Affine-Periodic Boundary Value Problem for a Fractional Differential Inclusion

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Abstract: In the article, affine-periodic boundary value problem involving fractional derivative is considered. Existence of solutions to a Caputo-type fractional differential inclusion is researched by some fixed-point theorems and set-valued analysis theory. Specifically, we consider two cases in which the multifunction has convex values and nonconvex values, respectively.

Keywords: affine-periodic; fractional derivative; differential inclusion; fixed-point theorem; set-valued analysis theory; boundary value condition

MSC: 34B15; 34A08; 26A33

1. Introduction

In recent years, the study of fractional calculus has aroused wide enthusiasm from scholars. Fractional calculus is widely used in physics, biology, control theory, celestial mechanics, economics and many other fields. The mathematical model established by fractional calculus can describe the phenomena in natural life more accurately and in more detail, so as to solve some problems that the integer order calculus mathematical model cannot solve. For the latest research on fractional calculus, we refer the readers to see [1–5].

Differential equation is a deterministic model used to describe systems in physics, engineering, mechanics, economics, etc. However, in real life and scientific practice, many phenomena cannot be described using the deterministic model, such as when describing some dynamic systems or uncertain objects. Differential inclusion is a dynamic system based on a certain but incomplete understanding of a system’s process, which is used to reveal the laws of uncertain dynamical systems and discontinuous dynamical systems. Differential inclusion theory is an important application in a lot of fields, for example, in automatic control systems, economic dynamic systems, adaptive control theory, etc. As a branch of the general theory of differential equation, differential inclusion theory is developing rapidly. For the research results of differential inclusion theory, readers can refer to [6–10].

The affine-periodic was firstly proposed by Professor Li in 2013 [11], which describes a physical phenomenon with symmetry in space and periodicity in time. The affine period is widely used in astrophysics. For some interesting results on the affine period, please refer to [12–15] and the references therein. In [12], using the lower and upper solutions method and topological degree theory, Xu et al. claimed that a Newton affine-periodic system admits an affine-periodic; In [13], Liu et al. showed that every first-order dissipative-\((T, \alpha)\)-affine-periodic system also has a dissipative-\((T, \alpha)\)-affine-periodic solution in \([0, \infty)\) using topological degree theory and the lower and upper solutions method; In [14], Xu et al. firstly gave some extremum principles for higher-order affine-periodic systems. Then, using these extremum principles, the authors studied the existence of affine-periodic solutions for \(n (n \in \mathbb{N})\)-order ordinary differential equations. A class of nonlinear fractional dynamical systems with affine-periodic boundary conditions were considered by Xu et al.
in [15]. Using the homotopy invariance of the Brouwer degree, the authors gained the existence of solutions to the fractional dynamical systems, while using Gronwall–Bellman inequality, the uniqueness of the solution was also obtained. However, fractional-order differential systems do not have affine-periodic solutions, but can only study the solutions with affine-periodic boundary value conditions. In [16], Gao et al. investigated the well-posedness of the affine-periodic boundary value solution to a sequential fractional differential equation. The existence results were obtained via Leray–Schauder and Krasnoselskii fixed-point theorems, while the uniqueness result was gained via the Banach contraction mapping principle.

Inspired by [16], in this article, we study the case when the nonlinear function is a multifunction. Precisely speaking, the \((K, \lambda)\)-affine-periodic boundary value problem of a fractional differential inclusion is expressed by

\[
\begin{align*}
\mathcal{C}D^\gamma z(t) + b\mathcal{C}D^\alpha z(t) & \in \mathcal{G}(t, z), \quad \text{for } t \in [0, K], \\
z(K) = \lambda z(0), & \quad z'(K) = \lambda z'(0),
\end{align*}
\]

where \(\mathcal{C}D^\alpha\) denotes the Caputo fractional derivative with the order \(0 < \alpha < 1\). \(b \in \mathbb{R}\) and \(\lambda\) are constants that satisfy \(\lambda \neq 1\), \(\lambda \neq e^{-bK}\).

The contribution of this article is to research the existence theorem of solutions to the \((K, \lambda)\)-affine-periodic boundary value problem of fractional differential inclusion. For the case of multifunction \(\mathcal{G}\) with convex and nonconvex values, we obtain the existence theorem of solutions by applying the Leray–Schauder alternative theorem and Covitz–Nadler fixed-point theorem, respectively.

The rest of this article is organized as follows. In Section 2, the definitions and useful lemmas are stated. The main results are presented in Section 3, and some examples are listed in Section 4. In the last Section 5, the conclusion is given.

2. Preliminaries

In this section, we put forward the definitions of fractional calculus and some basic theory of set-valued analysis. If the readers are interested, more details can be found in [17–20].

**Definition 1.** The Riemann–Liouville fractional integral of order \(\gamma > 0\) for a function \(u\) is defined as

\[
\mathcal{D}^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} u(s) ds, \quad t > 0,
\]

where \(\Gamma(\cdot)\) is the Gamma function.

**Definition 2.** The Caputo fractional derivative of order \(\gamma > 0\) for a function \(u\) can be written as

\[
\mathcal{C}D^\gamma u(t) = \begin{cases} 
\frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} u^{(m)}(s) ds, & \text{for } 1 < \gamma < m \quad (m \in \mathbb{N}^+), \\
\frac{1}{u^{(m)}(t)} \int_0^t (t-s)^{m-\gamma-1} u^{(m)}(s) ds, & \gamma = m \quad (m \in \mathbb{N}),
\end{cases}
\]

for \(t > 0\).

Throughout this paper, let \(C([0, K]; \mathbb{R})\) be a Banach space that is composed of all continuous functions \(x : [0, K] \to \mathbb{R}\) with the usual norm

\[
\|x\| = \sup_{t \in [0, K]} |x|.
\]
Let \(AC([0, K]; \mathbb{R})\) be the space of absolutely continuous functions and \(L^1([0, K]; \mathbb{R})\) be a Banach space of measurable functions \(x : [0, K] \to \mathbb{R}\) that are Lebesgue integrable and normed by

\[
\|x\|_{L^1} = \int_0^K |x(t)|dt.
\]

Furthermore, we introduce the notations:

\[
\mathcal{P}(\Theta) = \{A \subset \Theta : \text{\(A\) is nonempty.}\}
\]

\[
\mathcal{P}_f(\Theta) = \{A \in \mathcal{P}(\Theta) : \text{\(A\) is closed.}\}
\]

\[
\mathcal{P}_b(\Theta) = \{A \in \mathcal{P}(\Theta) : \text{\(A\) is bounded.}\}
\]

\[
\mathcal{P}_c(\Theta) = \{A \in \mathcal{P}(\Theta) : \text{\(A\) is compact.}\}
\]

\[
\mathcal{P}_c(\Theta) = \{A \in \mathcal{P}(\Theta) : \text{\(A\) is convex.}\}
\]

A multifunction \(\mathcal{F} : \Theta \to \mathcal{P}(\Theta)\) is closed(convex) valued, if, for all \(x \in \Theta\), \(\mathcal{F}(x)\) is closed(convex); a multifunction \(\mathcal{F} : \Theta \to \mathcal{P}(\Theta)\) is bounded on bounded sets, if, for all \(\mathcal{B} \in \mathcal{P}_b(\Theta)\), \(\mathcal{F}(\mathcal{B}) = \bigcup_{x \in \mathcal{B}} \mathcal{F}(x)\) is bounded in \(\Theta\).

**Definition 3.** A multifunction \(\mathcal{F} : \Theta \to \mathcal{P}(\Theta)\) is said to be
(i) completely continuous, if \(\mathcal{F}\) maps the bounded closed set into the relatively compact set;
(ii) upper semicontinuous (u.s.c.), if, for each closed set \(Q \subseteq \mathcal{P}(\Theta)\), the set \(\mathcal{F}^-(Q) = \{x \in \Theta : \mathcal{F}(x) \cap Q \neq \emptyset\}\) is closed in \(\Theta\);
(iii) lower semicontinuous (l.s.c.), if, for each closed set \(Q \subseteq \mathcal{P}(\Theta)\), the set \(\mathcal{F}^+(Q) = \{x \in \Theta : \mathcal{F}(x) \subseteq Q\}\) is closed in \(\Theta\).

**Proposition 1.** If the multifunction \(\mathcal{F}\) is completely continuous with nonempty compact values, then \(\mathcal{F}\) is u.s.c. if, and only if, \(\mathcal{F}\) has a closed graph.

**Definition 4.** A multifunction \(\mathcal{F} : [0, K] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})\) is called Carathéodory if
(i) for each \(z \in \mathbb{R}\), \(t \to \mathcal{F}(t, z)\) is measurable;
(ii) for almost every \(t \in [0, K]\), \(z \to \mathcal{F}(t, z)\) is u.s.c.

Moreover, a Carathéodory function \(\mathcal{F}\) is called \(L^1\)-Carathéodory if
(iii) for each \(\gamma > 0\), there exists \(\varphi_\gamma \in L^1([0, K], \mathbb{R}^+)\) such that

\[
\|\mathcal{F}(t, z)\| = \sup\{|v| : v \in \mathcal{F}(t, z)\} \leq \varphi_\gamma(t)
\]

for all \(\|z\| \leq \gamma\) and for a.e. \(t \in [0, K]\).

A multifunction \(\mathcal{F} : [0, K] \to \mathcal{P}(\mathbb{R})\) is measurable if the function

\[
t \mapsto d(z, \mathcal{F}(t)) = \inf\{|z - v| : v \in \mathcal{F}(t)\}, \quad \text{for each } z \in \mathbb{R},
\]

is measurable.

The set of selections of \(\mathcal{F}\) is defined by

\[
S_{\mathcal{F}} := \{f \in L^1([0, K]; \mathbb{R}) : f(t) \in \mathcal{F}(t, z(t)) \text{ for a.e. } t \in [0, K]\},
\]

for each \(z \in C([0, K]; \mathbb{R})\). Applying Aumann’s selection theorem ([21]), for a measurable multifunction \(\mathcal{F} : \Theta \to \mathcal{P}(\Theta)\), if

\[
\omega \to \inf\{\|v\| : v \in \mathcal{F}(t)\} \in L^1_{\mathcal{P}}
\]

the set \(S_{\mathcal{F}}\) is nonempty. Let \(U\) be a subset of \([0, K] \times \mathbb{R}\). \(U\) is \(L \otimes B\) measurable if \(U\) belongs to the \(\sigma\)-algebra generated by all sets of the form \(T \times Q\), where \(T\) is Lebesgue measurable in \([0, K]\) and \(Q\) is Borel measurable in \(\mathbb{R}\). A subset \(U\) of \(L^1([0, K]; \mathbb{R})\) is decomposable if
$T \subset [0, K]$ is measurable and, for all $u_1, u_2 \in U$, the function $u_1\chi_T + u_2\chi_{\bar{T}^c} \in U$, where $\chi_T$ denotes the characteristic function of $T$. Let $(\Theta, d)$ be a metric space; for $U, V \subset \Theta$, the Hausdorff metric is gained by

$$d_H(U, V) = \max\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v)\},$$

where $d(U, V) = \inf_{u \in U} d(u, v)$ and $d(U, V) = \inf_{v \in V} d(u, v)$.

**Definition 5.** A multifunction $\mathcal{F} : \Theta \to \mathcal{P}_f(\Theta)$ is $\epsilon-$Lipschitz if and only if there exists $\epsilon > 0$ such that

$$d_H(\mathcal{F}(u), \mathcal{F}(v)) \leq \epsilon d(u, v),$$

for each $u, v \in \Theta$, and if $\epsilon < 1$, $\mathcal{F}$ is a contraction.

The multifunction $\mathcal{F}$ has a fixed point if there is $u \in \Theta$ such that $u \in \mathcal{F}(u)$. Next, we propose the following lemmas, which are crucial to our research.

**Lemma 1 (Leray–Schauder alternative theorem [22]).** Let $\Theta$ be a Banach space, $\mathcal{P} \subseteq \Theta$ be closed and convex, $\Omega$ be an open set of $\mathcal{P}$ and $0 \in \Omega$. If $\mathcal{F} : \overline{\Omega} \to \mathcal{P}_c(\mathcal{P})$ is compact and u.s.c., then either $\mathcal{F}$ has a fixed point in $\overline{\Omega}$ or there exist $x \in \partial \Omega$ and $\mu \in (0, 1)$ such that $x \in \mu \mathcal{F}(x)$.

**Lemma 2 (Bressan–Colombo selection theorem [23]).** Let $X$ be a measurable and separable Banach space, and $(\Omega, \Sigma, \mu)$ be a finite measure space. Suppose the multifunction $\mathcal{F} : \Theta \to L^p(\Omega, \Theta)$ is l.s.c. and has closed decomposable values. Then, $\mathcal{F}$ has a continuous selection.

**Lemma 3 (Covitz–Nadler fixed-point theorem [24]).** Let $\Theta$ be a complete metric space. Assume that $\mathcal{F} : \Theta \to \mathcal{P}_f(\Theta)$ is a contraction, then, $\mathcal{F}$ has a fixed point $u \in \Theta$ such that $u \in \mathcal{F}(u)$.

**Lemma 4 ([25]).** Let $\Theta$ be a Banach space. Let $\mathcal{F} : [0, K] \times \Theta \to \mathcal{P}_{\leq c}(\Theta)$ be an $L^1$-Carathéodory multifunction and let $P : L^1([0, K]; \Theta) \to C([0, K]; \Theta)$ be a continuous linear operator. Then, the operator

$$P \circ S_\mathcal{F} : C([0, K]; \Theta) \to \mathcal{P}_{\leq c}(C([0, K]; \Theta))$$

$$x \mapsto (P \circ S_\mathcal{F})(x) = P(S_\mathcal{F}(x, (t)))$$

is a closed graph operator in $C([0, K]; \Theta) \times C([0, K]; \Theta)$.

**Lemma 5 ([16]).** For every $\varphi(t) \in C([0, K], \Theta)$, the boundary value problem

$$\left\{ \begin{array}{ll}
C\mathcal{D}^{\alpha+1}z(t) + b^\alpha C\mathcal{D}^\alpha z(t) = \varphi(t), & t \in [0, K], \\
z(K) = \lambda z(0), & z'(K) = \lambda z'(0),
\end{array} \right. \tag{2}$$

has a unique solution which is denoted by

$$z(t) = \int_0^t \int_0^\infty e^{-b(t-v)} \frac{(v-u)^{\alpha-1}}{\Gamma(\alpha)} \varphi(v) dv du + \mu_1(t) \int_0^K \int_0^\infty e^{-b(K-v)} \frac{(v-u)^{\alpha-1}}{\Gamma(\alpha)} \varphi(v) dv du + \mu_2(t) \int_0^K \frac{(K-u)^{\alpha-1}}{\Gamma(\alpha)} \varphi(u) du, \tag{3}$$

where $\mu_1(t) = \frac{e^{-bt}}{\lambda - e^{-bt}}$, $\mu_2(t) = \frac{1}{b[\lambda-e^{-bt}]} - \frac{e^{-bt}}{b[\lambda-e^{-bt}]}$ and $\lambda \neq 1$, $\lambda \neq e^{-bK}$.
There exists a function $\phi$ and the multifunction $G$ where

$$\sup_{t \in [0,K]} |\mu_1(t)|$$

Assume that $\theta$ is a positive constant and $M$ is a constant given in (5).

Step 1. Let us consider the operator $\Phi$ defined as the set of functions $y(t)$ such that $g(t) \in G(t,z(t))$ a.e. on $[0,K]$ and

$$\inf_{t \in [0,K]} |\mu_2(t)| = 0.$$

3. Main Results

In this section, we will use fixed-point theorems to prove the existence results of the problem (1); for convenience, we let $M = \frac{K^\alpha}{\Gamma(\alpha + 1)} \left[ (1 + \overline{\mu}_1) \left( \frac{1 - bK}{b} \right) + \overline{\mu}_2 \right].$ (5)

where $\overline{\mu}_1 = \sup_{t \in [0,K]} |\mu_1(t)|$ and $\overline{\mu}_2 = \sup_{t \in [0,K]} |\mu_2(t)|.$

Now, we give our first result:

**Theorem 1.** Assume that

(A1) The multifunction $G : [0,K] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is Carathéodory and convex;

(A2) There exists a function $\phi : \mathbb{R} \cup \{0\} \rightarrow \mathbb{R}^+$, which is continuous and nondecreasing, and a function $q(t) \in L^1([0,K];\mathbb{R}^+), satisfying

$$\|G(t,z)\| := \sup\{|y| : y \in G(t,z)\} \leq q(t)\phi(\|z\|),$$

for all $t \in [0,K]$ and $z \in C([0,K];\mathbb{R})$.

Then, the problem (1) with a $(K,\lambda)$-affine-periodic boundary value condition admits at least one solution on $[0,K]$, if

$$M\|q\|\phi(\eta) < \eta,$$

where $\eta$ is a positive constant and $M$ is a constant given in (5).

**Proof.** Let us consider the operator $\Phi : C([0,K];\mathbb{R}) \rightarrow \mathcal{P}(C([0,K];\mathbb{R}))$, where $\Phi(z)$ is defined as the set of functions $y(t) \in C([0,K];\mathbb{R})$ such that

$$y(t) = \int_0^t \int_0^e e^{-(t-v)(\alpha - 1)} g(u)dvdu$$

where $g \in S_{G(t,z(t))}$. What follows is to transform problem (1) into a fixed-point problem $z \in \Phi(z)$ and prove the existence of the fixed point. The proof is divided into four steps: Step 1. $\Phi$ is convex.
Let \( y_1, y_2 \in \Phi(z) \). For each \( t \in [0, K] \), there exist \( g_1, g_2 \in S_{G(z, \cdot(z))} \), so that

\[
y_k(t) = \int_0^t \int_0^\alpha e^{-b(t-u)} \frac{(v-u)^{a-1}}{\Gamma(a)} g_k(u) dudv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g_k(u) dudv + \mu_2(t) \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} g_k(u) du, \quad (k = 1, 2).
\]

Let \( \beta \in [0, 1] \); for every \( t \in [0, K] \), we have

\[
[\beta y_1 + (1 - \beta) y_2](t) = \int_0^t \int_0^\alpha e^{-b(t-u)} \frac{(v-u)^{a-1}}{\Gamma(a)} (\beta g_1 + (1 - \beta) g_2)(u) dudv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} (\beta g_1 + (1 - \beta) g_2)(u) dudv + \mu_2(t) \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} (\beta g_1 + (1 - \beta) g_2)(u) du.
\]

Because \( G \) is convex, we know that \( S_{G(z, \cdot(z))} \) is convex, which follows that \( \beta y_1 + (1 - \beta) y_2 \in \Phi(z) \), then \( \Phi \) is convex.

Step 2. \( \Phi \) is completely continuous.

Let \( \Omega_\eta = \{ z \in C([0, K]; \mathbb{R}) : \|z\| < \eta \} \), where \( \eta \) is a positive constant satisfying \( M\|q\|\Phi(\eta) < \eta \). What follows is to show that \( \Phi(\Omega_\eta) \subset \Omega_\eta \) in \( C([0, K]; \mathbb{R}) \). Thus, for each \( y \in \Phi(z), z \in \Omega_\eta \), there exists \( g \in S_{G(z, \cdot(z))} \), satisfying

\[
y(t) = \int_0^t \int_0^\alpha e^{-b(t-u)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) dudv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) dudv + \mu_2(t) \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} g(u) du,
\]

and by (A2)

\[
|y(t)| \leq \left| \int_0^t \int_0^\alpha e^{-b(t-u)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) dudv \right| + |\mu_1(t)| \left| \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) dudv \right| + |\mu_2(t)| \left| \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} g(u) du \right|
\]

\[
\leq \int_0^t \int_0^\alpha e^{-b(t-u)} \frac{(v-u)^{a-1}}{\Gamma(a)} |g(u)| dudv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} |g(u)| dudv + \mu_2(t) \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} |g(u)| du
\]

\[
\leq \left( \int_0^t \int_0^\alpha e^{-b(t-v)} \frac{v^a}{\Gamma(a + 1)} dv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{v^a}{\Gamma(a + 1)} dv + \mu_2(t) \frac{K^a}{\Gamma(a + 1)} \right) |q(t)| \Phi(\|z\|)
\]

\[
\leq \frac{K^a}{\Gamma(a + 1)} \left[ 1 - e^{-bK} \frac{1}{b} + \mu_1(t) \left( 1 - e^{-bK} \frac{1}{b} \right) + \mu_2(t) \frac{K^a}{\Gamma(a + 1)} \right] |q(t)| \Phi(\|z\|),
\]

that is

\[
\|y(t)\| \leq M \|q\| \Phi(\eta) < \eta,
\]

i.e., \( \Phi(\Omega_\eta) \subset \Omega_\eta \).
Next, we claim that $\Phi(\Omega_y)$ is equicontinuous in $C([0,K];\mathbb{R})$. Let $0 \leq t_1 < t_2 \leq K$ and $z \in \Omega_y$, for every $y \in \Phi(z)$, we obtain

$$y(t_2) - y(t_1) \leq \left| \int_0^{t_2} \int_0^0 e^{-b(t_2-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) du dv - \int_0^{t_1} \int_0^0 e^{-b(t_1-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) du dv \right| + |\mu_1(t_2) - \mu_1(t_1)| \left| \int_0^K \int_0^0 e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) du dv \right| + |\mu_2(t_2) - \mu_2(t_1)| \left| \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} g(u) du \right| \to 0,$$

as $t_1 \to t_2$. Consequently, the completely continuous $\Phi : C([0,K];\mathbb{R}) \to \mathcal{P}(C([0,K];\mathbb{R}))$ is obtained using the Arzela–Ascoli theorem.

Step 3. $\Phi$ has a closed graph.

Let $z_n \to z_*$, $y_n \in \Phi(z_n)$ and $y_n \to y_*$. We claim that $y_* \in \Phi(z_*)$. For every $n$, select $g_n \in \mathcal{G}_{(z_n,\varepsilon_n)})$ such that

$$y_n(t) = \int_0^t \int_0^0 e^{-b(t-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g_n(u) du dv + \mu_1(t) \int_0^K \int_0^0 e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g_n(u) du dv + \mu_2(t) \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} g_n(u) du.$$

Let a continuous linear operator $P : L^1([0,K];\mathbb{R}) \to C([0,K];\mathbb{R})$ be defined by

$$g \mapsto P(g) = \int_0^t \int_0^0 e^{-b(t-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) du dv + \mu_1(t) \int_0^K \int_0^0 e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g(u) du dv + \mu_2(t) \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} g(u) du.$$

$P \circ \mathcal{G}_{(z_*,\varepsilon)}$ is an operator, which has closed graph, by Lemma 4. As $z_n \to z_*$, and $y_n(t) \in P(\mathcal{G}_{(z_n,\varepsilon_n)})$, for all $n$, there exists $g_* \in \mathcal{G}_{(z_*,\varepsilon_*)}$ such that

$$y_*(t) = \int_0^t \int_0^0 e^{-b(t-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g_*(u) du dv + \mu_1(t) \int_0^K \int_0^0 e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} g_*(u) du dv + \mu_2(t) \int_0^K \frac{(K-u)^{a-1}}{\Gamma(a)} g_*(u) du,$$
that is, \( y_s \in \Phi(z_s) \).

Step 4. \( \Phi \) has a fixed point.

Let \( z \) be a solution to problem (1). There exists \( g \in L^1([0, K]; \mathbb{R}) \) such that

\[
z(t) = \int_0^t \int_0^s e^{-\beta(t-v)} \frac{(\nu - u)^{\alpha-1}}{\Gamma(\alpha)} g(u) du dv + \mu_1(t) \int_0^K \int_0^s e^{-\beta(K-v)} \frac{(\nu - u)^{\alpha-1}}{\Gamma(\alpha)} g(u) du dv + \mu_2(t) \int_0^K (K - u)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} g(u) du,
\]

for \( t \in [0, K] \).

Taking account of (A2), for any \( t \in [0, K] \), we gain that

\[
|z(t)| \leq \frac{K^a}{\Gamma(\alpha + 1)} \left( \frac{1 - e^{-bK}}{b} + \beta_1 \frac{1 - e^{-bK}}{b} + \beta_2 \right) |q(t)| \phi(||z||),
\]

then,

\[
\|z(t)\| \leq M ||q|| \phi(||z||).
\]

Therefore, there exists \( \eta \) such that \( ||z|| = \eta \). Let \( \Lambda = \{ z \in C([0, K]; \mathbb{R}) : ||z|| < \eta + 1 \} \).

It is obvious that \( \Phi : \Lambda \rightarrow P(C([0, K]; \mathbb{R})) \) is upper semicontinuous and compact. Because of the selection of \( \Lambda \), there is without \( z \in \partial \Lambda \) and \( \beta \in (0, 1) \), such that \( z \in \beta \Phi(z) \). For Lemma 1 (Leray–Schauder alternative theorem), \( \Phi \) has a fixed point, \( z \in \Lambda \). This means that the fixed point \( z \) is the solution to the boundary value problem (1). The proof is completed. \( \Box \)

As the second result, we consider that the multifunction \( G \) is not necessarily convexvalued. Thanks to the Bressan–Colombo selection theorem and the Leray–Schauder alternative theorem, we gain the existence result of the problem (1).

**Theorem 2.** Suppose that

(A3) \( G : [0, K] \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \) is a nonempty compact multifunction satisfying

(i) \( (t, z) \rightarrow G(t, z) \) is \( \mathcal{L} \otimes \mathcal{B} \) measurable;

(ii) \( z \rightarrow G(t, z) \) is lower semicontinuous for each \( t \in [0, K] \).

(A4) For almost any \( t \in [0, K] \) and all \( ||z|| \leq \epsilon \), there exists \( \mu_e \in L^1([0, K]; \mathbb{R}^+) \), satisfying

\[
\|G(t, z)\| = \sup \{ |y| : y \in G(t, z) \} \leq \mu_e(t),
\]

where \( \epsilon \) is a positive constant.

Then, the solution set of the \((K, \lambda)\)-affine-periodic boundary value problem (1) is nonempty.

**Proof.** Define the Nemytskii operator \( N : C([0, K]; \mathbb{R}) \rightarrow L^1([0, K]; \mathbb{R}) \) associated with \( G \) as

\[
N(z) = \{ g \in L^1([0, K]; \mathbb{R}) : g(t) \in G(t, z), \text{ for a.e. } t \in [0, K] \}.
\]

By virtue of (A3), (A4) and Theorem 3.2 in [26], it is clear that \( N(\cdot) \) is nonempty, decomposable, closed and lower semicontinuous. By Lemma 2 (Bressan–Colombo selection theorem), for all \( z \in C([0, K]; \mathbb{R}) \), a continuous function \( g : C([0, K]; \mathbb{R}) \rightarrow L^1([0, K]; \mathbb{R}) \) exists to satisfy \( g(z) \in S_{G, z} \).

Let us consider the boundary value problem as follows:

\[
\begin{cases}
C^{\alpha+1} z(t) + \beta C^{\alpha} z(t) = g(z(t)), & \text{for a.e. } t \in [0, K], \\
z(K) = \lambda z(0), z'(K) = \lambda z'(0).
\end{cases}
\]  

(8)

It is clear that if \( z \in AC([0, K]; \mathbb{R}) \) is a solution to (8), then \( z \) is a solution to problem (1).
Let us define the operator \( \Phi \) to transform problem (8) into a fixed-point problem,

\[
\Phi(z)(t) = \int_0^t \int_0^\alpha e^{-b(t-v)} \frac{(v-u)^{n-1}}{\Gamma(n)} g(u)du dv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{n-1}}{\Gamma(n)} g(u)du dv + \mu_2(t) \int_0^K \frac{(K-u)^{n-1}}{\Gamma(n)} g(u)du.
\]

(9)

Similar to the previous analysis, it is easy to know that \( \Phi \) is convex, completely continuous and has a closed graph. The proof process is analogous to Theorem 1, so we do not repeat it here. \( \Box \)

As the last result of this article, we change the convex value condition of the multifunction \( G \) into the nonconvex value condition.

**Theorem 3.** Assume that

(A5) \( G : [0, K] \times \mathbb{R} \to \mathcal{P}_b(\mathbb{R}) \) is a multifunction that \( G(\cdot, z) : [0, K] \to \mathcal{P}_b(\mathbb{R}) \) is measurable for each \( z \in \mathbb{R} \);

(A6) For almost every \( t \in [0, K] \) and \( z_1, z_2 \in C([0, K]; \mathbb{R}) \), there exists a function \( l(t) \in L^1([0, K], \mathbb{R}^+) \), satisfying

\[
d_H(G(t, z_1), G(t, z_2)) \leq l(t)|z_1 - z_2|,
\]

where \( d_H(\cdot, \cdot) \) is the Hausdorff metric.

Then, the \( (K, \lambda) \)-affine-periodic boundary value problem (1) has a solution on \([0, K] \) if \( M\|l\|_{L^1} < 1 \).

**Proof.** As \( G(\cdot, z) \) is measurable, \( S_{G(\cdot, z(\cdot))} \) is nonempty for each \( z \in C([0, K]; \mathbb{R}) \), which implies that \( G \) has a measurable selection. We divided the proof process into two steps:

Step 1. \( \Phi \) is closed.

Let \( \{z_n\}_{n \geq 0} \) be a sequence where \( z_n \) is convergent to \( z_\ast \) in \( C([0, K]; \mathbb{R}) \), then \( z_\ast \in C([0, K]; \mathbb{R}) \). For any \( t \in [0, K] \), there exists \( x_n \in S_{G(\cdot, z_n(\cdot))} \) such that,

\[
z_n(t) = \int_0^t \int_0^\alpha e^{-b(t-v)} \frac{(v-u)^{n-1}}{\Gamma(n)} x_n(u)du dv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{n-1}}{\Gamma(n)} x_n(u)du dv + \mu_2(t) \int_0^K \frac{(K-u)^{n-1}}{\Gamma(n)} x_n(u)du.
\]

In view of \( G \) as compact, we may pass to a subsequence to understand that \( x_n \to x_\ast \in L^1([0, K]; \mathbb{R}) \). It is easy to obtain that \( x_\ast \in S_{G(\cdot, z_\ast(\cdot))} \) and

\[
z_n(t) \to z_\ast(t) = \int_0^t \int_0^\alpha e^{-b(t-v)} \frac{(v-u)^{n-1}}{\Gamma(n)} x_\ast(u)du dv + \mu_1(t) \int_0^K \int_0^\alpha e^{-b(K-v)} \frac{(v-u)^{n-1}}{\Gamma(n)} x_\ast(u)du dv + \mu_2(t) \int_0^K \frac{(K-u)^{n-1}}{\Gamma(n)} x_\ast(u)du.
\]

for each \( t \in [0, K] \). That is, \( z_\ast \in \Phi(z) \), which means that \( \Phi \) is closed.

Step 2. \( \Phi \) is a contractive multifunction.
Let \( z_1, z_2 \in C([0, K]; \mathbb{R}) \) and \( \tilde{y}_1 \in \Phi(z_1) \). Therefore, there exists \( m_1(t) \in G(t, z_1(t)) \) such that

\[
\tilde{y}_1(t) = \int_0^t \int_0^\nu e^{-b(t-\nu)} \frac{(v-u)^{a-1}}{\Gamma(a)} m_1(u) du dv \\
+ \mu_1(t) \int_0^\nu \int_0^{b(K-v)} e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} m_1(u) du dv \\
+ \mu_2(t) \int_0^{b(K-u)} \frac{(K-u)^{a-1}}{\Gamma(a)} m_1(u) du, \quad \text{for } t \in [0, K].
\]

Applying (A3), there exists \( \omega \in G(t, z_2) \) such that

\[
|m_1(t) - \omega| \leq l(t)|z_1 - z_2|, \quad t \in [0, K].
\]

Consider an operator \( \mathcal{F} : [0, K] \to \mathcal{P}(\mathbb{R}) \), which is defined as

\[
\mathcal{F}(t) = \{ \omega \in \mathbb{R} : |m_1(t) - \omega| \leq l(t)|z_1 - z_2| \}.
\]

Let \( m_2(t) \) be the measurable selection for \( \mathcal{F} \). We obtain \( m_2(t) \in G(t, z_2(t)) \) for the measurable of the multivalued operator \( \mathcal{F}(t) \cap G(t, z_2(t)) \). Then, one gains

\[
|m_1(t) - m_2(t)| \leq l(t)|z_1 - z_2|, \quad \text{for every } t \in [0, K].
\]

For every \( t \in [0, K] \), we define

\[
\tilde{y}_2(t) = \int_0^t \int_0^\nu e^{-b(t-\nu)} \frac{(v-u)^{a-1}}{\Gamma(a)} m_2(u) du dv \\
+ \mu_1(t) \int_0^\nu \int_0^{b(K-v)} e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} m_2(u) du dv \\
+ \mu_2(t) \int_0^{b(K-u)} \frac{(K-u)^{a-1}}{\Gamma(a)} m_2(u) du.
\]

As a consequence,

\[
\left| \tilde{y}_1(t) - \tilde{y}_2(t) \right| \leq \left| \int_0^t \int_0^\nu e^{-b(t-\nu)} \frac{(v-u)^{a-1}}{\Gamma(a)} |m_3(u) - m_2(u)| du dv \right| \\
+ \left| \mu_1(t) \int_0^\nu \int_0^{b(K-v)} e^{-b(K-v)} \frac{(v-u)^{a-1}}{\Gamma(a)} |m_3(u) - m_2(u)| du dv \right| \\
+ \left| \mu_2(t) \int_0^{b(K-u)} \frac{(K-u)^{a-1}}{\Gamma(a)} |m_3(u) - m_2(u)| du \right| \\
\leq \frac{K^a}{\Gamma(a+1)} \left( \frac{1 - e^{-bK}}{b} + \Phi_1 \left( \frac{1 - e^{-bK}}{b} \right) + \Phi_2 \right) \int_0^K |m_3(u) - m_2(u)| du \\
\leq M \int_0^K l(t)|z_1 - z_2| du.
\]

Thus,

\[
\| \tilde{y}_1(t) - \tilde{y}_2(t) \| \leq M\|l\|_1 \|z_1 - z_2\|.
\]

Analogously, it follows that

\[
d_H(\Phi(z_1), \Phi(z_2)) \leq M\|l\|_1 \|z_1 - z_2\|.
\]
As $M\|l\|_{1} < 1$, then $\Phi$ is a contraction. It is easy to understand that $\Phi$ has a fixed point by Lemma 3 (Covitz–Nadler fixed-point theorem), so the $(K, \lambda)$-affine-periodic boundary value problem (10) has at least one solution on $[0, K]$. \hfill \Box

4. Example

Example 1. Let us consider the $(1,2)$-affine-periodic problem:

\[
\begin{cases}
C^\alpha D^\alpha x(t) + C^\lambda D^\lambda x(t) \in G(t, x), & \text{for } t \in [0, 1], \\
x(1) = 2x(0), x'(1) = 2x'(0),
\end{cases}
\]

Here $\alpha = \frac{1}{2}$, $b = 1$, $K = 1$, $\lambda = 2$ and the multifunction $G : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is defined by

\[
x \to G(t, x) = \left[ \frac{z^2}{z^2 + 2} + t^2 + 4, \frac{z}{z + 1} + \sin t + t^3 \right].
\]

Noting that, $\|G(t, z)\| = \sup\{|z| : z \in G\} \leq 6 = q(t)\phi(\|z\|)$, $z \in \mathbb{R}$, where $q(t) = 1$, $\phi(\|z\|) = 6$. With the above assumptions, we can obtain $\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}, \Pi_1 = \sup_{t \in [0, 1]} \frac{1}{z^2-e^{-1}} \approx 0.6127, \Pi_2 = \sup_{t \in [0, 1]} |1 - e^{-1}z^2| \approx 0.7746, M = \frac{1}{\Gamma(\frac{1}{2})} [(1 + \Pi_1)(1 - e^{-1}) + \Pi_2] \approx 2.0243$. Further, using the condition $M\|q(\phi(\eta)) < \eta$, we can find that $\eta > 12.1458$. As shown, all the assumptions of Theorem 1 are held. Then, problem (10) admits at least one solution on $[0, 1]$.

5. Conclusions

This article is devoted to research the existence of a $(K, \lambda)$-affine-periodic boundary value solution to a fractional differential inclusion. Applying the Leray–Schauder alternative theorem and Covita and Nadler fixed-point theorem, we consider two cases of the multifunction $G$ with a convex value and nonconvex value. Nevertheless, here, we are dealing with the Caputo fractional derivative. There are also other types of fractional derivative, such as the Hilfer fractional derivative, Hadamard fractional derivative and so on. Moreover, the function we consider in this paper is in one-dimensional space. In the future, we will study the fractional order affine-periodic boundary value problem involving other types of fractional derivatives in N-dimensional space.

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