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Adaptive Control for Finite-Time Cluster Synchronization of Fractional-Order Fully Complex-Valued Dynamical Networks

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Abstract: This paper aims to address finite-time cluster synchronization (FTCS) issues for fractional-order fully complex-valued dynamical networks (FFCVDNs) with time delay. To compensate for the limited application of one controller, the delay-dependent and delay-independent adaptive controllers with regard to quadratic and absolute-valued norms are developed, respectively. Based on the finite-time stability theorem and auxiliary inequality techniques, detailed Lyapunov analysis is provided to ensure that FFCVDNs can achieve FTCS, and the settling times (STs) are estimated on the basis of system and control parameters characterized by system models to decrease the conservativeness of the existing results. Finally, simulation examples are provided to verify the correctness of theoretical analysis.

Keywords: complex-valued dynamical networks; finite-time stability; adaptive control; cluster synchronization



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1. Introduction

Recently, a strong upsurge in investigations of cluster synchronization issues of dynamical networks (DNs) have been witnessed because this technique is able to describe some very relevant complex natural and social phenomena in multiple fields, such as engineering control, communication engineering, communication engineering, etc. Its core idea is that, based on the different characteristics and functions of nodes, the nodes can be classified into distinct clusters to satisfy the distinct synchronization behaviors in the different clusters [1–6]. It is worth noting that, in application, achieving synchronization within a finite time can better meet practical needs [7]. Therefore, some admirable achievements with respect to FTCS for DNs have been obtained [8–13].

However, the network models in the above-mentioned literature [8–13] focus on the integer order case and the real state space. Note that the fractional order network systems can characterize the quality of memory and heredity of nodes [14–16]. This inspires us to explore the dynamical behaviors of fractional-order complex-valued dynamical networks (FCVDNs) [17–22]. To date, several noticeable results in FTCS of complex DNs have been acquired. For example, by utilizing the separation method to convert complex-valued networks into two real-valued, synchronization problems for fractional-order complex-valued neural networks were studied [23,24]. It should be pointed out that the separation method may increase the complexity of theoretical analysis and calculation. To overcome this drawback, some norms about the sign function of complex numbers were introduced to address the synchronization issue of FCVDNs [25,26].

Furthermore, time delay is utilized to describe the dependence of the current state change rate on past states, a technique that is employed widely in communication networks, power systems, drug circulation systems, and so on. The existence of the time delay can increase the difficulty of analyzing system dynamic behaviors and reduce system stability [27,28]. In recent important works, the authors only considered no delay, a

single time delay, or a single coupling delay [25,26,29,30], which differs from dynamical systems in the actual world. Hence, the question of how to eliminate the impact of both time and coupling delays to achieve the cluster synchronization of FFCVDNs is a valuable and challenging issue. On the other hand, due to the complexities of nodes' dynamics and topological structures, complex dynamical networks are not always able to synchronize by themselves. Therefore, different effective control approaches such as state feedback control [8,9] and adaptive control [10,11] have been applied to achieve finite-time cluster synchronization. As a feasible and effective control strategy, adaptive control has fascinated a considerable number of scholars in recent years, since it can automatically adjust the control parameters based on the operating environment changes to achieve optimal control. Therefore, it is necessary to design an adaptive control strategy to solve the FTCS. Regrettably, to the best of our knowledge, few results for the FTCS of FFCVDNs with time delay under the adaptive control scheme have been published.

Motivated by the above discussion, this paper focuses on the FTCS issue of FFCVDNs with internal delay as well as non-delayed and delayed couplings. The main contributions of this paper can be summarized as follows: (1) A general fractional order complex network model is investigated, where the state space is extended to the complex number field. In the meantime, the internal delay and couplings delay are introduced to describe the complex dynamic behaviors of nodes. (2) A non-separation approach is adopted to simplify the design of the adaptive control strategy and decrease the computational costs. (3) The delay-dependent and delay-independent adaptive control schemes are utilized to realize the FTCS; less conservative results are obtained compared with the existing works.

The content of this paper is listed as follows: in Section 2, preliminaries and a model are offered; in Section 3, the delay-dependent and delay-independent adaptive controllers are utilized to realize FTCS for the delayed FFCVDNs, respectively; in Section 4, some numerical simulation examples are provided; Section 5 concludes this study.

Notations. In this study, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, $\mathbb{C}^n$, and $\mathbb{C}^{m \times n}$ denote the real field, positive real field, complex field, n -dimensional complex space, and $m \times n$ -dimensional complex matrices, respectively. For any $\mathcal{K} = \mathcal{D} + \mathbf{i}\mathcal{Q} \in \mathbb{C}$, $\bar{\mathcal{K}} = \mathcal{D} - \mathbf{i}\mathcal{Q}$, $|\mathcal{K}|_1 = |\mathcal{D}| + |\mathcal{Q}|$, $|\mathcal{K}|_2 = \sqrt{\bar{\mathcal{K}}\mathcal{K}}$, $[\mathcal{K}] = \text{sign}(\text{Re}(\mathcal{K})) + \mathbf{i}\text{sign}(\text{Im}(\mathcal{K}))$, $\mathbf{i} = \sqrt{-1}$. For any $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_n) \in \mathbb{C}^n$, $\mathcal{K} = \text{Re}(\mathcal{K}) + \mathbf{i}\text{Im}(\mathcal{K})$, $\|\mathcal{K}\|_1 = \|\text{Re}(\mathcal{K})\|_1 + \|\text{Im}(\mathcal{K})\|_1$, $\|\mathcal{K}\|_2 = \sqrt{\mathcal{K}^H \mathcal{K}}$. $[\mathcal{K}] = (\text{sign}(\text{Re}(\mathcal{K}_1)) + \mathbf{i}\text{sign}(\text{Im}(\mathcal{K}_1)), \dots, \text{sign}(\text{Re}(\mathcal{K}_n)) + \mathbf{i}\text{sign}(\text{Im}(\mathcal{K}_n)))^T$ represents the signum function vector of $\mathcal{K}$. $\mathcal{K}^H$ denotes conjugate transposition. $\Delta(k) = 1, 2, \dots, k$. $I_N = (1, 1, \dots, 1)_N^T$. $\text{diag}(l_1, l_2, \dots, l_n)$, E_N and $C^n([t_0, +\infty))$ denote an $n \times n$ diagonal matrix, the n -dimensional identity matrix, and the set of all continuous n -differential functions from $[t_0, +\infty)$ into $\mathbb{C}$, respectively. $\Gamma(\cdot)$ is the gamma function.

2. Preliminaries and Modeling

Definition 1 ([31,32]). The fractional integral of order $0 < \alpha < 1$ for $f(t) : [t_0, +\infty) \rightarrow \mathbb{C}$ is defined by:

$${}_{t_0}I_t^\alpha = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2 ([31,32]). For $f \in C^n([t_0, +\infty), \mathbb{C})$, the Caputo fractional-order derivative α -order is defined as:

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t \frac{f'(s)}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1.$$

In this work, we consider the delayed FFCVDNs composed of r communities and N nodes; the dynamic equation of the i -th ($i \in \Delta(N)$) model is described by:

$${}^C D_t^\alpha x_i(t) = f_{\zeta_i}(x_i(t), x_i(t - \pi_1)) + c_1 \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} a_{ij} G_1 x_j(t) + c_2 \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} b_{ij} G_2 x_j(t - \pi_2), \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbb{C}^n$; $\mathcal{C}_k$ represents a set of nodes belonging to the k th cluster; $\varsigma_i : \Delta(N) \rightarrow \Delta(r)$ satisfies that if $i \in \mathcal{C}_k$, then $\varsigma_i = k$; $f_{\varsigma_i} : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a nonlinear function; $c_1, c_2 \in \mathbb{C}$ represent the coupling strength; π_1 and π_2 denote the internal delay and the coupling delay, respectively; $G_l = \text{diag}(\delta_1^{(l)}, \delta_2^{(l)}, \dots, \delta_n^{(l)}) \in \mathbb{C}^{n \times n}$ ($l = 1, 2$) is the inner matrix; and $A = (a_{ij})_{N \times N}$ and $B = (b_{ij})_{N \times N}$ denote the topological structure of the FFCVDNs without and with delay, respectively, satisfying the following rules. If nodes i and j exist a connection, then $a_{ij}, b_{ij} \neq 0 \in \mathbb{C}$, $a_{ii} = -\sum_{j=1, j \neq i}^N a_{ij}$, $b_{ii} = -\sum_{j=1, j \neq i}^N b_{ij}$; otherwise, $a_{ij} = b_{ij} = 0$ ($i \neq j$).

Suppose N nodes are divided into r nonempty communities indicated by $\Delta(N) = \bigcup_{v=1}^r \mathcal{C}_v$, where $\mathcal{C}_1 = \{1, \dots, q_1\}$, $\mathcal{C}_2 = \{q_1 + 1, \dots, q_2\}$, $\dots$, $\mathcal{C}_r = \{q_{r-1} + 1, \dots, N\}$. Then, A and B can be respectively expressed as:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rr} \end{bmatrix}.$$

Assumption 1. Matrices A_{pm} and B_{pm} satisfy zero row sum matrices $p, m = 1, 2, \dots, r$.

Remark 1. According to Assumption 1, there exists a cooperative relationship between nodes in the same community, while there exists a cooperative or competitive relationship between nodes in different communities.

The isolated system is described by:

$${}^C_{t_0} D_t^\alpha \mathfrak{s}_{\varsigma_i}(t) = f_{\varsigma_i}(\mathfrak{s}_{\varsigma_i}(t), \mathfrak{s}_{\varsigma_i}(t - \pi)) \quad (2)$$

and satisfies $\mathfrak{s}_{\varsigma_i}(t) \neq \mathfrak{s}_{\varsigma_j}(t)$ ($\varsigma_i \neq \varsigma_j$).

Definition 3. To achieve the FTCS goal of (1) and (2), define $\mathfrak{e}_i(t) = x_i(t) - \mathfrak{s}_{\varsigma_i}(t)$ as the error variables, $\mathfrak{e}(t_0) = x_i(0) - \mathfrak{s}_{\varsigma_i}(0)$. Under the controller $u_i(t)$, (1) and (2) are said to realize FTCS if there exists an ST function $\mathbb{T}(\mathfrak{e}(t_0))$ such that $\lim_{t \rightarrow \mathbb{T}(\mathfrak{e}(t_0))} \|\mathfrak{e}_i(t)\| = 0$, and $\|\mathfrak{e}_i(t)\| \equiv 0$ for $t > \mathbb{T}(\mathfrak{e}(t_0))$.

From Assumption 1, we infer that $\sum_{j \in \mathcal{C}_k} a_{ij} G_1 \mathfrak{s}_k(t) = 0$ and $\sum_{j \in \mathcal{C}_k} b_{ij} G_2 \mathfrak{s}_k(t - \pi_2) = 0$. Further, the error system can be expressed as:

$${}^C_{t_0} D_t^\alpha \mathfrak{e}_i(t) = F_{\varsigma_i}(\mathfrak{e}_i(t), \mathfrak{e}_i(t - \pi_1)) + c_1 \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} a_{ij} G_1 \mathfrak{e}_j(t) + c_2 \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} b_{ij} G_2 \mathfrak{e}_j(t - \pi_2) + u_i(t), \quad (3)$$

where $u_i(t)$ is the control law, and $F_{\varsigma_i}(\mathfrak{e}_i(t), \mathfrak{e}_i(t - \pi_1)) = f_{\varsigma_i}(x_i(t), x_i(t - \pi_1)) - f_{\varsigma_i}(\mathfrak{s}_{\varsigma_i}(t), \mathfrak{s}_{\varsigma_i}(t - \pi_1))$.

Assumption 2. For $x(t), y(t) \in \mathbb{C}^n$, $t \geq 0$, there exist constants $\vartheta_k^{(p)} > 0$, $\zeta_k^{(p)} > 0$ ($p = 1, 2$) such that:

$$\|\hat{f}_k(x(t), x(t - \pi_1)) - \hat{f}_k(y(t), y(t - \pi_1))\|_p \leq \vartheta_k^{(p)} \|x(t) - y(t)\|_p + \zeta_k^{(p)} \|x(t - \pi_1) - y(t - \pi_1)\|_p.$$

Remark 2. It is not difficult to verify that many chaotic dynamical systems satisfy Assumption 2, such as the Lorenz system, the Chua system, delayed DNs, etc. [33–35].

Lemma 1 ([36]). Let $f(t)$ be a continuously differentiable function; then:

$${}^C_{t_0} D_t^\alpha f^2(t) \leq 2f(t) {}^C_{t_0} D_t^\alpha f(t), t \in [t_0, \infty), \alpha \in (0, 1).$$

Lemma 2 ([25,37]). Let $\aleph(t) \in \mathbb{C}^n$ be differentiable; then:

$${}^C D_t^\alpha \aleph^H(t) \aleph(t) \leq \aleph^H(t) {}^C D_t^\alpha \aleph(t) + ({}^C D_t^\alpha \aleph^H(t)) \aleph(t), \alpha \in (0, 1), t \geq t_0.$$

Lemma 3 ([25,30]). Let $\alpha \in (0, 1)$, $\|(\sqcup), \aleph(t) \in \mathbb{C}^n$ and $\wp \in \mathbb{C}$; then:

- (a) ${}^C D_t^\alpha (\aleph^H(\sqcup) [\aleph(t)] + [\aleph(t)]^H \aleph(t)) \leq [\aleph(t)]^H {}^C D_t^\alpha \aleph(t) + ({}^C D_t^\alpha \aleph^H(\sqcup)) [\aleph(t)]$
- (b) $\aleph^H(\sqcup) [\aleph(t)] + [\aleph(t)]^H \aleph(t) = 2\|\aleph(t)\|_1 \geq 2\|\aleph(t)\|_2.$
- (c) $\|{}^H(\sqcup) [\aleph(t)] + [\aleph(t)]^H\|(\sqcup) \leq 2\|(\sqcup)\|_1.$
- (d) $\wp + \bar{\wp} = 2\text{Re}(\wp) \leq |\wp|_2 \leq |\wp|_1.$

Lemma 4 ([25]). If there is a $\Lambda > 0$ that satisfies:

$${}^C D_t^\alpha V(t) \leq -\Lambda, \quad V(t) \in \mathbb{R}^+, \quad (4)$$

then for all $t \geq \mathbb{T}^*$, $\lim_{t \rightarrow \mathbb{T}^*} V(t) = 0$ and $V(t) \equiv 0$, where:

$$\mathbb{T}^* = t_0 + \left(\frac{V(t_0)\Gamma(\alpha + 1)}{\Lambda} \right)^{\frac{1}{\alpha}}. \quad (5)$$

3. Main Results

The FTCS issue for the considered FFCVDNs is addressed by designing delay-dependent and delay-independent adaptive controllers, respectively, in this section.

3.1. Delay-Dependent Adaptive Control

Theorem 1. Suppose that Assumptions 1–2 hold. Considering systems (1) and (2), the controller is designed as:

$$u_i(t) = \begin{cases} -d_i(t)\epsilon_i(t) - \Lambda \frac{[\epsilon_i(t)]}{\|\epsilon_i(t)\|_1} - \frac{1}{2} \sum_{p=1}^2 \epsilon_i^{(p)} \epsilon_i(t - \pi_p), & \epsilon_i(t) \neq 0, \\ 0, & \epsilon_i(t) = 0, \end{cases} \quad (6)$$

where the feedback strength $d_i(t)$ is adapted based on the following updated law:

$${}^C D_t^\alpha d_i(t) = q_i \epsilon_i^H(t) \epsilon_i(t), \quad (7)$$

and $q_i, \Lambda, \check{\theta}_i^{(1)}, \check{\theta}_i^{(2)} > 0$, satisfying:

$$\begin{cases} \zeta^{(2)} - \Theta_{(1)} \leq 0, \\ c_2 \delta_l^{(2)} B - \Theta_{(2)} \leq 0, \\ \bar{c}_2 \delta_l^{(2)} B^H - \Theta_{(2)} \leq 0, \end{cases} \quad (8)$$

where $l = \Delta(n)$, $\zeta^{(2)} = \text{diag}(\underbrace{\zeta_1^{(2)}, \dots, \zeta_1^{(2)}}_{1, \dots, q_1}, \underbrace{\zeta_2^{(2)}, \dots, \zeta_2^{(2)}}_{q_1+1, \dots, q_2}, \dots, \underbrace{\zeta_r^{(2)}, \dots, \zeta_r^{(2)}}_{q_{r-1}+1, \dots, N})$,

$\Theta^{(1)} = \text{diag}(\check{\theta}_1^{(1)}, \check{\theta}_2^{(1)}, \dots, \check{\theta}_N^{(1)})$, $\Theta^{(2)} = \text{diag}(\check{\theta}_1^{(2)}, \check{\theta}_2^{(2)}, \dots, \check{\theta}_N^{(2)})$; then, the FFCVDNs (1) and (2) can realize FTCS via controllers (6) and (7). Additionally, the ST is estimated as:

$$\mathbb{T} \leq \mathbb{T}_1 = t_0 + \left(\frac{V_1(t_0)\Gamma(\alpha + 1)}{\Lambda} \right)^{\frac{1}{\alpha}}, \quad (9)$$

where $V_1(t_0) = \frac{1}{2} \sum_{i=1}^N (\mathbf{e}_i^H(t_0) \mathbf{e}_i(t_0) + \frac{1}{\varrho_i} (\hat{d}_i(t_0) - d_i)^2)$, $\mathbf{e}_i(t_0)$ and $\hat{d}_i(t_0)$ are the initial values of $\mathbf{e}_i(t)$ and $d_i(t)$, respectively. d_i is a positive constant such that $2\vartheta^{(2)} + c_1 \delta_l^{(1)} A + \overline{c_1 \delta_l^{(1)}} A^H - 2\mathfrak{D} \leq 0$, $\vartheta^{(2)} = \text{diag}(\underbrace{\vartheta_1^{(2)}, \dots, \vartheta_1^{(2)}}_{1, \dots, q_1}, \underbrace{\vartheta_2^{(2)}, \dots, \vartheta_2^{(2)}}_{q_1+1, \dots, q_2}, \dots, \underbrace{\vartheta_r^{(2)}, \dots, \vartheta_r^{(2)}}_{q_{r-1}+1, \dots, N})$, $\mathfrak{D} = \text{diag}(d_1, d_2, \dots, d_N)$.

Proof. Construct the Lyapunov function:

$$V_1(e(t)) = \frac{1}{2} \sum_{i=1}^N \left(\frac{1}{\varrho_i} (d_i(t) - d_i)^2 + \mathbf{e}_i^H(t) \mathbf{e}_i(t) \right). \quad (10)$$

By employing Lemmas 1 and 2, the derivative of $V_1(t)$ and the solution of (3) are computed as:

$$\begin{aligned} {}^C_{t_0} D_t^\alpha V_1(e(t)) &\leq \frac{1}{2} \sum_{i=1}^N (\mathbf{e}_i^H(t) F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) + F_{\zeta_i}^H(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) \mathbf{e}_i(t)) \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in C_k} (\mathbf{e}_i^H(t) c_1 a_{ij} G_1 \mathbf{e}_j(t) + \mathbf{e}_j^H(t) \overline{c_1 a_{ij}} G_1^H \mathbf{e}_i(t)) \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in C_k} (\mathbf{e}_i^H(t) c_2 b_{ij} G_2 \mathbf{e}_j(t - \pi_2) + \mathbf{e}_j^H(t - \pi_2) \overline{c_2 b_{ij}} G_2^H \mathbf{e}_i(t)) \\ &\quad - \sum_{i=1}^N d_i(t) \mathbf{e}_i^H(t) \mathbf{e}_i(t) - \frac{1}{2} \sum_{p=1}^2 \sum_{i=1}^N \vartheta_i^{(p)} (\mathbf{e}_i^H(t) \mathbf{e}_i(t - \pi_p) + \mathbf{e}_i^H(t - \pi_p) \mathbf{e}_i(t)) \\ &\quad + \sum_{i=1}^N (d_i(t) - d_i) \mathbf{e}_i^H(t) \mathbf{e}_i(t) - \Lambda \sum_{i=1, \mathbf{e}_i(t) \neq 0}^N \frac{\mathbf{e}_i^H(t) [\mathbf{e}_i(t)] + [\mathbf{e}_i(t)]^H \mathbf{e}_i(t)}{\|\mathbf{e}_i(t)\|_1}. \end{aligned} \quad (11)$$

Based on Assumption 1 and Lemma 3, this yields that:

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^N (\mathbf{e}_i^H(t) F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) + F_{\zeta_i}^H(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) \mathbf{e}_i(t)) \\ &\leq \sum_{l=1}^n \mathbf{e}_l^H(t) \vartheta^{(2)} \mathbf{e}_l(t) + \frac{1}{2} \sum_{l=1}^n (\mathbf{e}_l^H(t) \zeta^{(2)} \mathbf{e}_l(t - \pi_1) + \mathbf{e}_l^H(t - \pi_1) \zeta^{(2)} \mathbf{e}_l(t)). \end{aligned} \quad (12)$$

By using Lemma 3 again, we derive:

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in C_k} (\mathbf{e}_i^H(t) c_1 a_{ij} G_1 \mathbf{e}_j(t) + \mathbf{e}_j^H(t) \overline{c_1 a_{ij}} G_1^H \mathbf{e}_i(t)) \\ &= \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t) (c_1 \delta_l^{(1)} A + \overline{c_1 \delta_l^{(1)}} A^H) \mathbf{e}_l(t), \end{aligned} \quad (13)$$

and

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in C_k} (\mathbf{e}_i^H(t) c_2 b_{ij} G_2 \mathbf{e}_j(t - \pi_2) + \mathbf{e}_j^H(t - \pi_2) \overline{c_2 b_{ij}} G_2^H \mathbf{e}_i(t)) \\ &= \frac{1}{2} \sum_{l=1}^n \left(\mathbf{e}_l^H(t) (c_2 \delta_l^{(2)} B) \mathbf{e}_l(t - \pi_2) + \mathbf{e}_l^H(t - \pi_2) (\overline{c_2 \delta_l^{(2)}} B^H) \mathbf{e}_l(t) \right). \end{aligned} \quad (14)$$

For $i \in \Delta(N)$, there is at least an index i , making $\mathbf{e}_i(t) \neq 0$ for any $t \geq t_0$. Correspondingly, we can obtain:

$$-\Lambda \sum_{i=1, \mathbf{e}_i(t) \neq 0}^N \frac{\mathbf{e}_i^H(t) [\mathbf{e}_i(t)] + [\mathbf{e}_i(t)]^H \mathbf{e}_i(t)}{\|\mathbf{e}_i(t)\|_1} \leq -\Lambda. \quad (15)$$

Substituting (12)–(16) into (11) gives:

$$\begin{aligned} {}^C_{t_0} D_t^\alpha V_1(t) &\leq \sum_{l=1}^n \left(\mathbf{e}_l^H(t) (c_2 \delta_l^{(2)} B - \Theta^{(2)}) \mathbf{e}_l(t - \pi_2) \right) + \sum_{l=1}^n \left(\mathbf{e}_l^H(t) (\zeta^{(2)} - \Theta^{(1)}) \mathbf{e}_l(t - \pi_1) \right) \\ &\quad + \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t) \left(-2\mathfrak{D} + 2\vartheta^{(2)} + c_1 \delta_l^{(1)} A + \overline{c_1 \delta_l^{(1)}} A^H \right) \mathbf{e}_l(t) - \Lambda \\ &\leq -\Lambda. \end{aligned} \quad (16)$$

On the basis of Lemma 4, the FFCVDNs (1) and (2) under controllers (6) and (7) can achieve FTCS within $\mathbb{T}_1$. This completes the proof. $\square$

Theorem 2. Suppose that Assumptions 1 and 2 hold. Considering systems (1) and (2), the controller is designed as:

$$u_i(t) = -d_i(t) \mathbf{e}_i(t) - \Lambda [\mathbf{e}_i(t)] - \frac{1}{2} \sum_{p=1}^2 \check{\theta}_i^{(p)} [\mathbf{e}_i(t)] \mathbf{e}_i^H(t - \pi_p) [\mathbf{e}_i(t - \pi_p)], \quad (17)$$

where the feedback strength $d_i(t)$ is adapted given the updated law mentioned below:

$${}^C_{t_0} D_t^\alpha d_i(t) = \frac{1}{2} \varrho_i ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]), \quad (18)$$

and $\varrho_i, \Lambda, \check{\theta}_i^{(1)}, \check{\theta}_i^{(2)} > 0, i = 1, 2, \dots, N$ satisfying:

$$\begin{cases} \zeta^{(1)} - \Theta^{(1)} \leq 0, \\ \Psi^{(l)} - \Theta^{(2)} \leq 0, \end{cases} \quad (19)$$

for $l = \Delta(n)$, where $\zeta^{(1)} = \text{diag}(\underbrace{\zeta_1^{(1)}, \dots, \zeta_1^{(1)}}_{1, \dots, q_1}, \underbrace{\zeta_2^{(1)}, \dots, \zeta_2^{(1)}}_{q_1+1, \dots, q_2}, \dots, \underbrace{\zeta_r^{(1)}, \dots, \zeta_r^{(1)}}_{q_{r-1}+1, \dots, N})$,

$\Theta^{(p)} = \text{diag}(\check{\theta}_1^{(p)}, \check{\theta}_2^{(p)}, \dots, \check{\theta}_N^{(p)}), (p = 1, 2), \Psi^{(l)} = (\Psi_{ij}^{(l)})_{N \times N}$ and $\Psi_{ij}^{(l)} = |\text{Re}(c_2 b_{ij} \delta_l^{(2)})| + |\text{Im}(c_2 b_{ij} \delta_l^{(2)})|$; then, the FFCVDNs (1) can achieve FNTS under controllers (17) and (18). Moreover, the ST is estimated by:

$$\mathbb{T} \leq \mathbb{T}_2 = t_0 + \left(\frac{V_2(t_0) \Gamma(\alpha + 1)}{\Lambda} \right)^{\frac{1}{\alpha}}, \quad (20)$$

where $V_2(t_0) = \frac{1}{2} \sum_{i=1}^N (\|\mathbf{e}_i(t_0)\|_1 + \frac{1}{\varrho_i} (\hat{d}_i(t_0) - d_i)^2)$. $d_i > 0$ satisfies $\vartheta^{(1)} + Y^{(l)} - \mathfrak{D} \leq 0$, $\vartheta^{(1)} = \text{diag}(\underbrace{\vartheta_1^{(1)}, \dots, \vartheta_1^{(1)}}_{1, \dots, q_1}, \underbrace{\vartheta_2^{(1)}, \dots, \vartheta_2^{(1)}}_{q_1+1, \dots, q_2}, \dots, \underbrace{\vartheta_r^{(1)}, \dots, \vartheta_r^{(1)}}_{q_{r-1}+1, \dots, N})$, $\mathfrak{D} = \text{diag}(d_1, d_2, \dots, d_N)$, $Y^{(l)} = (\tau_{ij}^{(l)})_{N \times N}$ and

$$\tau_{ij}^{(l)} = \begin{cases} \text{Re}(c_1 a_{ii} \delta_l^{(1)}) + |\text{Im}(c_1 a_{ii} \delta_l^{(1)})|, & i = j, \\ |c_1 a_{ij} \delta_l^{(1)}|_1, & i \neq j. \end{cases}$$

Proof. Consider the Lyapunov function:

$$V_2(t) = \frac{1}{2} \sum_{i=1}^N \left(\frac{1}{\varrho_i} (d_i(t) - d_i)^2 + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)] + [\mathbf{e}_i(t)]^H \mathbf{e}_i(t) \right). \quad (21)$$

By employing Lemmas 1 and 3, the derivative of $V_2(t)$ for the solution of (3) is computed as:

$$\begin{aligned}
 {}^C_{t_0}D_t^\alpha V_2(t) &\leq \frac{1}{2} \sum_{i=1}^N ([\mathbf{e}_i(t)]^H F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) + F_{\zeta_i}^H(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) [\mathbf{e}_i(t)]) \\
 &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} ([\mathbf{e}_i(t)]^H c_{1ij} G_1 \mathbf{e}_j(t) + \mathbf{e}_j^H(t) \overline{c_{1ij}} G_1^H [\mathbf{e}_i(t)]) \\
 &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} ([\mathbf{e}_i(t)]^H c_{2ij} G_2 \mathbf{e}_j(t - \pi_2) + \mathbf{e}_j^H(t - \pi_2) \overline{c_{2ij}} G_2^H [\mathbf{e}_i(t)]) \\
 &\quad - \frac{1}{2} \sum_{i=1}^N d_i(t) ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]) - \Lambda \sum_{i=1}^N [\mathbf{e}_i(t)]^H [\mathbf{e}_i(t)] \\
 &\quad - \frac{1}{2} \sum_{r=1}^2 \sum_{i=1}^N \check{\theta}_i^{(r)} ([\mathbf{e}_i(t - \pi_r)]^H \mathbf{e}_i(t - \pi_r) + \mathbf{e}_i^H(t - \pi_r) [\mathbf{e}_i(t - \pi_r)]) \\
 &\quad + \frac{1}{2} \sum_{i=1}^N (d_i(t) - d_i) ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]).
 \end{aligned} \tag{22}$$

In the light of Assumption 1 and Lemma 3, one obtains:

$$\begin{aligned}
 &\frac{1}{2} \sum_{i=1}^N ([\mathbf{e}_i(t)]^H F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) + F_{\zeta_i}^H(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) [\mathbf{e}_i(t)]) \\
 &\leq \sum_{i=1}^N \|F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1))\|_1 \\
 &= \sum_{l=1}^n I_N^T \boldsymbol{\vartheta}^{(1)} \widetilde{\mathbf{e}_l(t)} + \sum_{l=1}^n I_N^T \boldsymbol{\zeta}^{(1)} \widetilde{\mathbf{e}_l(t - \pi_1)},
 \end{aligned} \tag{23}$$

where $\widetilde{\mathbf{e}_l(t)} = (|e_{1l}(t)|_1, |e_{2l}(t)|_1, \dots, |e_{Nl}(t)|_1)^T$ and $\widetilde{\mathbf{e}_l(t - \pi_1)} = (|e_{1l}(t - \pi_1)|_1, |e_{2l}(t - \pi_1)|_1, \dots, |e_{Nl}(t - \pi_1)|_1)^T$.

Furthermore, from Lemma 3, it holds that: that

$$\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} ([\mathbf{e}_i(t)]^H c_{1ij} G_1 \mathbf{e}_j(t) + \mathbf{e}_j^H(t) \overline{c_{1ij}} G_1^H [\mathbf{e}_i(t)]) = \sum_{l=1}^n I_N^T \mathbf{Y}^{(l)} \widetilde{\mathbf{e}_l(t)}, \tag{24}$$

and

$$\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} ([\mathbf{e}_i(t)]^H c_{2ij} G_2 \mathbf{e}_j(t - \pi_2) + \mathbf{e}_j^H(t - \pi_2) \overline{c_{2ij}} G_2^H [\mathbf{e}_i(t)]) = \sum_{l=1}^n I_N^T \mathbf{\Psi}^{(l)} \widetilde{\mathbf{e}_l(t - \pi_2)}. \tag{25}$$

Additionally, it is not difficult to find:

$$-\Lambda \sum_{i=1}^N [\mathbf{e}_i(t)]^H [\mathbf{e}_i(t)] = -\Lambda \sum_{i=1}^N \sum_{l=1}^n \left((\text{sign}(\text{Re}(e_{il}(t))))^2 + (\text{sign}(\text{Im}(e_{il}(t))))^2 \right). \tag{26}$$

There is at least a pair of (i, l) for $i \in \Delta(N)$ and $l \in \Delta(n)$, such that $e_{il}(t) \neq 0$ for any $t \geq t_0$. Therefore, we obtain:

$$-\Lambda \sum_{i=1}^N [\mathbf{e}_i(t)]^H [\mathbf{e}_i(t)] \leq -\Lambda. \tag{27}$$

On grounds of Lemma 3, we have:

$$-\frac{1}{2} \sum_{i=1}^N d_i ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]) = - \sum_{l=1}^n I_N^T \mathfrak{D} \widetilde{\mathbf{e}_l(t)}, \quad (28)$$

and

$$-\frac{1}{2} \sum_{p=1}^2 \sum_{i=1}^N \varepsilon_i^{(p)} ([\mathbf{e}_i(t - \pi_p)]^H \mathbf{e}_i(t - \pi_p) + \mathbf{e}_i^H(t - \pi_p) [\mathbf{e}_i(t - \pi_p)]) = - \sum_{p=1}^2 \sum_{l=1}^n I_N^T \Theta^{(p)} \widetilde{\mathbf{e}_l(t - \pi_p)}. \quad (29)$$

Substituting (23)–(25) and (27)–(28) into (22) gives:

$$\begin{aligned} {}^C D_t^\alpha V_2(t) &\leq \sum_{l=1}^n I_N^T (\vartheta^{(1)} + Y^{(l)} - \mathfrak{D}) \widetilde{\mathbf{e}_l(t)} + \sum_{l=1}^n I_N^T (\zeta^{(1)} - \Theta^{(1)}) \widetilde{\mathbf{e}_l(t - \pi_1)} \\ &\quad + \sum_{l=1}^n I_N^T (\Psi^{(l)} - \Theta^{(2)}) \widetilde{\mathbf{e}_l(t - \pi_2)} - \Lambda \\ &\leq -\Lambda. \end{aligned} \quad (30)$$

Similarly, based on Lemma 4, the FFCVDNs (1) under controllers (17) and (18) can achieve FTCS within $\mathbb{T}_2$. This completes the proof. $\square$

Remark 3. It follows that $\|\mathbf{e}_i(t_0)\|_1 \geq \|\mathbf{e}_i(t_0)\|_2$ and that $\mathbb{T}_1 \leq \mathbb{T}_2$ for choosing the same Λ , q_i , and d_i . This indicates that, when assessing the ST under the same conditions, Theorem 1 provides a more accurate result than Theorem 2. In addition, from (9) and (34), it is clear that the ST is connected to the fractional-order α and the control parameter Λ .

3.2. Delay-Independent Adaptive Control

Superficially, the above delay-dependent adaptive controllers (6) and (17) are somewhat complicated and difficult to adopt in practice. Therefore, it is interesting and challenging to design delay-independent adaptive controllers to achieve FTCS of FFCVDNs.

Theorem 3. Suppose that Assumptions 1 and 2 hold. The control law is designed as:

$$u_i(t) = \begin{cases} -d_i(t) \mathbf{e}_i(t) - \Lambda \frac{[\mathbf{e}_i(t)]}{\|\mathbf{e}_i(t)\|_1}, & \mathbf{e}_i(t) \neq 0, \\ 0 & \mathbf{e}_i(t) = 0, \end{cases} \quad (31)$$

where the feedback strength $d_i(t)$ is adapted given the updated law mentioned below:

$${}^C D_t^\alpha d_i(t) = q_i \mathbf{e}_i^H(t) \mathbf{e}_i(t), \quad (32)$$

and $q_i, \Lambda > 0$. If the following conditions are satisfied:

$$\begin{cases} \zeta^{(2)} - 2\Theta^{(1)} \leq 0, \\ \Psi^{(l)} - 2\Theta^{(2)} \leq 0, \end{cases} \quad (33)$$

where $\check{\theta}_i^{(1)} > 0$, $\check{\theta}_i^{(2)} > 0$ are constants, $\zeta^{(2)} = \text{diag}(\underbrace{\zeta_1^{(2)}, \dots, \zeta_1^{(2)}}_{1, \dots, q_1}, \underbrace{\zeta_2^{(2)}, \dots, \zeta_2^{(2)}}_{q_1+1, \dots, q_2}, \dots,$

$\underbrace{\zeta_r^{(2)}, \dots, \zeta_r^{(2)}}_{q_{r-1}+1, \dots, N}$), $\Theta^{(p)} = \text{diag}(\varepsilon_1^{(p)}, \check{\theta}_2^{(p)}, \dots, \check{\theta}_N^{(p)})$, ($p = 1, 2$), $\Psi^{(l)} = (\Psi_{ij}^{(l)})_{N \times N}$ and

$\Psi_{ij}^{(l)} = |\operatorname{Re}(c_2 b_{ij} \delta_l^{(2)})| + |\operatorname{Im}(c_2 b_{ij} \delta_l^{(2)})|$, then the FFCVDNs (1) are said to achieve FNTS under controllers (31) and (32). Further, the ST can be estimated by:

$$\mathbb{T} \leq \mathbb{T}_3 = t_0 + \left(\frac{V_3(t_0) \Gamma(\alpha + 1)}{\Lambda} \right)^{\frac{1}{\alpha}}, \quad (34)$$

in which $V_3(t_0) = \sum_{i=1}^N \left(\frac{1}{2} \mathbf{e}_i^H(t_0) \mathbf{e}_i(t_0) + \sum_{p=1}^2 \check{\theta}_i^{(p)} \int_{t_0-\pi_p}^{t_0} \mathbf{e}_i^H(s) \mathbf{e}_i(s) ds + \frac{1}{2\varrho_i} (\hat{d}_i(t_0) - d_i)^2 \right)$.
 $d_i > 0$ such that $2\vartheta^{(2)} + \zeta^{(2)} + c_1 \delta_l^{(1)} A + \bar{c}_1 \delta_l^{(1)} A^H + 2\Theta^{(1)} + 2\Theta^{(2)} + \Psi^{(l)} - 2\mathfrak{D} \leq 0$,
 $\vartheta^{(2)} = \operatorname{diag}(\underbrace{\vartheta_1^{(2)}, \dots, \vartheta_1^{(2)}}_{1, \dots, q_1}, \underbrace{\vartheta_2^{(2)}, \dots, \vartheta_2^{(2)}}_{q_1+1, \dots, q_2}, \dots, \underbrace{\vartheta_r^{(2)}, \dots, \vartheta_r^{(2)}}_{q_{r-1}+1, \dots, N})$, $\mathfrak{D} = \operatorname{diag}(d_1, d_2, \dots, d_N)$.

Proof. Construct the following Lyapunov functional:

$$V_3(e(t)) = \sum_{i=1}^N \left(\frac{1}{2\varrho_i} (d_i(t) - d_i)^2 + \sum_{p=1}^2 \check{\theta}_i^{(p)} \int_{t-\pi_p}^t \mathbf{e}_i^H(s) \mathbf{e}_i(s) ds + \frac{1}{2} \mathbf{e}_i^H(t) \mathbf{e}_i(t) \right). \quad (35)$$

By employing Lemmas 1 and 2, the derivative of $V_3(t)$ for the solution of (3) is computed as:

$$\begin{aligned} {}_{t_0}^C D_t^\alpha V_3(e(t)) &\leq \frac{1}{2} \sum_{i=1}^N (\mathbf{e}_i^H(t) F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) + F_{\zeta_i}^H(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) \mathbf{e}_i(t)) \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} (\mathbf{e}_i^H(t) c_1 a_{ij} G_1 \mathbf{e}_j(t) + \mathbf{e}_j^H(t) \bar{c}_1 a_{ij} G_1^H \mathbf{e}_i(t)) \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} (\mathbf{e}_i^H(t) c_2 b_{ij} G_2 \mathbf{e}_j(t - \pi_2) + \mathbf{e}_j^H(t - \pi_2) \bar{c}_2 b_{ij} G_2^H \mathbf{e}_i(t)) \\ &\quad - \sum_{i=1}^N d_i(t) \mathbf{e}_i^H(t) \mathbf{e}_i(t) - \Lambda \sum_{i=1, \mathbf{e}_i(t) \neq 0}^N \frac{[\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + e_i^H(t) [\mathbf{e}_i(t)]}{\|\mathbf{e}_i(t)\|_1} \\ &\quad + \sum_{i=1}^N (d_i(t) - d_i) \mathbf{e}_i^H(t) \mathbf{e}_i(t) + \sum_{p=1}^2 \sum_{i=1}^N \check{\theta}_i^{(p)} \mathbf{e}_i^H(t) \mathbf{e}_i(t) \\ &\quad - \sum_{p=1}^2 \sum_{i=1}^N \check{\theta}_i^{(p)} \mathbf{e}_i^H(t - \pi_p) \mathbf{e}_i(t - \pi_p). \end{aligned} \quad (36)$$

Similar to the derivation of (12) and (14), we have

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^N (\mathbf{e}_i^H(t) F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) + F_{\zeta_i}^H(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) \mathbf{e}_i(t)) \\ &\leq \sum_{l=1}^n \mathbf{e}_l^H(t) \vartheta^{(2)} \mathbf{e}_l(t) + \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t) \zeta^{(2)} \mathbf{e}_l(t) + \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t - \pi_1) \zeta^{(2)} \mathbf{e}_l(t - \pi_1), \end{aligned} \quad (37)$$

and

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} (\mathbf{e}_i^H(t) c_2 b_{ij} G_2 \mathbf{e}_j(t - \pi_2) + \mathbf{e}_j^H(t - \pi_2) \overline{c_2 b_{ij} G_2} \mathbf{e}_i(t)) \\
& \leq \frac{1}{2} \sum_{l=1}^n \sum_{j=1}^N \sum_{l=1}^n \overline{\mathbf{e}_{il}(t)} (|\operatorname{Re}(c_2 b_{ij} \delta_l^{(2)})| + |\operatorname{Im}(c_2 b_{ij} \delta_l^{(2)})|) \mathbf{e}_{il}(t) \\
& \quad + \frac{1}{2} \sum_{l=1}^n \sum_{j=1}^N \sum_{l=1}^n \overline{\mathbf{e}_{jl}(t - \pi_2)} (|\operatorname{Re}(c_2 b_{ij} \delta_l^{(2)})| + |\operatorname{Im}(c_2 b_{ij} \delta_l^{(2)})|) \mathbf{e}_{jl}(t - \pi_2) \\
& = \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t) \Psi^{(l)} \mathbf{e}_l(t) + \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t - \pi_2) \Psi^{(l)} \mathbf{e}_l(t - \pi_2).
\end{aligned} \tag{38}$$

Substituting (13), (16), (37), and (38) into (36) gives:

$$\begin{aligned}
{}^C_{t_0} D_t^\alpha V_3(t) & \leq \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t) (2\vartheta^{(2)} + \zeta^{(2)} + c_1 \delta_l^{(1)} A + \overline{c_1 \delta_l^{(1)}} A^H + 2\Theta^{(1)} + 2\Theta^{(2)} + \Psi^{(l)} \\
& \quad - 2\mathfrak{D}) \mathbf{e}_l(t) - \Lambda + \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t - \pi_1) (\zeta^{(2)} - 2\Theta^{(1)}) \mathbf{e}_l(t - \pi_1) \\
& \quad + \frac{1}{2} \sum_{l=1}^n \mathbf{e}_l^H(t - \pi_2) (\Psi^{(l)} - 2\Theta^{(2)}) \mathbf{e}_l(t - \pi_2) \\
& \leq -\Lambda.
\end{aligned} \tag{39}$$

According to Lemma 4, the FFCVDNs (1) under controllers (31) and (32) can achieve FNTS within the time $\mathbb{T}_3$. The proof of Theorem 3 is complete. $\square$

Theorem 4. Suppose that Assumptions 1 and 2 hold. The control law is designed as:

$$u_i(t) = -d_i(t) \mathbf{e}_i(t) - \Lambda [\mathbf{e}_i(t)], \tag{40}$$

where the feedback strength $d_i(t)$ is adapted given the updated law mentioned below:

$${}^C_{t_0} D_t^\alpha d_i(t) = \frac{1}{2} q_i ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]), \tag{41}$$

and $q_i, \Lambda > 0$. If there exist constants $\check{\theta}_i^{(1)} > 0$ and $\check{\theta}_i^{(2)} > 0$ such that:

$$\begin{cases} \zeta^{(1)} - \Theta^{(1)} \leq 0, \\ \Psi^{(l)} - \Theta^{(2)} \leq 0, \end{cases} \tag{42}$$

where $\zeta^{(1)} = \operatorname{diag}(\underbrace{\zeta_1^{(1)}, \dots, \zeta_1^{(1)}}_{1, \dots, q_1}, \underbrace{\zeta_2^{(1)}, \dots, \zeta_2^{(1)}}_{q_1+1, \dots, q_2}, \dots, \underbrace{\zeta_r^{(1)}, \dots, \zeta_r^{(1)}}_{q_{r-1}+1, \dots, N})$, $\Theta^{(p)} = \operatorname{diag}(\check{\theta}_1^{(p)}, \check{\theta}_2^{(p)}, \dots, \check{\theta}_N^{(p)})$, $(p = 1, 2)$, $\Psi^{(l)} = (\psi_{ij}^{(l)})_{N \times N}$ and $\psi_{ij}^{(l)} = |\operatorname{Re}(c_2 b_{ij} \delta_l^{(2)})| + |\operatorname{Im}(c_2 b_{ij} \delta_l^{(2)})|$, then the FFCVDNs (1) can achieve FTCS under controllers (17) and (18). Moreover, the ST is estimated by:

$$\mathbb{T} \leq \mathbb{T}_4 = t_0 + \left(\frac{V_4(t_0) \Gamma(\alpha + 1)}{\Lambda} \right)^{\frac{1}{\alpha}}, \tag{43}$$

in which $V_4(t_0) = \frac{1}{2} \sum_{i=1}^N \left(\|\mathbf{e}_i(t_0)\|_1 + \frac{1}{q_i} (\hat{d}_i(t_0) - d_i)^2 + \sum_{p=1}^2 \check{\theta}_i^{(p)} \int_{t_0-\pi_p}^{t_0} \|\mathbf{e}_i(s)\|_1 ds \right)$.

$d_i > 0$ such that $\vartheta^{(1)} + \mathbf{Y}^{(l)} + \Theta^{(1)} + \Theta^{(2)} - \mathfrak{D} \leq 0$, $\mathfrak{D} = \text{diag}(d_1, d_2, \dots, d_N)$, $\vartheta^{(1)} = \text{diag}(\underbrace{\vartheta_1^{(1)}, \dots, \vartheta_1^{(1)}}_{1, \dots, q_1}, \underbrace{\vartheta_2^{(1)}, \dots, \vartheta_2^{(1)}}_{q_1+1, \dots, q_2}, \dots, \underbrace{\vartheta_r^{(1)}, \dots, \vartheta_r^{(1)}}_{q_{r-1}+1, \dots, N})$, $\mathbf{Y}^{(l)} = (\tau_{ij}^{(l)})_{N \times N}$ and

$$\tau_{ij}^{(l)} = \begin{cases} \text{Re}(c_1 a_{ij} \delta_l^{(1)}) + |\text{Im}(c_1 a_{ij} \delta_l^{(1)})|, & i = j, \\ |c_1 a_{ij} \delta_l^{(1)}|_1, & i \neq j. \end{cases}$$

Proof. Construct the following Lyapunov functional:

$$V_4(e(t)) = \frac{1}{2} \sum_{i=1}^N \left(\frac{1}{q_i} (d_i(t) - d_i)^2 + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)] + [\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \sum_{p=1}^2 \varepsilon_i^{(p)} \int_{t-\pi_p}^t (\mathbf{e}_i^H(s) [\mathbf{e}_i(s)] + [\mathbf{e}_i(s)]^H \mathbf{e}_i(s)) ds \right). \quad (44)$$

By employing Lemmas 1 and 3, the derivative of $V_4(t)$ for the solution of (3) is computed as:

$$\begin{aligned} {}^C_{t_0} D_t^\alpha V_4(t) &\leq \frac{1}{2} \sum_{i=1}^N ([\mathbf{e}_i(t)]^H F_{\zeta_i}(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) + F_{\zeta_i}^H(\mathbf{e}_i(t), \mathbf{e}_i(t - \pi_1)) [\mathbf{e}_i(t)]) \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} ([\mathbf{e}_i(t)]^H c_1 a_{ij} G_1 \mathbf{e}_j(t) + \mathbf{e}_j^H(t) \overline{c_1 a_{ij}} G_1^H [\mathbf{e}_i(t)]) \\ &\quad + \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^r \sum_{j \in \mathcal{C}_k} ([\mathbf{e}_i(t)]^H c_2 b_{ij} G_2 \mathbf{e}_j(t - \pi_2) + \mathbf{e}_j^H(t - \pi_2) \overline{c_2 b_{ij}} G_2^H [\mathbf{e}_i(t)]) \\ &\quad - \frac{1}{2} \sum_{i=1}^N d_i(t) ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]) - \Lambda \sum_{i=1}^N [\mathbf{e}_i(t)]^H [\mathbf{e}_i(t)] \\ &\quad + \frac{1}{2} \sum_{i=1}^N (d_i(t) - d_i) ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]) \\ &\quad + \frac{1}{2} \sum_{p=1}^2 \sum_{i=1}^N \check{\theta}_i^{(p)} ([\mathbf{e}_i(t)]^H \mathbf{e}_i(t) + \mathbf{e}_i^H(t) [\mathbf{e}_i(t)]) \\ &\quad - \frac{1}{2} \sum_{p=1}^2 \sum_{i=1}^N \check{\theta}_i^{(p)} ([\mathbf{e}_i(t - \pi_p)]^H \mathbf{e}_i(t - \pi_p) + \mathbf{e}_i^H(t - \pi_p) [\mathbf{e}_i(t - \pi_p)]). \end{aligned} \quad (45)$$

Similar to the proof of Theorem 2, this yields that:

$$\begin{aligned} {}^C_{t_0} D_t^\alpha V_4(t) &\leq \sum_{l=1}^n I_N^T (\vartheta^{(1)} + \mathbf{Y}^{(l)} + \Theta^{(1)} + \Theta^{(2)} - \mathfrak{D}) \widetilde{\mathbf{e}_l(t)} - \Lambda + \sum_{l=1}^n I_N^T (\zeta^{(1)} - \Theta^{(1)}) \widetilde{\mathbf{e}_l(t - \pi_1)} \\ &\quad + \sum_{l=1}^n I_N^T (\Psi^{(l)} - \Theta^{(2)}) \widetilde{\mathbf{e}_l(t - \pi_2)} \\ &\leq -\Lambda. \end{aligned} \quad (46)$$

By virtue of Lemma 4, the FFCVDNs (1) under controllers (40) and (41) can achieve FNTS within time $\mathbb{T}_4$. The proof of Theorem 4 is complete. $\square$

Remark 4. The controllers (7) and (18) include the time delays π_p , $-\frac{1}{2} \sum_{p=1}^2 \check{\theta}_i^{(p)} \mathbf{e}_i(t - \pi_p)$ or $-\frac{1}{2} \sum_{p=1}^2 \check{\theta}_i^{(p)} [\mathbf{e}_i(t)] \mathbf{e}_i^H(t - \pi_p) [\mathbf{e}_i(t - \pi_p)]$. This implies the control inputs are large. However, the conclusions derived from delay-dependent control are more accurate in estimating the ST under the same $d_i(t)$, Λ , q_i , and initial values. Similarly, in the adaptive update laws (7), (18), (32) and (41), the sufficiently small adaptive gains q_i can cause small control inputs, but the estimation for the upper bounds of ST may be quite large. Therefore, there is a trade-off between the control performance and the ST when using the adaptive control technique to handle FNTS of DNs. In selecting the adaptive gains of them, one should consider the real requirements.

Remark 5. Time delay widely exists in systems. To achieve FFCVDNs of FTCS, delay-dependent and delay-independent adaptive controllers have been designed in this paper. Delay-independent controllers are often easy to implement and analyze and also have lower control energy transmission. On the contrary, the delay-dependent controller with high control cost will usually achieve better finite-time synchronization. Therefore, in practical applications, different types of controllers can be selected based on task requirements.

Remark 6. In recent years, scholars have conducted research on the finite-time synchronization problem of FCVDNs [17,25,30,33]. However, there are some limitations in these studies, as listed below. Firstly, in [17,25], the system parameters consist of complex and real values. Compared to these, fully complex-valued systems are more practical and have a complex mathematical derivation. Secondly, time delay was not considered in [17,25,30], which is a common phenomenon in reality. Thirdly, when using state feedback control in [33], the feedback strength is fixed, which is less energy-efficient compared to the adaptive control strategy used in this paper.

Remark 7. Finite-time synchronization has a faster convergence rate compared with asymptotic and exponential synchronization, as the estimated ST is based on the initial states. In fact, when the initial state is unknown, the ST cannot be estimated. Therefore, it is interesting and necessary to study the FNTS of FFCVDNs. Next, we further explore the fixed time convergence results.

Remark 8. This article studies the FNTS of FFCVDNs by designing the adaptive controllers. DNs are considered fully complex-valued networks, which represent more general situations and have wider applications. These can be applied in research such as signal processing, audio analysis, and engineering control, etc.

4. Numerical Simulations

Consider FFCVDNs consisting of 3 communities: $\mathcal{C}_1 = \{1, 2, 3\}$, $\mathcal{C}_2 = \{4, 5, 6\}$, $\mathcal{C}_3 = \{7, 8, 9\}$ and 9 nodes, each 3-dimensional. The topology among the different clusters is shown in Figure 1. Set $c_1 = 8 - 2i$, $c_2 = 2 - 2i$, $\pi_1 = 0.2$, $\pi_2 = 0.1$, $B = 0.2A$,

$$G_1 = \begin{pmatrix} 1.5 + 1.5i & 0 & 0 \\ 0 & 1.4 + 1.5i & 0 \\ 0 & 0 & 1.4 + 1.5i \end{pmatrix}, G_2 = \begin{pmatrix} 1.8 + 1.8i & 0 & 0 \\ 0 & 1.9 + 1.9i & 0 \\ 0 & 0 & 1.9 + 2.1i \end{pmatrix},$$

$$A_{11} = \begin{pmatrix} -1 - 1i & 1 + 1i & 0 \\ 1 + 1i & -2 - 2i & 1 + 1i \\ 0 & 1 + 1i & -1 - 1i \end{pmatrix}, A_{12} = \begin{pmatrix} -1 - 1i & 0 & 1 + 1i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} 0 & 0 & 0 \\ -1 - 1i & 1 + 1i & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 + 1i & -1 - 1i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{22} = \begin{pmatrix} -1-1\mathbf{i} & 1+1\mathbf{i} & 0 \\ 1+1\mathbf{i} & -2-2\mathbf{i} & 1+1\mathbf{i} \\ 1+1\mathbf{i} & 1+1\mathbf{i} & -2-2\mathbf{i} \end{pmatrix}, A_{23} = \begin{pmatrix} -1-1\mathbf{i} & 1+1\mathbf{i} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_{32} = \begin{pmatrix} -1-1\mathbf{i} & 0 & 1+1\mathbf{i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_{33} = \begin{pmatrix} -1-1\mathbf{i} & 1+1\mathbf{i} & 0 \\ 1+1\mathbf{i} & -1-1\mathbf{i} & 0 \\ 0 & 1+1\mathbf{i} & -1-1\mathbf{i} \end{pmatrix}.$$

The nonlinear vector functions in the 3 communities are selected as:

$$\begin{aligned} f_1(x_i(t), x_i(t-\pi_1)) &= \tilde{D}_1 x_i(t) + g_1(x_i(t)) + h_1(x_i(t-\pi_1)), i=1,2,3, \\ f_2(x_i(t), x_i(t-\pi_1)) &= \tilde{D}_2 x_i(t) + g_2(x_i(t)) + h_2(x_i(t-\pi_1)), i=4,5,6, \\ f_3(x_i(t), x_i(t-\pi_1)) &= \tilde{D}_3 x_i(t) + g_3(x_i(t)) + h_3(x_i(t-\pi_1)), i=7,8,9, \end{aligned} \quad (47)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T \in \mathbb{C}^3$,

$$\begin{aligned} g_1(x_i(t)) &= (0, -x_{i1}x_{i3}, \frac{\overline{x_{i1}}x_{i2} + x_{i1}\overline{x_{i2}}}{2})^T, \\ g_2(x_i(t)) &= (0, -x_{i1}x_{i3}, \frac{\overline{x_{i1}}x_{i2} + x_{i1}\overline{x_{i2}}}{2})^T, \\ g_3(x_i(t)) &= (0, -x_{i1}x_{i3}, \frac{\overline{x_{i1}}x_{i2} + x_{i1}\overline{x_{i2}}}{2} + \mathbf{i}\operatorname{Re}(x_{i2})\operatorname{Im}(x_{i1}))^T, \\ h_1(x_i(t)) &= \left(0, (0.2 - 0.2\mathbf{i})(\cos(\operatorname{Re}(x_{i2})) + \sin(\operatorname{Im}(x_{i2}))\mathbf{i}), 0\right)^T, \\ h_2(x_i(t)) &= \left(0, (0.1 + 0.1\mathbf{i})(\cos(\operatorname{Re}(x_{i2})) + \sin(\operatorname{Im}(x_{i2}))\mathbf{i}), 0\right)^T, \\ h_3(x_i(t)) &= \left(0, (0.2 + 0.1\mathbf{i})(\cos(\operatorname{Re}(x_{i2})) + \sin(\operatorname{Im}(x_{i2}))\mathbf{i}), 0\right)^T, \\ \tilde{D}_1 &= \begin{pmatrix} -34.99 & 34.91 & 0 \\ -6.99 & 27.92 & 0 \\ 0 & 0 & -2.98 \end{pmatrix}, \tilde{D}_2 = \begin{pmatrix} -1.99 & 1.98 & 0 \\ 59.9 + 0.02\mathbf{i} & -1 - 0.06\mathbf{i} & 0 \\ 0 & 0 & -0.79 \end{pmatrix}, \\ \tilde{D}_3 &= \begin{pmatrix} -20.95 & 20.98 & 0 \\ 0 & 9.86 & 0 \\ 0 & 0 & -5.94 \end{pmatrix}. \end{aligned}$$

The initial conditions of system (1) are selected as:

$$\begin{cases} x_{i1}(\pi) = -1.5 - 0.5\mathbf{i} + (1.3 + 0.5\mathbf{i})\mathbf{i}, \\ x_{i2}(\pi) = 2.8 + 0.5\mathbf{i} - (1.1 + 0.5\mathbf{i})\mathbf{i}, \\ x_{i3}(\pi) = 4.1 + 0.5\mathbf{i} + (1 + 0.5\mathbf{i})\mathbf{i}, \end{cases} \quad (48)$$

where $\pi \in [-0.2, 0]$. Additionally, the trajectories of the system (2) are provided in Figure 2 with the initial values $\mathbf{s}_1(\pi) = (-1.5 + 1.1\mathbf{i}, 2.1 - 1.2\mathbf{i}, 3.9 + 0.5\mathbf{i})^T$, $\mathbf{s}_2(\pi) = (-2.7 + 2.5\mathbf{i}, 3.8 - 1.9\mathbf{i}, 5.3 + 2.6\mathbf{i})^T$, $\mathbf{s}_3(\pi) = (-3.8 + 5.1\mathbf{i}, 5 - 5.3\mathbf{i}, 5.8 + 5.9\mathbf{i})^T$, $\pi \in [-0.2, 0]$, when $\alpha = 0.97$.

Two examples are then given to demonstrate the correctness and flexibility of the outcomes.

4.1. Example 1

First, consider the FTCS of the FFCVDNs with time delay (1) with the controllers (17) and (18). Let $\hat{d}_1(t_0) = \hat{d}_2(t_0) = \hat{d}_3(t_0) = 69$, $\hat{d}_4(t_0) = \hat{d}_5(t_0) = \hat{d}_6(t_0) = 105$, $\hat{d}_7(t_0) = \hat{d}_8(t_0) = \hat{d}_9(t_0) = 42$, $\check{\theta}_1^{(1)} = \check{\theta}_2^{(1)} = \check{\theta}_3^{(1)} = 0.28$, $\check{\theta}_4^{(1)} = \check{\theta}_5^{(1)} = \check{\theta}_6^{(1)} = 0.15$, $\check{\theta}_7^{(1)} = \check{\theta}_8^{(1)} = \check{\theta}_9^{(1)} = 0.3$, $\check{\theta}_1^{(2)} = 3$, $\check{\theta}_2^{(2)} = 6$, $\check{\theta}_3^{(2)} = 3$, $\check{\theta}_4^{(2)} = 3.1$, $\check{\theta}_5^{(2)} = 6.1$, $\check{\theta}_6^{(2)} = 6.1$, $\check{\theta}_7^{(2)} = 3.3$, $\check{\theta}_8^{(2)} = 3.3$, $\check{\theta}_9^{(2)} = 3.3$, $\alpha = 0.97$, $\Lambda = 9.5$, $q_i = 10$, $d_1 = d_2 = d_3 = 70$, $d_4 = d_5 =$

$d_6 = 107$, $d_7 = d_8 = d_9 = 45$, $i \in \Delta(9)$. According to Theorem 2, FFCVDNs can achieve FNTS. Figure 3 depicts the trajectories of the feedback strength $d_i(t)$. Apparently, feedback strength $d_i(t)$ converges to the fixed values at the end. Figure 4 shows the trajectories of synchronization errors (2) via controllers (17) and (18). Obviously, the synchronization errors of nodes within the same community converge to 0. Figure 4 shows the trajectories of synchronization errors (2) via controllers (17) and (18). Obviously, the same community's synchronization errors converge to zero.

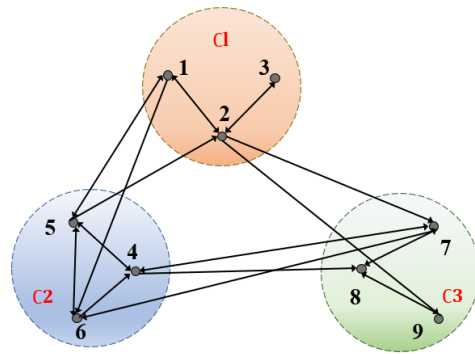


Figure 1. The topology of system (1) with 3 communities. C_k represents different communities, $k = \Delta(3)$.

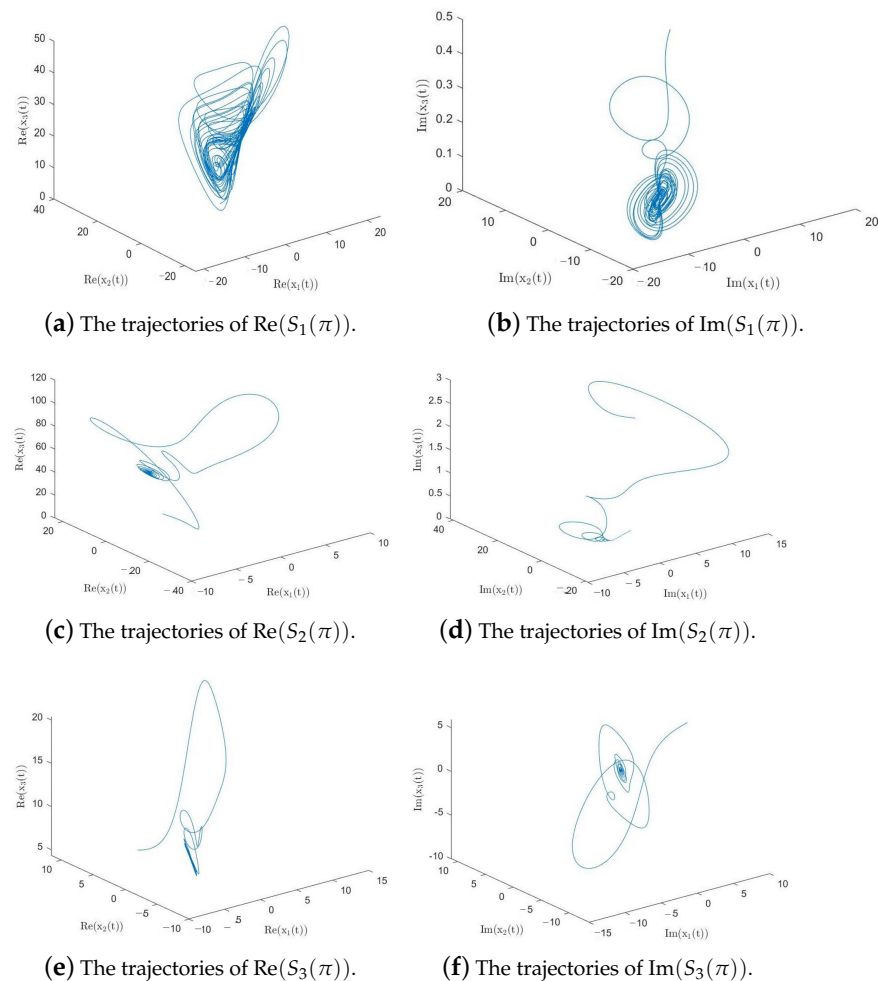


Figure 2. State trajectories for system (2) with 3 different communities. Obviously, the internal status of nodes belonging to different communities varies.

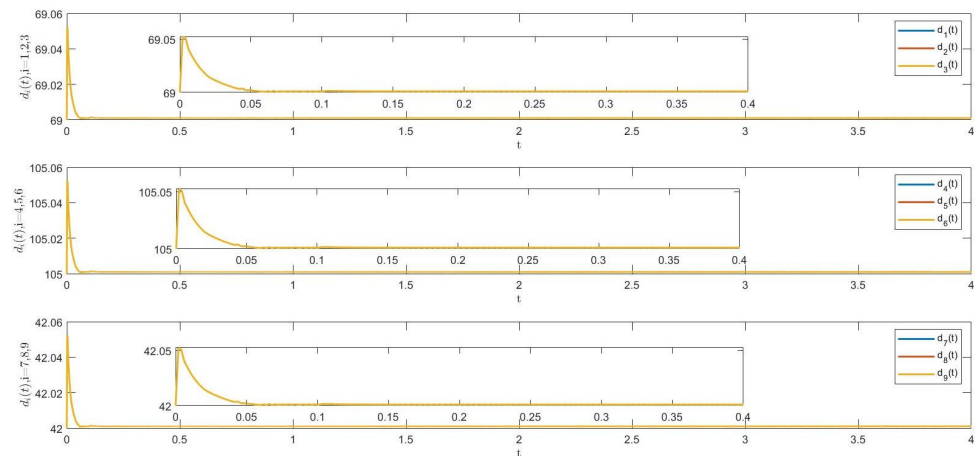


Figure 3. The curves between feedback strength $d_i(t)$ and time t under the initial values $d_1(t_0) = d_2(t_0) = d_3(t_0) = 69$, $d_4(t_0) = d_5(t_0) = d_6(t_0) = 105$, $d_7(t_0) = d_8(t_0) = d_9(t_0) = 42$, $i = \Delta(9)$. Feedback strength $d_i(t)$ converges to fixed values at the end.

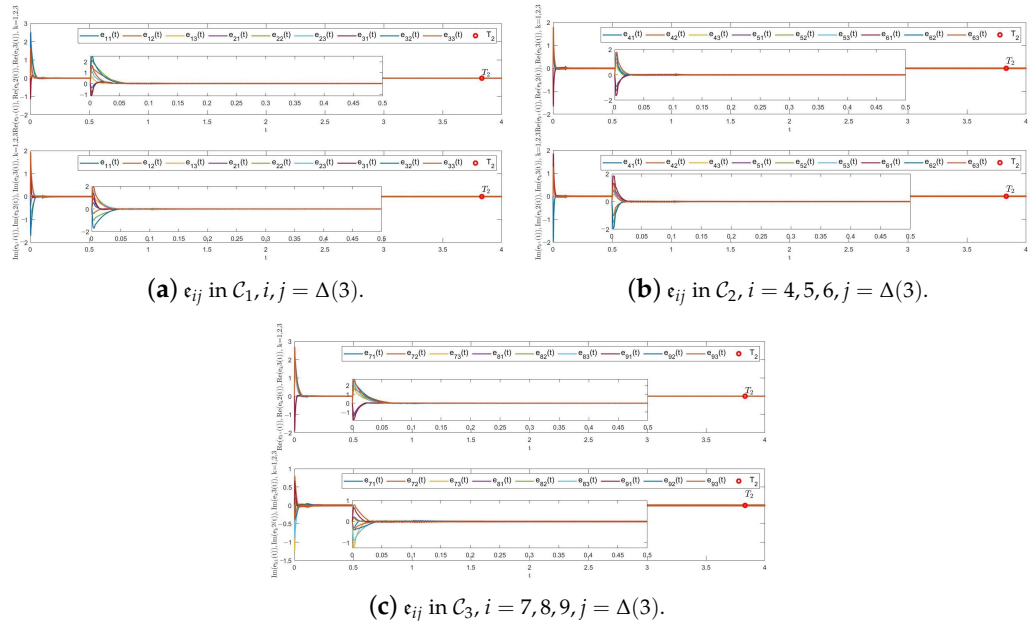


Figure 4. The solutions to synchronization errors e_{ij} over time t , $i = \Delta(9)$, $j = \Delta(3)$. It is easy to see that e_{ij} converges to zero in the same community, which indicates that the internal nodes within the same community can achieve synchronization within $\mathbb{T}_2 = 3.8318$.

In FFCVDNs, there exists three communities. Figure 5 depicts the trajectories of state variables x_{i1} , x_{i2} , and x_{i3} ($i = \Delta(9)$) of system (1) via controllers (17) and (18). It is clear that nodes from the same community have similar dynamic behaviors, whereas nodes from other communities display different dynamic behaviors. This means that the system has achieved FTCS. The ST can be estimated as $\mathbb{T}_2 \leq 3.8318$.

4.2. Example 2

Next, consider the FTCS of proposed network (1) with controllers (40) and (41). Let $\hat{d}_1(t_0) = \hat{d}_2(t_0) = \hat{d}_3(t_0) = 90$, $\hat{d}_4(t_0) = \hat{d}_5(t_0) = \hat{d}_6(t_0) = 107$, $\hat{d}_7(t_0) = \hat{d}_8(t_0) = \hat{d}_9(t_0) = 44$, $\check{\theta}_1^{(1)} = \check{\theta}_2^{(1)} = \check{\theta}_3^{(1)} = 0.28$, $\check{\theta}_4^{(1)} = \check{\theta}_5^{(1)} = \check{\theta}_6^{(1)} = 0.15$, $\check{\theta}_7^{(1)} = \check{\theta}_8^{(1)} = \check{\theta}_9^{(1)} = 0.3$, $\check{\theta}_1^{(2)} = 3$, $\check{\theta}_2^{(2)} = 6$, $\check{\theta}_3^{(2)} = 3$, $\check{\theta}_4^{(2)} = 3.1$, $\check{\theta}_5^{(2)} = 6.1$, $\check{\theta}_6^{(2)} = 6.1$, $\check{\theta}_7^{(2)} = 3.3$, $\check{\theta}_8^{(2)} = 3.3$, $\check{\theta}_9^{(2)} = 3.3$, $\alpha = 0.97$, $\Lambda = 9.5$, $q_i = 70$, $d_1 = d_2 = d_3 = 99$, $d_4 = d_5 = d_6 = 114$, $d_7 = d_8 = d_9 = 49$, $i = \Delta(9)$. Then, the conditions in Theorem 4 are met, which means

FFCVDNs can achieve FNTS. Figure 6 depicts the trajectories of the feedback strength $d_i(t)$. From Figure 6, we can see that the feedback strength $d_i(t)$ converges to fixed values at the end. Figure 7 describes the trajectories of synchronization errors (2) via controllers (40) and (41). Obviously, the synchronization errors of nodes within the same community converge to 0.

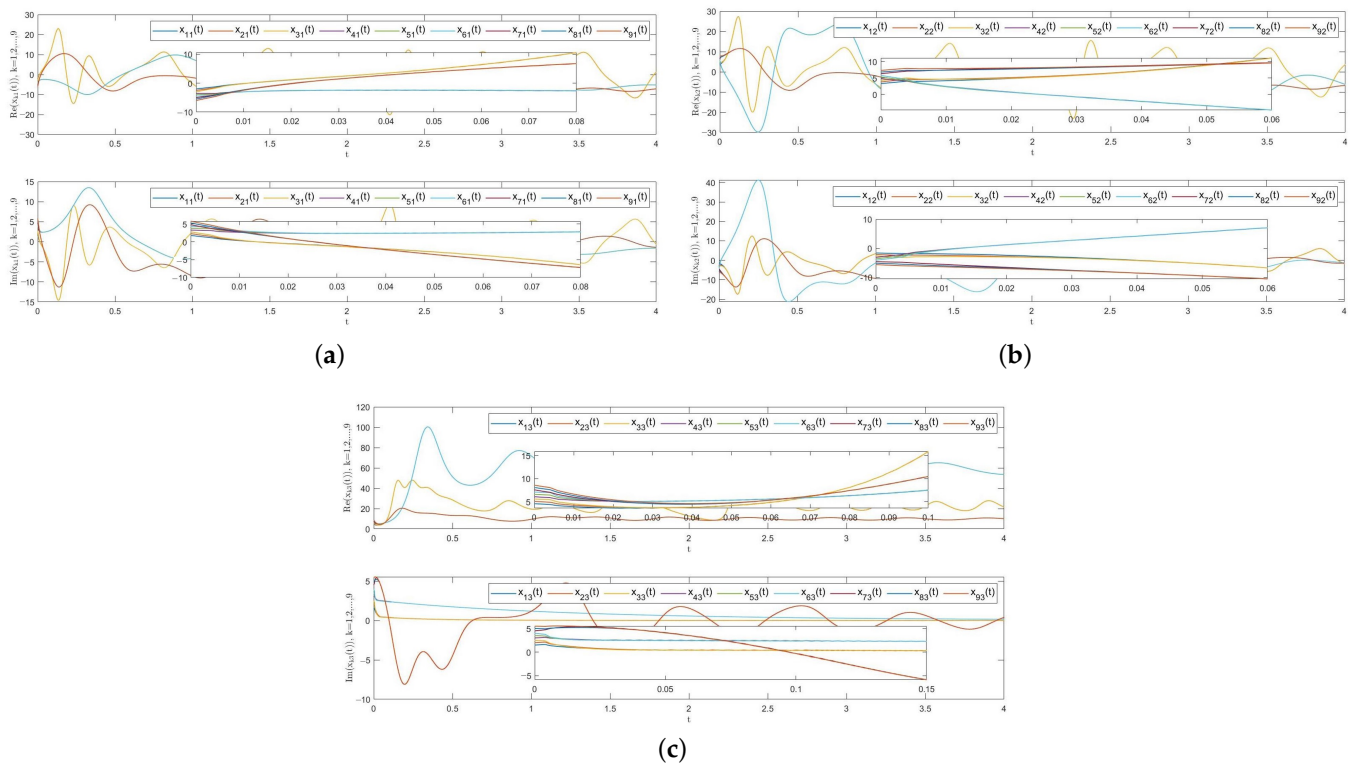


Figure 5. Trajectories of state variables x_{ij} of system (1) under controllers (17) and (18), $i = \Delta(9)$, $j = \Delta(3)$. In each community, the trajectories of nodes can converge to the corresponding target trajectories. (a) Trajectories of x_{i1} under controllers (17) and (18), $i = \Delta(9)$. (b) Trajectories of x_{i2} under controllers (17) and (18), $i = \Delta(9)$. (c) Trajectories of x_{i3} under controllers (17) and (18), $i = \Delta(9)$.

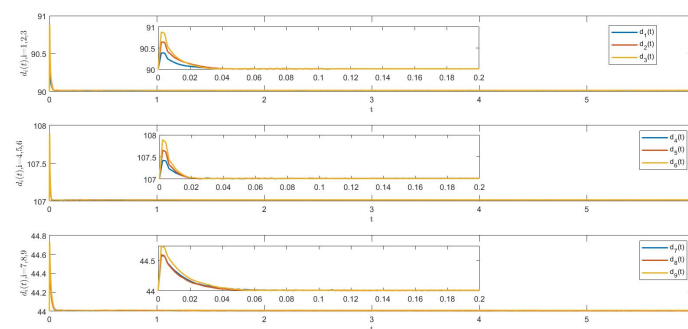


Figure 6. The curves between feedback strength $d_i(t)$ and time t under the initial values $d_1(t_0) = d_2(t_0) = d_3(t_0) = 90, d_4(t_0) = d_5(t_0) = d_6(t_0) = 107, d_7(t_0) = d_8(t_0) = d_9(t_0) = 44$, $i = \Delta(9)$. Feedback strength $d_i(t)$ converges to fixed values at the end.

The trajectories of x_{i1} , x_{i2} , and x_{i3} ($i = \Delta(9)$) of system (1) via controllers (40) and (41) are shown in Figure 7. It is easy to see that the nodes from the same community have similar dynamic behaviors, whereas nodes from other communities display different dynamic behaviors. According to Figures 7 and 8, system (1) can achieve FTCS via controllers (40) and (41), and the ST can be estimated as $\mathbb{T}_4 \leq 5.6279$.

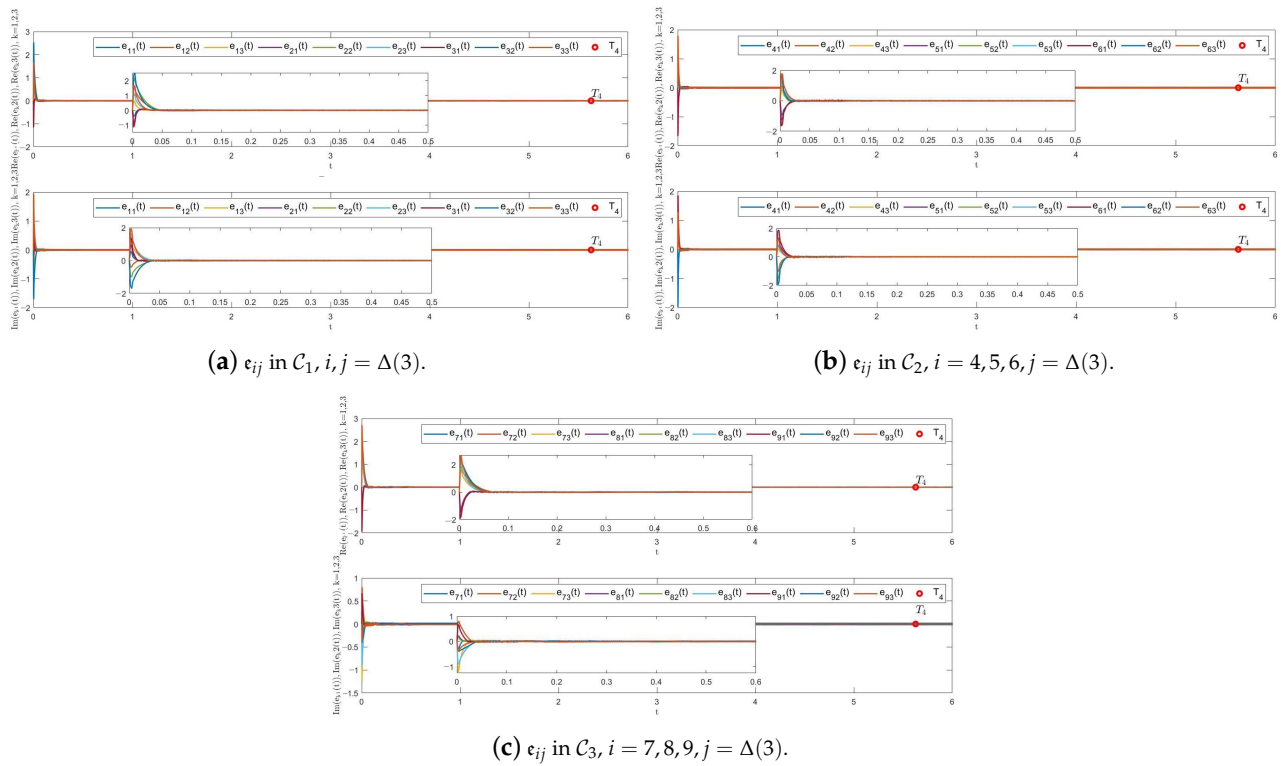


Figure 7. The solutions to synchronization errors e_{ij} over time t , $i = \Delta(9)$, $j = \Delta(3)$. It is easy to see that e_{ij} converges to zero in the same community, which indicates that the internal nodes within the same community can achieve synchronization within $\mathbb{T}_4 = 5.6279$.

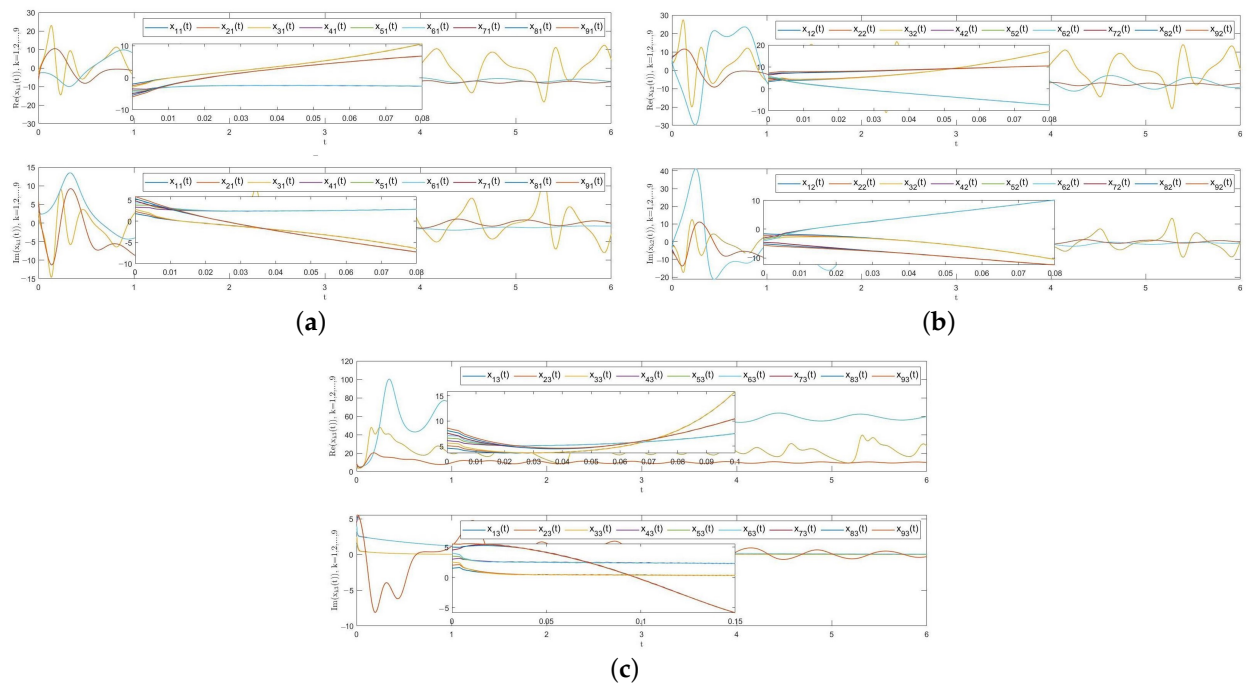


Figure 8. Trajectories of state variables x_{ij} of system (1) under controllers (40) and (41), $i = \Delta(9)$, $j = \Delta(3)$. In each community, the trajectories of nodes can converge to the corresponding target trajectories. (a) Trajectories of x_{i1} under controllers (40) and (41), $i = \Delta(9)$. (b) Trajectories of x_{i2} under the controllers (40) and (41), $i = \Delta(9)$. (c) Trajectories of x_{i3} under controllers (40) and (41), $i = \Delta(9)$.

Remark 9. From Figures 4, 5, 7, and 8, it can be seen that the real ST is shorter than the evaluated ST and that there exists an estimation error. On the one hand, it is because the use of Lemmas 1–3 amplifies the estimation of ST, and on the other hand, it is prone to errors when simulating actual systems. Based on the above discussion, finding a more accurate ST estimation will be one of our future research topics.

Remark 10. As shown in Figures 3 and 6, the feedback strength $d_i(t)$ automatically adjusts with changes to the operating environment. Adaptive control can reduce the conservatism on state feedback control in which the feedback strength is fixed; thus, the strength must be maximal, which means a kind of waste in practice.

5. Conclusions

In this paper, the FTCS issue for FFCVDNs with time delay have been studied. The delay-dependent and delay-independent controllers have been designed based on the quadratic norm and the absolute-valued norm instead of transforming the complex-valued networks into two independent real-valued systems. Some synchronization criteria have been established to ensure the realization of FNTS, where the ST is related to the fractional order α and fractional-order power law Λ . Finally, some illustrative simulations show the correctness of the theoretical results. The results can be applied to the analysis of dynamical networks, such as traffic control, communication engineering, distributed computing, etc. Significantly, in practical application, some phenomena can be better characterized by discontinuous systems [38,39]. This inspires us to study the FTCS of discontinuous systems in the future.

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References

1. Gan, Q.T.; Xiao, F.; Qin, Y.; Yang, J. Fixed-time cluster synchronization of discontinuous directed community networks via periodically or aperiodically switching control. *IEEE Access*. **2019**, *7*, 83306–83318. [\[CrossRef\]](#)
2. Boonsatit, N.; Rajendran, S.; Lim, C.P.; Jirawattanapanit, A.; Mohandas, P. New Adaptive Finite-Time Cluster Synchronization of Neutral-Type Complex-Valued Coupled Neural Networks with Mixed Time Delays. *Fractal Fract.* **2022**, *6*, 515. [\[CrossRef\]](#)
3. Fan, H.; Shi, K.; Zhao, Y. Pinning impulsive cluster synchronization of uncertain complex dynamical networks with multiple time-varying delays and impulsive effects. *Phys. A* **2022**, *587*, 126534. [\[CrossRef\]](#)
4. He, Q.; Ma, Y. Quantized adaptive pinning control for fixed/preassigned-time cluster synchronization of multi-weighted complex networks with stochastic disturbances. *Nonlinear Anal. Hybrid Syst.* **2022**, *44*, 101157. [\[CrossRef\]](#)
5. Yang, S.; Hu, C.; Yu, J.; Jiang, H.J. Finite-time cluster synchronization in complex-variable networks with fractional-order and nonlinear coupling. *Neural Netw.* **2021**, *135*, 212–224. [\[CrossRef\]](#)
6. Yang, Q.; Wu, H.; Cao, J.D. Pinning exponential cluster synchronization for fractional-order complex dynamical networks with switching topology and mode-dependent impulses. *Neurocomputing* **2021**, *428*, 182–194. [\[CrossRef\]](#)
7. Bhat, S.P.; Bernstein, D.S. Finite-time stability of homogeneous system. In Proceedings of the American Control Conference, Albuquerque, NM, USA, 6 June 1997; pp. 2513–2514.
8. Cai, S.; Zhou, F.; He, Q. Fixed-time cluster lag synchronization in directed heterogeneous community networks. *Phys. A* **2019**, *525*, 128–142. [\[CrossRef\]](#)

9. Cui, W.; Fang, J.A.; Zhang, W.B.; Wang, X. Finite-time cluster synchronisation of Markovian switching complex networks with stochastic perturbations. *IET Control Theory Appl.* **2014**, *8*, 30–41. [\[CrossRef\]](#)
10. Jiang, S.Q.; Lu, X.B.; Cai, G.L.; Cai, S.M. Adaptive finite-time control for overlapping cluster synchronization in coupled complex networks. *Neurocomputing* **2017**, *266*, 188–195. [\[CrossRef\]](#)
11. Jiang, S.Q.; Lu, X.B.; Cai, G.L.; Cai, S.M. Adaptive fixed-time control for cluster synchronisation of coupled complex networks with uncertain disturbances. *Int. J. Syst. Sci.* **2017**, *48*, 3382–3390. [\[CrossRef\]](#)
12. Liu, X.W.; Chen, T.P. Finite-time and fixed-time cluster synchronization with or without pinning control. *IEEE Trans. Cybern.* **2018**, *48*, 240–252. [\[CrossRef\]](#)
13. Yang, X.S.; Ho, D.W.; Lu, J.Q.; Song, Q. Finite-time cluster synchronization of T-S fuzzy complex networks with discontinuous subsystems and random coupling delays. *IEEE Trans. Fuzzy Syst.* **2015**, *23*, 2302–2316. [\[CrossRef\]](#)
14. Li, H.L.; Cao, J.D.; Jiang, H.J.; Alsaedi, A. Finite-time synchronization and parameter identification of uncertain fractional-order complex networks. *Phys. A* **2019**, *587*, 533. [\[CrossRef\]](#)
15. Li, Y.; Kao, Y.G.; Wang, C.H.; Xia, H.W. Finite-time synchronization of delayed fractional-order heterogeneous complex networks. *Neurocomputing* **2020**, *384*, 368–375. [\[CrossRef\]](#)
16. Peng, X.; Wu, H.Q.; Cao, J.D. Global nonfragile synchronization in finite time for fractional-order discontinuous neural networks with nonlinear growth activations. *IEEE Trans. Neural Netw. Learn. Syst.* **2019**, *30*, 2123–2137. [\[CrossRef\]](#)
17. Xu, Y.; Li, W.X. Finite-time synchronization of fractional-order complex-valued coupled systems. *Phys. A* **2020**, *549*, 123903. [\[CrossRef\]](#)
18. Li, H.L.; Cao, J.D.; Jiang, H.J.; Alsaedi, A. Finite-time synchronization of fractional-order complex networks via hybrid feedback control. *Neurocomputing* **2018**, *320*, 69–75. [\[CrossRef\]](#)
19. Li, H.L.; Cao, J.D.; Jiang, H.J.; Alsaedi, A. Graph theory-based finite-time synchronization of fractional-order complex dynamical networks. *J. Frankl. Inst.* **2018**, *355*, 5771–5789. [\[CrossRef\]](#)
20. Lu, J.Y.; Guo, Y.P.; Ji, Y.D.; Fan, S.S. Finite-time synchronization for different dimensional fractional-order complex dynamical networks. *Chaos Solitons Fract.* **2020**, *130*, 109433. [\[CrossRef\]](#)
21. Jia, Y.; Wu, H.; Cao, J.D. Non-fragile robust finite-time synchronization for fractional-order discontinuous complex networks with multi-weights and uncertain couplings under asynchronous switching. *Appl. Math. Comput.* **2020**, *370*, 124929. [\[CrossRef\]](#)
22. Zhang, L.Z.; Zhong, J.; Lu, J.Q. Intermittent control for finite-time synchronization of fractional-order complex networks. *Neural Netw.* **2021**, *144*, 11–22. [\[CrossRef\]](#) [\[PubMed\]](#)
23. Jian, J.; Wang, P. Lagrange α -exponential stability and α -exponential convergence for fractional-order complex-valued neural networks. *Neural Netw.* **2017**, *91*, 1–10. [\[CrossRef\]](#)
24. Kaslik, E.; Radulescu, I.R. Dynamics of complex-valued fractional-order neural networks. *Neural Netw.* **2017**, *89*, 39–49. [\[CrossRef\]](#)
25. Hou, T.Q.; Yu, J.; Hu, C.; Jiang, H.J. Finite-time synchronization of fractional-order complex-variable dynamic networks. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 4297–4307. [\[CrossRef\]](#)
26. Zheng, B.B.; Hu, C.; Yu, J.; Jiang, H.J. Finite-time synchronization of fully complex-valued neural networks with fractional-order. *Neurocomputing* **2020**, *373*, 70–80. [\[CrossRef\]](#)
27. Long, C.Q.; Zhang, G.D.; Zeng, Z.G.; Hu, J.H. Finite-time stabilization of complex-valued neural networks with proportional delays and inertial terms: A non-separation approach. *Neural Netw.* **2022**, *148*, 86–95. [\[CrossRef\]](#)
28. Das, S.; Das, P.; Das, P. Dynamics and control of multidrug-resistant bacterial infection in hospital with multiple delays. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *89*, 105279. [\[CrossRef\]](#)
29. Ren, Y.; Jiang, H.J.; Li, J.R.; Lu, B.L. Finite-time synchronization of stochastic complex networks with random coupling delay via quantized aperiodically intermittent control. *Neurocomputing* **2021**, *420*, 337–348. [\[CrossRef\]](#)
30. Xiong, K.L.; Yu, J.; Hu, C.; Jiang, H.J. Synchronization in finite/fixed time of fully complex-valued dynamical networks via nonseparation approach. *J. Frankl. Inst.* **2020**, *357*, 473–493. [\[CrossRef\]](#)
31. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: New York, NY, USA, 2006.
32. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
33. Kang, Q.; Yang, Q.; Yang, J.; Gan, Q.; Li, R. Synchronization in finite-time of delayed fractional-order fully complex-valued dynamical networks via non-separation method. *Entropy* **2022**, *24*, 1460. [\[CrossRef\]](#)
34. Xia, W.G.; Cao, J.D. Pinning synchronization of delayed dynamical networks via periodically intermittent control. *Chaos* **2009**, *19*, 013120. [\[CrossRef\]](#) [\[PubMed\]](#)
35. Cai, S.M.; He, Q.B.; Hao, J.J.; Liu, Z.R. Exponential synchronization of complex networks with nonidentical time-delayed dynamical nodes. *Phys. A* **2010**, *374*, 2539–2550. [\[CrossRef\]](#)
36. Yu, J.; Hu, C.; Jiang, H.J. Corroendum to “Projective synchronization for fractional neural networks”. *Neural Netw.* **2015**, *67*, 152–154. [\[CrossRef\]](#)
37. Xu, Q.; Zhuang, S.X.; Liu, S.J.; Xiao, J. Decentralized adaptive coupling synchronization of fractional-order complex-variable dynamical networks. *Neurocomputing* **2016**, *186*, 119–126. [\[CrossRef\]](#)

38. Danca, M. Controlling chaos in discontinuous dynamical systems. *Chaos Solitons Fract.* **2014**, *22*, 605–612. [[CrossRef](#)]
39. Forti, M.; Grazzini, M.; Nistri, P.; Pancioni, L. Generalized Lyapunov approach for convergence of neural networks with discontinuous or non-Lipschitz activations. *Phys. D* **2006**, *214*, 88–99. [[CrossRef](#)]

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