Analysis of the Corneal Geometry of the Human Eye with an Artificial Neural Network

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Abstract: In this paper, a hybrid cuckoo search technique is combined with a single-layer neural network (BHCS-ANN) to approximate the solution to a differential equation describing the curvature shape of the cornea of the human eye. The proposed problem is transformed into an optimization problem such that the $L_2$-error remains minimal. A single hidden layer is chosen to reduce the sink of the local minimum values. The weights in the neural network are trained with a hybrid cuckoo search algorithm to refine them so that we obtain a better approximate solution for the given problem. To show the efficacy of our method, we considered six different corneal models. For validation, the solution with Adam’s method is taken as a reference solution. The results are presented in the form of figures and tables. The obtained results are compared with the fractional order Darwinian particle swarm optimization (FO-DPSO). We determined that results obtained with BHCS-ANN outperformed the ones acquired with other numerical routines. Our findings suggest that BHCS-ANN is a better methodology for solving real-world problems.

Keywords: nonlinear biosystem; corneal geometry; neural network; artificial intelligence; cuckoo search; medical image; human eye

1. Introduction

Eyesight is an indispensable sense for every organism that does not only connect with their surrounding but also keeps them safe and regulate mental sharpness. In the eye, the cornea is the outer transparent layer tissue which help us to see clearly and serves as a barrier against various infections. Corneal disorders lead to severe eyesight problems including keratitis, myopia, corneal dystrophy, dry eye, astigmatism, etc. To understand the underlying topography of corneal mechanisms, mathematical models can provide value insight for ophthalmologists. The basic mechanism of the eye is well known to everyone. The main functions and working procedures of the eye are explained at the high-school level. We all know that the cornea works like a window for the eye. When a light beam is focused on an eye, it directly passes through the cornea, which has a diameter of 0.5 inches. The geometry of the human eye is described in Figure 1. The mathematical form of the $n$-dimensional equation for the mean curvature is given as follows:

$$\text{div}\left(\frac{\nabla \psi(t)}{\sqrt{1 + |\nabla \psi(t)|^2}}\right) = c\psi - \frac{d}{\sqrt{1 + |\nabla \psi(t)|^2}} \quad \text{in } \Omega$$

$\psi(t) = 0 \text{ on } \partial \Omega.$
Here, $c, d > 0$, and $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is the bounded Lipschitz domain. Recently, Corsato et al. [1] studied Equation (1) for describing regularity, existence, and uniqueness. They described the mean anisotropic curvature considering the Dirichlet conditions. The one-dimensional analysis of Equation (1) was carried out by Okrasiński and Płociniczak [2]. They used the linearization technique on the divergence control operator by considering $[0, 1]$ as a bounded domain. Their relationship to the cornea is described as follows:

$$\psi'(0) = \psi(1) = 0$$

A more detailed analysis related to Equation (2) can be found in Coelho et al. [3]. In the early 1990s, Komai and Ushiki [4] analyzed the three-dimensional collagen fibrils in the human eye. Later on, this work was extended by Peh et al. [5] and Almubrad and Akhtar [6]. They discussed the various impacts and briefly described the expansion of the cornea and its effects on tree shrews.

The exact solution to nonlinear problems is always a point of interest for researchers; however, due to nonlinearity and complexity including high dimensionality, closed-form solutions are very rare to find. One of the best attempts to explain the closed-form solution is made by Okrasiński and Płociniczak [2]. In addition to their unique analysis, they showed that their results are quite impressive, up to 1% absolute error. In this work, they approximated nonlinearity through a hyperbolic cosine function. The work of Płociniczak et al. [7] opened the door to the application of semianalytical methods. He [8] introduced a new idea to these methods by adding Taylor’s series. This suggestion has made great improvements in the solution as well as in the reduction of the absolute error. Methods like Adomian decomposition [9], spline interpolation [10], variational iteration [11], Green’s function approach [12], and differential transformation [9] are normally considered very effective for boundary value problems (BVPs). Płociniczak et al. [13] further explained the formulation of the corneal curvature with the help of pseudo-time derivatives. The transformed partial differential equations (PDEs) are further solved numerically, and the results are compared with the available literature. Griffiths et al. [14] analyzed the radial basis function by considering the mesh-free approach. A brief survey can be found in the references [15–17]. Researchers frequently use Green’s function-based approaches to solve BVPs. For instance, Zur [18–20] provided a series of works employing Green’s and quasi-Green’s functions to investigate the vibration of thin circular plates, thin circular plates with variable thickness, and elastically supported functionally graded annular plates. The exact Green’s function for each given rectangular potential was determined by Andrade [21]. The Ahyoune et al. [22] method of solving quasi-static partial element equivalent circuits made use of a weighted combination of 2D and 3D analytical Green’s functions. The BVPs arise in heat transfer, strong nonlinear oscillation, electroanalytical chemistry, and Bratu problems, and are solved using linked Green’s function with a fixed-point iteration approach [16,23].

In recent years, artificial-intelligence-based methods have been introduced in the literature for solving various complex problems [24]. The applications of artificial neural networks (ANNs) can be seen in circuit and electromagnetic theory [25,26], the fuel ignition model [27], motor induction models [28], the Thomas–Fermi model [29], doubly singular nonlinear systems [28], nanotechnology [30], nanofluidics [31], nonlinear prey–predator models [32], nonlinear equations [33], Troesch’s problem [34], optimal control [35], signal processing [36], and the modeling and control of particle accelerators [37]. Pinsky and Datye [38] used the finite element method for the analysis of the incised cornea of the human eye. Pandolfi and Manganiello [39] analyzed the corneal shape model (CSM) by using a numerical procedure. Płociniczak et al. [7] studied the nonlinear CSM and presented a detailed description of the CSM-based boundary value problem (BVP). Ahmad et al. [40] studied the nonlinear CSM by using the neuroevolution-based approach. Płociniczak and Okrasinski [41] presented a detailed overview of the CSM model and explained the physical impact of each parameter encountered in the study. Recently, Erturk et al. [42]
used the concept of fractional derivatives to analyze the noninteger behavior of the CSM. Sáez-Gutiérrez et al. [43] investigated the surface geometry of the cornea by using the evolutionary algorithm. The use of machine learning is increasing day by day and has covered a substantial area of research [44]. Aziz et al. [45] used the cuckoo search (CS) algorithm for the analysis of the fish image. A more recent survey on CS can be found in the references [46–48].

Figure 1. Geometry of human corneal model.

This evidence inspired the authors of this paper to design new computing standards for solving the CSM based on the power of artificial intelligence (AI) using neural network (NN) modeling, and hybrid cuckoo search (BHCS) as a global search algorithm.

2. Our Contribution

In view of the above literature, we propose a new methodology known as BHCS-ANN, which has the following features.

• To the best of our knowledge, this is the first study to tackle the curvature model of the eye by using the hybrid cuckoo search with the neural network.
• The proposed BHCS-ANN is applied to six different corneal geometries to show the accuracy of the method.
• The proposed BHCS-ANN is more accurate as compared to other state-of-the-art.
• Based on statistical analysis performed, BHCS-ANN outperformed other techniques.

Section 3 of this article explains the model proposed and the corresponding methodology that we chose for solution purposes. The results are described through statistical parameters presented in Section 5. The results achieved are displayed through graphs and tables in Section 6. Finally, the conclusion is provided in Section 7.

3. Mathematical Model

The description of a physical problem always asks for a mathematical relation that best describes certain parameters describing the problem. These problems are normally complex in nature, describing the nonlinear phenomenon, as shown in Figure 1. Researchers are well interested in the solution of these problems. The literature presented above shows that various semianalytical and numerical techniques are adopted for the solution of such types of problems. The proposed problem in this study is given by Equation (1). In our research, we separated our methodology into two distinct parts. First, we transformed the CSM differential equation into an optimization problem together with its boundary conditions. In the second phase, BHCS is used to minimize the mean squared error that assists ANN in determining the unknown weights in the network. This whole procedure is explained in Figures 2 and 3, which depict the abstract and the architecture of ANN for CSM.
Neural Network Modeling for Differential Equations

ANN is widely used for approximating the solution of differential equations [49]. The use of ANN is extended to fractional and some other nonlinear problems, as shown in the references [50,51]. We take continuous mapping by addressing the derivatives in the form of a series.

\[
\hat{\psi}(t) = \sum_{i=1}^{M} \tilde{\alpha}_i \sigma(\phi_i t + \beta_i),
\]

(3)

\[
\hat{\psi}^{(m)}(t) = \sum_{i=1}^{M} \tilde{\alpha}_i \sigma^{(m)}(\phi_i t + \beta_i),
\]

(4)
where \( \hat{\psi}(t) \) is the approximate solution and \( \phi, \beta, \) and \( \hat{a} \) are weights. Similarly, \( M \) is the number of neurons and \( m \) is the order of the derivative. The radial basis function given by \( \sigma(t) = \frac{1}{1+e^{-rt}} \) is introduced into Equations (3) and (4), and we have

\[
\hat{\psi}(t) = \sum_{i=1}^{M} \hat{a}_i \left( \frac{1}{1+e^{-\left(\phi_i + \beta_i\right)t}} \right),
\]

for \( m = 2 \), we have

\[
\hat{\psi}''(t) = \sum_{j=1}^{M} \hat{a}_j \phi_j^2 \left[ \frac{2e^{-2(\phi_j + \beta_j)} - e^{-2(\phi_j + \beta_j)} - e^{-2(\phi_j + \beta_j)}}{\left(1+e^{-2(\phi_j + \beta_j)}\right)^3} \right].
\]

We need to reduce the \( L_2 \) error; therefore, using the above equations for constructing the optimization function as given below,

\[
\epsilon = \epsilon_1 + \epsilon_2,
\]

where the \( L_2 \) for the equations and the boundary conditions are given by \( \epsilon_1 \) and \( \epsilon_2 \), and are defined as under

\[
\epsilon_1 = \frac{1}{N} \sum_{k=1}^{N} \left( \sqrt{1+\psi'^2(t)} \left( \hat{\psi}''(t) - c \sqrt{1+\psi'^2(t)} \hat{\psi}(t) + d \right) \right)^2,
\]

\[
M = \frac{1}{N}, \quad \hat{\psi}_k = \hat{\psi}(t_k), \quad t_k = kh,
\]

\[
\epsilon_2 = \frac{1}{2} \left( \left( \hat{\psi}_0 \right)^2 + \left( \hat{\psi}_M \right)^2 \right).
\]

4. Proposed Algorithm

4.1. Proposed BHCS Algorithm

In this work, we use the biogeography-based heterogeneous cuckoo search (BHCS) technique that utilizes the Lévy flights’ approach in search by adopting the cuckoo search (CS) algorithm. The biogeography-based approach utilizes the migration operator, which is particularly adopted in the local area, to produce new solutions.

The heterogeneous cuckoo search algorithm uses the Lévy flights approach. This technique was introduced by Ding et al. [52] and further extended by Cheung et al. [53] and is explained under

\[
t_{\text{new}} = \begin{cases} 
\frac{t_{\text{old}} + \hat{a} \cdot (t_i - t_g)}{1 + L \cdot (1 - \hat{a})}, & \text{Lévy (} \hat{\beta} \text{)} \quad \frac{2}{3} < sr \leq 1 \\
\frac{t_{\text{old}} + \epsilon \cdot (t_i - t_g)}{1 + L \cdot (1 - \hat{a})}, & \epsilon, \text{else}, \\
\end{cases}
\]

Here, the \( \epsilon = \delta \epsilon^0, L = \delta \ln(1/\eta), t_g \) represents the best solution in the present version; \( \hat{a} \) symbolizes the mean of all options. As random numbers in the range \([0, 1], \eta, \) and \( sr, \hat{a} \) is a constant. Equation (11) shows that three distinct updating equations with equal probabilities are used in heterogeneous BHCS. The Lévy flight serves as the foundation for the first update equation in the original CS. The second and third updating equations, on the other hand, are based on the quantum mechanism. Using inhomogeneous updating limitations can generate multiple possibilities for expanding the flying and search toward the actual global region in an optimistic manner.
4.2. Biogeography-Based Revelation Operator

The second phase of BHCS shows an impact of changing the search operator for the development of unique solutions.

A biogeography-based migration operator enables the host bird to recognize alien eggs with a high degree of certainty, abandon old nests, and produce new nests. Following that, the population is ranked from best to worst, and emigration rates are ascribed to each response. Here, emigration rates $\mu$ are defined as

$$\mu_i = \frac{E S_i}{MP}.$$  

In the above equation, the maximum emigration rate is $E = 1$ and the number of species in the solution is $S_i = MPI$. The first algorithm describes the biogeography-based operator for discovering the $i$th solution. Solutions with higher fitness might share more characteristics with other solutions in the biogeography-based discovery operator, which is beneficial for exploitation enhancement.

5. Statistical Evaluation

This section will introduce some statistical concepts and formulas for explaining the performance and other graphical explanations. Some of the well-known concepts are the mean absolute deviation (MAD), global mean absolute deviation (GMAD), GFIT, TIC, ENSE, and GENSE. In this study, the TIC, ENSE, and MAD performance matrices are considered for each CSM scenario. In addition, we conducted a statistical analysis of our findings by employing minimum, mean, and standard deviation. To demonstrate the robustness of our methodology, 100 separate runs are conducted to estimate the performance of GFIT, GMAD, GTIC, and GENSE.

$$\text{MAD} = \frac{1}{l} \sum_{m=1}^{l} |\psi(t_m) - \hat{\psi}(t_m)|$$  

$$\text{TIC} = \frac{\sqrt{\frac{1}{l} \sum_{m=1}^{l} (\psi(t_m) - \hat{\psi}(t_m))^2}}{\sqrt{\frac{1}{l} \sum_{m=1}^{l} (\psi(t_m))^2 + \frac{1}{l} \sum_{m=1}^{l} (\hat{\psi}(t_m))^2}}$$  

$$\text{NSE} = \left\{ 1 - \frac{\sum_{m=1}^{l} (\psi(t_m) - \hat{\psi}(t_m))^2}{\sum_{m=1}^{l} (\psi(t_m) - \bar{\psi}(t_m))^2} \right\}$$  

$$\bar{\psi}(t_m) = \frac{1}{l} \sum_{m=1}^{l} \psi(t_m)$$  

$$\text{ENSE} = 1 - \text{NSE}$$  

$$\text{GMAD} = \frac{1}{R} \sum_{r=1}^{R} \left( \frac{1}{m} \sum_{i=1}^{m} |\psi(t_i) - \hat{\psi}(t_i)| \right)$$  

$$\text{GFIT} = \frac{1}{R} \sum_{r=1}^{R} e_r$$  

$$\text{GENSE} = \frac{1}{R} \sum_{r=1}^{R} \left( \sqrt{\frac{1}{m} \sum_{i=1}^{m} (\psi(t_i) - \hat{\psi}(t_i))^2} \right)$$

TIC, MAD, and ENSE values approach zero for the best approximation solution that is close to the exact answer.
6. Results and Discussions

The section is devoted to the analysis of the results obtained for varying parameters \( c \) and \( d \) in Equation (2). We present this analysis by considering six different cases:

**Case 1:** Taking \( c = d = 0.1 \), we have, from Equation (2),

\[
\psi''(t) - 0.1\psi(t) + \frac{0.1}{\sqrt{1 + \psi'^2(t)}} = 0, \\
\psi'(0) = \psi(1) = 0, \ t \in [0, 1].
\] (20)

The corresponding objective function takes the form

\[
\zeta = \frac{1}{M} \sum_{k=1}^{M} \left( \sqrt{1 + \psi'^2(t_k)} \psi''(t_k) - 0.1 \sqrt{1 + \psi'^2(t_k)} \psi(t_k) + 0.1 \right)^2 + \frac{1}{2} \left( \left( \psi'_0 \right)^2 + \left( \psi'_M \right)^2 \right). \tag{21}
\]

Similarly, the remaining cases are given below:

**Case 2:** \( d = 0.1, c = 0.2 \).

**Case 3:** \( d = 0.1, c = 0.3 \).

**Case 4:** \( d = 0.2, c = 0.1 \).

**Case 5:** \( d = 0.3, c = 0.1 \).

**Case 6:** \( d = 0.4, c = 0.1 \).

The results for each case considered above are presented in Figure 4 for the state variable \( \psi \). The solution in each case is compared with the exact. The approximate solution and the exact solution comparison show that the proposed methodology has a high level of accuracy. In case 1, the solution curve drops down from 0.05 to 0 on the unit time interval. A quite similar trend can be observed in cases 2 and 3. The remaining cases 4, 5, and 6 are starting from different points of the state variables 0.099, 0.14, and 0.189, respectively. The trend of the approximate and the exact solution remains the same. From the analysis shown in case 5, we see that the radius of the cornea, and intramuscular pressure of the eye \((c = 0.1, \text{ and } d = 0.3)\) have the global minimal value 0.14 at \( t = 0 \) of the corneal curvature. The numerical results for all these cases are presented in the form of Tables 1 and 2. In the tables, the minimum values of the objective functions are compared with the standard deviation and the mean values for each scenario. The standard deviation shows a deviation of \(10^{-9}, 10^{-10}, \text{ and } 10^{-10} \) for cases 1, 2, and 3, respectively. The remaining cases 4, 5, and 6 have the same standard deviations up \(10^{-7} \). The minimum values of the state variable vary from \(10^{-9} \) to \(10^{-15} \), which is a very small quantity. This reduction is possible only due to the applied BHCS that minimizes the \(L^2\) error.

In Figure 5, the weight plots of the neural network are presented. In case 1, there are 9 neurons and 3 weight functions that range up to 4 in the third weight. In case 2, the number of neurons is 10, with the same number of weight functions that range up to 2. In cases 3 and 4 there are 7 neurons and three weight functions that are up to 4 and 2, respectively. Also, cases 5 and 6 have the same scenario but their ranges are different. These weights are chosen directly by the neural network for searching the best possible outcome for the objective function.

The absolute error in each case has been provided in Figure 6. The absolute errors in all six cases are very small and even vary up to \(10^{-10}\); for example, see case 4. In all six cases, the absolute error is computed at the grid points. The error in each case shows the accuracy of our implemented technique and its convergence.
The fitness function, MAD, TIC, and ENSE are all displayed in Figure 7 for all the cases. The fitness function minimum values in all the cases approach $10^{-11}$, while the worst of all the cases approach $10^{-4}$. The mean of all the values for the fitness function is also displayed through bar graphs. The mean is also observed to approach $10^{-6}$. The statistical analysis of the present results is also presented through bar graphs. The minimal results for the MAD remain below $10^{-6}$, while the mean values almost approach these points. In MAD, the worst cases are not too far from the mean and remain very near it. The bars of TIC and ENSE approach $10^{-6}$ and $10^{-7}$ in some cases, respectively. In the case of 4, these variations are very large, where the minimal values for ENSE approach $10^{-9}$. All this analysis shows that the fitness function, MAD, TIC, and ENSE are stable at various iterations and the range in each case shows the total performance of the computational process.

The fitness function is plotted with a bar graph by using the normal distribution, as shown in Figure 8 for each case. The number of independent runs is plotted on the $y$–axis, while the fitness is shown on the $x$–axis. In case 1, the values remain greater than zero and below $0.25 \times 10^{-5}$. A maximum of the values remain inside the middle of the bill shape. A quite similar trend can be seen in the case of 5. In case 2, the fitness function has variations up to $10^{-6}$, and the data remain inside the uniform distribution. In cases 3 and 4 again, the data remains inside the uniform bill and the variation extends up to $10^{-4}$ and $10^{-5}$, respectively. In the last case 6, all the values are contained in the uniform bill, while the fitness function varies up to $10^{-4}$.
The normal plots for MAD, TIC, and ENSE are presented in Figures 9–11. All six cases are comprehensively discussed against the number of runs. The results for MAD vary up to $10^{-5}$ for cases 1, 2, 4, and 5, respectively. The results for MAD remain inside the uniform distribution for all the cases and approach the midpoint. The results for cases 3 and 6 remain inside the range $10^{-4}$. The TIC and ENSE plots are shown in Figures 10 and 11. The results for TIC remain inside the unit bill and approach $10^{-5}$ in cases 2, 4, and 5.
respectively. Similarly, the results for cases 1 and 6 are bounded in $10^{-4}$, while for the remaining case 3, it lies in $10^{-3}$. The normal plots describing the bill shape distribution of the data for the ENSE show a variation up to $10^{-5}$ for cases 1, 2, 4, and 6. In case 3, this variation is $10^{-3}$, while in case 5 it is $10^{-6}$.

Figure 6. Absolute error obtained for various case studies.

Table 1. The statistical findings for every instance of the corneal shape model.

<table>
<thead>
<tr>
<th>t</th>
<th>Case 1</th>
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<th>Case 2</th>
<th></th>
<th>Case 3</th>
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<td>Min Mean STD</td>
<td>Min Mean STD</td>
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Figure 7. Various cases for best, worst, and mean.

Table 2. The statistical findings for every instance of the corneal shape model.

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<td>3.34 × 10⁻⁹</td>
<td>1.38 × 10⁻⁷</td>
<td>2.88 × 10⁻¹⁰</td>
</tr>
<tr>
<td>0.8</td>
<td>4.12 × 10⁻⁹</td>
<td>3.04 × 10⁻⁷</td>
<td>4.13 × 10⁻¹⁰</td>
</tr>
<tr>
<td>0.9</td>
<td>6.45 × 10⁻¹⁰</td>
<td>1.59 × 10⁻⁷</td>
<td>2.09 × 10⁻¹⁰</td>
</tr>
<tr>
<td>1</td>
<td>4.57 × 10⁻⁹</td>
<td>4.79 × 10⁻⁷</td>
<td>7.69 × 10⁻¹⁰</td>
</tr>
</tbody>
</table>
The fitness function minimum, mean, and standard deviation values for each are presented in Table 1. All these variations are shown for a step size of 0.1. In case 1, the minimum value for the fitness function is \(2.10 \times 10^{-15}\), while the standard deviation remains up to \(1.59 \times 10^{-9}\). In case 2, the minimum value is \(5.00 \times 10^{-15}\), while the standard deviation touches \(1.32 \times 10^{-10}\). Similarly, in cases 3–6, the minimum values of the fitness functions are \(8.68 \times 10^{-15}\), \(8.43 \times 10^{-11}\), \(7.71 \times 10^{-13}\), and \(1.34 \times 10^{-12}\), respectively. The standard deviations for all these cases touch \(10^{-10}\) and \(10^{-7}\).

The global performances of TIC, MAD, FIT, and ENSE are shown in Table 3. The minimum value of GTIC exists in the case of 6, which is \(2.38 \times 10^{-5}\). These values for GMAD, GFIT, and GENSE are \(3.19 \times 10^{-6}\), \(2.38 \times 10^{-5}\), \(7.69 \times 10^{-8}\), and \(6.71 \times 10^{-7}\). All these show the mean of the minimum values. The corresponding standard deviation for each case is plotted to check the stability.
Figure 9. Normal plots of MAD for different cases.

Table 3. Corneal shape model’s global performance indices and their respective results.

<table>
<thead>
<tr>
<th>Case</th>
<th>GTIC Mean</th>
<th>GTIC STD</th>
<th>GMAD Mean</th>
<th>GMAD STD</th>
<th>GFTI Mean</th>
<th>GFTI STD</th>
<th>GENSE Mean</th>
<th>GENSE STD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.63 \times 10^{-5}$</td>
<td>$2.00 \times 10^{-5}$</td>
<td>$3.19 \times 10^{-6}$</td>
<td>$2.92 \times 10^{-5}$</td>
<td>$3.68 \times 10^{-7}$</td>
<td>$1.45 \times 10^{-5}$</td>
<td>$1.17 \times 10^{-6}$</td>
<td>$3.71 \times 10^{-5}$</td>
</tr>
<tr>
<td>2</td>
<td>$4.59 \times 10^{-5}$</td>
<td>$1.58 \times 10^{-5}$</td>
<td>$3.19 \times 10^{-6}$</td>
<td>$2.21 \times 10^{-5}$</td>
<td>$7.69 \times 10^{-8}$</td>
<td>$7.19 \times 10^{-8}$</td>
<td>$2.30 \times 10^{-8}$</td>
<td>$2.31 \times 10^{-8}$</td>
</tr>
<tr>
<td>3</td>
<td>$1.20 \times 10^{-4}$</td>
<td>$2.09 \times 10^{-4}$</td>
<td>$1.39 \times 10^{-5}$</td>
<td>$2.58 \times 10^{-5}$</td>
<td>$1.88 \times 10^{-6}$</td>
<td>$3.37 \times 10^{-6}$</td>
<td>$1.31 \times 10^{-4}$</td>
<td>$3.48 \times 10^{-4}$</td>
</tr>
<tr>
<td>4</td>
<td>$2.24 \times 10^{-5}$</td>
<td>$1.48 \times 10^{-5}$</td>
<td>$5.68 \times 10^{-6}$</td>
<td>$4.11 \times 10^{-5}$</td>
<td>$2.02 \times 10^{-6}$</td>
<td>$9.41 \times 10^{-6}$</td>
<td>$6.71 \times 10^{-7}$</td>
<td>$1.84 \times 10^{-5}$</td>
</tr>
<tr>
<td>5</td>
<td>$1.81 \times 10^{-4}$</td>
<td>$2.59 \times 10^{-4}$</td>
<td>$9.16 \times 10^{-5}$</td>
<td>$1.29 \times 10^{-4}$</td>
<td>$3.62 \times 10^{-7}$</td>
<td>$1.73 \times 10^{-7}$</td>
<td>$2.39 \times 10^{-7}$</td>
<td>$5.38 \times 10^{-7}$</td>
</tr>
<tr>
<td>6</td>
<td>$2.38 \times 10^{-5}$</td>
<td>$5.39 \times 10^{-5}$</td>
<td>$9.18 \times 10^{-6}$</td>
<td>$1.29 \times 10^{-4}$</td>
<td>$5.68 \times 10^{-6}$</td>
<td>$9.60 \times 10^{-6}$</td>
<td>$1.81 \times 10^{-4}$</td>
<td>$2.59 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

The execution time and the number of executed functions are presented in Table 4. For validity, the standard deviation is also presented. A total of 200,010 functions for each case were computed with a minimal mean time of 17.7791 for case 1. The generation means remains 2000 in all the cases.
Figure 10. Normal plots of TIC for different cases.

Table 4. Complexity analysis for each corneal shape model case.

<table>
<thead>
<tr>
<th>Case</th>
<th>Execution Time Mean</th>
<th>STD</th>
<th>Generation Mean</th>
<th>STD</th>
<th>Function Counts Mean</th>
<th>STD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17.7791</td>
<td>3.0599</td>
<td>2000</td>
<td>0</td>
<td>200,010</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>27.5776</td>
<td>2.9333</td>
<td>2000</td>
<td>0</td>
<td>200,010</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>27.1637</td>
<td>2.448</td>
<td>2000</td>
<td>0</td>
<td>200,010</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>27.421</td>
<td>3.0261</td>
<td>2000</td>
<td>0</td>
<td>200,010</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>26.8823</td>
<td>3.2615</td>
<td>2000</td>
<td>0</td>
<td>200,010</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>27.0509</td>
<td>3.888</td>
<td>2000</td>
<td>0</td>
<td>200,010</td>
<td>0</td>
</tr>
</tbody>
</table>

The proposed BHCS-ANN has better performance as compared to the available results of FO-DPSO presented by Waseem et al. [54]. In their work, they compared their results with the PSO-ASA. In this study, the BHCS-ASA results are compared with FO-DPSO in Tables 5 and 6. In each case, our results are better than the available literature that proves the performance of our implemented method.
Figure 11. Normal plots of ENSE for different cases.

Table 5. Comparison of the min. values for the approximate function in different cases.

<table>
<thead>
<tr>
<th>t</th>
<th>Min Case 1 BHCS-ANN</th>
<th>Min Case 2 BHCS-ANN</th>
<th>Min Case 2 FO-DPSO [54]</th>
<th>Min Case 3 BHCS-ANN</th>
<th>Min Case 3 FO-DPSO [54]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.04 × 10⁻¹²</td>
<td>1.04 × 10⁻¹¹</td>
<td>3.88 × 10⁻¹³</td>
<td>1.74 × 10⁻¹¹</td>
<td>2.09 × 10⁻¹⁴</td>
</tr>
<tr>
<td>0.1</td>
<td>1.03 × 10⁻¹¹</td>
<td>1.03 × 10⁻¹¹</td>
<td>1.09 × 10⁻¹²</td>
<td>1.21 × 10⁻¹⁰</td>
<td>5.51 × 10⁻¹⁴</td>
</tr>
<tr>
<td>0.2</td>
<td>1.18 × 10⁻¹³</td>
<td>8.26 × 10⁻¹²</td>
<td>1.28 × 10⁻¹³</td>
<td>3.58 × 10⁻¹²</td>
<td>2.09 × 10⁻¹⁴</td>
</tr>
<tr>
<td>0.3</td>
<td>2.08 × 10⁻¹²</td>
<td>9.73 × 10⁻¹⁴</td>
<td>3.91 × 10⁻¹⁰</td>
<td>7.32 × 10⁻¹¹</td>
<td>1.65 × 10⁻¹⁴</td>
</tr>
<tr>
<td>0.4</td>
<td>7.19 × 10⁻¹⁵</td>
<td>3.18 × 10⁻¹²</td>
<td>6.67 × 10⁻¹³</td>
<td>1.52 × 10⁻¹¹</td>
<td>1.10 × 10⁻¹⁴</td>
</tr>
<tr>
<td>0.5</td>
<td>1.44 × 10⁻¹⁵</td>
<td>5.28 × 10⁻¹²</td>
<td>2.91 × 10⁻¹⁴</td>
<td>1.65 × 10⁻¹¹</td>
<td>1.41 × 10⁻¹⁴</td>
</tr>
<tr>
<td>0.6</td>
<td>4.93 × 10⁻¹⁵</td>
<td>1.62 × 10⁻¹⁵</td>
<td>4.71 × 10⁻¹⁰</td>
<td>4.76 × 10⁻¹¹</td>
<td>3.81 × 10⁻¹⁴</td>
</tr>
<tr>
<td>0.7</td>
<td>4.19 × 10⁻¹⁴</td>
<td>2.66 × 10⁻¹³</td>
<td>7.29 × 10⁻¹⁰</td>
<td>6.16 × 10⁻¹²</td>
<td>1.21 × 10⁻¹³</td>
</tr>
<tr>
<td>0.8</td>
<td>3.17 × 10⁻¹⁴</td>
<td>3.68 × 10⁻¹²</td>
<td>5.00 × 10⁻¹⁵</td>
<td>1.96 × 10⁻¹¹</td>
<td>8.68 × 10⁻¹⁵</td>
</tr>
<tr>
<td>0.9</td>
<td>5.76 × 10⁻¹⁵</td>
<td>7.29 × 10⁻¹³</td>
<td>1.52 × 10⁻¹²</td>
<td>3.33 × 10⁻¹¹</td>
<td>1.30 × 10⁻¹³</td>
</tr>
<tr>
<td>1</td>
<td>2.10 × 10⁻¹⁵</td>
<td>2.84 × 10⁻¹²</td>
<td>3.48 × 10⁻¹³</td>
<td>1.94 × 10⁻¹¹</td>
<td>1.26 × 10⁻¹⁴</td>
</tr>
</tbody>
</table>
Table 6. Comparison of the min. values for the approximate function in different cases.

<table>
<thead>
<tr>
<th>t</th>
<th>BHCS-ANN</th>
<th>FO-DPSO [54]</th>
<th>BHCS-ANN</th>
<th>FO-DPSO [54]</th>
<th>BHCS-ANN</th>
<th>FO-DPSO [54]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$4.38 \times 10^{-10}$</td>
<td>$7.61 \times 10^{-10}$</td>
<td>$1.18 \times 10^{-10}$</td>
<td>$1.21 \times 10^{-10}$</td>
<td>$7.58 \times 10^{-10}$</td>
<td>$4.40 \times 10^{-10}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$1.11 \times 10^{-9}$</td>
<td>$8.83 \times 10^{-10}$</td>
<td>$7.68 \times 10^{-11}$</td>
<td>$7.74 \times 10^{-11}$</td>
<td>$8.78 \times 10^{-10}$</td>
<td>$1.06 \times 10^{-9}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$8.43 \times 10^{-11}$</td>
<td>$6.64 \times 10^{-10}$</td>
<td>$9.67 \times 10^{-11}$</td>
<td>$9.70 \times 10^{-11}$</td>
<td>$6.58 \times 10^{-10}$</td>
<td>$8.38 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$1.99 \times 10^{-9}$</td>
<td>$1.26 \times 10^{-12}$</td>
<td>$7.71 \times 10^{-13}$</td>
<td>$7.66 \times 10^{-13}$</td>
<td>$1.34 \times 10^{-12}$</td>
<td>$2.00 \times 10^{-9}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$2.36 \times 10^{-9}$</td>
<td>$3.49 \times 10^{-9}$</td>
<td>$7.18 \times 10^{-11}$</td>
<td>$7.23 \times 10^{-11}$</td>
<td>$3.53 \times 10^{-10}$</td>
<td>$2.40 \times 10^{-9}$</td>
</tr>
<tr>
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<td>$5.09 \times 10^{-10}$</td>
<td>$1.72 \times 10^{-10}$</td>
<td>$1.68 \times 10^{-10}$</td>
<td>$5.12 \times 10^{-10}$</td>
<td>$4.62 \times 10^{-10}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$4.69 \times 10^{-10}$</td>
<td>$1.32 \times 10^{-10}$</td>
<td>$1.62 \times 10^{-10}$</td>
<td>$1.59 \times 10^{-10}$</td>
<td>$1.32 \times 10^{-7}$</td>
<td>$4.72 \times 10^{-10}$</td>
</tr>
<tr>
<td>0.7</td>
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<td>$3.23 \times 10^{-11}$</td>
<td>$1.09 \times 10^{-10}$</td>
<td>$1.12 \times 10^{-10}$</td>
<td>$3.19 \times 10^{-11}$</td>
<td>$3.35 \times 10^{-9}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$4.12 \times 10^{-9}$</td>
<td>$2.94 \times 10^{-10}$</td>
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<td>$7.08 \times 10^{-11}$</td>
<td>$2.87 \times 10^{-10}$</td>
<td>$4.18 \times 10^{-9}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$6.45 \times 10^{-10}$</td>
<td>$1.64 \times 10^{-10}$</td>
<td>$1.41 \times 10^{-12}$</td>
<td>$1.36 \times 10^{-12}$</td>
<td>$1.62 \times 10^{-10}$</td>
<td>$6.54 \times 10^{-10}$</td>
</tr>
<tr>
<td>1</td>
<td>$4.57 \times 10^{-9}$</td>
<td>$1.45 \times 10^{-10}$</td>
<td>$7.09 \times 10^{-10}$</td>
<td>$7.11 \times 10^{-10}$</td>
<td>$1.43 \times 10^{-10}$</td>
<td>$4.60 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

7. Conclusions

In this work, the BHCS-based neural network is applied for the first time to the CSM model. The results are presented in the form of figures and tables, where Adam’s method is taken as a reference method. We observed the following on the basis of our analysis.

- This approach effectively minimized the fitness function and provided the best approximation of the solution to the problem.
- The proposed approach demonstrated efficacy in all CSM scenarios and identified the optimum approximation for the CSM geometry in all scenarios.
- In figures, the results were compared with Adam’s numerical solution, where the BHCS-ANN showed a better trend.
- The statistical evaluations such as MAD, TIC, and ENSE were evaluated in 100 different runs, and the results showed that the proposed approach outperformed the current state of the art.
- The obtained results for the minimum approximated functions were compared with the FO-DPSO algorithm, where BHCS-ANN performed better in all the cases.


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Data Availability Statement: No data are used in this study.

Acknowledgments: Researchers Supporting Project number (RSPD2023R1060), King Saud University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

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