fractal and fractional

Article
A Fractional (q, q') Non-Extensive Information Dimension for Complex Networks

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Simple Summary: The fractional (q, q')-information dimension for complex networks is introduce, and a dual version of the (q, q')-entropy, called (q, q')-extropy, is proposed. Experiments reveal that the fractional (q, q')-information dimension is less than the classical one (based on Shannon entropy) for both real-world and synthetic networks.

Abstract: This article introduces a new fractional approach to the concept of information dimensions in complex networks based on the (q, q')-entropy proposed in the literature. The q parameter measures how far the number of sub-systems (for a given size ε) is from the mean number of overall sizes, whereas q' (the interaction index) measures when the interactions between sub-systems are greater ($q' > 1$), lesser ($q' < 1$), or equal to the interactions into these sub-systems. Computation of the proposed information dimension is carried out on several real-world and synthetic complex networks. The results for the proposed information dimension are compared with those from the classic information dimension based on Shannon entropy. The obtained results support the conjecture that the fractional (q, q')-information dimension captures the complexity of the topology of the network better than the information dimension.

Keywords: complex networks; measures of information; fractional-order entropy

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1. Introduction

Entropy—introduced by Clausius [1] in the context of thermodynamics—is a crucial measure of the uncertainty of the state in a physical system, allowing for specification of the state of disorder, randomness, or uncertainty in the micro-structure of the system. Due to this fact, researchers in many scientific fields have continually extended, interpreted, and applied the notion of entropy.

Several generalizations of the celebrated Shannon entropy, originally related to information processes [2], have been introduced in the literature. For a deeper review of entropy measures, the reader is referred to [3–8].

Given a probability distribution $P = \{p_1, p_2, \ldots, p_N\}$ under a probability space $X = \{x_1, x_2, \ldots, x_N\}$, the Shannon entropy under P (see [9]) is generated as:

$$I = \lim_{t \to -1} \frac{d}{dt} \sum_{i=1}^{N} p_i^{-t} = -\sum_{i=1}^{N} p_i \ln p_i, \tag{1}$$

where N is the total number of (microscopic) probabilities p_i and $\sum_{i=1}^{N} p_i = 1$.

Similarly, the Tsallis entropy (also called q-entropy) [10–12] is generated by the same procedure but using Jackson's q-derivative operator $D_q^t f(t) = \dfrac{f(qt) - f(t)}{(q-1)t}$, $t \neq 0$, [13] (see also [9,14,15]), given by

$$I_T = \lim_{t \to -1} D_q^t \sum_{i=1}^{N} p_i^{-t} = -\sum_{i=1}^{N} p_i ln_q p_i, \tag{2}$$

where the $q-$logarithm is defined by

$$ln_q(p_i) = \frac{p_i^{1-q} - 1}{1-q}, \tag{3}$$

$(p_i > 0, q \in \mathbb{R}, q \neq 1, ln_1 p_i = ln p_i)$.

The Tsallis entropy is connected to the Shannon entropy through the limit

$$\lim_{q \to 1} I_T = I, \tag{4}$$

which is why it is considered a parameter extension of Shannon entropy.

Several entropy measures have been revealed following the same procedure above, using appropriate fractional-order differentiation operators on the generative function $\sum_{i=1}^{N} p_i^{-t}$ with respect to the variable t and then letting $t \to -1$ (see, e.g., [16–23]).

A new measure of information, called extropy, has been introduced by Lad, Sanfilippo, and Agrò [24] as the dual version of Shannon entropy. In the literature, this measure of uncertainty has received considerable attention in recent years [25–27]. The entropy and extropy of a binary distribution ($N = 2$) are identical.

In recent years, complex networks and systems have been extensively studied, as they are helpful tools for modelling complex systems in various interdisciplinary fields, such as mathematics, statistical physics, computer science, sociology, economics, biology, and so on (see [28–36], to name just a few).

The dimension of a network is a crucial concept for understanding the underlying architecture, complex topology, and dynamic processes of the network, which are difficult to understand. The dependence of model behaviour on the dimension of the system leads to the occurrence of critical phenomena.

For an updated survey on the fractal dimensions of networks and other related theoretical topics, we refer the reader to [37–44].

In the study of the structure of complex networks, the fractional-order information dimension has been developed by combining the fractional order entropy and information dimensio (see, for instance, Refs. [45,46] and the references given therein).

This article proposes a fractional (q, q')-information dimension for complex networks derived by applying the fractional order entropy introduced in [47]. This new information dimension is computed on several networks, both those gathered from real-world fields and synthetic complex networks. The results provide evidence that the fractional two-parameter non-extensive information dimension describes the complexity of the topology of the network better than the information dimension. This is corroborated by statistical analysis and data mining techniques.

The remainder of this paper is structured as follows. Section 2 introduces a fractional entropy measure and the information dimension of complex networks. Then, the proposed fractional information dimension measure is introduced in Section 3. Section 4 focuses on applying this new measure to various complex networks. Finally, the findings of this study and our conclusions are given in Section 4.

2. Preliminaries

2.1. Fractional $(q, q')-$*Entropy*

Following the same procedure used to obtain the Shannon and Tsallis entropies, a generalized non-extensive two-parameter entropy, named fractional $(q, q')-$entropy, was developed in [47], obtained by the action of a derivative operator previously proposed by Chakrabarti and Jagannathan [48]:

$$I_{q,q'} := \lim_{t \to -1} D^t_{q,q'} \sum_{i=1}^{N} p_i^{-t} = \sum_{i=1}^{N} \frac{p_i^{q'} - p_i^{q}}{q - q'}, \tag{5}$$

where $D^t_{q,q'}$ of a function f is given by $D^t_{q,q'} f(t) = \dfrac{f(qt) - f(q't)}{(q - q')t}$.

Following the general idea that extropy is the complementary dual version of entropy, we present the $(q, q')-$extropy for a discrete random variable X as

$$J_{q,q'} = \sum_{i=1}^{N} \frac{(1 - p_i)^{q'} - (1 - p_i)^{q}}{q - q'}. \tag{6}$$

An easy computation shows that Equation (5) can be expressed in terms of the Tsallis entropy:

$$I_{q,q'} = \frac{(1 - q')I_T - (1 - q)I_T}{q - q'}. \tag{7}$$

Note that $I_{q,q'} \geq 0 \ \forall q, q'$ and $I_{q,q'} = \dfrac{W^{1-q} - W^{1-q'}}{q' - q}$ for $p_i = 1/W \ \forall i$. Consider a system composed of two independent sub-systems A and B with factorized probabilities $p_{i,A}$ and $p_{i,B}$; then,

$$I_{q,q'} = I^A_{q,q'} + I^B_{q,q'} + (1 - q) I^A_{q,q'} I^B_{q,1} + (1 - q') I^B_{q,q'} I^A_{q,1}, \tag{8}$$

where the $I(q, 1)$ entropy resembles the Tsallis entropy in Equation (2) and $I(1, 1)$ is the Shannon entropy in Equation (1). Thus, $I(q, q')$ is non-additive for $q, q' \neq 1$.

2.2. Information Dimension of Networks

The information dimension measuring the topological complexity of a given network is sketched briefly in the following.

The definition of the information dimension was introduced in [49], considering the Shannon entropy in Equation (1), as follows:

$$d_I = -\lim_{\varepsilon \to 0} \frac{I(\varepsilon)}{\ln \varepsilon} = \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{N_b} p_i(\varepsilon) \ln p_i(\varepsilon)}{\ln \varepsilon}, \tag{9}$$

where $p_i(\varepsilon) = \dfrac{n_i(\varepsilon)}{n}$, $n_i(\varepsilon)$ are the nodes into the ith box of size ε, n is the total number of nodes in the network, and N_b is the number of boxes required to cover the network. The reader may consult [50,51] for in-depth details on obtaining N_b.

Applying Equation (9), we can assert that

$$I(\varepsilon) \sim -d_I \ln \varepsilon + \beta, \tag{10}$$

for some constant β, where ε is the diameter of the boxes covering the network.

3. Fractional (q, q') Information Dimension of Complex

Now, we proceed to the primary goal of this article, which is to introduce the fractional $(q, q')-$information dimension of complex network, which is denoted by $d_{q,q'}$ and given by:

$$d_{qq'} = -\lim_{\varepsilon \to 0} \frac{I_{q,q'}(\varepsilon)}{\ln \varepsilon} = \lim_{\varepsilon \to 0} \frac{\frac{\sum_{i=1}^{N_b} p_i^{q'}(\varepsilon) - p_i^{q}(\varepsilon)}{q-q'}}{\ln \varepsilon}, \tag{11}$$

where $p_i(\varepsilon) = \dfrac{n_i(\varepsilon)}{n}$, $n_i(\varepsilon)$ are the nodes in the ith box of size ε, n is the total number of nodes in the network, and N_b is the number of boxes required to cover the network. The parameters q and q' depend on the minimal covering of the network; thus, the maximal entropy minimal covering principle was adopted, as in the previous research on complex networks [45,46,52–54], for the computation of $\varepsilon = [2, \Delta]$, where Δ denotes the diameter of the network.

For some constant β, Equation (12) can be deduced from Equation (11):

$$I_{q,q'}(\varepsilon) \sim -d_{q,q'} \ln \varepsilon + \beta. \tag{12}$$

Computation of q, q'

The computation of q relies on the idea that a network can be considered as a system that can be divided into several sub-systems. This division is based on the formation of minimum boxes by the box-covering heuristic. Hence, the number of sub-systems is equal to the number of boxes N_b for a given size ε.

For a given box size ε, the value of q is determined as the average of q_ε, denoted by $\overline{q}_\varepsilon$, where

$$q_\varepsilon := \frac{(\Delta - 1) N_b(\varepsilon)}{\sum_{\varepsilon=2}^{\Delta} N_b(\varepsilon)}. \tag{13}$$

Note that $q_\varepsilon = (q_2, q_3, \ldots, q_\Delta)$.

This approximation measures how far the number of sub-systems (for a given size ε) is from the mean number of overall sizes, which is the baseline.

Now, to quantify the interactions among the elements that form the sub-systems (nodes) and among these sub-systems (boxes), the parameters α and β were introduced in [46]:

$$\alpha_{\varepsilon,i} = 1 - \frac{|S_i| indeg(S_i)}{n \sum_{i=1}^{N_b} indeg(S_i)}, \tag{14}$$

$$\beta_{\varepsilon,i} = 1 - \frac{outdeg(S_i)\varepsilon}{\Delta \sum_{i=1}^{N_b} outdeg(S_i)}, \tag{15}$$

where $|S_i|$ is the number of nodes in S_i, n is the number of nodes of the network, $indeg(G_i)$ are the edges among the nodes that are in S_i, $outdeg(S_i)$ are the edges among the sub-networks S_i, ε is the diameter of the box that covers the sub-network S_i, and Δ is the diameter of the network.

Finally, q' is defined by

$$q' = \frac{\overline{\beta}_{\varepsilon,i}}{\overline{\alpha}_{\varepsilon,i}}, \tag{16}$$

where $\overline{\beta}_{\varepsilon,i}$, $\overline{\alpha}_{\varepsilon,i}$ are the mean of $\beta_{\varepsilon,i}$ and $\alpha_{\varepsilon,i}$, respectively, as they are vectors of type $(a_{\varepsilon,1}, a_{\varepsilon,2}, \ldots, a_{\varepsilon,N_b})$. Equation (16) defines the interaction index [46], which indicates whether β is equal to $(q' = 1)$, greater than $(q' > 1)$ or less than $(q' < 1)$ α. Hence, it reflects which type of interaction is stronger, that is, either inner sub-system interactions (α) or outer interactions (β) are stronger, or both are balanced.

Figure 1 shows examples of how α and β are computed. Once a box covering is obtained (using the approach in [51]) for $\varepsilon = 2$, see Figure 1a, the re-normalization agglomerates the nodes in the boxes into super-nodes (sub-systems) S_1 and S_2, as shown in Figure 1b. As $\Delta = 2$, in the example, $q = \overline{q}_{\varepsilon} = q_2 = 1$. Furthermore, $indeg(S_1) = 3$, $indeg(S_2) = 1$, $outdeg(S_1) = 1$, and $outdeg(S_2) = 1$, which reflect the degrees of the nodes of the re-normalized network in Figure 1b.

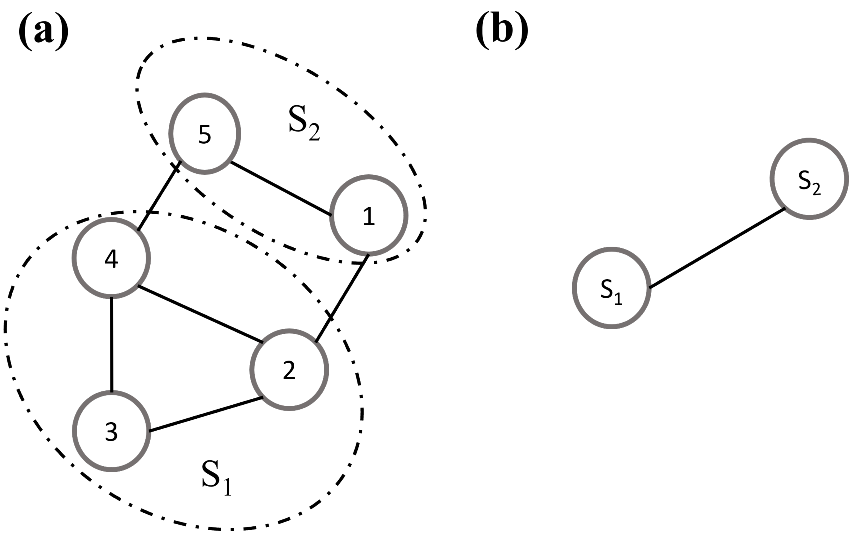

Figure 1. (**a**) Box covering of a network and (**b**) network re-normalization for $\varepsilon = 2$.

As $n = 5$, the results for Equation (14) are $\alpha_{2,S_1} = 0.550$ and $\alpha_{2,S_2} = 0.900$, whereas those from Equation (15) are $\beta_{2,S_1} = 0.50$, and $\beta_{2,S_2} = 0.500$; thus, $q' = 0.689$. The included networks have a diameter greater than the example shown above and, so, the steps to estimate q_{ε} and q'_{ε} are repeated for each box size ε, resulting in two vectors that are averaged to obtain q and q'. Once q and q' have been obtained, then $d_{q,q'}$ can be computed by approximating Equation (12) to ε vs. $I_{q,q'}(\varepsilon)$ through non-linear regression [55].

4. Results

4.1. Real-World Networks

The fractional $(q, q')-$information dimension Equation (11) and the classical information dimension Equation (9) were computed for 28 real-world networks gathered from [46,56] (see Table 1 for the diameter, source, and number of nodes and edges for each network). These networks cover several fields, such as biological, social, technological, and communications fields, and so, they can be considered to be representative.

Table 1. Diameter, number of nodes, source, d_I, and $d_{q,q'}$ of real-world networks.

Network	Full Name	Source	Diameter	Nodes	Edges
ACF	American college football	[46]	4	115	613
BCEPG	Bio-CE-PG	[46]	8	1692	47,309
BGP	Bio-grid-plant	[46]	26	1272	2726
BGW	Bio-grid-worm	[46]	12	16,259	762,774
CEN	*C. elegans* neural network	[46]	5	297	2148
CNC	Ca-netscience	[46]	17	379	914
COL	SocfbColgate88	[56]	6	3482	155,044
DRO	Drosophilamedulla1	[56]	6	1770	33,635

Table 1. *Cont.*

Network	Full Name	Source	Diameter	Nodes	Edges
DS	Dolphins social network	[46]	8	62	159
ECC	*E. coli* cellular network	[46]	18	2859	6890
EM	Email	[46]	8	1133	5451
IOF	Infopenflights	[56]	14	2905	30,442
JM	Jazz-musician	[46]	6	198	2742
JUN	Jung2015	[56]	16	2989	31,548
LAS	Lada Adamic's network	[46]	8	350	3492
LDU	Labanderiadunne	[56]	6	700	6444
MAR	Marvel	[56]	11	19,365	96,616
MIT	SocfbMIT	[56]	8	6402	251,230
PAIR	Pairdoc	[56]	14	8914	25,514
PG	Power grid network	[46]	46	4941	6594
PGP	Techpgp	[56]	24	10,680	24,340
POW	Powerbcspwr10	[56]	49	5300	13,571
PRI	SocfbPrinceton12	[56]	9	6575	293,307
TC	Topology of communications	[46]	7	174	557
USAA	USA airport network	[46]	7	500	2980
WHO	TechWHOIS	[56]	8	7476	56,943
YEAST	Protein interaction	[46]	11	2223	7046
ZCK	Zachary's karate club	[46]	5	34	78

Next, the models of Equations (10) and (12)—which correspond to the classical information dimension and the $(q,q')-$information model, respectively—were approximated by carrying out non-linear regression [55] in MATLAB R2022a. The best model was selected according to the summed Bayesian information criterion with bonuses ($SBICR$) [57]. The $SBICR$ penalizes overly complex models (which were estimated independently) and the size of the data set employed to approximate the parameters; hence, the model with the largest $SBICR$ score should be selected.

Table 2 shows the fit values for the information model Equation (10) and fractional (q,q') model Equation (12) with respect to the information and $(q,q')-$information, respectively. The results of $SBICR$, d_I, $d_{q,q'}$, q, and q' computations are also provided. The columns $SBICR_I$ and $SBICR_{q,q'}$ indicate that Equation (12) performed better than Equation (10) for all networks except PG and POW (in bold). Additionally, $q > 1$ indicates that the number of sub-systems for a given ε was higher than the baseline (i.e., mean sub-systems found for all ε). On the other hand, for 12 networks, the interaction between sub-systems ($q' > 1$; in bold) was stronger; furthermore, for 16 networks, the inner interactions between the elements of the sub-systems ($q' < 1$) were higher than those between sub-systems (i.e., outer interactions).

Table 2. The $SBICR$, d_I, $d_{q,q'}$, q, and q' values obtained for the information model Equation (10) and the fractional $(q,q')-$information model Equation (12).

Network	$SBICR_I$	$SBICR_{(q,q')}$	d_I	$d_{q,q'}$	q	q'
ACF	−10.135	−7.348	1.913	0.930	2.976	0.442
BCEPG	−35.380	−20.988	1.828	1.004	6.109	1.814
BGP	−95.919	−95.442	1.591	1.037	3.558	0.817
BGW	−64.525	−42.927	1.949	1.006	2.437	1.219
CEN	−13.897	−12.103	1.822	0.988	3.253	0.805
CNC	−58.166	−49.747	1.835	1.014	3.606	0.733
COL	−25.187	−18.343	2.679	1.001	3.655	1.364
DRO	−23.34	−18.202	1.443	0.998	5.856	0.662
DS	−17.868	−14.616	1.498	0.989	3.959	0.788
ECC	−87.426	−66.706	1.625	1.029	5.044	1.442
EM	−34.459	−31.15	1.547	1.001	4.663	0.915
IOF	−65.328	−51.656	1.736	1.028	4.846	1.112
JM	−19.449	−13.532	2.805	0.986	2.659	0.726
JUN	−63.643	−58.511	2.485	1.006	5.95	1.105
LAS	−30.269	−21.898	2.103	1.020	3.459	0.365
LDU	−20.750	−20.091	1.850	0.998	2.879	0.864
MAR	−52.612	−46.428	1.644	1.003	3.976	0.900
MIT	−39.208	−25.351	2.295	1.006	4.353	1.165
PAIR	−68.947	−57.065	1.582	1.004	2.726	0.670
PG	−186.322	−199.049	1.463	0.999	5.221	1.324
PGP	−117.159	−100.827	1.573	1.013	2.930	1.666
POW	−186.721	−206.691	1.589	0.994	5.176	0.489
PRI	−45.866	−27.325	2.533	1.016	6.338	1.234
TC	−22.759	−22.025	1.57	0.991	5.494	1.143
USAA	−26.655	−18.497	1.678	0.996	2.358	0.626
WHO	−36.125	−29.873	1.675	1.002	4.751	1.280
YEAST	−48.576	−43.622	1.533	1.006	6.500	0.594
ZCK	−9.77	−3.672	1.500	0.950	5.094	0.696

Figure 2a shows the results for the SocfbPrinceton12 network, where the fractional (q,q') model (dotted line) is closer to the fractional $(q,q')-$information (+) than the information model (solid line) to information (*). This is rather difficult to appreciate in Figure 2b, making the $SBICR$ a valuable tool for analysis. The opposite scenario can be seen in Figure 2c, where the information model performed better than the fractional (q,q') model as the value of $SBICR_I$ was higher than the value of $SBICR_{(q,q')}$ for the Power grid network (PG) (see Table 2).

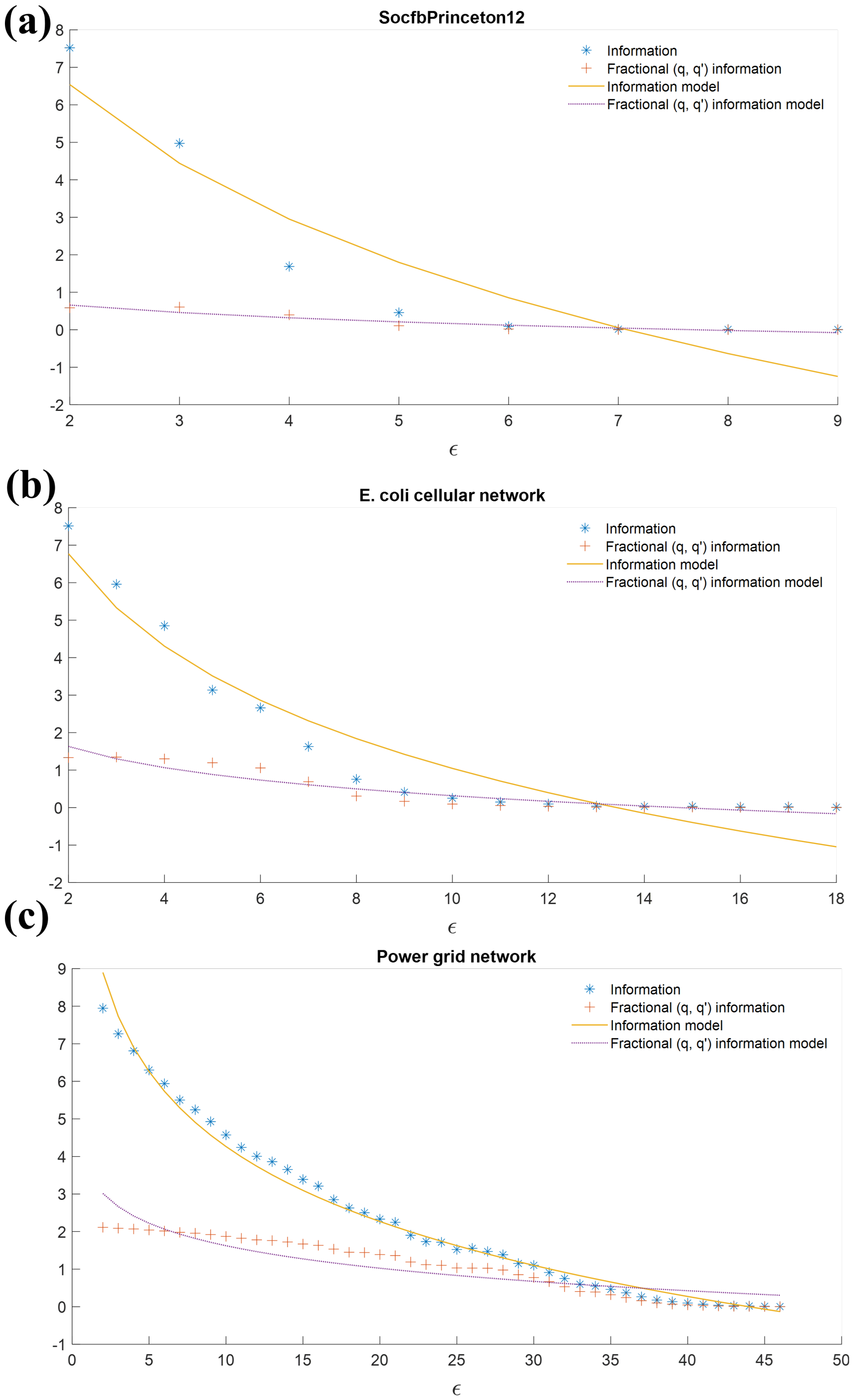

Figure 2. Fit of Equations (10) and (11) of (**a**) SocfbPrinceton12, (**b**) *E. coli* cellular, and (**c**) Power grid network.

4.2. Synthetic Networks

A similar procedure was followed on the networks generated using the Barabasi–Albert (BA) [58], Song, Havlin, and Makse (SHM) [59], and Watts and Strogatz (WS) models [60]. First, d_I and $d_{q,q'}$ were computed, following which the best models using Equations (10) and (12) were chosen based on the $SBICR$. There were 225 BA–based networks, 211 SHM networks, and 216 WS networks. Table 3 summarises the nodes, edges, d_I, $d_{q,q'}$, and the information model selected between Equations (10) and (12). See the Supplementary Materials for details on the parameters of each model used to generate the networks, as well as the specific $SBICR_I$, $SBICR_{(q,q')}$, d_I, $d_{q,q'}$, q, and q' values.

Table 3. The nodes (max–min), edges (max–min), d_I (max–min), $d_{q,q'}$ (max–min), and the percentage of the information model for synthetic networks. I = Equation (10) and q, q' = Equation (12).

Network Model	Nodes	Edges	d_I	$d_{q,q'}$	Model
BA	2000–4500	2685–40,455	3.930–7.412	0.864–1.729	q, q' (100%)
SHM	10–36,480	9–880,475	0.831–12.689	0.005–2.432	q, q' (70.83%)
WS	2000–4000	4000–40,000	0.955–7.363	0.015–1.540	q, q' (100%)

A remarkable finding on the real and synthetic networks was that $d_{q,q'} < d_I$. The fractional (q, q') model fitted all BA and WS networks and about 71% of the SHM networks better. Table 4 summarises the parameters of the SHM model that produced 29%(153) networks for which the information model fit better (see the Supplementary Materials for the meaning of each parameter). The values of the SHM parameters are influenced by the assortativity ($MODE = 1$) and hub repulsion ($MODE = 2$), such that the only conditions that intersected on $MODE = 1$ and $MODE = 2$ were $G = 2$ $M = 3$ $IB = 0$ $BB = 0.400$, $G = 3$ $M = 2$ $IB = 0$ $BB = 1$, and $G = 3$ $M = 2$ $IB = 0.400$ $BB = 0.800$.

Table 4. The parameters of the SHM model that produced networks for which the information model fit better.

G	M	IB	BB	$MODE$
2	2	0	0.400	1
2	2	0.400	0.400	1
2	3	0	0.400	1
2	3	0.400	$[0, 1]$	1
2	4	0	0.800	1
3	2	0	$[0, 1]$	1
3	2	0.400	$[0.200, 0.800]$	1
3	3	$[0, 0.400]$	≤ 0.800	1
3	4	0	1	1
4	$[2, 3]$	0	0.400	1
2	2	0	$[0, 0.200, 0.800]$	2
2	2	0.400	$[0, 1]$	2
2	2	1	≤ 0.800	2
2	3	0	≤ 0.400	2
2	3	0.400	0.200	2
3	2	0	$[2, 0.600, 1]$	2
3	2	0.400	0.800	2
3	4	0	0.400	2
4	2	0	≤ 0.200	2
4	2	0.400	$[0.400, 0.200]$	2
4	3	0	0.800	2

Additionally, for BA, setting the average node degree (ad) equal to 1 produced networks with stronger outer interactions than inner ones ($q' > 1$). This occurred regardless of the number of initial nodes ($n0$) and total nodes (n) (see Table S1 in the Supplementary

Materials). On the other hand, three SHM networks (SHM_G–3 M–4 IB–0.400 BB–0.000 MODE–2, SHM_G–4 M–3 IB–0.000 BB–0.400 MODE–2, SHM_G–4 M–3 IB–0.400 BB–0.000 MODE–2) and one WS network (WS–2000–2–0.400) obtained $q' > 1$; (see Tables S2 and S3). These results suggest that the fractional $(q, q')-$information dimension captures the complexity of the network topology, as the SHM model tunes the links between nodes into the boxes (IB) and the connections between boxes through BB. The BA and WS models do not possess this capability.

Next, a Kruskal–Wallis test was conducted on d_I and $d_{q.q'}$. The results demonstrated that, for $\alpha \leq 0.001$, a significant difference was found ($H(2) = 112.568$, $p < 0.0001$). However, a deeper analysis conducted using the Mann–Whitney U test revealed no difference between the d_I of SHM ($mdn = 3.401$) and WS ($mdn = 4.190$); see Figure 3a. On the other hand, a significant difference ($H(2) = 216.667$, $p < 0.0001$) was found between SHM ($mdn = 1.102$), WS ($mdn = 0.577$), and BA ($mdn = 1.451$) for $d_{q.q'}$; see Figure 3b. The detailed pairwise comparison results of the Mann–Whitney U test are presented in Tables S4 and S5.

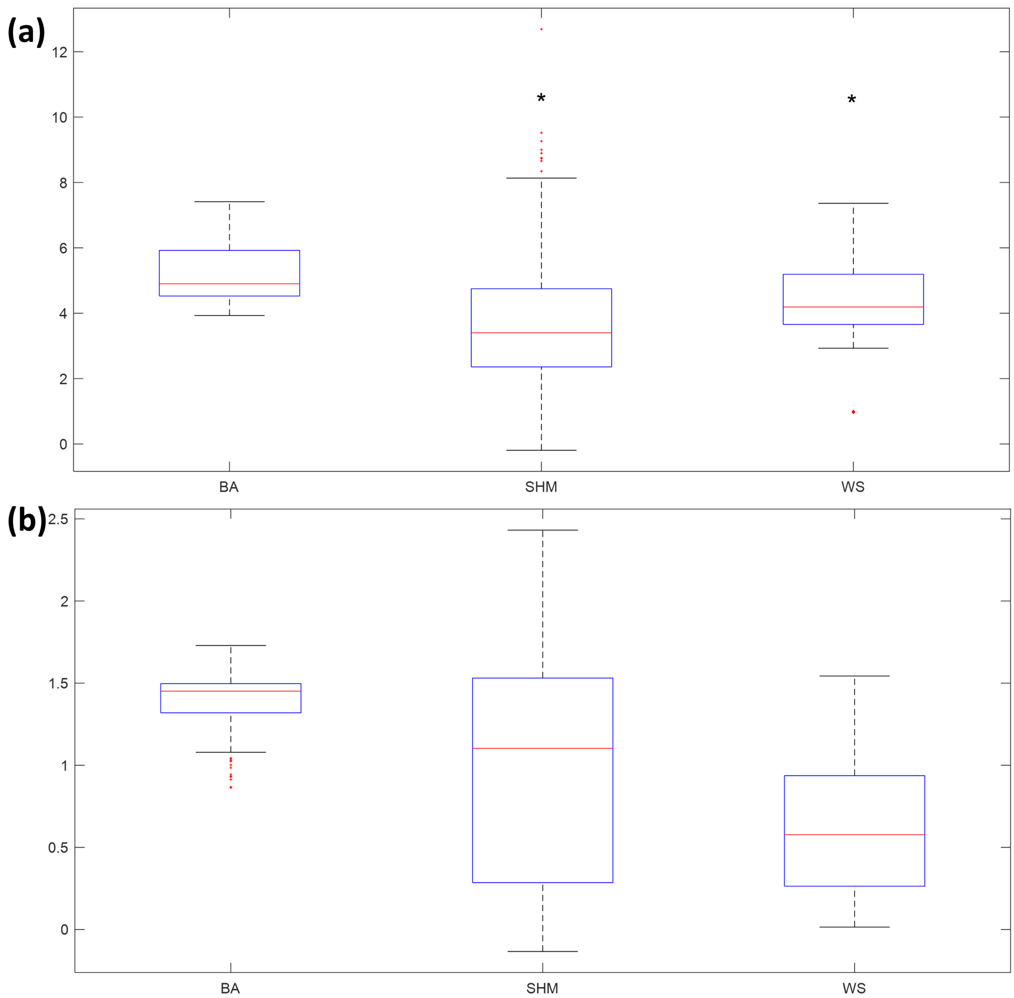

Figure 3. (**a**) d_I and (**b**) $d_{q.q'}$ of BA, SHM, and WS networks. Mann–Whitney U test revealed a significant difference in $d_{q.q'}$ for all types of networks. The * symbol indicates no statistical difference between them.

Additionally, a C4.5 decision tree, implemented in WEKA [61] as J48, was constructed using three data sets: (1) d_I, (2) $d_{q.q'}$, (3) $d_{q.q'}$, q, and q'. Each model obtained from these data sets was trained and tested using 10-fold cross-validation. The accuracy (ACC) and Matthew's correlation coefficient (MCC) were used as metrics to evaluate the classification performance. The model built from $d_{q.q'}$ obtained an ACC = 0.682 and MCC = 0.632; both higher than those (ACC = 0.653 and MCC = 0.514) obtained by the model built using d_I. Additionally, the models constructed using $d_{q.q'}$, q, and q' obtained the best performance (ACC = 0.915, MCC = 0.960), as can be seen from Figure 4. These results suggest that the fractional $(q, q')-$information dimension and the q and q' parameters better describe the complex topology of the synthetic networks than the information dimension d_I.

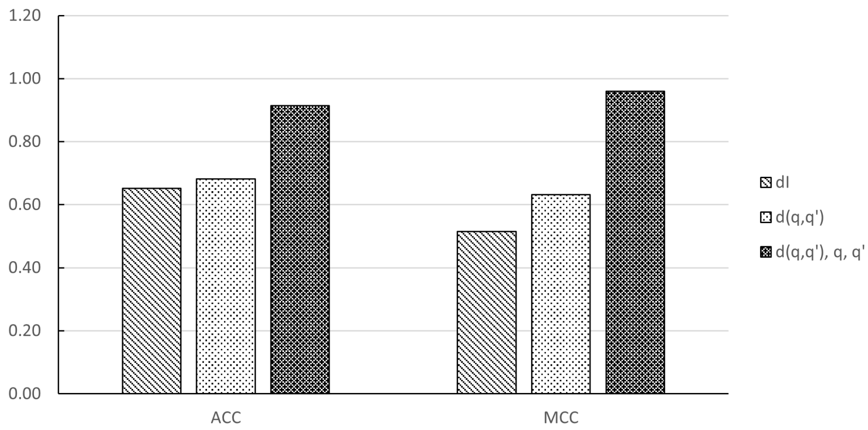

Figure 4. Accuracy (ACC) and Matthew's correlation coefficient of decision trees built on data sets: (1) d_I, (2) $d_{q.q'}$, and (3) $d_{q.q'}$, q, and q'.

5. Conclusions

This article introduced a new fractional $(q, q')-$information dimension for complex networks. The rationale of the proposed definition is that a network can be divided into several sub-systems. Hence, q measures how far the number of sub-systems (for a given size ε) is from the mean number of overall sizes, which is treated as the baseline. On the other hand, q' (interaction index) measures whether the interactions between sub-systems are greater ($q' > 1$), lesser ($q' < 1$), or equal to the interactions into these sub-systems ($q' = 1$).

Starting from experimental results on real and synthetic networks, a glance at the interactions between sub-systems indicates that clear interconnection patterns emerge, especially in the networks generated using the SHM model, the parameters of which play a crucial role in obtaining networks that the information model best fit. The initial node parameter of the BA model led to the generation of networks where the outer interactions were stronger than inner ones (i.e., $q' > 1$), no matter the values of the remaining parameters. Finally, our experiments revealed that $d_{q,q'} < d_I$ in both types of network.

Additionally, the $d_{q,q'}$ values differed between the synthetic networks generated using the BA, SHM, and WS models. The $d_{q,q'}$ value, the mean number of sub-systems of the network (q), and the interaction index (q') capture the complex topological features of synthetic networks, allowing for their classification with a good performance, even outperforming the classic information dimension d_I. For future work, extending the network's classification using a long short-term memory fed with $I(\varepsilon)$, q_ε, and $\frac{\overline{\beta}_{\varepsilon,i}}{\overline{\alpha}_{\varepsilon,i}}$ might allow us to achieve better results.

There is evidence that the fractional $(q, q')-$information dimension of complex networks based on $(q, q')-$extropy seems to be a complementary dual statistical index of the fractional $(q, q')-$information dimension. It is an exciting area for future research, and we hope to prove the extent to which these new formulations will be helpful.

Supplementary Materials: The following supporting information can be downloaded at: https://www.mdpi.com/article/10.3390/fractalfract7100702/s1, Table S1: The $SBICR$, d_I, $d_{q,q'}$, q, and q' values for the information model Equation (10) and the fractional (q, q') information model Equation (12) on BA networks; Table S2: The $SBICR$, d_I, $d_{q,q'}$, q, and q' values for the information model Equation (10) and the fractional (q, q') information model Equation (12) on SHM networks; Table S3: The $SBICR$, d_I, $d_{q,q'}$,

q, and q' values for the information model Equation (10) and the fractional (q, q') information model Equation (12) on WS networks; Table S4: Mann–Whitney U test using adjusted alpha $*\alpha = 3.333 \times 10^{-4}$ for d_I; Table S5: Mann–Whitney U test using adjusted alpha $\alpha = 3.333 \times 10^{-4}$ for $d_{(q,q')}$.

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Abbreviations

The following abbreviations are used in this manuscript:

ACC	Accuracy
MCC	Matthew's Correlation Coefficient
MDPI	Multidisciplinary Digital Publishing Institute
SBICR	Bayesian Information Criterion with Bonuses
BA	Barabasi–Albert
SHM	Song, Havlin, and Makse
WS	Watts and Strogatz

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