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Precise Conditions on the Unique Solvability of the Linear Fractional Functional Differential Equations Related to the ς-Nonpositive Operators

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Abstract: Exact conditions for the existence of the unique solution of a boundary value problem for linear fractional functional differential equations related to ς-nonpositive operators are established. The exact solvability conditions are based on the a priori estimation method. All theoretical investigations are illustrated by an example of the pantograph-type model from electrodynamics.

Keywords: fractional order functional differential equations; nonpositive operator; unique solvability; Caputo derivative; exact conditions; the pantograph-type model from electrodynamics

MSC: 26A33; 34A08; 34K37; 47H07

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1. Introduction

Fractional functional differential equations (FFDEs) are a relatively new branch of differential equations that have been developed intensively in recent decades. One can find sound investigations in this area in monographs [1–5]. Initial and boundary value problems for FFDEs are studied in [6–9]; also, there are many publications that studied the unique solvability conditions of the functional differential equations and their properties (see [10–14]). Also, the theory of FFDEs is often used in applied sciences. The model with discrete memory effect is studied in [15–17]; the pantograph-type model from electrodynamics is investigated in [15,18,19]; mathematical model for studying the COVID-19 infection described by piecewise fractional differential equations is considered in [20]. The present paper is motivated mainly by [11,13]. The precise conditions for the unique solvability of FFDE related to the ς-positive operators can be found in [15–18].

The main interest of our investigations is the FFDEs related to the ς-nonpositive operators with boundary value condition $x(a) + \alpha x(b) = c$, on the interval $[a, b]$, where $-\infty < a < b < +\infty$, $\alpha, c \in \mathbb{R}^n$. The objective of our investigation was to establish precise conditions on the unique solvability of the FFDEs with the boundary value condition that can be more easily used in application to concrete problems. Mainly, we obtained conditions that do not require calculation of fractional derivatives. The second main aim of our investigation was to obtain the condition on $x(a)$ when the solution depends on boundary value property $x(a) + \alpha x(b) = c$. As an example, we study the pantograph model from electrodynamics (see [19,21–23]).

The paper is organized in the following way. We give the necessary notation and definitions in Section 1.1, formulate the problem in Section 2, auxiliary statements are summarized in Section 3, a restriction on the initial value $x(a)$ is obtained in Section 4, the precise requirements on the unique solvability of the linear FFDEs are formulated and proved in Section 5 (here, we used the a priori estimation method), and the example of the pantograph-type model from electrodynamics on the results referred to earlier can be found in Section 6.

1.1. Notation

In the paper, we use the following notation:

- $q \in (0,1)$ is an order of the Caputo fractional derivative D_a^q;
- The interval $I_{ab} = [a,b]$, accordingly $I_{at_*} = [a,t_*]$;
- $\mathbb{R} := (-\infty,\infty)$; $x := \|x\|_\infty := \max_{1\le i\le n} |x_i|$ for $x = (x_i)_{i=1}^n \in \mathbb{R}^n$;
- $L_1(I_{ab},\mathbb{R}^n)$ is the Banach space of all Lebesgue integrable vector functions $x : I_{ab} \to \mathbb{R}^n$ with the norm
$$L_1(I_{ab},\mathbb{R}^n) \ni x \longmapsto \|x\|_{L_1} = \int_a^b \|x(s)\|ds;$$
- $L_\infty(I_{ab},\mathbb{R}^n)$ is the Banach space of all Lebesgue integrable vector functions $x : I_{ab} \to \mathbb{R}^n$ with the norm
$$L_\infty(I_{ab},\mathbb{R}^n) \ni x \longmapsto \|x\|_{L_\infty} = \operatorname*{ess\,sup}_{t\in I_{ab}} \|x(s)\|;$$
- $AC(I_{ab},\mathbb{R}^n)$ is the Banach space of absolutely continuous functions $x : I_{ab} \to \mathbb{R}^n$ with the norm
$$AC(I_{ab},\mathbb{R}^n) \ni x \longmapsto \|x\|_{AC} := \int_a^b \|x'(s)\|\, ds + \lim_{t\to a+} \|x(t)\|;$$
- For fixed $\sigma_i \subset \{-1,1\}$, $i = 1,2,\ldots,n$,
$$\varsigma = \begin{pmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_n \end{pmatrix} \tag{1}$$
we set
$$\mathbb{R}^n_\varsigma := \sigma_1\mathbb{R}_+ \times \sigma_2\mathbb{R}_+ \times \cdots \times \sigma_n\mathbb{R}_+.$$

2. Problem Formulation

We consider an FFDE
$$D_a^q\big(x(t)\big) = l(x)(t) + r(t), \quad t \in I_{ab} \tag{2}$$
subject to the boundary condition
$$x(a) + \alpha x(b) = c, \tag{3}$$
where D_a^q is the Caputo fractional derivative of order $q \in (0,1)$, and $l \in AC(I_{ab},\mathbb{R}^n) \to L_1(I_{ab},\mathbb{R}^n)$ is a linear operator, function $r \in L_1(I_{ab},\mathbb{R}^n)$ and $\alpha, c \in \mathbb{R}^n$.

Definition 1 (Sections 2.1 and 2.4, formulae (2.1.8), (2.1.9) and (2.4.4), (2.4.5) and Corollary 2.3, [3])**.** *For $x(t) \in AC(I_{ab},\mathbb{R}^n)$ the Caputo fractional derivative $D_a^q x(t)$ exists almost everywhere on I_{ab} and*
$$D_a^q\big(x(t)\big) = \frac{1}{\Gamma(1-q)}\left(\frac{d}{dt}\right)\int_a^t (t-s)^{-q}\Big(x(s) - x(a)\Big)ds, \quad 0 < q < 1, \tag{4}$$
where $\Gamma(q) : [0,\infty) \to \mathbb{R}$ is the Gamma function and
$$\Gamma(q) := \int_0^\infty t^{q-1}e^{-t}dt. \tag{5}$$

Definition 2 (Formula (2.1.1), p. 69, [3]). *The fractional integral* $\mathcal{I}_{at}^{q}$ *of order* q, $q \in (0,1)$, *for* $x(t) \in L_1(I_{ab}, \mathbb{R}^n)$, *is defined by*

$$\mathcal{I}_{at}^{q}(x(t)) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} x(s) ds, \quad t > a. \tag{6}$$

Definition 3. *For certain given* $\sigma_i \subset \{-1, 1\}$, $i = 1, 2, \ldots, n$, *an operator* $l : AC(I_{ab}, \mathbb{R}^n) \to L_1(I_{ab}, \mathbb{R}^n)$ *is* ς*-nonpositive operator, if the fact that the relation*

$$\varsigma x(t) \geq 0, \quad t \in I_{ab} \tag{7}$$

is true implies that

$$\varsigma l(x)(t) \leq 0, \quad \text{almost everywhere on} \quad t \in I_{ab}, \tag{8}$$

where ς *is defined by* (1).

3. Auxiliary Statements

The following statements are essential to the investigation.

Lemma 1 (Lemma 2.21 and Lemma 2.22, [3]). *The next properties are true:*

(i) *Assume that* $0 < q < 1$ *and* $x(t) \in L_\infty(I_{ab}, \mathbb{R}^n)$, *then*

$$D_a^q\Big(\mathcal{I}_{at}^{q}(x(t))\Big) = x(t) \quad \text{almost everywhere on} \quad I_{ab},$$

where $\mathcal{I}_a^q$ *and Gamma-function are defined by* (6) *and* (5) *correspondingly;*

(ii) *Assume that* $0 < q < 1$ *and* $x(t) \in AC(I_{ab}, \mathbb{R}^n)$, *then*

$$\mathcal{I}_{at}^{q}\Big(D_a^q(x(t))\Big) = x(t) - x(a) \quad \text{almost everywhere on} \quad I_{ab}, \tag{9}$$

where $\mathcal{I}_{at}^{q}$ *is defined by* (6).

Lemma 2. *The set of absolutely continuous solutions to the FFDE* (2) *coincides with the set of absolutely continuous solutions to the fractional integral equation:*

$$x(t) = x(a) + \mathcal{I}_{at}^{q}\Big(l(x)(t) + r(t)\Big), \quad t \in I_{ab}, \tag{10}$$

where $\mathcal{I}_{at}^{q}$ *is defined by* (6).

Proof. Assume that $x \in AC(I_{ab}, R^n)$. Let us integrate FFDE (2) from a to t taking into account Definition 2:

$$\mathcal{I}_{at}^{q}\Big(D_a^q(x(t))\Big) = \mathcal{I}_{at}^{q}\Big(l(x)(t) + r(t)\Big), \quad t \in I_{ab}.$$

Then, in view of Lemma 1 (Formula (9)) we obtain that

$$x(t) - x(a) = \mathcal{I}_{at}^{q}\Big(l(x)(t) + r(t)\Big), \quad t \in I_{ab}.$$

The last relation is Formula (10). □

4. Conditions at the Initial Point $x(a)$.

In the following theorem, we express $x(a)$ in terms of (3), where $|\alpha| \neq -1$.

Theorem 1. *The FFDE (2) has a boundary value property (3) if and only if*

$$x(a) = \frac{1}{\alpha+1}\Big(c - \alpha \mathcal{I}^q_{ab}\big(l(x)(t) + r(t)\big)\Big). \tag{11}$$

Proof of Theorem 1. We know from Lemma 2 that

$$x(b) = x(a) + \mathcal{I}^q_{ab}\Big(l(x)(t) + r(t)\Big); \tag{12}$$

on the other hand, from the boundary value condition (3), we have that

$$x(b) = \frac{1}{\alpha}(c - x(a));$$

then, we can replace (12):

$$\frac{c}{\alpha} - \frac{x(a)}{\alpha} = x(a) + \mathcal{I}^q_{ab}\Big((lx)(t) + r(t)\Big).$$

Next,

$$x(a)(\frac{1}{\alpha} + 1) = \frac{c}{\alpha} - \mathcal{I}^q_{ab}\Big(l(x)(t) + r(t)\Big),$$

hence

$$x(a) = \frac{\alpha}{\alpha+1}\Big(\frac{c}{\alpha} - \mathcal{I}^q_{ab}\big(l(x)(t) + r(t)\big)\Big).$$

Therefore, (11) is true. □

The next Lemma is an analogue to the Fredholm alternative.

Lemma 3 (The Fredholm alternative, Corollary from Theorem VI.14, [24]; and Sections 2.1 and 3.9, [25])**.** *If the homogeneous boundary value problem*

$$D^q_a(x(t)) = l(x)(t), \quad t \in I_{ab}, \tag{13}$$

$$x(a) + \alpha x(b) = 0, \tag{14}$$

only has the trivial solution, then the nonhomogeneous initial value problem (2), (3) is uniquely solvable.

5. Exact Conditions on the Unique Solvability of the Linear FFDE

Here, we consider a linear operator $l : AC(I_{ab}, \mathbb{R}^n) \to L_1(I_{ab}, \mathbb{R}^n)$ with the property

$$|\varsigma l(x)(t)| \leq \eta(t)\|x\|_{AC}, \quad \text{a. e.} \quad t \in I_{ab}, \quad x \in AC(I_{ab}, \mathbb{R}^n), \tag{15}$$

where function $\eta \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$.

$$\alpha \leq 0 \tag{16}$$

and

$$\varsigma(1 - \operatorname{sign}\alpha)c \geq 0. \tag{17}$$

Definition 4 ([13])**.** *We will say that the operator l with property (15) is a t_*-Volterra operator, where $t_* \in I_{ab}$, if for arbitrary $a_1 \in [a, t_*], a_2 \in [t_*, b]$, $a_1 \neq a_2$, and $x \in C(I_{ab}, R^n)$ satisfying the condition*

$$x(t) = 0 \quad t \in [a_1, a_2],$$

we have

$$l(x)(t) = 0 \quad \textit{for almost all } t \in [a_1, a_2].$$

For further investigation, we need the following auxiliary Lemma. Note, that we assume that the problem (2) and (3) has a nontrivial solution due to Theorem 3 and Remark 2.1 in [13].

Lemma 4. *Assume that* $|\alpha| \leq 1$*, and operator* l *is* ς*-nonpositive (see Definition 3), and* l *is a* a*-Volterra operator. Also, suppose that* x *is a nontrivial solution to the problem (2), (3), where* $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$ *and* $c \in \mathbb{R}^n$ *is such that the inequality (17) holds, satisfying*

$$\min_{t \in I_{ab}} \varsigma x(t) < 0. \tag{18}$$

Then, there exist $t_* \in (a, b]$ *and* $t^* \in [a, t_*)$ *such that*

$$\varsigma x(t_*) = \min_{t \in I_{ab}} \varsigma x(t), \tag{19}$$

$$\varsigma x(t^*) = \max_{t \in I_{at_*}} \varsigma x(t) > 0. \tag{20}$$

We remind you that $I_{at_*} = [a, t_*]$ is described in the Notation (Section 1.1).

Proof of Lemma 4. We set

$$m = -\min_{t \in I_{ab}} \varsigma x(t), \tag{21}$$

$$J = \{t \in I_{ab} : \varsigma x(t) = -m\}, \quad t_* = \sup J.$$

We see that

$$m > 0,$$

$$\varsigma x(t_*) = -m. \tag{22}$$

Taking in to account (2), (16), and (17), we have that

$$\text{if } a \in J, \ \text{ then } |\alpha| = 1, \ c = 0, \ \text{and } t_* = b. \tag{23}$$

So, $t_* \in (a, b]$.

Let us show that

$$\max_{t \in I_{t_*}} \varsigma x(t) > 0.$$

Suppose the opposite, that

$$\varsigma x(t) \leq 0, \ t \in I_{at_*}. \tag{24}$$

Taking into account the fact that l is an a-Volterra operator and Lemma 2, the integration of (2) from a to t_* gives that

$$\varsigma x(t_*) - \varsigma x(a) = \varsigma \mathcal{I}^q_{at_*}\Big(|(lx)(t)| + r(t)\Big),$$

in view of (24) and the hypotheses that l is ς-nonpositive operator and $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$, we obtain that

$$\varsigma x(t_*) - \varsigma x(a) = \varsigma \mathcal{I}^q_{at_*}\Big(|(lx)(t)| + r(t)\Big) \geq 0, \tag{25}$$

where $\mathcal{I}^q_{at^*}\Big(|l(x)(t)| + r(t)\Big) = \frac{1}{\Gamma(q)}\int_a^{t_*}(t-s)^{q-1}|l(x)(s)|ds + \frac{1}{\Gamma(q)}\int_a^{t_*}(t-s)^{q-1}r(s)ds.$

From (21), (22), and (25) we obtain that $a \in J$ and, thus, it follows from (23) that $|\alpha| = 1$, $c = 0$, and $t_* = b$. According to (25) we determine $r \equiv 0$ and $l(x) \equiv 0$, i.e.,

$$\varsigma x(t) = \varsigma x(a) = -m, \quad t \in I_{ab}.$$

Therefore, (25) indicates that

$$0 = \|l(-m)(t)\|_{L_1} = m\|l(1)(t)\|_{L_1}.$$

Following this, as we suppose that the operator l is nontrivial for $|\alpha| = 1$, the last equality shows that $m = 0$, which is a contradiction. □

Theorem 2. *Assume that $|\alpha| \leq 1$ and linear operator l is ς-nonpositive operator (see Definition 3), l is an a-Volterra operator, and let there exists a function $\beta \in AC(I_{ab}, \mathbb{R}^n_\varsigma)$ satisfying*

$$\varsigma D_a^q(\beta(t)) \leq \varsigma l(\beta)(t), \quad t \in I_{ab}, \tag{26}$$

$$\varsigma\beta(t) > 0, \quad t \in [a, b). \tag{27}$$

Then, the homogeneous problem (13), (14) only has a trivial solution and, for every $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$ and $c \in \mathbb{R}^n$ satisfying (17), the solution to the nonhomogeneous boundary value problem (2), (3) is unique and has property (7).

Proof of Theorem 2. Assume that x is a nontrivial solution to problem (2), (3), where $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$ and $c \in \mathbb{R}^n$ is such that Inequality (17) holds. Now, let us prove that (7) is fulfilled. Suppose the contrary, that Inequality (18) holds. According to Lemma 4, there exist $t_* \in (a, b]$ and $t^* \in [a, t_*)$ such that (20) is valid. It is clear that there exists $t_2 \in (t^*, t_*)$ such that

$$x(t_2) = 0, \quad \text{and} \quad a \leq t^* < t_2 < t_* \leq b. \tag{28}$$

Put

$$\varsigma v(t) = \varsigma p\beta(t) - \varsigma x(t), \quad t \in I_{ab}, \quad \text{where} \quad p = \max_{t\in[a,t_2]} \frac{\varsigma x(t)}{\varsigma\beta(t)}.$$

From (20), we know that

$$p > 0 \tag{29}$$

and there exists $t_1 \in [a, t_2)$ such that

$$v(t_1) = 0. \tag{30}$$

Also, it holds that

$$\varsigma v(t) \geq 0, \quad t \in [a, t_2]. \tag{31}$$

Taking into account (2), (26), and (29), we get

$$\varsigma D_a^q(v(t)) \leq \varsigma l(v)(t) - \varsigma r(t), \quad t \in I_{ab}.$$

Hence, by virtue of (31), the assumptions that l is ς-nonpositive and $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$, and the fact that l is an a-Volterra operator and property (4), we obtain

$$\varsigma D_a^q(v(t)) \leq 0, \quad t \in [a, t_2].$$

Accordingly, taking into account (30),

$$\varsigma v(t) \leq 0, \quad t \in [t_1, t_2],$$

next, in addition to (27)–(29), we find $0 < \varsigma v(t_2) \leq 0$, a contradiction. We have established that if x is a nontrivial solution to problem (2), (3), where $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$ and $c \in \mathbb{R}^n$ is such that the inequality (17) is fulfilled, then the inequality (7) is true.

Now, assume that the homogeneous problem (13), (14) has a nontrivial solution x_0. From the linearity of the operator l we have that $-x_0$ is also a nontrivial solution to the problem (13), (14), and, according to the findings above, we get

$$\varsigma x_0(t) \geq 0, \quad -\varsigma x_0(t) \geq 0, \quad t \in I_{ab}, \tag{32}$$

which is a contradiction, because from (32) we have that $x_0 = 0$, but we assumed that x_0 is the nontrivial solution. So, homogeneous problem (13), (14) only have the trivial solution and, in view of Lemma 3, the nonhomogeneous boundary value problem (2), (3) have a unique solution. □

Theorem 3. *Assume that $|\alpha| \leq 1$, l is ς-nonpositive, l is an a-Volterra operator, and the inequality*

$$\mathcal{I}^q_{ab}|l(1)(t)| \leq 1 \tag{33}$$

is fulfilled, where $\mathcal{I}^q_{ab}|l(1)(t)| = \frac{1}{\Gamma(q)}\int_a^b (t-s)^{q-1}|l(1)(s)|ds$.

Then, the assertion of Theorem 2 is true for the nonhomogeneous problem (2), (3), and homogeneous boundary value problem (13), (14).

Proof of Theorem 3. Assume that x is a nontrivial solution to problem (2), (3), where $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$ and $c \in \mathbb{R}^n$ is such that Inequality (17) is fulfilled. Let us show that (7) is true. Similarly to the proof of Theorem 2, we assume the opposite, that Inequality (18) is satisfied. In view of Lemma 4, there exist $t_* \in (a, b]$ and $t^* \in [a, t_*)$ such that (20) is valid. Taking into account Lemma 2 the integration of (2) from t^* to t_* provides

$$\varsigma x(t^*) - \varsigma x(t_*) = -\varsigma \mathcal{I}^q_{t^*t_*}\Big(l(x)(t) + r(t)\Big),$$

where $\mathcal{I}^q_{t^*t_*}\Big(l(x)(t) + r(t)\Big) = \frac{1}{\Gamma(q)}\int_{t^*}^{t_*}(t-s)^{q-1}l(x)(s)ds + \frac{1}{\Gamma(q)}\int_{t^*}^{t_*}(t-s)^{q-1}r(s)ds$.

Therefore, according to (19), the assumptions that l is ς-nonpositive, $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$, and the fact that l is an a-Volterra operator, we establish

$$\varsigma x(t^*) < \varsigma x(t^*) + |\varsigma x(t_*)| \leq \varsigma x(t^*)\mathcal{I}^q_{ab}|l(1)(t)|.$$

This inequality, together with (33), implies the contradiction $\varsigma x(t^*) < \varsigma x(t^*)$.

We verified that if x is a nontrivial solution to the problem (2), (3), where $r \in L_1(I_{ab}, \mathbb{R}^n_\varsigma)$ and $c \in \mathbb{R}^n$ is such that the inequality (17) is true, then the inequality (7) is fulfilled. Now let the homogeneous problem (13), (14) have a nontrivial solution x_0. It is obvious, that $-x_0$ is also a nontrivial solution to the problem (13), (14), and, according to the findings above, we obtain

$$\varsigma x_0(t) \geq 0, \quad -\varsigma x_0(t) \geq 0, \quad t \in I_{ab},$$

a contradiction. □

6. Example of Pantograph-Type Model

Here, we study the Pantograph-type model arising in electrodynamics [21,23]. The pantograph [22] is a device used in electric trains to collect electric current from the overload lines. The equation was first introduced by Ockendon and Tayler in 1971 (see [21]).

Example 1. *Let us consider the Pantograph-type model related to FFDE:*

$$D^q_0\big(x(t)\big) = \sum_{i=1}^{m} \kappa_i(t)x(\tau_i t) + r(t), \quad t \in [0, 1], \tag{34}$$

subjected to the boundary value condition

$$x(0) + \alpha x(1) = c \tag{35}$$

where $\alpha, c \in \mathbb{R}^n$, $r \in L_1([0,1], \mathbb{R}^n)$,

$$\kappa_i(t) = \begin{pmatrix} k^i_{11}(t) \; k^i_{12}(t) \dots k^i_{1n}(t) \\ k^i_{21}(t) \; k^i_{22}(t) \dots k^i_{2n}(t) \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ k^i_{n1}(t) \; k^i_{n2}(t) \dots k^i_{nn}(t) \end{pmatrix} \tag{36}$$

have summable components and $|\sum_{i=1}^m \varsigma\kappa_i(t)x(\tau_i t)| \leq \eta(t)\|x\|_{AC}$, $\eta \in L_1([0,1], \mathbb{R}^n_\varsigma)$, $\tau_i \in (0,\ 1]$.

Here, the interval $[a,b] = [0,\ 1]$ *and the FFDE* (34) *is a special case of the FFDE* (2) *with the operator*

$$l(x)(t) = \sum_{i=1}^m \kappa_i(t)x(\tau_i t) \quad \textit{for a. a. } t \in [0,1]. \tag{37}$$

We will need the following Lemmas.

Lemma 5. *The boundary value problem* (34), (35) *are equivalent to the FFDE*

$$x(t) = \mathcal{I}^q_{0t}(\sum_{i=1}^m \kappa_i(t)x(\tau_i t) + r(t)) + x(0),$$

where

$$\mathcal{I}^q_{0t}\Big(\sum_{i=1}^m \kappa_i(t)x(\tau_i t) + r(t)\Big) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\sum_{i=1}^m \kappa_i(s)x(\tau_i s) + r(s)\Big)ds \tag{38}$$

and

$$x(0) = \frac{1}{1+\alpha}\Big(c - \frac{\alpha}{\Gamma(q)}\int_0^1 (t-s)^{q-1}\Big(\sum_{i=1}^m \kappa_i(s)x(\tau_i s) + r(s)\Big)ds\Big). \tag{39}$$

Proof of Lemma 5. To prove Lemma 5, we use Lemmas 1 and 2, Theorem 1 with operator l defined by (37), the fractional integral $\mathcal{I}^q_{0t}$ defined by (38), and $x(0)$ defined by (39). □

Lemma 6. *If the inequality*

$$\varsigma \sum_{i=1}^m \kappa_i(t)\varsigma \leq 0 \quad \textit{for almost all } t \in [0,1] \tag{40}$$

is fulfilled, then the operator l *defined by* (37) *is* ς*-nonpositive.*

Proof of Lemma 6. Assume that vector function $x \in AC([0,1], \mathbb{R}^n_\varsigma)$ satisfies condition (7). In view of (8), where operator l is defined by (37), we have that

$$\varsigma l(x)(t) = \varsigma \sum_{i=1}^m \kappa_i(t)x(\tau_i t) = \varsigma \sum_{i=1}^m \kappa_i(t)\varsigma\varsigma x(\tau_i t), \quad t \in [0,1], \tag{41}$$

where ς is defined by (1) and κ_i, $i = 1, 2, \dots, m$ are defined by (36). From (8), (40), and (41) we obtain that

$$\varsigma l(x)(t) = \varsigma \sum_{i=1}^m \kappa_i(t)x(\tau_i t) \leq 0 \quad \text{for a. a. } t \in [0,1].$$

Thus, operator l given by formula (37) is ς-nonpositive (see Definition 3). So, Lemma 6 is proven. □

Theorem 4. *Let $|\alpha| \leq 1$, the condition (40) be fulfilled, $\tau_i \in (0,1]$, $t \in [0,1]$ and*

$$\mathcal{I}_{01}^{q}\Big(\sum_{i=1}^{m} \kappa_i(t)\Big) \leq 1. \tag{42}$$

Then, the homogeneous problem

$$x(0) + \alpha x(1) = 0$$

with FFDE

$$D_0^q(x(t)) = \sum_{i=1}^{m} \kappa_i(t) x(\tau_i t), \quad t \in [0,1],$$

only has the trivial solution and, for every $c \in \mathbb{R}^n$ and $r \in L_1([0,1], \mathbb{R}_\varsigma^n)$ satisfying (17), the solution to the nonhomogeneous problem (34), (35) has the property (7).

Proof of Theorem 4. To prove Theorem 4, we use Theorem 3 with the operator l defined by (37) and the fractional integral

$$\mathcal{I}_{01}^{q}\Big(\sum_{i=1}^{m} \kappa_i(t)\Big) = \frac{1}{\Gamma(q-1)} \int_0^1 (t-s)^{q-1} \Big(\sum_{i=1}^{m} \kappa_i(s)\Big) ds.$$

In such notation, we have that the inequality (33) from Theorem 3 is the inequality (42) from Theorem 4. □

7. Conclusions

In this paper, we studied the boundary value problem for FFDEs related to ς-nonpositive operators. The precise conditions on the unique solvability of the problem were obtained. The results represent improvement, because they extend the class of considered FFDEs expressed by ς-nonpositive operators to the boundary value problem $x(a) + \alpha x(b) = c$, in contrast to the initial value problems described in [15,17,18]. The calculated results can be easily applied to various problems, because these conditions do not require the calculation of fractional derivatives (Theorem 3). We expect that, in the future, there will be considerable interest in applying similar methods to the boundary value problem for nonlinear FFDEs.

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