fractal and fractional

Article

Nonlinear Integral Inequalities Involving Tempered Ψ-Hilfer Fractional Integral and Fractional Equations with Tempered Ψ-Caputo Fractional Derivative

Milan Medveď [1], Michal Pospíšil [1,2,*] and Eva Brestovanská [3]

[1] Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská Dolina, 842 48 Bratislava, Slovakia; milan.medved@fmph.uniba.sk
[2] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
[3] Department of International Management, Faculty of Management, Comenius University in Bratislava, Odbojárov 10, 831 04 Bratislava, Slovakia; eva.brestovanska@fm.uniba.sk
* Correspondence: michal.pospisil@fmph.uniba.sk or michal.pospisil@mat.savba.sk

Abstract: In this paper, the nonlinear version of the Henry–Gronwall integral inequality with the tempered Ψ-Hilfer fractional integral is proved. The particular cases, including the linear one and the nonlinear integral inequality of this type with multiple tempered Ψ-Hilfer fractional integrals, are presented as corollaries. To illustrate the results, the problem of the nonexistence of blowing-up solutions of initial value problems for fractional differential equations with tempered Ψ-Caputo fractional derivative of order $0 < \alpha < 1$, where the right side may depend on time, the solution, or its tempered Ψ-Caputo fractional derivative of lower order, is investigated. As another application of the integral inequalities, sufficient conditions for the Ψ-exponential stability of trivial solutions are proven for these kinds of differential equations.

Keywords: tempered Ψ-Hilfer fractional integral; generalized Henry–Gronwall inequality; tempered Ψ-Caputo fractional derivative; blowing-up solution; stability

MSC: 34A08; 34A40; 26A33; 26D10; 26D15

check for updates

Citation: Medveď, M.; Pospíšil, M.; Brestovanská, E. Nonlinear Integral Inequalities Involving Tempered Ψ-Hilfer Fractional Integral and Fractional Equations with Tempered Ψ-Caputo Fractional Derivative. *Fractal Fract.* **2023**, 7, 611. https://doi.org/10.3390/fractalfract7080611

Academic Editors: Sotiris K. Ntouyas, Ivanka Stamova, Jessada Tariboon and Bashir Ahmad

Received: 11 July 2023
Revised: 31 July 2023
Accepted: 6 August 2023
Published: 8 August 2023

1. Introduction and Preliminaries

Initial value problems for some semilinear parabolic differential equations can be studied in the framework of the theory of abstract evolution equations. Mild solutions of such equations are given by Volterra integral equations with weakly singular kernels, studied by Dan Henry in the famous monograph [1]. The basic tools for an analysis of these equations are integral inequalities with weakly singular kernels. Henry's lemmas ([1], Lemmas 7.1.1 and 7.1.2) relating to linear integral inequalities of this type are well known. D. Henry proved these lemmas by an iteration argument. However, this method is not applicable in nonlinear cases. In the papers [2,3], a new approach to an analysis, also suitable for the study of nonlinear integral inequalities with weakly singular kernels, was proposed. This method is useful also in the theory of parabolic PDEs, abstract evolution differential equations (see, e.g., [3–6]), and fractional differential equations (see, e.g., [7,8]). In the present paper, this method to nonlinear integral inequalities with the so-called tempered Ψ-Hilfer fractional integral is applied. It is a generalization of the Ψ-Hilfer fractional integral, which was introduced in [9].

Recently, fractional derivatives were used in the synchronization of fractional-order systems in cryptography and image encryption [10,11].

For the convenience of a reader, we recall the following definitions of the tempered Ψ-Hilfer fractional integral and a corresponding Caputo-like fractional derivative from [12].

Definition 1. *Let $\alpha > 0$, $\lambda \geq 0$, the real function $x(t)$ be continuous on $[a,b]$, and $\Psi \in C^1[a,b]$ be an increasing function. Then, the tempered Ψ-Hilfer fractional integral of order $\alpha > 0$ is defined by*

$$I_a^{\alpha,\lambda,\Psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} \, e^{-\lambda(\Psi(t)-\Psi(s))} \, \Psi'(s) \, x(s) \, ds$$

for $t \in [a,b]$, where $\Gamma(\cdot)$ is the Euler gamma function.

We recall that $I_a^{\alpha,\lambda,\Psi} x(t) = e^{-\lambda\Psi(t)} \, I_a^{\alpha,\Psi}(e^{\lambda\Psi(t)} \, x(t))$, where $I_a^{\alpha,\Psi}$ is the Ψ-Riemann–Liouville fractional integral (see [12]). For the purpose of this paper, we use the abbreviation

$$K(t,s,\alpha,\lambda) = (\Psi(t) - \Psi(s))^{\alpha-1} \, e^{-\lambda(\Psi(t)-\Psi(s))} \, \Psi'(s).$$

Definition 2. *Let $\Psi \in C^n[a,b]$ be such that $\Psi'(t) > 0$ for all $t \in [a,b]$. For $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\lambda \geq 0$, tempered Ψ-Caputo fractional derivative of order α is defined by*

$$^{C}D_a^{\alpha,\lambda,\Psi} x(t) = \frac{e^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{n-\alpha-1} \, \Psi'(s) \, x_{\lambda,\Psi}^{[n]}(s) \, ds \tag{1}$$

for $t \in [a,b]$, where

$$x_{\lambda,\Psi}^{[n]}(t) = \left[\frac{1}{\Psi'(t)} \frac{d}{dt}\right]^n \left(e^{\lambda\Psi(t)} \, x(t)\right).$$

This paper deals with the following Henry–Gronwall-type nonlinear integral inequality

$$u(t) \leq a(t) + b(t) \int_a^t K(t,s,\alpha,\lambda) \, F(s) \, \omega(u(s)) \, ds, \quad t \in [a,T). \tag{2}$$

To the best of our knowledge, at the present time, an inequality of the form (2) has not been studied in any known literature or published paper. Therefore, also our results on blowing-up solutions or stability are improving the existing knowledge and considered the brand-new.

For comparison, in [13], inequality (2) was proven with $\lambda = 0$, $F(t) \equiv 1$, $\omega(u) \equiv u$ (see [13], Theorem 3). In the papers [2,3], this inequality with $\Psi(t) \equiv t$ and $\lambda = 0$ was investigated. In paper [14], various forms of linear Henry–Gronwall-type integral inequalities, (2) with $\omega(u) \equiv u$, and their generalizations were proven. In [15], Gronwall-type integral inequality with the tempered Ψ-Hilfer fractional integral was stated (see Remark 1 for a comparison of this result with our version of linear integral inequality). Furthermore, the limit and other properties were investigated in [16].

In the following section, we prove an integral inequality of the Henry–Gronwall type for the right side depending on the tempered Ψ-Hilfer fractional integral, and its corollaries for particular cases of functions appearing in the inequality, including the linear integral inequality, and for multiple integrals of this type on the right-hand side. In Section 3, we present applications of the results of Section 2. More precisely, we prove new sufficient conditions for the nonexistence of blowing-up solutions of initial value problems corresponding to fractional differential equations with the tempered Ψ-Caputo fractional derivative and the right-side depending on the solution of this problem, its tempered Ψ-Caputo fractional derivatives of lower order, or its tempered Ψ-Hilfer fractional integrals of any positive order. In the second part of Section 3, we derive new sufficient conditions for the Ψ-exponential stability of the trivial solutions of fractional differential equations with the tempered Ψ-Caputo fractional derivative of order $0 < \alpha < 1$, and various kinds of nonlinearities. Finally, we summarize our results and sketch possible directions of future research in Section 4.

2. Integral Inequalities

In this section, we study the integral inequality (2) and provide some of its corollaries.

First, we formulate our main result on a nonlinear version of the Henry–Gronwall inequality.

Theorem 1. *Let $\lambda > 0$, $\alpha \in (0,1)$, $p > 1$, $p(\alpha-1)+1 > 0$, $q = \frac{p}{p-1}$, $a(t)$, $b(t)$, and $F(t)$ be non-negative, continuous functions on the interval $[a,T)$, where $a < T \le \infty$, and $\Psi \in C^1[a,\infty)$ be a non-negative, increasing function with $\Psi'(t) > 0$ for all $t \in (a,\infty)$, $\lim_{t\to\infty}\Psi(t) = \infty$. Let $\omega\colon [0,\infty) \to [0,\infty)$ be a continuous, positive, non-decreasing function and $u(t)$ be a non-negative continuous function on $[a,T)$, satisfying the inequality (2). Then*

$$u(t) \le \left[\Omega^{-1}\left(\Omega(A(t)) + B(t)\int_a^t F(s)^q\,\Psi'(s)\,ds\right)\right]^{1/q} \tag{3}$$

for all $t \in [a,T)$, where

$$\begin{gathered} A(t) = 2^{q-1}\sup_{a\le s\le t} a(s)^q, \qquad B(t) = 2^{q-1}M_p^q \sup_{a\le s\le t} b(s)^q, \\ M_p = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\lambda)^{p(\alpha-1)+1}}\right)^{1/p}, \end{gathered} \tag{4}$$

$\Omega(v) = \int_{v_0}^{v}\frac{d\sigma}{[\omega(\sigma^{1/q})]^q}$ for $v_0, v > 0$, and Ω^{-1} is the inverse of Ω.

Proof. Since $\frac{1}{p}+\frac{1}{q} = 1$, we can write $\Psi'(t)$ as the product $\Psi'(t) = \Psi'(t)^{1/p}\,\Psi'(t)^{1/q}$ and using the Hölder inequality, from (2), we obtain

$$\begin{aligned} &\int_a^t K(t,s,\alpha,\lambda)\,F(s)\,\omega(u(s))\,ds \\ &\le \left(\int_a^t [\Psi(t)-\Psi(s)]^{p(\alpha-1)}\,\mathrm{e}^{-p\lambda[\Psi(t)-\Psi(s)]}\,\Psi'(s)\,ds\right)^{1/p} \\ &\quad\times\left(\int_a^t F(s)^q\,\omega(u(s))^q\,\Psi'(s)\,ds\right)^{1/q}. \end{aligned} \tag{5}$$

Using the transformation $\sigma = \Psi(t)-\Psi(s)$, we obtain

$$\begin{aligned} &\left(\int_a^t [\Psi(t)-\Psi(s)]^{p(\alpha-1)}\,\mathrm{e}^{-p\lambda[\Psi(t)-\Psi(s)]}\,\Psi'(s)\,ds\right)^{1/p} \\ &= \left(\int_0^{\Psi(t)-\Psi(a)} \sigma^{p(\alpha-1)}\,\mathrm{e}^{-p\lambda\sigma}\,d\sigma\right)^{1/p} \le \left(\int_0^{\infty}\sigma^{p(\alpha-1)}\,\mathrm{e}^{-p\lambda\sigma}\,d\sigma\right)^{1/p} = M_p. \end{aligned} \tag{6}$$

Inequalities (2) and (6) yield

$$u(t) \le a(t) + M_p\,b(t)\left(\int_a^t F(s)^q\,[\omega(u(s))]^q\,\Psi'(s)\,ds\right)^{1/q}. \tag{7}$$

Since $(a_1+a_2)^q \le 2^{q-1}(a_1^q + a_2^q)$ for any $a_1, a_2 \ge 0$, from inequality (7), we obtain

$$u(t)^q \le \tilde{a}(t) + 2^{q-1}M_p^q\,b(t)^q\int_a^t F(s)^q\,[\omega(u(s))]^q\,\Psi'(s)\,ds, \tag{8}$$

where $\tilde{a}(t) = 2^{q-1}a(t)^q$.

If $v(t) = u(t)^q$, then the inequality (8) can be written in the form

$$v(t) \le \tilde{a}(t) + 2^{q-1}M_p^q\,b(t)^q\int_a^t F(s)^q\,[\omega(v(s)^{1/q})]^q\,\Psi'(s)\,ds. \tag{9}$$

From the Butler and Rogers theorem ([17], Theorem, p. 78), it follows

$$v(t) \le \Omega^{-1}\left(\Omega(A(t)) + B(t)\int_a^t F(s)^q\,\Psi'(s)\,ds\right), \tag{10}$$

where functions A, B, Ω, Ω^{-1} are as in the statement. This means that the inequality (3) holds. □

Next, we present a couple of particular cases of the above theorem as corollaries. The first of them deals with a linear version of inequality (2).

Corollary 1. *Let the assumptions of Theorem 1 be fulfilled with $\omega(u) \equiv u$, i.e., the inequality*

$$u(t) \le a(t) + b(t)\int_a^t K(t,s,\alpha,\lambda)\,F(s)\,u(s)\,ds, \quad t \in [a,T), \tag{11}$$

holds. Then

$$u(t) \le A(t)^{\frac{1}{q}} \exp\left\{\frac{B(t)}{q}\int_a^t F(s)\,\Psi'(s)\,ds\right\} \tag{12}$$

for all $t \in [a,T)$, where $A(t)$ and $B(t)$ are given by (4).

Proof. The form of function ω implies that $\Omega(v) = \ln\frac{v}{v_0}$ for $v_0, v > 0$, and $\Omega^{-1}(v) = v_0\,\mathrm{e}^{v}$ for $v_0 > 0$, $v \in \mathbb{R}$. The statement is then easily obtained from (3). □

Remark 1. *In [15], Gronwall-type integral inequality was proved assuming (11) with $F(t) \equiv 1$ and $a(t)$ was not supposed to be non-negative. The statement from [15] involved an infinite series, whereas the exponential in (12) is reminiscent of the classic Bellman result [18].*

Remark 2. *Except for $\omega(u) \equiv u$, when $\Omega(v) \xrightarrow{v\to\infty} \infty$, one can often meet the power case, $\omega(u) \equiv u^\rho$ for $\rho > 1$ (see Theorem 6 below). Then, $\Omega(v) = \frac{1}{\rho-1}\left(\frac{1}{v_0^{\rho-1}} - \frac{1}{v^{\rho-1}}\right) \xrightarrow{v\to\infty} \frac{v_0^{1-\rho}}{\rho-1} < \infty$ and (3) has the form*

$$u(t) \le \left[A(t)^{1-\rho} - (\rho-1)B(t)\int_a^t F(s)^q\,\Psi'(s)\,ds\right]^{-\frac{1}{q(\rho-1)}}.$$

On the other side, it is not always necessary to be able to express $\Omega(v)$ explicitly in order to derive interesting results (see, e.g., Theorem 2). This is the case of the function $\omega(u) \equiv (\ln(u^\rho+2))^{\frac{1}{\rho}}$ for $\rho > 1$. Setting $q = \rho$ (while all other assumptions of Theorem 1 are satisfied) yields $\Omega(v) = \int_{v_0}^{v}\frac{d\sigma}{\ln(\sigma+2)} \xrightarrow{v\to\infty} \infty$, which fulfills condition (18).

Corollary 2. *Let the assumptions of Theorem 1 be fulfilled and $a(t)$, $b(t)$ be non-decreasing functions. Then, the statement of Theorem 1 holds with*

$$A(t) = 2^{q-1}a(t)^q, \qquad B(t) = 2^{q-1}M_p^q\,b(t)^q.$$

The following corollary of Theorem 1 considers multiple Ψ-Hilfer fractional integrals on the right side.

Corollary 3. *Let $n \in \mathbb{N}$, $\lambda > 0$, $\alpha_i \in (0,1)$ for $i = 1,2,\ldots,n$, $p > 1$, be such that $p(\alpha_i - 1) + 1 > 0$ for each $i = 1,2,\ldots,n$, $q = \frac{p}{p-1}$, $a(t)$, $b_i(t)$, and $F_i(t)$, $i = 1,2,\ldots,n$, be non-negative,* *continuous functions on the interval $[a, T)$, where $a < T \leq \infty$, and functions Ψ, ω be as in Theorem 1. If $u(t)$ is a non-negative continuous function on $[a, T)$, satisfying the inequality*

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t) \int_a^t K(t,s,\alpha_i,\lambda)\, F_i(s)\, \omega_i(u(s))\, ds, \;\; t \in [a,T), \tag{13}$$

then (3) *holds for all $t \in [a, T)$, where*

$$F = \max_{i=1,2,\ldots,n} F_i, \qquad \omega = \max_{i=1,2,\ldots,n} \omega_i,$$

$$A(t) = 2^{q-1} \sup_{a \leq s \leq t} a(s)^q, \qquad B(t) = 2^{q-1} \sup_{a \leq s \leq t} \left(\sum_{i=1}^{n} M_{p,i}\, b_i(s) \right)^q,$$

$$M_{p,i} = \left(\frac{\Gamma(p(\alpha_i - 1) + 1)}{(p\lambda)^{p(\alpha_i - 1)+1}} \right)^{1/p}, \; i = 1, 2, \ldots, n,$$

$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q}$ *for $v_0, v > 0$, and Ω^{-1} is the inverse of Ω.*

Proof. From (13), we obtain

$$u(t) \leq a(t) + \sum_{i=1}^{n} b_i(t) \int_a^t K(t,s,\alpha_i,\lambda)\, F(s)\, \omega(u(s))\, ds, \;\; t \in [a,T).$$

Following the proof of Theorem 1, one derives

$$u(t) \leq a(t) + \left(\sum_{i=1}^{n} M_{p,i}\, b_i(t) \right) \left(\int_a^t F(s)^q\, [\omega(u(s))]^q\, \Psi'(s)\, ds \right)^{1/q}. \tag{14}$$

Consequently (see (8)),

$$u(t)^q \leq \tilde{a}(t) + 2^{q-1} \left(\sum_{i=1}^{n} M_{p,i}\, b_i(t) \right)^q \int_a^t F(s)^q\, [\omega(u(s))]^q\, \Psi'(s)\, ds,$$

where $\tilde{a}(t) = 2^{q-1} a(t)^q$. Applying ([17], Theorem, p. 78) completes the proof. □

3. Applications to Fractional Differential Equations with Tempered Ψ-Caputo Derivative

In this section, we use our results from Section 2 to show the properties of the solutions of initial value problems for fractional differential equations with a tempered Ψ-Caputo derivative of order $\alpha \in (0, 1)$ and with various right sides. More precisely, we prove the results with the nonexistence of blowing-up solutions and the stability of a trivial solution.

3.1. Blowing-Up Solutions

First, we consider the following initial value problem

$$ {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) = f(t, x(t)), \quad t \geq a, \tag{15}$$

$$x_{\lambda,\Psi}^{[0]}(a) = x_a^0 \tag{16}$$

for some constant $x_a^0 \in \mathbb{R}^N$, where $0 < \alpha < 1$, $\lambda \geq 0$, $\Psi \in C^1[a, \infty)$ is such that $\Psi'(t) > 0$ for all $t \geq a$, and $f \in C([a, \infty) \times \mathbb{R}^N, \mathbb{R}^N)$. Let us recall that $x_{\lambda,\Psi}^{[0]}(t) = e^{\lambda \Psi(t)}\, x(t)$. A solution of (15), (16) is understood in the sense of the following definition.

Definition 3. *A function $x \in C^1([a, \infty), \mathbb{R}^N)$ is a solution of the initial value problem* (15), (16) *if ${}^{C}D_a^{\alpha,\lambda,\Psi} x(t)$ exists and is continuous on (a, ∞), $x(t)$ fulfills Equation* (15) *and initial condition* (16).

Theorem 2. *Let $x_a^0 \in \mathbb{R}^N$, $\lambda > 0$, $\alpha \in (0,1)$, $f \in C([a,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$ be such that*

$$\|f(t,u)\| \leq F(t)\,\omega(\|u\|) \tag{17}$$

for some non-decreasing $F \in C[a,\infty)$, functions Ψ, ω be as in Theorem 1, and

$$\int_{v_0}^{\infty} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q} = \infty \tag{18}$$

for some $v_0 > 0$, with $q = \frac{p}{p-1}$ for $p > 1$ such that $p(\alpha - 1) + 1 > 0$. Then, any non-extensible solution of (15), (16) is global, i.e., there is no blowing-up solution of this initial value problem.

Proof. Let x be a non-extensible solution of (15), (16) which is defined on $[a,b)$ for some $b < \infty$, i.e., $\lim_{t\to b^-} \|x(t)\| = \infty$. From ([19], Theorem 2), x satisfies the integral equation

$$x(t) = \mathrm{e}^{-\lambda\Psi(t)}\,x_a^0 + \frac{1}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\,f(s,x(s))\,ds$$

for all $t \in [a,b)$. Using estimation (17), we obtain

$$\|x(t)\| \leq \mathrm{e}^{-\lambda\Psi(t)}\,\|x_a^0\| + \frac{1}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\,F(s)\,\omega(\|x(s)\|)\,ds$$

which is of the form (2). Theorem 1 yields

$$\Omega(\|x(t)\|^q) \leq \Omega(A(t)) + B(t)\int_a^t F(s)^q\,\Psi'(s)\,ds \tag{19}$$

for all $t \in [a,b)$, where

$$A(t) = 2^{q-1}\Big(\mathrm{e}^{-\lambda\Psi(a)}\,\|x_a^0\|\Big)^q, \qquad B(t) = \frac{2^{q-1}M_p^q}{\Gamma(\alpha)^q},$$
$$M_p = \left(\frac{\Gamma(p(\alpha-1)+1)}{(p\lambda)^{p(\alpha-1)+1}}\right)^{1/p}, \tag{20}$$

$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q}$ for $v_0, v > 0$, and Ω^{-1} is the inverse of Ω. Taking $\lim_{t\to b^-}$ in (19), we obtain a contradiction since the left-hand side tends to ∞ and the right-hand side is bound. Therefore, $b = \infty$ and the proof is complete. □

If the right-hand side of (15) also depends on the tempered Ψ-Caputo derivatives of a lower order of solution, we have the following result.

Theorem 3. *Let $n \in \mathbb{N}$, $x_a^0 \in \mathbb{R}^N$, $\lambda > 0$, $\alpha \in (0,1)$, $0 < \alpha_i < \alpha$, $i = 1,2,\ldots,n$, $f \in C([a,\infty) \times \mathbb{R}^{(n+1)N}, \mathbb{R}^N)$ be such that*

$$\|f(t,u,v_1,\ldots,v_n)\| \leq F(t)\,\big(\omega_0(\|u\|) + \omega_1(\|v_1\|) + \ldots + \omega_n(\|v_n\|)\big) \tag{21}$$

for some non-decreasing $F \in C[a,\infty)$, function Ψ be as in Theorem 1, $\omega_i \colon [0,\infty) \to [0,\infty)$ be a continuous, positive, non-decreasing function for each $i = 0,1,\ldots,n$, and condition (18) holds for $\omega = \max_{i=0,1,\ldots,n} \omega_i$ and $q = \frac{p}{p-1}$ for $p > 1$ such that $p(\alpha - \alpha_i - 1) + 1 > 0$ for each $i = 1,2,\ldots,n$. Then, any non-extensible solution of

$${}^{C}D_a^{\alpha,\lambda,\Psi}x(t) = f(t, x(t), {}^{C}D_a^{\alpha_1,\lambda,\Psi}x(t), \ldots, {}^{C}D_a^{\alpha_n,\lambda,\Psi}x(t)), \quad t \geq a, \tag{22}$$
$$x_{\lambda,\Psi}^{[0]}(a) = x_a^0 \tag{23}$$

is global, i.e., there is no blowing-up solution of this initial value problem.

To prove this theorem, we need an auxiliary lemma.

Lemma 1. *Let $0 < \alpha < \beta < 1$, $\lambda \geq 0$, and $\Psi \in C^1[a,b]$ be such that $\Psi'(t) > 0$ for all $t \in [a,b]$. Then*

$$I_a^{\alpha,\lambda,\Psi}\,{}^C D_a^{\beta,\lambda,\Psi} x(t) = {}^C D_a^{\beta-\alpha,\lambda,\Psi} x(t)$$

for any $x \in C^1[a,b]$.

Proof. First, we rewrite the left-hand side using the Ψ-Riemann–Liouville fractional integral and the Ψ-Caputo fractional derivative as in [12],

$$\begin{aligned} I_a^{\alpha,\lambda,\Psi}\,{}^C D_a^{\beta,\lambda,\Psi} x(t) &= \mathrm{e}^{-\lambda\Psi(t)}\, I_a^{\alpha,\Psi}\left(\mathrm{e}^{\lambda\Psi(t)}\left[\mathrm{e}^{-\lambda\Psi(t)}\,{}^C D_a^{\beta,\Psi}\left(\mathrm{e}^{\lambda\Psi(t)}\,x(t)\right)\right]\right) \\ &= \mathrm{e}^{-\lambda\Psi(t)}\, I_a^{\alpha,\Psi}\,{}^C D_a^{\beta,\Psi}\left(\mathrm{e}^{\lambda\Psi(t)}\,x(t)\right). \end{aligned}$$

Next, using ([20], Theorem 3), we write

$${}^C D_a^{\beta,\Psi} y(t) = I_a^{1-\beta,\Psi}\,\frac{y'(t)}{\Psi'(t)}$$

for a C^1-function y. Then, using the composition of the fractional integrals [21] (see also [22]), we obtain

$$I_a^{\alpha,\Psi}\,{}^C D_a^{\beta,\Psi} y(t) = I_a^{\alpha,\Psi} I_a^{1-\beta,\Psi}\,\frac{y'(t)}{\Psi'(t)} = I_a^{1+\alpha-\beta,\Psi}\,\frac{y'(t)}{\Psi'(t)} = {}^C D_a^{\beta-\alpha,\Psi} y(t)$$

since $0 < \beta - \alpha < 1$. Hence, we obtain

$$I_a^{\alpha,\lambda,\Psi}\,{}^C D_a^{\beta,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)}\,{}^C D_a^{\beta-\alpha,\Psi}\left(\mathrm{e}^{\lambda\Psi(t)}\,x(t)\right) = {}^C D_a^{\beta-\alpha,\lambda,\Psi} x(t).$$

This completes the proof. □

Proof of Theorem 3. Let us again suppose that x is a solution of (22), (23) defined on $[a,b)$, satisfying $\lim_{t\to b^-}\|x(t)\| = \infty$. For short, we denote

$$\tilde{f}(t) = f(t, x(t), {}^C D_a^{\alpha_1,\lambda,\Psi} x(t), \ldots, {}^C D_a^{\alpha_n,\lambda,\Psi} x(t)).$$

Then, by ([19], Theorem 2), function x solves

$$x(t) = \mathrm{e}^{-\lambda\Psi(t)}\, x_a^0 + \frac{1}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\,\tilde{f}(s)\,ds \tag{24}$$

for all $t \in [a,b)$. For the norm, we have the estimation

$$\|x(t)\| \leq \mathrm{e}^{-\lambda\Psi(t)}\,\|x_a^0\| + \frac{1}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\,\|\tilde{f}(s)\|\,ds$$

for all $t \in [a,b)$. Let us define

$$\begin{aligned} z(t) = \mathrm{e}^{-\lambda\Psi(t)}\,\|x_a^0\| + \sum_{i=0}^{n}\frac{1}{\Gamma(\alpha-\alpha_i)}\int_a^t K(t,s,\alpha-\alpha_i,\lambda) \\ \times F(s)\left(\omega_0(\|x(s)\|) + \omega_1(\|{}^C D_a^{\alpha_1,\lambda,\Psi} x(s)\|) + \ldots + \omega_n(\|{}^C D_a^{\alpha_n,\lambda,\Psi} x(s)\|)\right) ds, \end{aligned}$$

where α_0 is set to 0. Then, by (21), $\|x(t)\| \leq z(t)$ for all $t \in [a,b)$.

Next, let $i \in \{1, 2, \ldots, n\}$ be arbitrary and fixed. Applying the operator $I_a^{\alpha-\alpha_i,\lambda,\Psi}$ to Equation (22), by Lemma 1, we obtain

$$ {}^C D_a^{\alpha_i,\lambda,\Psi} x(t) = I_a^{\alpha-\alpha_i,\lambda,\Psi} f(t, x(t), {}^C D_a^{\alpha_1,\lambda,\Psi} x(t), \ldots, {}^C D_a^{\alpha_n,\lambda,\Psi} x(t)). $$

For the norm, we have

$$ \| {}^C D_a^{\alpha_i,\lambda,\Psi} x(t) \| \leq \frac{1}{\Gamma(\alpha - \alpha_i)} \int_a^t K(t, s, \alpha - \alpha_i, \lambda) \, \|\tilde{f}(s)\| \, ds \leq z(t) $$

for all $t \in [a, b)$. Consequently, using the fact that each ω_i, $i = 0, 1, \ldots, n$, is non-decreasing, we derive $\omega_0(\|x(t)\|) \leq \omega(z(t))$ and $\omega_i(\| {}^C D_a^{\alpha_i,\lambda,\Psi} x(t)\|) \leq \omega(z(t))$ for each $i = 1, 2, \ldots, n$, and all $t \in [a, b)$. This yields

$$ z(t) \leq \mathrm{e}^{-\lambda\Psi(t)} \, \|x_a^0\| + \sum_{i=0}^{n} \frac{n+1}{\Gamma(\alpha - \alpha_i)} \int_a^t K(t, s, \alpha - \alpha_i, \lambda) \, F(s) \, \omega(z(s)) \, ds. $$

Note that the assumption on p implies $p(\alpha - 1) + 1 > 0$. By Corollary 3, we obtain

$$ \|x(t)\| \leq z(t) \leq \left[\Omega^{-1} \left(\Omega(A(t)) + B(t) \int_a^t F(s)^q \, \Psi'(s) \, ds \right) \right]^{1/q} \tag{25} $$

with

$$ A(t) = 2^{q-1} \left(\mathrm{e}^{-\lambda\Psi(a)} \, \|x_a^0\| \right)^q, \qquad B(t) = 2^{q-1} \left(\sum_{i=0}^{n} \frac{M_{p,i}(n+1)}{\Gamma(\alpha - \alpha_i)} \right)^q, $$
$$ M_{p,i} = \left(\frac{\Gamma(p(\alpha - \alpha_i - 1) + 1)}{(p\lambda)^{p(\alpha - \alpha_i - 1) + 1}} \right)^{1/p}, \; i = 0, 1, \ldots, n, \tag{26} $$

$\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q}$ for $v_0, v > 0$, and Ω^{-1} is the inverse of Ω. Equivalently, we have (19) for any $t \in [a, b)$. For $x \to b^-$, we obtain a contradiction. This completes the proof. $\square$

In the next step, we add dependency on tempered Ψ-Hilfer fractional integrals of a solution.

Theorem 4. *Let $n, m \in \mathbb{N}$, $x_a^0 \in \mathbb{R}^N$, $\lambda > 0$, $\alpha \in (0, 1)$, $0 < \alpha_i < \alpha$, $i = 1, 2, \ldots, n$, $\beta_j > 0$, $j = 1, 2, \ldots, m$, $f \in C([a, \infty) \times \mathbb{R}^{(n+m+1)N}, \mathbb{R}^N)$ be such that*

$$ \begin{aligned} \|f(t, u, v_1, \ldots, v_n, w_1, \ldots, w_m)\| \leq F(t) \, (&\omega_0(\|u\|) + \omega_1(\|v_1\|) + \ldots + \omega_n(\|v_n\|) \\ &+ \omega_{-1}(\|w_1\|) + \ldots + \omega_{-m}(\|w_m\|)) \end{aligned} \tag{27} $$

for some non-decreasing $F \in C[a, \infty)$, function Ψ be as in Theorem 1, $\omega_i \colon [0, \infty) \to [0, \infty)$ be a continuous, positive, non-decreasing function for each $i = -m, -m+1, \ldots, n$, and condition (18) holds for $\omega = \max_{i=-m,-m+1,\ldots,n} \omega_i$, and $q = \frac{p}{p-1}$ for $p > 1$ such that $p(\alpha - \alpha_i - 1) + 1 > 0$ for each $i = 1, 2, \ldots, n$. Then, any non-extensible solution of

$$ \begin{aligned} {}^C D_a^{\alpha,\lambda,\Psi} x(t) = f(t, x(t), &{}^C D_a^{\alpha_1,\lambda,\Psi} x(t), \ldots, {}^C D_a^{\alpha_n,\lambda,\Psi} x(t), \\ &I_a^{\beta_1,\lambda,\Psi} x(t), \ldots, I_a^{\beta_m,\lambda,\Psi} x(t)), \quad t \geq a, \end{aligned} \tag{28} $$
$$ x_{\lambda,\Psi}^{[0]}(a) = x_a^0 \tag{29} $$

is global, i.e., there is no blowing-up solution of this initial value problem.

Proof. In some parts, the proof is analogous to the proof of Theorem 3, so we skip several steps. Let $x\colon [a,b) \to \mathbb{R}^N$ be a solution of (28), (29) such that $\lim_{t\to b^-} \|x(t)\| = \infty$. For short, we denote

$$\tilde{f}(t) = f(t, x(t), {}^C D_a^{\alpha_1,\lambda,\Psi} x(t), \dots, {}^C D_a^{\alpha_n,\lambda,\Psi} x(t), \\ I_a^{\beta_1,\lambda,\Psi} x(t), \dots, I_a^{\beta_m,\lambda,\Psi} x(t)).$$

Then, x has the form (24) and satisfies $\|x(t)\| \le z(t)$ for all $t \in [a,b)$, where

$$\begin{aligned} z(t) = a(t)\|x_a^0\| + \int_a^t \sum_{i=0}^n \bigg(&\frac{1}{\Gamma(\alpha-\alpha_i)} (\Psi(t)-\Psi(s))^{\alpha-\alpha_i-1} \\ &+ \sum_{j=1}^m \frac{(\Psi(t)-\Psi(a))^{\beta_j}}{\Gamma(\alpha+\beta_j)} (\Psi(t)-\Psi(s))^{\alpha-1} \bigg) \mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))} \\ \times \Psi'(s)\, F(s) \big(\omega_0(\|x(s)\|) &+ \omega_1(\|{}^C D_a^{\alpha_1,\lambda,\Psi} x(s)\|) + \ldots + \omega_n(\|{}^C D_a^{\alpha_n,\lambda,\Psi} x(s)\|) \\ &+ \omega_{-1}(\|I_a^{\beta_1,\lambda,\Psi} x(s)\|) + \ldots + \omega_{-m}(\|I_a^{\beta_m,\lambda,\Psi} x(s)\|)\big)\, ds, \end{aligned}$$

where

$$a(t) = \mathrm{e}^{-\lambda\Psi(t)} \max_{j=1,2,\dots,m} \left\{ 1, \frac{(\Psi(t)-\Psi(a))^{\beta_j}}{\Gamma(\beta_j+1)} \right\}.$$

Applying the operator $I_a^{\alpha-\alpha_i,\lambda,\Psi}$ to Equation (28), one can show that

$$\|{}^C D_a^{\alpha_i,\lambda,\Psi} x(t)\| \le z(t)$$

for each $i = 1,2,\dots,n$, and all $t \in [a,b)$. Next, for arbitrary fixed $j = 1,2,\dots,m$, applying the operator $I_a^{\beta_j,\lambda,\Psi}$ to (24), we have

$$\begin{aligned} I_a^{\beta_j,\lambda,\Psi} x(t) &= I_a^{\beta_j,\lambda,\Psi} \left(\mathrm{e}^{-\lambda\Psi(t)}\, x_a^0 + I_a^{\alpha,\lambda,\Psi} \tilde{f}(t) \right) \\ &= I_a^{\beta_j,\lambda,\Psi} (\mathrm{e}^{-\lambda\Psi(t)}\, x_a^0) + I_a^{\beta_j,\lambda,\Psi} I_a^{\alpha,\lambda,\Psi} \tilde{f}(t). \end{aligned}$$

Now, using substitution $\sigma = \Psi(t) - \Psi(s)$, we obtain

$$\begin{aligned} I_a^{\beta_j,\lambda,\Psi} (\mathrm{e}^{-\lambda\Psi(t)}\, x_a^0) &= \mathrm{e}^{-\lambda\Psi(t)}\, I_a^{\beta_j,\Psi} x_a^0 \\ = \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(\beta_j)} \int_a^t (\Psi(t)-\Psi(s))^{\beta_j-1} \Psi'(s)\, x_a^0\, ds &= \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(\beta_j)} \int_0^{\Psi(t)-\Psi(a)} \sigma^{\beta_j-1} x_a^0\, ds \\ = \frac{\mathrm{e}^{-\lambda\Psi(t)} (\Psi(t)-\Psi(a))^{\beta_j} x_a^0}{\beta_j \Gamma(\beta_j)} &= \frac{\mathrm{e}^{-\lambda\Psi(t)} (\Psi(t)-\Psi(a))^{\beta_j} x_a^0}{\Gamma(\beta_j+1)}. \end{aligned}$$

Furthermore, by the composition of the Ψ-Riemann–Liouville fractional integrals [22], we derive

$$\begin{aligned} I_a^{\beta_j,\lambda,\Psi} I_a^{\alpha,\lambda,\Psi} \tilde{f}(t) &= \mathrm{e}^{-\lambda\Psi(t)}\, I_a^{\beta_j,\Psi} I_a^{\alpha,\Psi} (\mathrm{e}^{\lambda\Psi(t)}\, \tilde{f}(t)) \\ &= \mathrm{e}^{-\lambda\Psi(t)}\, I_a^{\alpha+\beta_j,\Psi} (\mathrm{e}^{\lambda\Psi(t)}\, \tilde{f}(t)) = I_a^{\alpha+\beta_j,\lambda,\Psi} \tilde{f}(t). \end{aligned}$$

Thus, for the norm of the fractional integral of a solution, we have the following estimation

$$\begin{aligned} \|I_a^{\beta_j,\lambda,\Psi} x(t)\| &\le \frac{\mathrm{e}^{-\lambda\Psi(t)} (\Psi(t)-\Psi(a))^{\beta_j} \|x_a^0\|}{\Gamma(\beta_j+1)} \\ &+ \frac{1}{\Gamma(\alpha+\beta_j)} \int_a^t K(t,s,\alpha+\beta_j,\lambda)\, \|\tilde{f}(s)\|\, ds \le z(t) \end{aligned}$$

for each $j = 1, 2, \dots, m$, and all $t \in [a, b)$.

Using the fact that each ω_i, $i = -m, -m+1, \dots, n$, is non-decreasing, we derive

$$\begin{aligned} \omega_0(\|x(t)\|) &\le \omega(z(t)), \\ \omega_i(\|{}^C D_a^{\alpha_i,\lambda,\Psi} x(t)\|) &\le \omega(z(t)), \quad i = 1, 2, \dots, n, \\ \omega_{-j}(\|I_a^{\beta_j,\lambda,\Psi} x(t)\|) &\le \omega(z(t)), \quad j = 1, 2, \dots, m, \end{aligned}$$

for all $t \in [a, b)$. Hence,

$$\begin{aligned} z(t) \le a(t)\|x_a^0\| &+ \sum_{i=0}^{n} \frac{n+m+1}{\Gamma(\alpha-\alpha_i)} \int_a^t K(t,s,\alpha-\alpha_i,\lambda)\, F(s)\, \omega(z(s))\, ds \\ &+ \sum_{j=1}^{m} \frac{(n+m+1)(\Psi(t)-\Psi(a))^{\beta_j}}{\Gamma(\alpha+\beta_j)} \int_a^t K(t,s,\alpha,\lambda)\, F(s)\, \omega(z(s))\, ds. \end{aligned}$$

Note that $p(\alpha-1)+1 > 0$ by the assumption on p. By Corollary 3, we obtain the estimation (25) with

$$A(t) = 2^{q-1}\left(\sup_{a\le s\le t} a(s)\, \|x_a^0\|\right)^q,$$

$$B(t) = 2^{q-1}\left(\sum_{i=0}^{n} \frac{M_{p,i}(n+m+1)}{\Gamma(\alpha-\alpha_i)} + M_p \sum_{j=1}^{m} \frac{(n+m+1)(\Psi(t)-\Psi(a))^{\beta_j}}{\Gamma(\alpha+\beta_j)}\right)^q$$

for M_p, $M_{p,i}$ given by (20), (26), respectively, $\Omega(v) = \int_{v_0}^{v} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q}$ for $v_0, v > 0$, and Ω^{-1} is the inverse of Ω. Equivalently, we have (19) for any $t \in [a, b)$. For $x \to b^-$, we obtain a contradiction. This completes the proof. □

3.2. Stability

In this part, we investigate the stability of a trivial solution of the scalar fractional differential equation

$$ {}^C D_a^{\alpha,\lambda,\Psi} x(t) = -\aleph x(t) + f(t, x(t)), \quad t \ge a, \tag{30}$$

with $\aleph > 0$, $0 < \alpha < 1$, $\lambda \ge 0$, $\Psi \in C^1[a,\infty)$ such that $\Psi'(t) > 0$ for all $t \ge a$, and $f \in C([a,\infty) \times \mathbb{R}, \mathbb{R})$ satisfying $f(t, 0) = 0$ for all $t \ge a$, in the sense of the following definition.

Definition 4. *The trivial solution, $x(t) \equiv 0$, of Equation (30) is called locally Ψ-exponentially stable if there exist constants $c_1, c_2, \delta > 0$ depending on α, λ, Ψ, $\aleph$, and f such that any solution $y\colon [a,\infty) \to \mathbb{R}$ of Equation (30) satisfying $y_{\lambda,\Psi}^{[0]}(a) = y_a^0 \in \mathbb{R}$ with $|y_a^0| < \delta$, fulfills $|y(t)| \le c_1\, e^{c_2\Psi(t)}$ for all $t \ge a$.*

It is worth noting that, in this section, we do not discuss the existence of solutions of (30), (16), or any analogous initial value problem. It is assumed that they exist and an appropriate convergence is proven to the trivial solution.

First, we consider the nonlinearity of order $o(|x(t)|)$, where o is the Landau symbol.

Theorem 5. *Let $\aleph > 0$, $\lambda > 0$, $\alpha \in (0, 1)$, Ψ be as in Theorem 1. Moreover, let $f \in C([a,\infty) \times \mathbb{R}, \mathbb{R})$ satisfy $f(t, x) = o(|x|)$, i.e., for any $P > 0$, there is $\delta > 0$ such that*

$$|x| < \delta \quad \Longrightarrow \quad |f(t, x)| \le P|x|.$$

Then, the trivial solution of Equation (30) is locally Ψ-exponentially stable.

Proof. Let x be a solution of Equation (30) satisfying (16) for some x_a^0 sufficiently close to 0 (it is specified later how close) defined on $[a, \infty)$. Then, by ([19], Theorem 3), x is given by

$$\begin{aligned} x(t) &= \mathrm{e}^{-\lambda\Psi(t)}\, E_\alpha(-(\Psi(t)-\Psi(a))^\alpha \aleph)\, x_a^0 \\ &+ \int_a^t K(t,s,\alpha,\lambda)\, E_{\alpha,\alpha}(-(\Psi(t)-\Psi(s))^\alpha \aleph)\, f(s,x(s))\, ds, \end{aligned} \tag{31}$$

where

$$E_{u,v}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(uj+v)}$$

for $u, v > 0$, is the Mittag–Leffler function (cf. [23], §18.1) and $E_u(z) = E_{u,1}(z)$. From [24] (see also [25]), $E_{u,v}$ is completely monotonous, i.e., if $0 < u \le 1$, $u \le v$, then

$$(-1)^m \frac{d^m}{dz^m}[E_{u,v}(-z)] \ge 0$$

for each $m = 0, 1, \ldots$ and all $z \ge 0$. Hence,

$$E_\alpha(-(\Psi(t)-\Psi(a))^\alpha \aleph) \ge 0, \qquad E_{\alpha,\alpha}(-(\Psi(t)-\Psi(s))^\alpha \aleph) \ge 0.$$

Let us fix $P > 0$ and assume that $|x(t)| < \delta$ for all $t \ge a$ (this is established later). Moreover,

$$\begin{aligned} \frac{d}{dt}[E_\alpha(-(\Psi(t)-\Psi(a))^\alpha \aleph)] &= \underbrace{\frac{d}{dz}[E_\alpha(-z)]_{z=(\Psi(t)-\Psi(a))^\alpha \aleph}}_{\le 0} \underbrace{\frac{\alpha\, \aleph\, \Psi'(t)}{(\Psi(t)-\Psi(a))^{1-\alpha}}}_{\ge 0} \le 0, \\ \frac{d}{ds}[E_{\alpha,\alpha}(-(\Psi(t)-\Psi(s))^\alpha \aleph)] &= \underbrace{\frac{d}{dz}[E_{\alpha,\alpha}(-z)]_{z=(\Psi(t)-\Psi(s))^\alpha \aleph}}_{\le 0} \underbrace{\frac{-\alpha\, \aleph\, \Psi'(s)}{(\Psi(t)-\Psi(s))^{1-\alpha}}}_{\le 0} \ge 0 \end{aligned}$$

for all $a < s < t$. Hence,

$$\begin{aligned} E_\alpha(-(\Psi(t)-\Psi(a))^\alpha \aleph) &\le E_\alpha(-(\Psi(t)-\Psi(a))^\alpha \aleph)\big|_{t=a} = E_\alpha(0) = \frac{1}{\Gamma(\alpha)}, \\ E_{\alpha,\alpha}(-(\Psi(t)-\Psi(s))^\alpha \aleph) &\le E_{\alpha,\alpha}(-(\Psi(t)-\Psi(s))^\alpha \aleph)\big|_{s=t} = E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)} \end{aligned}$$

for all $a \le s \le t$. Therefore,

$$\begin{aligned} |x(t)| &\le \mathrm{e}^{-\lambda\Psi(t)}\, E_\alpha(-(\Psi(t)-\Psi(a))^\alpha \aleph)\, |x_a^0| \\ &+ P \int_a^t K(t,s,\alpha,\lambda)\, E_{\alpha,\alpha}(-(\Psi(t)-\Psi(s))^\alpha \aleph)\, |x(s)|\, ds \\ &\le \mathrm{e}^{-\lambda\Psi(t)} \frac{|x_a^0|}{\Gamma(\alpha)} + \frac{P}{\Gamma(\alpha)} \int_a^t K(t,s,\alpha,\lambda)\, |x(s)|\, ds \end{aligned} \tag{32}$$

for all $t \ge a$. Now, denote $u(t) = \mathrm{e}^{\frac{\lambda\Psi(t)}{2}}\, |x(t)|$. This function can be estimated as

$$\begin{aligned} u(t) &\le \mathrm{e}^{-\frac{\lambda\Psi(t)}{2}} \frac{|x_a^0|}{\Gamma(\alpha)} + \frac{P}{\Gamma(\alpha)} \int_a^t K\left(t,s,\alpha,\frac{\lambda}{2}\right) u(s)\, ds \\ &\le \frac{|x_a^0|}{\Gamma(\alpha)} + \frac{P}{\Gamma(\alpha)} \int_a^t K\left(t,s,\alpha,\frac{\lambda}{2}\right) u(s)\, ds, \quad t \ge a. \end{aligned}$$

Now, we apply Corollary 1 with any $p, q > 1$ satisfying its assumptions and

$$A = A(t) = 2^{q-1}\left(\frac{|x_a^0|}{\Gamma(\alpha)}\right)^q, \qquad B = B(t) = 2^{q-1}\left(\frac{M_p P}{\Gamma(\alpha)}\right)^q,$$
$$M_p = \left(\frac{\Gamma(p(\alpha-1)+1)}{\left(\frac{p\lambda}{2}\right)^{p(\alpha-1)+1}}\right)^{1/p},$$

to obtain

$$u(t) \leq 2^{\frac{q-1}{q}}\frac{|x_a^0|}{\Gamma(\alpha)}\,\mathrm{e}^{\frac{B(\Psi(t)-\Psi(a))}{q}} \leq 2^{\frac{q-1}{q}}\frac{|x_a^0|}{\Gamma(\alpha)}\,\mathrm{e}^{\frac{B\,\Psi(t)}{q}}, \quad t \geq a.$$

Notice that $B \to 0^+$ as $P \to 0^+$. For the original solution x, we have

$$|x(t)| \leq 2^{\frac{q-1}{q}}\frac{|x_a^0|}{\Gamma(\alpha)}\,\mathrm{e}^{\left(\frac{B}{q}-\frac{\lambda}{2}\right)\Psi(t)}, \quad t \geq a. \tag{33}$$

Now, if $P > 0$ is so small that $\frac{B}{q} - \frac{\lambda}{2} < 0$ and $|x_a^0| < \delta$ is such that $|x_a^0| < 2^{\frac{1-q}{q}}\Gamma(\alpha)\,\delta$, then $|x(t)| < \delta$ for all $t \geq a$. So, the estimation of $|f(t, x(t))|$ in (32) is justified. Therefore, (33) means the local Ψ-exponential stability of the trivial solution. □

When the nonlinearity is of a higher order, we have a similar result; however, the Ψ-exponential stability is obtained more easily (there is no restriction on P in the proof).

Theorem 6. *Let $\aleph > 0$, $\lambda > 0$, $\alpha \in (0,1)$, Ψ be as in Theorem 1. Moreover, let $f \in C([a,\infty) \times \mathbb{R}, \mathbb{R})$ satisfy $f(t,x) = o(|x|^\rho)$ for some $\rho > 1$, i.e., for any $P > 0$, there is $\delta > 0$ such that*

$$|x| < \delta \quad \Longrightarrow \quad |f(t,x)| \leq P|x|^\rho.$$

Then, the trivial solution of Equation (30) is locally Ψ-exponentially stable.

Proof. Solution x of initial value problem (30), (16) is given by (31). Let $P > 0$ be fixed and assume that $|x(t)| \leq \delta$ for all $t \geq a$. Similarly to the proof of Theorem 5, we have

$$|x(t)| \leq \mathrm{e}^{-\lambda\Psi(t)}\frac{|x_a^0|}{\Gamma(\alpha)} + \frac{P}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\,|x(s)|^\rho\,ds$$

for all $t \geq a$. Let us denote $u(t) = \mathrm{e}^{\frac{\lambda\Psi(t)}{2\rho}}\,|x(t)|$. Then, using $1 - \frac{1}{2\rho} > 0$, we derive

$$u(t) \leq \frac{|x_a^0|}{\Gamma(\alpha)} + \frac{P}{\Gamma(\alpha)}\int_a^t K\left(t,s,\alpha,\frac{\lambda(2\rho-1)}{2\rho}\right)\mathrm{e}^{\frac{\lambda(1-\rho)}{2\rho}\Psi(s)}\,u(s)^\rho\,ds, \quad t \geq a.$$

We apply Theorem 1 with

$$F(t) = \mathrm{e}^{\frac{\lambda(1-\rho)}{2\rho}\Psi(t)}, \qquad A = A(t) = 2^{q-1}\left(\frac{|x_a^0|}{\Gamma(\alpha)}\right)^q, \qquad B = B(t) = 2^{q-1}\left(\frac{M_p P}{\Gamma(\alpha)}\right)^q,$$
$$M_p = \left(\frac{\Gamma(p(\alpha-1)+1)}{\left(\frac{p\lambda(2\rho-1)}{2\rho}\right)^{p(\alpha-1)+1}}\right)^{1/p}, \qquad \Omega(v) = \int_{v_0}^v \frac{d\sigma}{\sigma^\rho}, \quad v_0, v > 0.$$

So, we obtain

$$u(t) \leq \left[\Omega^{-1}\left(\Omega(A) + B\int_a^t \Psi'(s)\,\mathrm{e}^{\frac{q\lambda(1-\rho)}{2\rho}\Psi(s)}\,ds\right)\right]^{\frac{1}{q}}, \quad t \geq a. \tag{34}$$

Furthermore, we estimate the inner integral as

$$\int_a^t \Psi'(s)\,\mathrm{e}^{\frac{q\lambda(1-\rho)}{2\rho}\Psi(s)}\,ds = \int_{\Psi(a)}^{\Psi(t)} \mathrm{e}^{-\frac{q\lambda(\rho-1)s}{2\rho}}\,ds \leq \int_0^\infty \mathrm{e}^{-\frac{q\lambda(\rho-1)s}{2\rho}}\,ds = \frac{2\rho}{q\lambda(\rho-1)} < \infty.$$

Thus, the right-hand side of (34) is bounded from above by a finite constant. Returning back to x, we obtain

$$|x(t)| \leq \left[\Omega^{-1}\left(\Omega\left(2^{q-1}\left(\frac{|x_a^0|}{\Gamma(\alpha)}\right)^q\right) + \left(\frac{M_p\,P}{\Gamma(\alpha)}\right)^q \frac{2^q\rho}{q\lambda(\rho-1)}\right)\right]^{\frac{1}{q}} \mathrm{e}^{-\frac{\lambda\Psi(t)}{2\rho}}$$

for all $t \geq a$. This was to be proven. To be able to use the estimation of $|f(t,x(t))|$, it only remains to verify the smallness of $|x(t)|$ for all $t \geq a$. Having a fixed arbitrary $P > 0$, it can be achieved by taking $|x_a^0| < \delta$ so small that

$$\Omega\left(2^{q-1}\left(\frac{|x_a^0|}{\Gamma(\alpha)}\right)^q\right) < \Omega(\delta^q) - \left(\frac{M_p\,P}{\Gamma(\alpha)}\right)^q \frac{2^q\rho}{q\lambda(\rho-1)}.$$

This is always possible since $\Omega(v) \to -\infty$ as $v \to 0^+$. $\square$

Finally, we consider the case when the nonlinearity depends on a tempered Ψ-Caputo derivative of the lower order of a solution.

Theorem 7. *Let $\aleph > 0$, $\lambda > 0$, $0 < \beta < \alpha < 1$, $2\alpha - \beta < 1$, Ψ be as in Theorem 1. Moreover, let $f \in C([a,\infty) \times \mathbb{R}^2, \mathbb{R})$ satisfy $f(t,x,y) = o(|x| + |y|)$, i.e., for any $P > 0$ there is $\delta > 0$ such that*

$$|x|, |y| < \delta \quad \Longrightarrow \quad |f(t,x,y)| \leq P(|x| + |y|).$$

Then, the trivial solution of equation

$$ {}^C D_a^{\alpha,\lambda,\Psi} x(t) = -\aleph x(t) + f(t, x(t), {}^C D_a^{\beta,\lambda,\Psi} x(t)), \quad t \geq a, \tag{35}$$

is locally Ψ-exponentially stable.

Proof. By ([19], Theorem 3), the solution x of the initial value problem (35), (15) has the form

$$\begin{aligned} x(t) &= \mathrm{e}^{-\lambda\Psi(t)}\,E_\alpha(-(\Psi(t)-\Psi(a))^\alpha \aleph)\,x_a^0 \\ &+ \int_a^t K(t,s,\alpha,\lambda)\,E_{\alpha,\alpha}(-(\Psi(t)-\Psi(s))^\alpha\aleph)\,f(s,x(s),{}^C D_a^{\beta,\lambda,\Psi} x(s))\,ds \end{aligned}$$

for $t \geq a$. Let us define the auxiliary function

$$\begin{aligned} z(t) &= \mathrm{e}^{-\lambda\Psi(t)}\,\frac{|x_a^0|}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\,|f(s,x(s),{}^C D_a^{\beta,\lambda,\Psi} x(s))|\,ds \\ &+ \frac{1}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\,|f(s,x(s),{}^C D_a^{\beta,\lambda,\Psi} x(s))|\,ds \end{aligned}$$

for $t \geq a$. Using the complete monotonicity of the Mittag–Leffler function (see the proof of Theorem 5), it is easy to see that $x(t) \leq z(t)$ for all $t \geq a$. Using Lemma 1, we derive

$$\begin{aligned} {}^C D_a^{\beta,\lambda,\Psi} x(t) &= I_a^{\alpha-\beta,\lambda,\Psi}\,{}^C D_a^{\alpha,\lambda,\Psi} x(t) = I_a^{\alpha-\beta,\lambda,\Psi}\left(-\aleph x(t) + f(t,x(t),{}^C D_a^{\beta,\lambda,\Psi} x(t))\right) \\ &= \frac{1}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\left(-\aleph x(s) + f(s,x(s),{}^C D_a^{\beta,\lambda,\Psi} x(s))\right)ds. \end{aligned}$$

Hence,

$$
\begin{aligned}
|{}^{C}D_a^{\beta,\lambda,\Psi}x(t)| \le{}& \frac{\aleph}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\,|x(s)|\,ds \\
&+\frac{1}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\,|f(s,x(s),{}^{C}D_a^{\beta,\lambda,\Psi}x(s))|\,ds \\
\le{}& z(t)+\frac{\aleph}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\,z(s)\,ds, \quad t\ge a.
\end{aligned} \tag{36}
$$

Let $P>0$ be fixed and $|x(t)|, |{}^{C}D_a^{\beta,\lambda,\Psi}x(t)|<\delta$ for all $t\ge a$ (this is assured at the end of the proof). Then, using the estimations of $|x(t)|$ and $|{}^{C}D_a^{\beta,\lambda,\Psi}x(t)|$, function $z(t)$ can be estimated as follows:

$$
\begin{aligned}
z(t) \le{}& \mathrm{e}^{-\lambda\Psi(t)}\,\frac{|x_a^0|}{\Gamma(\alpha)}+\frac{P}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\,z(s)\,ds+\frac{P}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\,z(s)\,ds \\
&+\frac{P}{\Gamma(\alpha)}\int_a^t K(t,s,\alpha,\lambda)\left(z(s)+\frac{\aleph}{\Gamma(\alpha-\beta)}\int_a^s K(s,\sigma,\alpha-\beta,\lambda)\,z(\sigma)\,d\sigma\right)ds \\
&+\frac{P}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\left(z(s)+\frac{\aleph}{\Gamma(\alpha-\beta)}\int_a^s K(s,\sigma,\alpha-\beta,\lambda)\,z(\sigma)\,d\sigma\right)ds
\end{aligned} \tag{37}
$$

for all $t\ge a$. We rewrite the first of the two double integrals by changing the order of integration,

$$
\begin{aligned}
I_1 :={}& \int_a^t K(t,s,\alpha,\lambda)\left(\int_a^s K(s,\sigma,\alpha-\beta,\lambda)\,z(\sigma)\,d\sigma\right)ds \\
={}& \int_a^t\left(\int_\sigma^t (\Psi(t)-\Psi(s))^{\alpha-1}(\Psi(s)-\Psi(\sigma))^{\alpha-\beta-1}\,\Psi'(s)\,ds\right) \\
&\times\Psi'(\sigma)\,\mathrm{e}^{-\lambda(\Psi(t)-\Psi(\sigma))}\,z(\sigma)\,d\sigma.
\end{aligned}
$$

In the inner integral, we substitute $\xi=\frac{\Psi(s)-\Psi(\sigma)}{\Psi(t)-\Psi(\sigma)}$ to obtain

$$
\begin{aligned}
&\int_\sigma^t (\Psi(t)-\Psi(s))^{\alpha-1}(\Psi(s)-\Psi(\sigma))^{\alpha-\beta-1}\,\Psi'(s)\,ds \\
&=\int_0^1 \xi^{\alpha-\beta-1}(1-\xi)^{\alpha-1}\,d\xi\,(\Psi(t)-\Psi(\sigma))^{2\alpha-\beta-1} \\
&=B(\alpha-\beta,\alpha)\,(\Psi(t)-\Psi(\sigma))^{2\alpha-\beta-1},
\end{aligned}
$$

where $B(\cdot,\cdot)$ is the Euler beta function. Thus,

$$
I_1 = B(\alpha-\beta,\alpha)\int_a^t K(t,\sigma,2\alpha-\beta,\lambda)\,z(\sigma)\,d\sigma.
$$

Analogously, one can rewrite the other double integral

$$
\begin{aligned}
I_2 :={}& \int_a^t K(t,s,\alpha-\beta,\lambda)\left(\int_a^s K(s,\sigma,\alpha-\beta,\lambda)\,z(\sigma)\,d\sigma\right)ds \\
={}& B(\alpha-\beta,\alpha-\beta)\int_a^t K(t,\sigma,2(\alpha-\beta),\lambda)\,z(\sigma)\,d\sigma.
\end{aligned}
$$

Using the formula $B(u,v)=\frac{\Gamma(u)\,\Gamma(v)}{\Gamma(u+v)}$, from (37), we obtain

$$
\begin{aligned}
z(t) \le{}& \mathrm{e}^{-\lambda\Psi(t)}\,|x_a^0|+2P\int_a^t K(t,s,\alpha,\lambda)\,z(s)\,ds+2P\int_a^t K(t,s,\alpha-\beta,\lambda)\,z(s)\,ds \\
&+\aleph P\int_a^t K(t,s,2\alpha-\beta,\lambda)\,z(s)\,ds+\aleph P\int_a^t K(t,s,2(\alpha-\beta),\lambda)\,z(s)\,ds, \quad t\ge a.
\end{aligned}
$$

Here, we use $\Gamma(u) > 1$ for all $u \in (0,1)$ along with the assumption of the theorem on α, β. For $w(t) = \mathrm{e}^{\frac{\lambda\Psi(t)}{2}} z(t)$, we have the following estimation

$$\begin{aligned} w(t) \le |x_a^0| + 2P\int_a^t K\left(t,s,\alpha,\frac{\lambda}{2}\right) w(s)\,ds + 2P\int_a^t K\left(t,s,\alpha-\beta,\frac{\lambda}{2}\right) w(s)\,ds \\ + \aleph P\int_a^t K\left(t,s,2\alpha-\beta,\frac{\lambda}{2}\right) w(s)\,ds + \aleph P\int_a^t K\left(t,s,2(\alpha-\beta),\frac{\lambda}{2}\right) w(s)\,ds, \quad t \ge a. \end{aligned}$$

Now, we apply Corollary 3 with any $p, q > 1$ satisfying its assumptions, and $F(t) \equiv 1$, $\omega(u) \equiv u$, $\Omega(v) = \ln\frac{v}{v_0}$ for $v_0, v > 0$, $\Omega^{-1}(v) = v_0\,\mathrm{e}^v$ for $v_0 > 0$, $v \in \mathbb{R}$, $b_{1,2} = 2P$, $b_{3,4} = \aleph P$,

$$A = A(t) = 2^{q-1}|x_a^0|^q, \qquad B = B(t) = 2^{q-1}\left(\sum_{i=1}^{n} M_{p,i}\, b_i\right)^q,$$

$$M_{p,1} = \left(\frac{\Gamma(p(\alpha-1)+1)}{\left(\frac{p\lambda}{2}\right)^{p(\alpha-1)+1}}\right)^{1/p}, \qquad M_{p,2} = \left(\frac{\Gamma(p(\alpha-\beta-1)+1)}{\left(\frac{p\lambda}{2}\right)^{p(\alpha-\beta-1)+1}}\right)^{1/p},$$

$$M_{p,3} = \left(\frac{\Gamma(p(2\alpha-\beta-1)+1)}{\left(\frac{p\lambda}{2}\right)^{p(2\alpha-\beta-1)+1}}\right)^{1/p}, \qquad M_{p,4} = \left(\frac{\Gamma(p(2\alpha-2\beta-1)+1)}{\left(\frac{p\lambda}{2}\right)^{p(2\alpha-2\beta-1)+1}}\right)^{1/p}.$$

As a result, we obtain

$$w(t) \le A^{\frac{1}{q}}\,\mathrm{e}^{\frac{B\,\Psi(t)}{q}}, \quad t \ge a,$$

or

$$|x(t)| \le z(t) = \mathrm{e}^{-\frac{\lambda\Psi(t)}{2}}\, w(t) \le 2^{\frac{q-1}{q}}|x_a^0|\,\mathrm{e}^{\left(\frac{B}{q}-\frac{\lambda}{2}\right)\Psi(t)}, \quad t \ge a. \tag{38}$$

Since $B \to 0^+$ as $P \to 0^+$, we can take $P > 0$ so small that $\frac{B}{q} - \frac{\lambda}{2} < 0$. Now, it only remains to show that it is possible to find $0 < |x_a^0| < \delta$ so small that $|x(t)| < \delta$ and $|{}^C D_a^{\beta,\lambda,\Psi} x(t)| < \delta$ for all $t \ge a$. Then, estimation (38) proves the statement.

Clearly, if $|x_a^0| < 2^{\frac{1-q}{q}}\delta$, then (38) assures that $|x(t)| < \delta$ for all $t \ge a$, provided that $|{}^C D_a^{\beta,\lambda,\Psi} x(t)| < \delta$ for all $t \ge a$. But from (36), we know that

$$\begin{aligned} |{}^C D_a^{\beta,\lambda,\Psi} x(t)| &\le 2^{\frac{q-1}{q}}|x_a^0| + \frac{\aleph}{\Gamma(\alpha-\beta)}\int_a^t K(t,s,\alpha-\beta,\lambda)\, 2^{\frac{q-1}{q}}|x_a^0|\,ds \\ &\le 2^{\frac{q-1}{q}}|x_a^0|\left(1 + \frac{\aleph}{\lambda^{\alpha-\beta}}\right) \end{aligned}$$

since

$$\int_a^t K(t,s,\alpha-\beta,\lambda)\,ds \le \frac{\Gamma(\alpha-\beta)}{\lambda^{\alpha-\beta}}$$

(for details, see (6)). Thus, it is sufficient to take $|x_a^0| < 2^{\frac{1-q}{q}}\left(1 + \frac{\aleph}{\lambda^{\alpha-\beta}}\right)^{-1}\delta$. This completes the proof. $\square$

Remark 3. *To be more precise, the latter proof works this way: one takes any $P > 0$ such that $\frac{B}{q} - \frac{\lambda}{2} < 0$. This prescribes $\delta > 0$ by the assumption of the theorem. Then, one uses $|x_a^0| > 0$ so small as desired at the end of the proof. The continuity of $x(t)$ and ${}^C D_a^{\beta,\lambda,\Psi} x(t)$ assures that they are smaller than δ on some interval $[a, a+\varepsilon_1)$ for $\varepsilon_1 > 0$. Then, estimations (36) and (38) are valid which makes $|x(t)|$ and $|{}^C D_a^{\beta,\lambda,\Psi} x(t)|$ even smaller. So, we can extend the interval where the estimations hold by another $\varepsilon_2 > 0$, etc. Similar findings were obtained for the proofs of Theorems 5 and 6.*

4. Discussion

In this paper, we proved a new nonlinear version of the Henry–Gronwall-type integral inequality involving tempered Ψ-Hilfer fractional integral. We also discussed its particular cases, including the linear case, and proved a corollary for multiple fractional integrals of this type. These results generalize the recently published Gronwall-type integral inequality from [15], and present a natural step in the evolution of a new type of fractional calculus, namely tempered Ψ-fractional calculus.

To illustrate the applicability of the integral inequalities, we considered blowing-up solutions of initial value problems for differential equations with a tempered Ψ-Caputo fractional derivative and the right side depending on time, the solution itself, its tempered Ψ-Caputo fractional derivatives of lower orders, or tempered Ψ-Hilfer fractional integrals of the solution. Sufficient conditions for the nonexistence of this type of solutions were proven.

We also applied the new integral inequalities to prove sufficient conditions for the Ψ-exponential stability of the trivial solution of differential equations with tempered Ψ-Caputo fractional derivative and various right-hand sides.

Further stability results can be proven using our integral inequalities by considering more difficult nonlinearities, e.g., depending on the Ψ-Hilfer fractional integrals of the solution, or other kinds of fractional integrals and derivatives. Other possible applications include controllability, observability, existence of solutions, and other asymptotic properties of solutions.

Author Contributions: Conceptualization, M.M. and M.P.; methodology, M.M. and M.P.; investigation, M.M., M.P. and E.B.; validation, M.P., M.M. and E.B.; writing—original draft preparation, M.P.; writing—review and editing, M.P., M.M. and E.B.; supervision, M.M. All authors have read and agreed to the published version of the manuscript.

Funding: M.M. was supported by the Slovak Grant Agency VEGA grant No. 1/0084/23. M.P. was partially supported by the Slovak Grant Agency VEGA grant No. 1/0084/23 and No. 2/0127/20. A 100% discount on the APC for this invited paper was gladly accepted.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors thank the anonymous reviewers for their valuable comments which improved the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Henry, D. *Geometric Theory of Semilinear Parabolic Equations*; Springer: Berlin/Heidelberg, Germany; New York, NY, USA, 1981.
2. Medved', M. A new approach to an analysis of Henry type integral inequalities and their Bihari type versions. *J. Math. Anal. Appl.* **1997**, *214*, 349–366. [CrossRef]
3. Medved', M. Integral inequalities and global solutions of semilinear evolution equations. *J. Math. Anal. Appl.* **2002**, *267*, 643–650. [CrossRef]
4. Kirane, M.; Tatar, N.E. Global existence and stability of some semilinear problems. *Arch. Math. (Brno)* **2000**, *36*, 33–44.
5. Medved', M. Singular integral inequalities and stability of semilinear parabolic equations. *Arch. Math. (Brno)* **1998**, *34*, 183–190.
6. Tatar, N.E. The role of an integral inequality in the study of certain differential equations. *J. Ineq. Pure Appl. Math.* **2005**, *6*, 1–13.
7. Medved', M.; Pospíšil, M. On the existence and exponential stability for differential equations with multiple delays and nonlinearity depending on fractional substantial integrals. *Electron. J. Qual. Theory Differ. Equ.* **2019**, *43*, 1–17. [CrossRef]
8. Alsaedi, A.; Ahmad, B.; Kirane, M. A survey of useful inequalities in fractional calculus. *Fract. Calc. Appl. Anal.* **2017**, *20*, 574–594. [CrossRef]
9. Sousa, J.C.; De Oliveira, E.C. On the Ψ-Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *60*, 72–91. [CrossRef]
10. Roohi, M.; Zhang, C.; Chen, Y. Adaptive model-free synchronization of different fractional-order neural networks with an application in cryptography. *Nonlinear Dyn.* **2020**, *100*, 3979–4001.
11. Taheri, M.; Zhang, C.; Berardehi, Z.R.; Chen, Y.; Roohi, M. No-chatter model-free sliding mode control for synchronization of chaotic fractional-order systems with application in image encryption. *Multimed. Tools Appl.* **2022**, *81*, 24167–24197. [CrossRef]

12. Medved', M.; Brestovanská, E. Differential equations with tempered Ψ-Caputo fractional derivative. *Math. Model. Anal.* **2021**, *26*, 631–650. [CrossRef]
13. Sousa, J.C.; De Oliveira, E.C. A Gronwall inequality and the Cauchy-type problems by means of Ψ-Hilfer operator. *Differ. Equ. Appl.* **2019**, *11*, 87–16.
14. Kahouli, O.; Boucenna, D.; Makhlouf, A.B.; Alruwaily, Y. Some new weakly singular integral inequalities with applications to differential equations in frame of tempered χ-fractional derivatives. *Mathematics* **2022**, *10*, 3792.
15. Mali, A.D.; Kucche, K.D.; Fernandez, A.; Fahad, H.M. On tempered fractional calculus with respect to functions and the associated fractional differential equations. *Math. Meth. Appl. Sci.* **2022**, *45*, 11134–11157. [CrossRef]
16. Kucche, K.D.; Mali, A.D.; Fernandez, A.; Fahad, H.M. On tempered Hilfer fractional derivative with respect to functions and the associated fractional differential equations. *Chaos Solitons Fractals* **2022**, *163*, 112547. [CrossRef]
17. Butler, G.; Rogers, T. A generalization of a lemma of Bihari and applications to piecewise estimates for integral equations. *J. Math. Anal. Appl.* **1971**, *33*, 77–81.
18. Bellman, R. The stability of solutions of linear differential equations. *Duke Math. J.* **1943**, *10*, 643–647. [CrossRef]
19. Medved', M.; Pospíšil, M. Generalized Laplace transform and tempered Ψ-Caputo fractional derivative. *Math. Model. Anal.* **2023**, *28*, 146–162. [CrossRef]
20. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]
21. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
22. Almeida, R.; Malinowska, A.B.; Monteiro, T.T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Math. Meth. Appl. Sci.* **2018**, *41*, 336–352.
23. Erdélyi, A. *Higher Transcendental Functions*; McGraw-Hill Book Company, Inc.: New York, NY, USA, 1955; Volume 3.
24. Schneider, W.R. Completely monotone generalized Mittag–Leffler functions. *Expo. Math.* **1996**, *14*, 3–16.
25. Miller, K.S.; Samko, S.G. A note on the complete monotonicity of the generalized Mittag–Leffler function. *Real Anal. Exch.* **1997**, *23*, 753–755. [CrossRef]