}_{p,q}(\mathbb{R}^N)}). \end{aligned}$$

For another consideration of the following initial-boundary value problem,

$$\begin{cases} \partial_t u = (1+\gamma\partial_t^{\mu})\Delta u + f(t,x), \quad \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0, \quad \text{in } \partial\Omega \times (0,T), \\ u(0,x) = u_0(x), \quad \text{in } \Omega, \end{cases} \tag{19}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain and boundary $\partial\Omega$ is C^2, $\gamma > 0$. It is clear that A_p is a realization of the Laplace operator $A = \Delta$ on $L^p(\Omega)$ $(1 < p < \infty)$ under Neumann type boundary condition $\frac{\partial u}{\partial \nu} = 0$. It is known that $-A_p$ generates an analytic semigroup on $L^p(\Omega)$ of the spectral angle that is less than $\pi/2$; by (4) and the definition of $S(t)$, the problem (19) can be reformulated as problem (1). Then, by a direct application of Theorem 6, we know that for any $u_0 \in D(A_p)$, $f \in C^{\vartheta}([0,T];D(A_p))$ for $0 < \vartheta < 1$, problem (19) possesses a unique classical solution in the function space $C^{\mu\wedge(1-\mu)}([0,T];L^p(\Omega)) \cap C^1((0,T];L^p(\Omega))$ with the estimate

$$\|u\|_{C^{\mu\wedge(1-\mu)}([0,T];L^p(\Omega))} \leq C(\|f\|_{C^{\vartheta}([0,T];D(A_p))} + \|A_p u_0\|).$$

In particular, we consider an initial-boundary value problem with a Dirichlet boundary value condition as follows:

$$\begin{cases} \partial_t u(t,x) = (1+\gamma\partial_t^{\mu})u_{xx} + f(t,x), \quad t \in [0,1],\ x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in [0,1], \\ u(0,x) = 0, \quad x \in [0,\pi], \end{cases} \tag{20}$$

where $\gamma > 0$ and $f(t,x) = \left(2t + t^2 + \frac{2\gamma}{\Gamma(3-\mu)}t^{2-\mu}\right)\sin x$, $t \in [0,1]$.

Let $A = \partial_{xx}$ and let us consider the spectral problem on $L^2[0,\pi]$, i.e., $Ae_k = -\lambda_k e_k$, $e_k(0) = e_k(\pi) = 0$. It is known that $\lambda_k = k^2$ is the eigenvalue of $-A$ corresponding to the eigenfunction $e_k(x) = \sqrt{2/\pi}\sin(kx)$. According to the discussion in [14,19], the problem (20) can be rewritten as the sum of the Fourier series for problem

$$u_k'(t) + \lambda_k u(t) + \gamma\lambda_k D_t^{\mu} u_k(t) = f_k(t), \quad t > 0; \quad u_k(0) = 0.$$

It follows that the analytic resolvent $S(t)$ is given by

$$S(t)u = \sum_{k=1}^{\infty} S_k(t)(u,e_k)e_k, \quad u \in L^2[0,\pi],$$

where

$$S_k(t) = \frac{1}{\pi}\int_0^{\infty} e^{-zt}\frac{\gamma\lambda_k z^{\alpha}\sin(\mu\pi)}{(-z+\lambda_k\gamma\cos(\mu\pi)+\lambda_k)^2 + (\lambda_k\gamma z^{\alpha}\sin(\mu\pi))^2}dz.$$

Since $f \in C^1([0,1];L^2[0,\pi])$, the Corollary 2 shows that problem (20) has a strict solution. In fact, one can check that $u(t,x) = t^2\sin(x)$ meets all equations in (20) and u is indeed the strict solution. Also, in view of $f(0) = 0$, it is clear that the strict solution $u, u_{xx} \in C^{\mu\wedge(1-\mu)}([0,T];L^2[0,\pi])$ and $u' \in C^{\frac{1}{2}\wedge\mu\wedge(1-\mu-\frac{1}{q})}([0,T];L^2[0,\pi])$ for a suitable number $q > \frac{1}{1-\mu}$ by Theorem 7.

6. Conclusions

In this paper, we considered an abstract fractional differential equation that involves the fractional Rayleigh–Stokes problem in the abstract version. First of all, we proved that the interpolation space constructed by the analytic resolvent is isometric isomorphic to a classical real interpolation space. Secondly, we obtained the existence and uniqueness of the mild solution. And then, by using some properties of the analytic resolvent and the interpolation space, we established the Hölder regularities of the mild solution. In addition, we showed that the mild solution becomes a classical solution. Finally, based on the efficient conditions and the interpolation space, we obtained the Hölder regularities of the classical solution and the strict solution. The obtained properties of this type of fractional differential equation will be further made clear to understand the structure of solutions to the fractional Rayleigh–Stokes problem.

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