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Article

Positive Solutions for Some Semipositone Fractional Boundary Value Problems on the Half-Line

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Abstract: Our goal is to address the question of existence and uniqueness of a positive continuous solution to some semipositone fractional boundary value problems on the half-line. Global estimates on this solution are given. This kind of problems, where the nonlinearity is allowed to be sign-changing, are often difficult to solve analytically and becomes more challenging specially when we are looking for positive solutions. The main result is obtained by means of the properties of the Green function and fixed point theorem.

Keywords: semipositone fractional boundary value problems; positive solution; Green's function; half-line

MSC: 34B15; 34B40; 34B18; 34B27

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1. Introduction

In recent years, it has been shown that fractional differential equations serve as powerful tools for modeling real-world phenomena which occur in scientific and engineering disciplines (see, i.e., [1,2]). The existence and/or uniqueness of solutions (or positive solutions) to boundary value issues for fractional-order differential equations have been the subject of extensive investigation (see, i.e., [3–10] and the references therein).

In [10], under sufficient conditions, the authors proved, by using a fixed point theorem, the existence of solutions to the problem

$$\begin{cases} -\mathfrak{D}^{\alpha}\varphi(x) = F(x,\varphi(x)), \quad x > 0,\ 1 < \alpha \leq 2, \\ \varphi(0) = 0,\ \lim\limits_{x\to\infty} \mathfrak{D}^{\alpha-1}\varphi(x) = \beta\varphi(\xi), \end{cases} \tag{1}$$

where $\beta \in \mathbb{R}$, $0 < \xi < \infty$ and $\mathfrak{D}^{\alpha}$ is the Riemann-Liouville fractional derivative (see Definition 2).

In [11], by using Leray-Schauder theorem, the authors established the existence of nonnegative solutions for the problem

$$\begin{cases} -\mathfrak{D}^{\alpha}\varphi(z) = F(z,\varphi(z)), \quad z \in (0,\infty),\ 1 < \alpha \leq 2, \\ I_0^{\alpha-2}\varphi(0) = 0,\ \lim\limits_{z\to\infty} \mathfrak{D}^{\alpha-1}\varphi(z) = \beta I_0^{\alpha-1}(\xi), \end{cases} \tag{2}$$

where $F \in C([0,\infty)\times\mathbb{R},[0,\infty))$ and $0 < \beta, \xi < \infty$. Here I_0^{γ} is the Riemann-Liouville fractional integral (see Definition 1).

In [12], the authors studied the problem

$$\begin{cases} -\mathfrak{D}^{\alpha}\varphi(t) = F(t,\varphi(t)), \quad t > 0,\ 1 < \alpha \leq 2, \\ \lim\limits_{t\to 0} t^{2-\alpha}\varphi(t) = \lim\limits_{t\to\infty} \mathfrak{D}^{\alpha-1}\varphi(t) = \int_0^{\infty} g(r)\varphi(r)dr, \end{cases} \tag{3}$$

where $F \in C((0,\infty) \times [0,\infty), [0,\infty))$ with $F(z,0) \neq 0$ on $(0,\infty)$ and $g \in L^1([0,\infty))$ with $\int_0^\infty [\frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} + \zeta^{\alpha-2}]g(\zeta)d\zeta < 1$.

The existence of positive solutions is obtained by using the monotone iterative technique.

In [6], by applying the Karamata theory and the Schauder fixed point theorem, the authors established the existence and uniqueness of a positive continuous solution to the problem

$$\begin{cases} -\mathfrak{D}^\alpha \varphi(z) = a(z)\varphi^\sigma, \quad z \in (0,\infty),\ 1 < \alpha \leq 2, \\ \lim\limits_{z\to 0} z^{2-\alpha}\varphi(z) = 0,\ \lim\limits_{z\to\infty} z^{1-\alpha}\varphi(z) = 0, \end{cases} \tag{4}$$

where $\sigma \in (-1,1)$, and $a \in C((0,\infty),[0,\infty))$.

Natural phenomena in scientific and engineering areas can be described by semipositone problems, see for example [13]. Such problems involving ordinary and fractional differential equations have been considered by many authors for both finite and infinite intervals, see for example [14–20] and the references therein.

For instance, in [18], the authors used Bifurcation theory to prove the existence of positive solutions of some classes of semi-positone problems with nonlinear boundary conditions

$$\begin{cases} -\varphi''(z) = \lambda F(z,\varphi(z)), \quad z \in (0,1), \\ \varphi(0) = 0,\ \varphi'(1) + c((\varphi(1)))\varphi(1) = 0, \end{cases} \tag{5}$$

where c is a nondecreasing function in $C([0,\infty),[0,\infty))$ with $c(\infty) < \infty$ and $F \in C([0,1] \times [0,\infty), \mathbb{R})$ with $F(z,0) < 0$, for $z \in [0,1]$.

In [14], by using the Krasnosel'skii fixed point theorem, the authors established, for small positive λ, the existence of positive solutions to the superlinear semi-positone boundary value problem

$$\begin{cases} -(p(z)\varphi')'(z) = \lambda F(z,\varphi(z)), \quad z \in (r,R), \\ a\varphi(r) - bp(r)\varphi'(r) = c\varphi(R) + dp(R)\varphi'(R) = 0, \end{cases} \tag{6}$$

where $a,b,c,d \geq 0$ with $ac + ad + bc > 0$, p is a positive continuous function in $[r,R]$ and $F \in C([r,R] \times [0,\infty), \mathbb{R})$ satisfying some adequate conditions.

On the other hand, in [21], by applying the fixed point theory and the upper and lower solutions method, the authors obtained a new result on the existence of at least three distinct nonnegative solutions for the following nonlocal fractional boundary value problem

$$\begin{cases} {}^C\mathfrak{D}^\beta(p(z)\varphi'(z)) + q(z)F(z,\varphi(z)) = 0, \quad z \in (0,\infty), \\ p(0)\varphi'(0) = 0,\ \lim\limits_{z\to\infty}\varphi(z) = \int_0^\infty g(r)\varphi(r)dr, \end{cases} \tag{7}$$

where ${}^C\mathfrak{D}^\beta$ is the standard Caputo derivative (see [1,2,22]), $0 < \beta < 1$ is a constant, F, g, p and q are given functions.

By using Schauder's fixed point theorem combined with the diagonalization method, Arara et al. (see [23]) studied the existence of solutions for boundary value problems for fractional differential equations of the form

$$\begin{cases} {}^C\mathfrak{D}^\alpha\varphi(z) = F(z,\varphi(z)), \quad z \in (0,\infty),\ 1 < \alpha \leq 2, \\ \varphi(0) = z_0 \in \mathbb{R},\ \varphi \text{ is bounded on } [0,\infty), \end{cases} \tag{8}$$

where ${}^C\mathfrak{D}^\alpha$ is the standard Caputo derivative of order $1 < \alpha \leq 2$ and $F : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [24], the authors have considered

$$\begin{cases} \mathfrak{D}^{\beta}(\varphi - \varphi(0))(z) = F(z, \varphi(z)), \quad z \in (0, \infty), \\ \varphi(0) = z_0 \in \mathbb{R}, \end{cases} \tag{9}$$

where $\mathfrak{D}^{\beta}$ is the Riemann-Liouville fractional derivative of order $0 < \beta < 1$ and $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying: there exists the continuous function $q \in C([0, \infty), [0, \infty))$ such that

$$|F(z, y_1) - F(z, y_2)| \le q(z)|y_1 - y_2|, \text{ for all } z \ge 0 \text{ and } y_1, y_2 \in \mathbb{R}.$$

They have showed, by using the contraction principle, that the initial value problem (9) has a unique solution defined in $C([0, \infty), \mathbb{R})$.

However, it seems, there are few works concerning the existence of positive solutions for semipositone fractional problems on the half-line, see [25,26].

In this paper, we deal with the following semipositone Riemann-Liouville fractional (see Definition 2) boundary value problem

$$\begin{cases} -\mathfrak{D}^{\alpha}\varphi(z) = \mathfrak{p}(z) + \lambda F(z, \varphi(z)), \quad z \in (0, \infty), \ 1 < \alpha \le 2, \\ \lim\limits_{z \to 0^+} z^{2-\alpha}\varphi(z) = 0, \ \lim\limits_{z \to \infty} z^{1-\alpha}\varphi(z) = 0, \end{cases} \tag{10}$$

where $\lambda \ge 0$, $\mathfrak{p} \in C((0, \infty), [0, \infty))$ which may be singular at 0 satisfying $\int_0^{\infty} \min(1, z)$ $\mathfrak{p}(z)dz < \infty$ and $F : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous semipositone function.

The main purpose of this paper is to provide sufficient conditions to guarantee that the semipositone problem (10) admits a unique positive continuous solution φ on $(0, \infty)$ satisfying

$$\frac{1}{c}\gamma(z) \le \varphi(z) \le c\gamma(z), \text{ for } z > 0,$$

where c is a positive constant and

$$\gamma(z) := \int_0^{\infty} \mathbb{G}_{\alpha}(z, \zeta)\mathfrak{p}(\zeta)d\zeta. \tag{11}$$

Here, $\mathbb{G}_{\alpha}(z, \zeta)$ is the the Green's function of the operator $\varphi \to -\mathfrak{D}^{\alpha}\varphi$ with boundary conditions $\lim\limits_{z \to 0} z^{2-\alpha}\varphi(z) = 0$ and $\lim\limits_{z \to \infty} z^{1-\alpha}\varphi(z) = 0$.
From Lemma 11 [6], the explicit expression of $\mathbb{G}_{\alpha}(z, \zeta)$ is given by

$$\mathbb{G}_{\alpha}(z, \zeta) = \frac{1}{\Gamma(\alpha)}[z^{\alpha-1} - (\max(z - \zeta, 0))^{\alpha-1}], \ z, \zeta \in [0, \infty). \tag{12}$$

By means of the properties of the Green function and the contraction principle on a convenient complete metric space, we have proved our main result. We emphasize that the approach adapted in this paper can be applied in many other problems involving ordinary, elliptic or fractional differential equations.

In the rest of this paper, for $\alpha \in (1, 2]$, we use the following notations:

(i) $\mathcal{B}^+((0, \infty))$ denotes the set of nonnegative Borel measurable functions in $(0, \infty)$.
(ii) For $g_1, g_2 \in \mathcal{B}^+((0, \infty))$, the notation $g_1 \asymp g_2$ on a set S means there exists $c > 0$ such that $\frac{1}{c}g_2(\zeta) \le g_1(\zeta) \le cg_2(\zeta)$, for all $\zeta \in S$.
(iii) Let $C((0, \infty), \mathbb{R})$ (resp. $C^+((0, \infty), \mathbb{R})$) be the set of all (resp. nonnegative) continuous functions on $(0, \infty)$.
(iv) $C_{2-\alpha}([0, \infty)) = \{h \in C((0, \infty), \mathbb{R}); z \to z^{2-\alpha}h(z) \text{ is continuous on } [0, \infty)\}$.
(v) $\mathcal{J} = \{h \in C^+((0, \infty), \mathbb{R}) : \int_0^{\infty} \min(1, \zeta)h(\zeta)d\zeta < \infty\}$.

(vi) $\mathcal{M} = \{h \in \mathcal{B}^+((0,\infty)) : \int_0^\infty r^{\alpha-1} h(r)dr < \infty\}$.

(vii) For $q \in \mathcal{M}$, we let

$$\mathbb{M}_q := \int_0^\infty r^{\alpha-1} q(r)dr \text{ and } \xi_q := \sup_{z,\zeta \in (0,\infty)} \int_0^\infty \frac{\mathbb{G}_\alpha(z,r)\mathbb{G}_\alpha(r,\zeta)}{\mathbb{G}_\alpha(z,\zeta)} q(r)dr. \tag{13}$$

We will prove (see Lemma 1) that

$$\frac{1}{\Gamma(\alpha)}\mathbb{M}_q \le \xi_q \le \frac{1}{(\alpha-1)\Gamma(\alpha)}\mathbb{M}_q.$$

In particular, $\xi_q < \infty$

To study problem (10), we assume that:

$(A1)$ $\mathfrak{p}$ is a nontrivial function in $\mathcal{J}$.

$(A2)$ $F : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists $k \ge 0$ such that

$$|F(z,0)| \le k\mathfrak{p}(z), \text{ for } z > 0.$$

$(A3)$ there exists a function $q \in \mathcal{M}$ such that

$$|F(z,y_1) - F(z,y_2)| \le q(z)|y_1 - y_2|, \text{ for all } z > 0 \text{ and } y_1, y_2 \in \mathbb{R}.$$

Our main result is the following:

Theorem 1. *Under assumptions $(A1)$–$(A3)$, then there exists $\lambda^* > 0$ such that for $\lambda \in [0,\lambda^*)$, the semipositone problem (10) admits a unique positive solution $\varphi \in C_{2-\alpha}([0,\infty))$ such that*

$$2\frac{(\lambda^* - \lambda(k\lambda^* + 1))}{(2\lambda^* - \lambda)}\gamma(z) \le \varphi(z) \le 2\lambda^*(\frac{1+\lambda k}{2\lambda^* - \lambda})\gamma(z), \text{ for } z > 0, \tag{14}$$

where $\gamma(z)$is given by (11).

Remark 1. *Let $\mathfrak{p}$ be a nontrivial function in $\mathcal{J}$. Then from Proposition 15 [6], the unique solution of the linear problem*

$$\begin{cases} -\mathfrak{D}^\alpha \varphi(z) = \mathfrak{p}(z), \quad z \in (0,\infty),\ 1 < \alpha \le 2, \\ \lim_{z\to 0^+} z^{2-\alpha}\varphi(z) = 0,\ \lim_{z\to\infty} z^{1-\alpha}\varphi(z) = 0, \end{cases}$$

is given by $\gamma(z)$, see (11).

In Theorem 1, under sufficient conditions, we have proved that by adding a small perturbation of the above linear problem, we still have a unique solution which globally behaves like $\gamma(z)$.

The paper is organized as follows. In Section 2, we recall and establish some technical estimates related to $\mathbb{G}_\alpha(z,\zeta)$, which will be very useful. In Section 3, we prove our main result and we give an example.

2. Preliminaries

We recall the following basic definitions (see [1,2,22]).

Definition 1. *For $\beta > 0$ and $h : (0,\infty) \to \mathbb{R}$, we let*

$$I_0^\beta h(z) := \frac{1}{\Gamma(\beta)} \int_0^z (z-\zeta)^{\beta-1} h(\zeta)d\zeta, \ \ z > 0,$$

where Γ is the Euler Gamma function.

Definition 2. *The Riemann-Liouville fractional derivative of order $\beta > 0$ for a function $h : (0,\infty) \to \mathbb{R}$ is given by*

$$\mathfrak{D}^{\beta} h(z) := (\frac{d}{dz})^{n} I_0^{n-\beta} h(z), \;\; z > 0,$$

where $n = [\beta] + 1$ and $[\beta]$ is the integer part of β.

The following estimates on the Green function $\mathbb{G}_{\alpha}(z,\zeta)$ given by (12) hold.

Proposition 1 ([5])**.** *Let $1 < \alpha \leq 2$ and $\mathbb{G}_{\alpha}(z,\zeta)$ given by (12), then we have*
(i) $\mathbb{G}_{\alpha}(z,\zeta)$ is continuous on $[0,\infty) \times [0,\infty)$ with

$$(\alpha-1)z^{\alpha-2}\min(z,\zeta) \leq \Gamma(\alpha)\mathbb{G}_{\alpha}(z,\zeta) \leq z^{\alpha-2}\min(z,\zeta). \tag{15}$$

In particular

$$(\alpha-1)z^{\alpha-2}\min(1,z)\min(1,\zeta) \leq \Gamma(\alpha)\mathbb{G}_{\alpha}(z,\zeta) \leq z^{\alpha-2}\max(1,z)\min(1,\zeta). \tag{16}$$

(ii) For all $z, r, \zeta \in (0,\infty)$,

$$\frac{\mathbb{G}_{\alpha}(z,r)\mathbb{G}_{\alpha}(r,\zeta)}{\mathbb{G}_{\alpha}(z,\zeta)} \leq \frac{1}{(\alpha-1)\Gamma(\alpha)} r^{\alpha-1}. \tag{17}$$

Lemma 1. *Let $q \in \mathcal{M}$, then*

$$\frac{1}{\Gamma(\alpha)}\mathbb{M}_q \leq \xi_q \leq \frac{1}{(\alpha-1)\Gamma(\alpha)}\mathbb{M}_q, \tag{18}$$

where $\mathbb{M}_q$ and ξ_q are given in (13).
In particular, $\xi_q < \infty$.

Proof. Since $q \in \mathcal{M}$, then from (13) and (17), we have

$$\xi_q \leq \frac{1}{(\alpha-1)\Gamma(\alpha)}\mathbb{M}_q < \infty.$$

On the other hand, from (12), we derive that

$$\lim_{\zeta\to\infty}\frac{\mathbb{G}_{\alpha}(r,\zeta)}{\mathbb{G}_{\alpha}(z,\zeta)} = \frac{r^{\alpha-1}}{z^{\alpha-1}} \text{ and } \lim_{z\to 0}\frac{\mathbb{G}_{\alpha}(z,r)}{z^{\alpha-1}} = \frac{1}{\Gamma(\alpha)}.$$

Hence by applying Fatou's lemma, we obtain

$$\frac{1}{\Gamma(\alpha)}\mathbb{M}_q = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} r^{\alpha-1}q(r)dr \leq \liminf_{z\to 0}\int_0^{\infty}\frac{\mathbb{G}_{\alpha}(z,r)}{z^{\alpha-1}} r^{\alpha-1}q(r)dr$$

and

$$\begin{aligned}\int_0^{\infty}\frac{\mathbb{G}_{\alpha}(z,r)}{z^{\alpha-1}} r^{\alpha-1}q(r)dr &\leq \liminf_{\zeta\to\infty}\int_0^{\infty}\frac{\mathbb{G}_{\alpha}(z,r)\mathbb{G}_{\alpha}(r,\zeta)}{\mathbb{G}_{\alpha}(z,\zeta)}q(r)dr \\ &\leq \xi_q.\end{aligned}$$

So $\frac{1}{\Gamma(\alpha)}\mathbb{M}_q \leq \xi_q$. □

Define $\mathbb{V}$ on $B^{+}((0,\infty))$, by

$$\mathbb{V}f(z) := \int_0^{\infty}\mathbb{G}_{\alpha}(z,\zeta)f(\zeta)d\zeta, \; z \in (0,\infty). \tag{19}$$

The next proposition is due to (Proposition 15) [6].

Proposition 2. *Assume that $\zeta \to \min(1,\zeta)f(\zeta) \in C((0,\infty),\mathbb{R}) \cap L^1(0,\infty)$, then $\mathbb{V}f$ is the unique solution in $C_{2-\alpha}([0,\infty))$ of the problem*

$$(\mathcal{H}_f)\begin{cases} -\mathfrak{D}^\alpha \varphi(z) = f(z), \ \ z \in (0,\infty), \ 1 < \alpha \leq 2, \\ \\ \lim\limits_{z\to 0} z^{2-\alpha}\varphi(z) = 0 \text{ and } \lim\limits_{z\to\infty} z^{1-\alpha}\varphi(z) = 0. \end{cases} \tag{20}$$

Remark 2. *Assume $(A1)$ and let $\gamma(z)$ given by (11).*

Then, by Proposition 2, $\gamma \in C_{2-\alpha}([0,\infty))$ and it is the unique solution of problem $(\mathcal{H}_{\mathfrak{p}})$. Furthermore, from (16), we have

$$(\alpha-1)C_{\alpha,\mathfrak{p}} z^{\alpha-2}\min(1,z) \leq \gamma(z) \leq C_{\alpha,\mathfrak{p}} z^{\alpha-2}\max(1,z), \text{ for } z > 0, \tag{21}$$

where $C_{\alpha,\mathfrak{p}} := \frac{1}{\Gamma(\alpha)}\int_0^\infty \min(1,\zeta)\mathfrak{p}(\zeta)d\zeta$.

Lemma 2. *Let $q \in \mathcal{M}$, then*

$$\mathbb{V}(q\gamma)(z) \leq \xi_q \gamma(z), \text{ for } z > 0. \tag{22}$$

Proof. Let $q \in \mathcal{M}$, then from (19), (11), the Fubini-Tonelli theorem and (13), we obtain for $z > 0$,

$$\begin{aligned} \mathbb{V}(q\gamma)(z) &= \int_0^\infty \mathbb{G}_\alpha(z,\zeta)q(\zeta)\Big(\int_0^\infty \mathbb{G}_\alpha(\zeta,r)\mathfrak{p}(r)dr\Big)d\zeta \\ &= \int_0^\infty \mathfrak{p}(r)\Big(\int_0^\infty \mathbb{G}_\alpha(z,\zeta)\mathbb{G}_\alpha(\zeta,r)q(\zeta)d\zeta\Big)dr \\ &\leq \xi_q \int_0^\infty \mathbb{G}_\alpha(z,r)\mathfrak{p}(r)dr \\ &= \xi_q\gamma(z). \end{aligned}$$

□

Proposition 3. *Let $\alpha \in (1,2]$, $\nu < 2$ and $\mu > 1$. Set $b(r) = \dfrac{1}{r^\nu(1+r)^{\mu-\nu}}$, for $r > 0$. Then $b \in \mathcal{J}$ and*

$$\gamma_0(z) := \mathbb{V}b(z) \asymp z^{\alpha-2}\psi_\nu(\min(z,1))\phi_\mu(\max(z,1)),$$

where for $z \in (0,1]$,

$$\psi_\nu(z) = \begin{cases} z^{2-\nu} & \text{if } 1 < \nu < 2, \\ z\ln(\frac{2}{z}) & \text{if } \nu = 1, \\ z & \text{if } \nu < 1, \end{cases}$$

and for $z \geq 1$

$$\phi_\mu(z) = \begin{cases} z^{2-\mu} & \text{if } 1 < \mu < 2, \\ \ln(1+z) & \text{if } \mu = 2, \\ 1 & \text{if } \mu > 2. \end{cases}$$

Proof. From Proposition 1 (i), we have

$$\begin{aligned} z^{2-\alpha}\gamma_0(z) &\asymp \int_0^2 \min(z,\zeta)\zeta^{-\nu}d\zeta + \int_2^\infty \min(z,\zeta)\zeta^{-\mu}d\zeta \\ : &= \mathbb{I}(z) + \mathbb{J}(z). \end{aligned}$$

Case 1: Assume that $0 < z \leq 1$.

Clearly, we have

$$\mathbb{J}(z) \asymp z. \tag{23}$$

On the other hand,

$$\begin{aligned} \mathbb{I}(z) &= \int_0^z \zeta^{1-\nu} d\zeta + z \int_z^2 \zeta^{-\nu} d\zeta \\ &= \mathbb{I}_1(z) + \mathbb{I}_2(z). \end{aligned}$$

Therefore, we obtain

$$\mathbb{I}_1(z) \asymp z^{2-\nu},$$

and

$$\mathbb{I}_2(z) \asymp \begin{cases} z^{2-\nu} & \text{if } 1 < \nu < 2, \\ z \ln(\frac{2}{z}) & \text{if } \nu = 1, \\ z & \text{if } \nu < 1, \end{cases}$$

Hence, it follows that

$$\mathbb{I}(z) \asymp \begin{cases} z^{2-\nu} & \text{if } 1 < \nu < 2, \\ z \ln(\frac{2}{z}) & \text{if } \nu = 1, \\ z & \text{if } \nu < 1. \end{cases} \tag{24}$$

Combining (23) and (24), we deduce that for $0 < z \leq 1$,

$$z^{2-\alpha}\gamma_0(z) \asymp \psi_\nu(z).$$

Case 2: Assume that $z \geq 3$.

Clearly, we have

$$\mathbb{I}(z) \asymp 1. \tag{25}$$

On the other hand,

$$\begin{aligned} \mathbb{J}(z) &= \int_2^z \zeta^{1-\mu} d\zeta + z \int_z^\infty \zeta^{-\mu} d\zeta \\ &= \mathbb{J}_1(z) + \mathbb{J}_2(z). \end{aligned}$$

Therefore, we obtain

$$\mathbb{J}_2(z) \asymp z^{2-\mu},$$

and

$$\mathbb{J}_1(z) \asymp \begin{cases} z^{2-\mu} & \text{if } 1 < \mu < 2, \\ \ln(1+z) & \text{if } \mu = 2, \\ 1 & \text{if } \mu > 2. \end{cases}$$

Hence, it follows that

$$\mathbb{J}(z) \asymp \begin{cases} z^{2-\mu} & \text{if } 1 < \mu < 2, \\ \ln(1+z) & \text{if } \mu = 2, \\ 1 & \text{if } \mu > 2. \end{cases} \tag{26}$$

Combining (25) and (26), we deduce that for $z \geq 3$,

$$\begin{aligned} z^{2-\alpha}\gamma_0(z) &\asymp \begin{cases} z^{2-\mu} & \text{if } 1<\mu<2, \\ \ln(\frac{z}{2}) & \text{if } \mu=2, \\ 1 & \text{if } \mu>2. \end{cases} \\ &\asymp \begin{cases} z^{2-\mu} & \text{if } 1<\mu<2, \\ \ln(1+z) & \text{if } \mu=2, \\ 1 & \text{if } \mu>2. \end{cases} \\ &\asymp \phi_\mu(z). \end{aligned}$$

Now since the functions $z \to \phi_\mu(z)$ and $z \to z^{2-\alpha}\gamma_0(z)$ are positive and continuous on $[1,3]$, we deduce that

$$z^{2-\alpha}\gamma_0(z) \asymp \phi_\mu(z), \text{ on } [1,3].$$

Hence

$$z^{2-\alpha}\gamma_0(z) \asymp \phi_\mu(z), \text{ on } [1,\infty).$$

Finally, we obtain the result by combining the above two cases. □

3. Proof of Main Result

We recall that $\gamma(z)$ is given by (11).

Lemma 3. *Assume that $(A1)$–$(A3)$ hold. Let $\varphi \in C_{2-\alpha}([0,\infty))$ with $\varphi(z) \asymp \gamma(z)$. Then φ is a solution of problem (10) if and only if*

$$\varphi(z) = \gamma(z) + \lambda \int_0^\infty \mathbb{G}_\alpha(z,\zeta)F(\zeta,\varphi(\zeta))d\zeta, \text{ for } z>0. \tag{27}$$

Proof. Assume that φ satisfies (27).

From $(A1)$ and Remark 2, we have $\gamma(z) \in C_{2-\alpha}([0,\infty))$ and it is a solution of problem $(\mathcal{H}_{\mathfrak{p}})$ (see (20)).

By using $(A2)$ and $(A3)$, we obtain

$$\begin{aligned} |F(z,\varphi(z))| &\leq |F(z,\varphi(z)) - F(z,0)| + |F(z,0)| \\ &\leq cq(z)\gamma(z) + k\mathfrak{p}(z), \end{aligned}$$

for some constant $c>0$.

On the other hand, from (21), it follows that

$$\min(1,z)q(z)\gamma(z) \leq C_{\alpha,\mathfrak{p}} z^{\alpha-2} q(z)\min(1,z)\max(1,z) = C_{\alpha,\mathfrak{p}} z^{\alpha-1}q(z).$$

Since $\mathfrak{p} \in \mathcal{J}$ and $q \in \mathcal{M}$, we deduce that the map $z \to \min(1,z)F(z,\varphi(z))$ is integrable on $(0,\infty)$. Therefore, by Proposition 2, we conclude that $z \to v(z) := \mathbb{V}F(.,\varphi)(z) \in C_{2-\alpha}([0,\infty))$ and satisfies

$$\begin{cases} -\mathfrak{D}^\alpha v(z) = F(z,\varphi(z)), \ z \in (0,\infty), \\ \lim\limits_{z\to 0} z^{2-\alpha}v(z) = 0 \text{ and } \lim\limits_{z\to\infty} z^{1-\alpha}v(z) = 0. \end{cases}$$

So φ is a solution of problem (10).

Conversely, let $\varphi \in C_{2-\alpha}([0,\infty))$ be a solution of problem (10) with $\varphi(z) \asymp \gamma(z)$. Then $w(z) := \varphi(z) - \gamma(z) - \lambda \int_0^\infty \mathbb{G}_\alpha(z,\zeta) F(\zeta, \varphi(\zeta)) d\zeta$ satisfies

$$\begin{cases} \mathfrak{D}^\alpha w(z) = 0, \quad z \in (0,\infty), \\ \lim\limits_{z\to 0} z^{2-\alpha} w(z) = 0 \text{ and } \lim\limits_{z\to\infty} z^{1-\alpha} w(z) = 0. \end{cases}$$

From Theorem 2.4 [2],

$$w(z) = c_1 z^{\alpha-1} + c_2 z^{\alpha-2}, \text{ for some } c_1, c_2 \in \mathbb{R}.$$

Using $\lim\limits_{z\to 0} z^{2-\alpha} w(z) = 0$ and $\lim\limits_{z\to\infty} z^{1-\alpha} w(z) = 0$, we conclude that $c_2 = c_1 = 0$. That is $w(z) \equiv 0$.

Hence φ satisfies (27). □

Proof of Theorem 1. Consider the Banach space of functions

$$E = \{v \in C_{2-\alpha}([0,\infty)) : \sup_{z>0} \frac{|v(z)|}{\gamma(z)} < +\infty\}$$

with the γ-norm

$$\|v\|_\gamma = \sup_{z>0} \frac{|v(z)|}{\gamma(z)}. \tag{28}$$

Hence (E, d) is a complete metric space, with

$$d(v_1, v_2) := \|v_1 - v_2\|_\gamma.$$

Set $\lambda^* := \frac{1}{2\xi_q}$ and for $\lambda \in [0, \lambda^*)$, let Ω be the non-empty closed subset of (E, d) defined by

$$\Omega = \{v \in E,\ A\gamma(z) \le v(z) \le B\gamma(z), \text{ for } z > 0\},$$

where $A := 2\frac{(\lambda^* - \lambda(k\lambda^* + 1))}{(2\lambda^* - \lambda)}$ and $B := 2\lambda^* (\frac{1+\lambda k}{2\lambda^* - \lambda})$.

Define $\mathbb{T}$ on Ω by

$$\mathbb{T}v(z) = \gamma(z) + \lambda \int_0^\infty \mathbb{G}_\alpha(z,\zeta) F(\zeta, v(\zeta)) d\zeta, \text{ for } z > 0.$$

We claim that $\mathbb{T}$ is a contraction operator from (Ω, d) into itself.

Following the proof of Lemma 3, we conclude that for all $v \in \Omega$, $\mathbb{T}v(z) \in C_{2-\alpha}([0,\infty))$.

By using $(A1)$–$(A3)$ and Lemma 2, we obtain for all $v \in \Omega$ and $z > 0$

$$\begin{aligned} \left| \int_0^\infty \mathbb{G}_\alpha(z,\zeta) F(\zeta, v(\zeta)) d\zeta \right| &\le \int_0^\infty \mathbb{G}_\alpha(z,\zeta) |F(\zeta, v(\zeta)) - F(\zeta, 0)| d\zeta + \int_0^\infty \mathbb{G}_\alpha(z,\zeta) |F(\zeta, 0)| d\zeta \\ &\le (\frac{B}{2\lambda^*} + k)\gamma(z) \end{aligned}$$

So $\mathbb{T}(\Omega) \subset \Omega$.

Now for $v_1, v_2 \in \Omega$, by using $(A2)$–$(A3)$, $(A1)$ and Lemma 2, we obtain for $z > 0$

$$\begin{aligned} |\mathbb{T}v_1(z) - \mathbb{T}v_2(z)| &\le \lambda \int_0^\infty \mathbb{G}_\alpha(z,\zeta) |F(\zeta, v_1(\zeta)) - F(\zeta, v_2(\zeta))| d\zeta \\ &\le \lambda \int_0^\infty \mathbb{G}_\alpha(z,\zeta) q(\zeta) |v_1(\zeta) - v_2(\zeta)| d\zeta \\ &\le \lambda d(v_1, v_2) \int_0^\infty \mathbb{G}_\alpha(z,\zeta) q(\zeta) \gamma(\zeta) d\zeta \\ &\le \frac{\lambda}{2\lambda^*} \xi_q d(v_1, v_2) \gamma(z). \end{aligned}$$

Hence

$$d(\mathbb{T}v_1, \mathbb{T}v_2) \leq \frac{\lambda}{2\lambda^*} d(v_1, v_2).$$

Since $\lambda < \lambda^*$, then $\mathbb{T}$ is a contraction operator from (Ω, d) into itself. So, there exists a unique $\varphi \in \Omega$ satisfying

$$\varphi(z) = \gamma(z) + \lambda \int_0^\infty \mathbb{G}_\alpha(z,\zeta) F(\zeta, \varphi(\zeta)) d\zeta. \tag{29}$$

Due to Lemma 3, we conclude that φ is the unique solution to problem (10) satisfying (14). □

Example 1. *Let* $1 < \alpha \leq 2$ *and* $s < \alpha + 1$. *Let* $\nu < 2$, $\mu > 1$ *and* $\mathfrak{p}(z) = \dfrac{1}{z^\nu (1+z)^{\mu-\nu}}$, *for* $z > 0$.

There exists $\lambda^* > 0$ *such that for* $\lambda \in [0, \lambda^*)$, *the problem*

$$\begin{cases} -\mathfrak{D}^\alpha \varphi(z) = \mathfrak{p}(z) + \lambda z^{-s} e^{-z} \sin(z\varphi(z)),\ z > 0, \\ \lim\limits_{z \to 0^+} z^{2-\alpha} \varphi(z) = 0,\ \lim\limits_{z \to \infty} z^{1-\alpha} \varphi(z) = 0, \end{cases}$$

has a unique positive solution $\varphi \in C_{2-\alpha}([0, \infty))$ *satisfying*

$$\varphi(z) \asymp \gamma(z) \asymp z^{\alpha-2} \psi_\nu(\min(z,1)) \phi_\mu(\max(z,1)),$$

where for $z \in (0, 1]$,

$$\psi_\nu(z) = \begin{cases} z^{2-\nu} & \text{if } 1 < \nu < 2, \\ z \ln(\frac{2}{z}) & \text{if } \nu = 1, \\ z & \text{if } \nu < 1, \end{cases}$$

and for $z \geq 1$

$$\phi_\mu(z) = \begin{cases} z^{2-\mu} & \text{if } 1 < \mu < 2, \\ \ln(1+z) & \text{if } \mu = 2, \\ 1 & \text{if } \mu > 2. \end{cases}$$

Indeed, we may apply Theorem 1 (with $F(z, y) := z^{-s} e^{-z} \sin(zy)$ *and* $q(z) := z^{1-s} e^{-z}$*) and Proposition 3.*

4. Conclusions

In this paper, a semipositone fractional boundary value problems on the half-line is studied. Under sufficient conditions, we have proved the existence and uniqueness of a positive continuous solution with some global behavior. Natural phenomena in scientific and engineering areas can be described by semipositone problems. Such problem are more interesting and challenging due to the fact that the nonlinearity may change sign. The approach is based on a fixed point theorem. It will be interesting to investigate similar problems for others operators.

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