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A Quicker Iteration Method for Approximating the Fixed Point of Generalized \(\alpha\)-Reich-Suzuki Nonexpansive Mappings with Applications

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Abstract: Fixed point theory is a branch of mathematics that studies solutions that remain unchanged under a given transformation or operator, and it has numerous applications in fields such as mathematics, economics, computer science, engineering, and physics. In the present article, we offer a quicker iteration technique, the \(D^{**}\) iteration technique, for approximating fixed points in generalized \(\alpha\)-nonexpansive mappings and nearly contracted mappings. In uniformly convex Banach spaces, we develop weak and strong convergence results for the \(D^{**}\) iteration approach to the fixed points of generalized \(\alpha\)-nonexpansive mappings. In order to demonstrate the effectiveness of our recommended iteration strategy, we provide comprehensive analytical, numerical, and graphical explanations. Here, we also demonstrate the stability consequences of the new iteration technique. We approximately solve a fractional Volterra–Fredholm integro-differential problem as an application of our major findings. Our findings amend and expand upon some previously published results.

Keywords: fixed point; weak convergence; strong convergence; stability; fractional Volterra-Fredholm; integro-differential equations

MSC: 47H09; 47H10

1. Introduction

The fixed point theory is a basic field of mathematics with applications in many different scientific fields and real-world situations. It improves the identification of equilibrium states and the analysis of the strategic interactions between agents in both game theory and economics. Validating the convergence and stability of the iterative algorithms used in various numerical computations is useful in the field of computer science. It has been used in engineering as well, helping to comprehend the dynamic behavior of physical systems and aiding in the study and design of control systems. It is essential to understanding how quantum mechanical systems behave at equilibrium points and to the study of specific quantum phenomena in the field of physics. Furthermore, fixed point theory is useful for investigating continuous mappings, space transformations, and the characteristics of different operators in topology, geometry, and functional analysis. Its use in dynamical systems and differential equations makes it easier to analyze stability and long-term behavior. As in network theory and graph theory, it helps analyze the stability of the network structures and the presence of stable configurations; in mathematical economics, it helps investigate
market equilibriums and allocation difficulties. Fixed point theory is widely used in several disciplines to solve complicated issues and phenomena, demonstrating its importance in these domains. In this regard, we have added a short survey on fixed point applications in various fields (see Section 2).

Fixed point theory finds extensive applications in both theoretical and practical domains, spanning linear and nonlinear analysis, approximation theory, dynamic system theory, mathematical modeling, fractal mathematics, mathematical economics (including equilibrium problems, game theory, and optimization problems), differential equations, and integral equations. Within the framework of fixed-point theory, consider a Banach space denoted as $Y$, the set of positive integers represented by $\mathbb{Z}^+$, and the set of real numbers denoted by $\mathbb{R}$.

Let $Q$ be a nonempty subset of a Banach space. In this context, a fixed point of a self-mapping $\mathcal{F}$ defined using $Q$ is a point $j \in Q$ that satisfies the equation $j = \mathcal{F}j$. We denote the set of all fixed points of $\mathcal{F}$ as $\mathcal{F}_Q = \{ j \in Q : j = \mathcal{F}j \}$.

Furthermore, we categorize the mapping $\mathcal{F}$ as follows: $\mathcal{F}$ is referred to as a contraction if there exists $\sigma \in (0, 1)$ such that for all $j, k \in Q$, it satisfies the inequality $\| \mathcal{F}j - \mathcal{F}k \| \leq \sigma \| j - k \|$. $\mathcal{F}$ is termed nonexpansive if, for all $j, k \in Q$, it satisfies the inequality $\| \mathcal{F}j - \mathcal{F}k \| \leq \| j - k \|$. $\mathcal{F}$ is known as quasi-nonexpansive if, for all $j \in Q$ and $c \in \mathcal{F}_Q$, it satisfies the inequality $\| \mathcal{F}j - c \| \leq \| j - c \|$. The mapping $\mathcal{F} : Q \rightarrow Q$ satisfied the condition (C) if

$$\frac{1}{2} \| j - 3j \| \leq \| j - k \| \implies \| 3j - 3k \| \leq \| j - k \|, \quad \forall j, k \in Q.$$  

These definitions and concepts are fundamental within the framework of fixed point theory and have wide-ranging applications in diverse fields, contributing to the understanding and solution of various mathematical and real-world problems. In recent years, the wide-ranging applications of fixed point theory, particularly in the context of nonexpansive mappings, have ignited significant interest among researchers across various fields, including integral equations, differential equations, convex optimization, control theory, signal processing, and game theory. In the domain of Hilbert spaces, Browder’s seminal work marked a pivotal moment, as it provided the initial results regarding the existence of fixed points for nonexpansive mappings [1]. Building upon Browder’s foundation, Browder and Göhde independently extended these findings to uniformly convex Banach spaces in references [2,3]. Goebel and Kirk further expanded Browder’s results to reflexive Banach spaces, as documented in references [4,5]. Over the past two decades, the class of nonexpansive mappings has witnessed a series of insightful extensions and generalizations. Suzuki, in reference [6], introduced a crucial generalization known as Suzuki generalized nonexpansive mappings, often referred to as mappings satisfying condition (C). This development expanded the scope of nonexpansive mappings considerably. In reference [7], Aoyama and Kohsaka introduced a novel class of mappings known as $\alpha$-nonexpansive mappings, adding another layer of depth to this field. Furthermore, Pant and Shukla delved into the realm of generalized $\alpha$-nonexpansive mappings in reference [8], highlighting that this class of mappings holds broader applicability than those satisfying the criteria (C), as indicated by their research findings. These advancements in the theory of nonexpansive mappings have not only enriched the field of fixed point theory but also found practical utility in a multitude of disciplines, contributing to the ongoing evolution of mathematical and scientific knowledge.

In the domain of nonexpansive mappings, Pant and Pandey made a noteworthy contribution in their work [9] by introducing the class of Reich-Suzuki nonexpansive mappings. Their research revealed that this particular class of mappings possesses a broader scope compared to those adhering to condition (C). Furthermore, they provided compelling evidence by establishing fixed points and demonstrating the existence of these mappings, adding depth to the understanding of nonexpansive mappings.

More recently, a novel class of mappings has emerged, as documented by Pandy et al. in reference [10]. This new class of mappings is a result of a creative fusion, combining the
characteristics of generalized \( \alpha \)-nonexpansive mappings and Reich-Suzuki nonexpansive mappings. This innovative approach promises to expand the horizons of nonexpansive mapping theory, offering fresh perspectives and potential applications in various mathematical and scientific domains.

Park [11] introduced novel extensions of ordered fixed point theorems, expanding the boundaries of this theoretical framework. As documented in their research publication [12], Bakr et al. delved into various forms of cone metric spaces and explored their relevance in the realm of fixed point theory, shedding light on their practical applications. In [13], Liu et al. harnessed the power of fixed-point theory to investigate and establish novel existence theorems pertaining to solutions for a nonlinear fractional-order coupled delayed system, enriching the understanding of this complex problem. In the work presented in [14], Azm introduced a unique perspective by applying fixed point theory to triple-controlled metric-like spaces, utilizing numerical iteration techniques to pave the way for further exploration in this domain. Suwais and colleagues, as detailed in [15], conducted a comprehensive examination of fixed point theorems within symmetrically controlled M-metric type spaces, uncovering a multitude of new insights and conclusions. In their research published in [16], Omran et al. made a significant discovery by establishing Banach fixed-point theorems within the context of generalized metric spaces equipped with the Hadamard product, offering fresh perspectives on this fundamental concept. It is worth noting that a plethora of other noteworthy results related to fixed point theory and its myriad of applications can be found in references [17–25], collectively contributing to the ever-evolving landscape of fixed-point theory and its multifaceted implications.

Definition 1 ([10]). Consider a self-mapping \( Z \) defined on a nonempty subset \( Q \) within a Banach space, \( Y \). We categorize \( Z \) as a generalized \( \alpha \)-Reich-Suzuki nonexpansive mapping if there exists \( \alpha \in [0, 1) \) such that for all \( j, k \in Q \), the following occurs:

\[
\frac{1}{2} \| j - 3j \| \leq \| j - k \| \Rightarrow \| 3j - 3k \| \leq \max \{ U(j, k), V(j, k) \},
\]

where

\[
U(j, k) = \alpha \| j - 3j \| + \alpha \| k - 3k \| + (1 - 2\alpha) \| j - k \|,
\]

and

\[
V(j, k) = \alpha \| j - 3k \| + \alpha \| k - 3j \| + (1 - 2\alpha) \| j - k \|.
\]

A cornerstone result in the realm of metric spaces is the Banach contraction theorem, commonly referred to as the contraction principle. This theorem, initially formulated by Banach in 1922 [26], finds its application in the context of contraction mappings within a complete metric space, often employed in conjunction with the Picard iteration method. The Banach contraction principle ensures the existence and uniqueness of a fixed point for a given contraction mapping. However, it is worth noting that despite the guaranteed existence of fixed points for such mappings, the Picard iteration method exhibits convergence behavior towards fixed points of broader classes of mappings rather than just contraction mappings. Consequently, the Banach contraction principle does have its limitations. In order to address these limitations, numerous authors have recently devised various iterative approaches known for their simplicity and accelerated convergence rates. For comprehensive details, we refer the reader to [27–36]. Let us consider the control sequences \( g_p, h_p, \) and \( l_p \), where \( p \) is a positive integer. Within this context, we encounter well-established iterative processes often termed as follows: H. Aftab et al., S. Ishikawa, M. Aslam Noor, F. Gusay, R. P. Agaxwal et al., W. Robert Mann, U. Kfait et al., and A. Javid et al. established iterative methods to find fixed points, namely D, Ishikawa, Noor, Thakur, S, Mann, M, and F in [27,29,31,35,37–40], respectively. These iterative methods are notable for their utility in approximating fixed points and finding applications in various mathematical and scientific domains.
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proposed work for easy understanding. In Section 4, various results have been proposed.

2. A Short Survey on the Applications of Fixed Point Theory

Fixed point theory is an area of mathematics that has many applications in many
domains. Here are some prominent fixed point theory applications.

\[
\begin{align*}
\begin{cases}
  j_1 \in \mathbb{Q}, \\
  j_{p+1} = (1 - g_p)j_p + g_p3j_p, \\
  k_p = (1 - h_p)j_p + h_p3j_p, \\
  j_{p+1} = (1 - g_p)j_p + g_p3k_p.
\end{cases}
\end{align*}
\]

Equation (1)

In [27], Ali et al. show that the D iterative method (8) converges more quickly than
some of the iteration methods previously discussed, as well as numerous others in the
literature. In this regard, we established quick convergence iteration methods, namely the
D** iteration approach in Equation (9). Furthermore, we give comprehensive analytical,
numerical, and graphical explanations with previous existing iteration methods that are
defined in Equations (1)–(8).

\[
\begin{align*}
\begin{cases}
  j_1 \in \mathbb{Q}, \\
  l_p = 3((1 - h_p)j_p + h_p3j_p), \\
  k_p = 3((1 - g_p)j_p + g_p3j_p), \\
  j_{p+1} = \Theta^2k_p.
\end{cases}
\end{align*}
\]

Equation (9)

The rest of the work is arranged as follows: In Section 2, several applications of fixed
point theory are discussed. Section 3 is based on some basic terminologies related to our
proposed work for easy understanding. In Section 4, various results have been proposed
for comparison analysis and stability via our proposed iteration method. The convergence
analysis of our proposed method is analyzed in Section 5. An application is proposed for
the fractional 1-Fredholm Integro differential equation in Section 6. In Section 7, we give
the concluding remarks on our proposed work.
The asymptotic center of \( \{ p \} \) is defined as

\[
A(Q, \{ j_p \}) = \{ j \in Q : r(j, \{ j_p \}) = r(Q, \{ j_p \}) \}.
\]

It is known that in a uniformly convex Banach space, \( A(Q, \{ j_p \}) \) consists of exactly one point.

**Definition 5** ([50]). Assume the sequences \( \{ p \}_{p=1}^{\infty} \) and \( \{ q_p \}_{p=1}^{\infty} \) approximated by (09), which converge to \( c \) and \( \| p - c \| \leq u_p \) and \( \| q_p - c \| \leq v_p \), \( \forall p \geq 0 \). If \( \{ u_p \}_{p=1}^{\infty} \) and \( \{ v_p \}_{p=1}^{\infty} \) converge to \( u \) and \( v \) in an orderly way if

\[
\lim_{p \to \infty} \frac{\| u_p - u \|}{\| v_p - v \|} = 0.
\]

Thus, \( \{ p \}_{p=1}^{\infty} \) has a better convergence rate than \( \{ q_p \}_{p=1}^{\infty} \) to \( c \).
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Hence, let \( \| k - c \| \leq \| \mathcal{I}(1 - h)j + hj(j - c) - c \| \)
\[
\leq \| \mathcal{I}(1 - h)j + hj(j - c) - c \|
\leq \mathcal{E}(1 - h)\|j - c\| + \mathcal{E}h\|j(j - c)\|
\leq \mathcal{E}(1 - h)\|j - c\| + \mathcal{E}h\|j - c\|
\leq \mathcal{E}(1 - h)\|j - c\|\|j - c\|
\leq \mathcal{E}(1 - h)\|j - c\|\|j - c\|.
\]

Thus, the iterative process converges strongly to a unique fixed point in \( \mathbb{W}_3 \).

**Proof.** Let \( \mathcal{I} \) have a definite fixed point in \( Q \) since it is a contraction mapping in a Banach space, and assume that \( c \) is a fixed point. Now, by definition,
\[
\| k - c \| = \mathcal{E}(1 - g)\|j(j - c) + g\|j(j - c) - c \|
\leq \mathcal{E}(1 - g)\|j - c\|\|j - c\|
\leq \mathcal{E}(1 - g)\|j - c\|\|j - c\|.
\]

Hence,
\[
\| k - c \| = \mathcal{E}(1 - g)\|j(j - c) + g\|j(j - c) - c \|
\leq \mathcal{E}(1 - g)\|j - c\|\|j - c\|
\leq \mathcal{E}(1 - g)\|j - c\|\|j - c\|.
\]
Then,
\[
\|j_{p+1} - c\| = \|\mathcal{G}^2 k_p - 3c\|
\]
\[
= \|3(3k_p) - 3c\|
\]
\[
\leq \varepsilon \|3k_p - c\|
\]
\[
\leq \varepsilon^2 \|k_p - c\|
\]
\[
\leq \varepsilon^4 [1 - (g_p + \varepsilon g_p h_p)(1 - \varepsilon)] \| (j_p - c) \|
\]
After some repetition, we have
\[
\|j_p - c\| \leq \varepsilon^4 [1 - (g_p + \varepsilon g_p h_p)(1 - \varepsilon)] \| (j_1 - c) \|
\]
\[
\|j_{p-1} - c\| \leq \varepsilon^4 [1 - (g_{p-1} + \varepsilon g_{p-1} h_{p-1})(1 - \varepsilon)] \| (j_p - c) \|
\]
\[
\|j_{p-2} - c\| \leq \varepsilon^4 [1 - (g_{p-2} + \varepsilon g_{p-2} h_{p-2})(1 - \varepsilon)] \| (j_{p-1} - c) \|
\]
\[
\vdots
\]
\[
\|j_2 - c\| \leq \varepsilon^4 [1 - (g_1 + \varepsilon g_1 h_1)(1 - \varepsilon)] \| (j_1 - c) \|
\]
Therefore,
\[
\|j_{p+2} - c\| \leq \varepsilon^4 (p+2) \| (j_1 - c) \| \prod_{t=1}^{p} [1 - (g_t + \varepsilon g_t h_t)(1 - \varepsilon)].
\] (A)

where, \([1 - (g_p + \varepsilon g_p h_p)(1 - \varepsilon)]) \in (0, 1), \forall p \in Z^+\) (because, \(p, g_p, h_p \in (0, 1))\). Since, \(1 - j \leq e^{-j}\), for all \(j \in [0, 1]\). By using relation (A), we have
\[
\|j_{p+2} - c\| \leq \varepsilon^4 (p+2) \| (j_1 - c) \| e^{-(1-\varepsilon)} \sum_{t=0}^{p} \{g_t + \varepsilon g_t h_t\}.
\]

By taking the \(\lim_{p \to \infty}\) for both sides, we get \(\lim_{p \to \infty} \| j_p - c \| = 0\). After that, the \(D^{**}\) iteration approach is then analytically compared to a few existing iteration strategies. \(\square\)

**Theorem 2.** Let \(Y, Q, Z\), and \(\{j_p\}\) be as stated by Theorem 1, considering \(j_1 = d_1\). Let \(\{j_p\}_{p=1}^{\infty}\) and \(\{d_p\}_{p=1}^{\infty}\) be approximated by Equation (9) and Thakur [35]. Then, \(D^{**}\) converges to point \(c\) more quickly than Thakur [35] in \(\mathfrak{B}_3\).

**Proof.** By using Theorem 1 from [35], we have
\[
\|d_{p+2} - c\| \leq \varepsilon^2 (p+2) \| (d_1 - c) \| \prod_{t=1}^{p} [1 - gh(1 - \varepsilon)].
\]

Since \(g \leq g_p, \forall p \in Z^+\), we have \(\|d_{p+2} - c\| \leq \varepsilon^2 (p+2) \| (d_1 - c) \| [1 - gh(1 - \varepsilon)]^{p+2}\). Let \(u_p \varepsilon^2 (p+2) \| (d_1 - c) \| [1 - gh(1 - \varepsilon)]^{p+2}\). Now, by using Equation (A),
\[
\|j_{p+2} - c\| \leq \varepsilon^4 (p+2) \| (j_1 - c) \| \prod_{t=1}^{p} [1 - (g_t + \varepsilon g_t h_t)(1 - \varepsilon)].
\]
Again, on the condition that \(g \leq g_p \forall p \in Z^+,\)
\[
\|j_{p+2} - c\| \leq \varepsilon^4 (p+2) \| (j_1 - c) \| [1 - (g + \varepsilon gh)(1 - \varepsilon)]^{p+2}.
\]
Let \( v_p = q^{4(p+2)}||(j_1 - c)||[1 - (g + qgh)(1 - q)]^{p+2} \). Then,

\[
\frac{v_p}{u_p} = \frac{q^{4(p+2)}||(j_1 - c)||[1 - (g + qgh)(1 - q)]^{p+2}}{q^{2(p+2)}||(d_1 - c)||[1 - gh(1 - q)]^{p+2}}.
\]

Thus, taking the limit to be \( p \to \infty \lim_{p \to \infty} \frac{v_p}{u_p} = 0 \). Thus, the result under Definition 5 holds.

**Theorem 3.** Let \( Y, Q, 3, \) and \( \{ j_p \} \) behave as stated by Theorem 1, considering \( j_1 = d_1 \). Let \( \{ j_p \}_{p=1}^{\infty} \) and \( \{ d_p \}_{p=1}^{\infty} \) be approximated by Equation (9) and M [39]. Then, \( \{ j_p \} \) converges to point \( c \in B_3 \) more quickly than \( \{ d_p \} \).

**Proof.** By using Equation (A), we have

\[
||j_{p+2} - c|| \leq q^{4(p+2)}||(j_1 - c)||\prod_{i=1}^{p}||1 - (g_i + qgh_1)(1 - q)||
\]

Again, on the condition that \( g \leq g_p \forall p \in Z^+ \),

\[
||j_{p+2} - c|| \leq q^{4(p+2)}||(j_1 - c)||[1 - (g + qgh)(1 - q)]^{p+2}.
\]

Let \( u_p = q^{4(p+2)}||(j_1 - c)||[1 - (g + qgh)(1 - q)]^{p+2} \). Now, We define the M iteration:

\[
\begin{align*}
l_p &= (1 - g_p)j_p + g_p3j_p, \\
k_p &= 3l_p, \\
j_{p+1} &= 3k_p.
\end{align*}
\]

\[
||l_p - c|| = ||(1 - g_p)j_p + g_p3j_p - 3c||
\]

\[
\leq ||(1 - g_p)j_p + g_p3j_p - c||
\]

\[
\leq ||(1 - g_p)(j_p - c) + g_p(3j_p - c)||
\]

\[
\leq (1 - g_p)||j_p - c|| + g_p||3j_p - c||
\]

\[
\leq \{ (1 - g_p)||j_p - c|| + qgh_p||j_p - c|| \}
\]

\[
\leq q\{1 - g_p(1 - q)\}||j_p - c||.
\]

Now,

\[
||k_p - c|| \leq 3||l_p - c|| \leq q||l_p - c|| \leq q\{1 - g_p(1 - q)\}||j_p - c||.
\]

Therefore,

\[
||j_{p+1} - c|| \leq 3||k_p - c|| \leq q||k_p - c|| \leq q^2\{1 - g_p(1 - q)\}||j_p - c||.
\]

After some repetition,

\[
\begin{align*}
||j_p - c|| &\leq q^2\{1 - g_{p-1}(1 - q)\}||j_{p-1} - c|| \\
||j_{p-1} - c|| &\leq q^2\{1 - g_{p-2}(1 - q)\}||j_{p-2} - c|| \\
||j_{p-2} - c|| &\leq q^2\{1 - g_{p-3}(1 - q)\}||j_{p-3} - c|| \\
&\vdots \\
||j_2 - c|| &\leq q^2\{1 - g_1(1 - q)\}||j_1 - c||.
\end{align*}
\]
Thus,
\[ \|j_{p+2} - c\| \leq \epsilon^2(p+2) \| (j_1 - c) \| \prod_{i=1}^p \{1 - g_i(1 - \epsilon)\}. \]

Now, since \( g \leq g_p, \ \forall p \in \mathbb{Z}^+ \), we obtain
\[ \|j_{p+1} - c\| \leq \epsilon^2(p+2) \| (j_1 - c) \| \{1 - g(1 - \epsilon)\}^{p+2}. \]

As \( \{d_p\}_{p=0}^\infty \) are sequences generated by the M iterative method, we have
\[ \|d_{p+1} - c\| \leq \epsilon^2(p+2) \| (j_1 - c) \| \{1 - g(1 - \epsilon)\}^{p+2}. \]

Let us define
\[ v_p = \epsilon^2(p+2) \| (j_1 - c) \| \{1 - g(1 - \epsilon)\}^{p+2}. \]

Then,
\[ \frac{u_p}{v_p} = \frac{\epsilon^{4(p+2)} \| (j_1 - c) \| 1 - (g + \epsilon g h)(1 - \epsilon) \|^{p+2}}{\epsilon^{2(p+2)} \| (j_1 - c) \| \{1 - g(1 - \epsilon)\}^{p+2}}. \]

Thus, \( \lim_{t \to \infty} \frac{u_p}{v_p} = 0 \). The finding follows the definition in (5). \( \square \)

**Theorem 4.** Let \( Y, Q, 3, \) and \( \{j_p\} \) behave as stated by Theorem 1, considering \( j_1 = d_1 \). Let \( \{j_p\}_{p=1}^\infty \) and \( \{d_p\}_{p=1}^\infty \) be approximated by the newly designed iterative process defined in Equation (9) and \( F [40] \). Then, the convergence of \( F [35] \) follows \( D^{*} \).

**Proof.** According to Theorem 3.1 from [40],
\[ \|d_{p+2} - c\| \leq \epsilon^3(p+2) \| (d_1 - c) \| \prod_{i=1}^p \{1 - g_t(1 - \epsilon)\}. \]

Since \( g \leq g_p, \forall p \in \mathbb{Z}^+ \), we have
\[ \|d_{p+2} - c\| \leq \epsilon^3(p+2) \| (d_1 - c) \| \{1 - g(1 - \epsilon)\}^{p+2}. \]

Let \( u_p = \epsilon^3(p+2) \| (d_1 - c) \| \{1 - g(1 - \epsilon)\}^{p+2} \). Now, by using Equation (A),
\[ \|j_{p+2} - c\| \leq \epsilon^4(p+2) \| (j_1 - c) \| \prod_{i=1}^p \{1 - (g_t + \epsilon g h_t)(1 - \epsilon)\}. \]

Again, on the condition that \( g \leq g_p, \forall p \in \mathbb{Z}^+ \),
\[ \|j_{p+2} - c\| \leq \epsilon^4(p+2) \| (j_1 - c) \| \{1 - (g + \epsilon g h)(1 - \epsilon)\}^{p+2}. \]

Let \( v_p = \epsilon^4(p+2) \| (j_1 - c) \| \{1 - (g + \epsilon g h)(1 - \epsilon)\}^{p+2} \). Then,
\[ \frac{v_p}{u_p} = \frac{\epsilon^{4(p+2)} \| (j_1 - c) \| \{1 - (g + \epsilon g h)(1 - \epsilon)\}^{p+2}}{\epsilon^{3(p+2)} \| (d_1 - c) \| \{1 - g(1 - \epsilon)\}^{p+2}}. \]

Thus, the limit is \( p \to \infty \lim_{p \to \infty} \frac{v_p}{u_p} = 0 \). Therefore, \( \{j_p\} \) converges more quickly than \( \{d_p\} \). \( \square \)

**Theorem 5.** Let \( Y, Q, 3, \) and \( \{j_p\} \) behave as stated by Theorem 1, considering \( j_1 = d_1 \). Let \( \{j_p\}_{p=1}^\infty \) and \( \{d_p\}_{p=1}^\infty \) be approximated as per Equation (9) and \( D [27] \). Then, \( \{j_p\} \) converges to point \( c \in \mathcal{B}_3 \) more quickly than \( \{d_p\} \).
Proof. According to Equation (18) of Theorem 6 from [27], we have
\[
\|d_{p+2} - c\| \leq q^{3(p+2)} \| (d_1 - c) \| \prod_{t=1}^{p}(1 - (g_t + egh_t)(1 - q)).
\]
Since \( g \leq g_p \forall p \in Z^+ \), we have
\[
\|d_{p+2} - c\| \leq q^{2(p+2)} \| (d_1 - c) \| [1 - (gh(1 - q))]^{p+2}.
\]
Let \( u_p = q^{3(p+2)} \| (d_1 - c) \| [1 - (gh(1 - q))]^{p+2} \). Now, by using Equation (A),
\[
\|j_{p+2} - c\| \leq q^{3(p+2)} \| (j_1 - c) \| \prod_{t=1}^{p}(1 - (g_t + egh_t)(1 - q)).
\]
Again, on the condition that \( g \leq g_p \forall p \in Z^+ \),
\[
\|j_{p+2} - c\| \leq q^{4(p+2)} \| (j_1 - c) \| [1 - (gh(1 - q))]^{p+2}.
\]
Let \( v_p = q^{4(p+2)} \| (j_1 - c) \| [1 - (gh(1 - q))]^{p+2} \). Then,
\[
\frac{v_p}{u_p} = \frac{q^{4(p+2)} \| (j_1 - c) \| [1 - (gh(1 - q))]^{p+2}}{q^{3(p+2)} \| (d_1 - c) \| \prod_{t=1}^{p}(1 - (g_t + egh_t)(1 - q))} = q^{p+2}.
\]
Thus, \( \lim_{p \to \infty} \frac{v_p}{u_p} = 0 \), as \( p \to \infty \). Hence, \( D^{**} \) has a better convergence rate than \( D \) [27].

Theorem 6. Let \( Y, Q, 3, \) and \( \{j_p\} \) behave as stated in Theorem 1. The iterative process \( D^{**} \) is then stable.

Proof. Let \( \{j_p\} \) be a sequence in \( Q \). Consider the following: suppose the sequence produced by the new iteration process is defined by \( j_{p+2} = f(3, j_{p+1}) \) converging to the unique fixed point \( c \in 3 \). Set \( \epsilon_p = \|j_{p+2} - f(3, j_{p+1})\| \). Here, it can be shown that \( \lim_{p \to \infty} \epsilon_p = 0 \) if and only if \( \lim_{p \to \infty} j_{p+1} = c \). Let \( \lim_{p \to \infty} \epsilon_p = 0 \). Then, we have
\[
\|j_{p+2} - c\| \leq \|j_{p+2} - f(3, j_{p+1})\| + \|f(3, j_{p+1}) - c\| = \epsilon_p + \|j_{p+2} - c\|.
\]
According to Equation (A),
\[
\|j_{p+2} - c\| \leq q^{4(p+2)} \| (j_1 - c) \| [1 - (gh(1 - q))]^{p+2}.
\]
Since \( 0 < p < 1 \), and \( \lim_{p \to \infty} \epsilon_p = 0 \), then, according to Definition 6, we get \( \lim_{p \to \infty} \|j_p - c\| = 0 \). Hence, \( \lim_{p \to \infty} j_p = c \).

Conversely, let \( \lim_{p \to \infty} j_p = c \). Therefore,
\[
\epsilon_p = \|j_{p+2} - f(3, j_{p+1})\| \\
\leq \|j_{p+2} - c\| + \|f(3, j_{p+1}) - c\| \\
\leq \|j_{p+2} - c\| + q^{4(p+2)} \| (j_1 - c) \| [1 - (gh(1 - q))]^{p+2}.
\]
Hence, \( \lim_{p \to \infty} \epsilon_p = 0 \). Hence, by Definition 6, \( D^{**} \) iteration process is stable.

5. Convergence Analysis

Within the domain of uniformly convex Banach spaces, this section is dedicated to exploring both weak and strong convergence outcomes concerning the fixed points of generalized \( a \)-Reich-Suzuki nonexpansive mappings.
Theorem 7. Let $Y, Q, \mathcal{Z}$, and $j_p$ behave as is assumed in Theorem 1. If $\mathcal{B}_3 \neq \emptyset$ and $\mathcal{Z} : Q \to Q$ are generalized $\alpha$-Reich-Suzuki nonexpansive mappings, then $\lim_{p \to \infty} \| j_p - c \|$ exists, and $\forall c \in \mathcal{B}_3$.

Proof. Assume that $c \in \mathcal{B}_3$ and $j \in Q$. Given that $\mathcal{B}_3 \neq \emptyset$ and $\mathcal{Z}$ are generalized $\alpha$-Reich-Suzuki nonexpansive mappings, we then have

$$\| \mathcal{B}_3 j - \mathcal{B}_3 c \| \leq \| j - c \|.$$ 

Now, by applying the iterative method that is defined in Equation (9),

$$\| j_p - c \| = \| \mathcal{Z}(1 - h_p) j_p + h_p \mathcal{Z} j_p - c \|$$

$$\leq \| (1 - h_p) j_p + h_p \mathcal{Z} j_p - c \|$$

$$\leq \| (1 - h_p) (j_p - c) + h_p (\mathcal{Z} j_p - c) \|$$

$$\leq (1 - h_p) \| (j_p - c) \| + h_p \| (\mathcal{Z} j_p - c) \|$$

$$= \| (j_p - c) \|.$$ 

Hence,

$$\| k_p - c \| = \| \mathcal{Z}(1 - g_p) \mathcal{Z} j_p + g_p \mathcal{Z} j_p - c \|$$

$$\leq \| (1 - g_p) \mathcal{Z} j_p + g_p \mathcal{Z} j_p - c \|$$

$$\leq \| (1 - g_p) (j_p - c) + g_p (l_p - c) \|$$

$$\leq (1 - g_p) \| (j_p - c) \| + g_p \| (l_p - c) \|$$

$$= \| (j_p - c) \|.$$ 

Then,

$$\| j_{p+1} - c \| = \| \mathcal{S}^* k_p - c \|$$

$$= \| \mathcal{Z} k_p - c \|$$

$$\leq \| k_p - c \|$$

$$\leq \| (j_p - c) \|.$$ 

Thus, the sequence $\{ \| (j_p - c) \| \}$ does not increase and is bounded below. Hence, $\lim_{p \to \infty} \| j_p - c \|$ exists $\forall c \in \mathcal{B}_3$. \qed

Theorem 8. Let $Y, Q, \mathcal{Z}$, and $\{ j_p \}$ behave as is assumed in Theorem 1. Then, $\mathcal{B}_3 \neq \emptyset$ if and only if the sequence $\{ j_p \}$ is bounded and $\lim_{p \to \infty} \| j_p - \mathcal{Z} j_p \| = 0$.

Proof. According to Theorem 6, $\lim_{p \to \infty} \| j_p - c \|$ exists $\forall c \in \mathcal{B}_3$ and $\{ j_p \}$ is bounded. Let $\lim_{p \to \infty} \| j_p - c \| = q$. Now, by combining the above inequality and Theorem 6, $\lim_{p \to \infty} \sup\| I_p - c \| \leq \lim_{p \to \infty} \sup\| I_p - c \| = q$. Additionally,

$$\lim_{p \to \infty} \sup\| j_p - c \| \leq \lim_{p \to \infty} \sup\| j_p - c \| = q.$$
Now, according to the definition of our proposed iteration process,

\[ \|y_{p+1} - c\| = \|3^2 k_p - c\| \]
\[ = \|3(3k_p) - c\| \]
\[ \leq \|3k_p - c\| \]
\[ \leq \|k_p - c\| \]
\[ = \|3(3j_p + g_p 3l_p) - c\| \]
\[ \leq (1 - g_p)\|3j_p - c\| + g_p\|3l_p - c\| \]
\[ \leq (1 - g_p)\|j_p - c\| + g_p\|l_p - c\| \]
\[ \leq (1 - g_p)\|j_p - c\| + g_p\|l_p - c\| \]
\[ \leq \|j_p - c\| - g_p\|j_p - c\| + g_p\|l_p - c\| \]

Thus, \( \frac{\|y_{p+1} - c\| - \|y_p - c\|}{g_p} \leq \|l_p - c\| - \|j_p - c\| \). Since,

\[ \|j_{p+1} - c\| - \|j_p - c\| \leq \frac{\|y_{p+1} - c\| - \|y_p - c\|}{g_p} \leq \|l_p - c\| - \|j_p - c\| \]

Hence, \( \|j_{p+1} - c\| \leq \|l_p - c\| \). Therefore,

\[ q \leq \lim_{p \to \infty} \inf \|l_p - c\| \]

So, with the help of the above inequalities,

\[ q \lim_{p \to \infty} \inf \|l_p - c\| = \|3(1 - h_p)^2 j_p + h_p 3j_p - c\| \]
\[ \leq \| (1 - h_p)^2 j_p + h_p 3j_p - c\| \]
\[ \leq \| (1 - h_p) (j_p - c) + h_p (3j_p - c)\| \]
\[ \leq (1 - h_p) \| (j_p - c)\| + h_p \| (3j_p - c)\| \]
\[ \leq \| (1 - h_p) (j_p - c)\| + h_p \| (j_p - c)\| \]
\[ \leq q \]

Hence,

\[ \lim_{p \to \infty} \|j_p - 3j_p\| = 0 \]

Conversely, if \( \{j_p\} \) is bounded and \( \lim_{p \to \infty} \|j_p - 3j_p\| = 0 \), then \( c \in A(Q, \{j_p\}) \). Then, by using Lemma 1, we get

\[ r(3c, \{j_p\}) = \lim_{p \to \infty} \sup \|3j_p - c\| \]
\[ \leq \lim_{p \to \infty} \sup \left( \frac{3 + q}{1 - q} \right) \|j - 3j\| + \lim_{p \to \infty} \sup \|3j_p - c\| \]
\[ = \lim_{p \to \infty} \sup \|j_p - c\| \]
\[ = r(c, \{j_p\}) \]

Hence, \( 3c \in A(Q, \{j_p\}) \). Since \( Y \) is a uniformly convex Banach space, we now know that \( A(Q, \{j_p\}) \) is a set with only one element; thus, it stands to reason that \( 3c = c \). Thus, \( \mathcal{B}_3 \neq \emptyset \). \( \square \)

**Theorem 9.** Let \( Y, Q, 3, \) and \( \{j_p\} \) behave as is assumed in Theorem 1, such that \( \mathcal{B}_3 \neq \emptyset \). Consider that \( Y \) satisfies Opial’s condition; then, \( \{j_p\} \) is approximated by Equation (9) and converges weakly to \( c \in \mathcal{B}_3 \).
Proof. We have proven in Theorems 6 and 7 that \( \lim_{p \to \infty} \| j_p - c \| \) exists for \( \mathfrak{B}_3 \neq \emptyset \), and also \( \lim_{p \to \infty} \| j_p - 3 j_p \| = 0 \). We are going to now demonstrate that \( \{ j_p \} \) cannot have two weak subsequentials in \( \mathfrak{B}_3 \). Assume that \( c_1 \) and \( c_2 \) are the respective weak subsequential limits of \( \{ j_{p_k} \} \) and \( \{ j_{p_l} \} \). Then, as per Theorem 6, \( Z_{c_1} = c_1 \) and \( Z_{c_2} = c_2 \). We next demonstrate uniqueness. Consider \( c_1 \neq c_2 \), according to Opial's property:

\[
\lim_{p \to \infty} \| j_p - c_1 \| = \lim_{p \to \infty} \| j_{p_k} - c_1 \| < \lim_{p \to \infty} \| j_{p_k} - c_2 \| = \lim_{p \to \infty} \| j_p - c_2 \| = \lim_{p \to \infty} \| j_{p_l} - c_2 \| = \lim_{p \to \infty} \| j_{p_l} - c_1 \| = \lim_{p \to \infty} \| j_p - c_1 \| .
\]

which is a contradiction. So, \( c_1 = c_2 \). Thus, \( \{ j_p \} \) weakly converges to \( c \in \mathfrak{B}_3 \). □

Theorem 10. Let \( Y, Q, 3, \) and \( \{ j_p \} \) behave as is assumed in Theorem 1 such that \( \mathfrak{B}_3 \neq \emptyset \). Then, \( \{ j_p \} \) is approximated by (9) and converges strongly to a point in \( \mathfrak{B}_3 \).

Proof. We have proven that in Theorem 7, \( \lim_{p \to \infty} \| j_p - 3 j_p \| = 0 \). As \( Q \) is compact, then \( \{ j_{p_k} \} \) with limit \( b \) is a strongly convergent subsequence of \( \{ j_p \} \). With the help of Theorem 7, \( \| \{ j_{p_k} - 3b \} \| \leq (3 + 1) \| \{ j_{p_k} - 3 \{ j_{p_k} \} \| + \| 3 \{ j_{p_k} - b \} \| . As \( j_{p_k} \to 3b \) if \( p \to \infty \), then \( 3b = b \), where \( b \in \mathfrak{B}_3 \). Additionally, according to Theorem 6, \( \lim_{p \to \infty} \| j_p - b \| \) exists. Hence, \( b \) is a strong limit point in \( \mathfrak{B}_3 \). □

Next, by using a numerical example, we contrast the results of our iteration process computation with some known iteration processes using both tables and diagrams. The tables and graphs demonstrate the effectiveness of our iteration process.

Example 1. Let the self-contraction map \( 3 : \mathbb{R} \to \mathbb{R} \) be defined as \( 3(j) = (3j + 10)^{1/2} \). Let \( g_p = h_p = i_p = \frac{1}{2} \). The initial value \( j_1 = 3.1 \) is present in Table 1 and is graphically represented in Figure 1. This demonstrates the strong convergence of the iteration processes. The effectiveness of the suggested iteration process is demonstrated.

The numerical and graphical comparison between the Maan, Ishikawa, F, D, and \( D^{**} \) irititative methods are given in Table 1 and Figure 1, respectively.
Table 1. Numerical comparison between the Maan, Ishikawa, F, D, and $D^{**}$ irritative methods.

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Figure 1. Graphical compression between the Maan, Ishikawa, F, D, and $D^{**}$ irritative methods.

The numerical and graphical comparison between the Noor, S, Thukar, M, and $D^{**}$ irritative methods are given in Table 2 and Figure 2, respectively.
Table 2. Numerical comparison between the Noor, S, Thukar, M, and $D^{**}$ irritative methods.

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Figure 2. Graphical compression between the Noor, S, Thukar, M, and $D^{**}$ irritative methods.

6. Application to Fractional 1-Fredholm Integro Differential Equation

In a wide variety of engineering and applied scientific domains, fractional differential equations continue to play an important role as key modeling tools. It is well known that fractional models provide increased safety when compared to standard methods. In general, fractional differential equations may be used in various domains, such as physics, image processing, aerodynamics, economics, blood flow phenomena, and many more (see [53] and the references therein for further information and examples). The body of known research [54] includes examples of fractional calculus in a broad number of contexts. These fractional differential equations have been solved by using a wide range of analytical and numerical methods, as shown in [55]. Analytical methods are notoriously difficult to use when solving nonlinear fractional differential equations [56]. Now, using our iterative approach (9), we will solve a nonlinear fractional order differential equation.

Definition 7. According to Caputo, the fractional derivative of $f(t)$ is defined by

$$\frac{CD_t^m f(t)}{\Gamma(m-n)} = \frac{1}{\Gamma(m-n)} \int_0^t f^{(n)}(q)(t-q)^{m-1-n}dq, \quad (n-1 < m < n),$$
where the order of the derivation is denoted by \( m \), which may take on real or complex values, subject to the condition that \( \Re m > 0 \).

Here, we take into account a Volterra-Fredholm nonlinear fractional integro-differential equation:

\[
\frac{CD^m}{g(t)} = c(t)g(t) + r(t) + \int_0^t W_1(t, z)K_1(g(z))dz + \int_0^1 W_2(t, z)K_2(g(z))dz, 
\]

with starting point

\[
s'(0) = \eta_j, i = 0, 1, 2, \ldots n - 1. 
\]

where \( \frac{CD^m}{g} \) represents the Caputo fractional derivative, with \( n - 1 < m \leq n \) and \( m \in \mathbb{Z}^+ \), and where \( g : \mathbb{B} \rightarrow \mathbb{R} \) denotes an unknown continuous function over the interval \( \mathbb{B} = [0, 1] \), \( u : \mathbb{B} \rightarrow \mathbb{R} \) is a continuous function, and \( W_i : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{R} \) are also continuous functions; we introduce Lipschitz continuous functions \( K_i : \mathbb{R} \rightarrow \mathbb{R} \), with \( i = 1, 2 \). Now, let us consider the examination of the following hypotheses:

**Hypothesis 1** (H1). There are two constants, \( L_{K_1} \) and \( L_{K_2} \), that exist such that for every \( g_1 \) and \( g_2 \in C(\mathbb{B}, \mathbb{R}) \), we have

\[
|K_1(g_1(t)) - K_1(g_2(t))| \leq L_{K_1}|g_1 - g_2| 
\]

and

\[
|K_2(g_1(t)) - K_2(g_2(t))| \leq L_{K_2}|g_1 - g_2|. 
\]

**Hypothesis 2** (H2). \( \exists \) two functions \( W_1^* \), \( W_2^* \in C(\mathbb{D}, \mathbb{R}^+) \), \( \forall \) non-negative functions is continuous for \( \mathbb{D} = (t, v) \in \mathbb{R} \times \mathbb{R} : 0 \leq v \leq t \leq 1 \) such that \( W_1^* = \sup_{p, t \in [0,1]} \int_0^t |W_1(t, v)|dt < \infty, W_2^* = \sup_{t, v \in [0,1]} \int_0^1 |W_2(t, v)|dt < \infty \).

**Hypothesis 3** (H3). The function \( c, r : \mathbb{B} \rightarrow \mathbb{R} \) is continuous.

**Hypothesis 4** (H4). \( \frac{\|c\|_{\infty} + W_1^* L_{K_1} + W_2^* L_{K_2}}{(m+1)} < 1 \). If a function \( j^* \in (\mathbb{B}, \mathbb{R}) \) satisfies Equations (11) and (12), and it is regarded as the initial value problem’s solution. Finding the answers to (13) and (14) for \( j_0(u) \in (\mathbb{B}, \mathbb{R}) \) is equal to figuring out the answer to the following integral equation [57]:

\[
\begin{cases}
  j(t) = j_0 + \frac{1}{\Gamma(m)} \int_0^t (t - b)^{m-1} c(b)j(b)db + \frac{1}{\Gamma(m)} \int_0^1 (t - b)^{m-1} r(b)db \\
  +(\int_0^t W_1(b, \tau)K_1(j(\tau))d\tau) + \int_0^1 W_2(b, \tau)K_2(j(\tau))d\tau \bigg|_{t=0}^{t=1}
\end{cases}
\]

for each \( t \in Q \) and \( j_0 = \sum_{k=0}^{\infty} f^{(0)}(0^+)^{k} \). Under the Hypotheses (H1)-(H4), the solutions to problems (11) and (12) are unique. Here, we use an iteration process which is defined in Equation (9) to approximatively solve (11) and (12).

**Theorem 11.** Let \( \mathbb{Z} = (\mathbb{B}, \mathbb{R}) \) be a Banach space with the Chebyshev norm \( ||f - h||_\infty = \max_{t \in \mathbb{Z}} |f(y) - h(y)| \). Let \( j_p \) be the iteration process (10) for the self-map \( 3 : Q \rightarrow Q \), which is defined by

\[
\begin{cases}
  j(t) = j_0 + \frac{1}{\Gamma(m)} \int_0^t (t - b)^{m-1} c(b)j(b)db + \frac{1}{\Gamma(m)} \int_0^1 (t - b)^{m-1} r(b)db \\
  +(\int_0^t W_1(b, \tau)K_1(j(\tau))d\tau) + \int_0^1 W_2(b, \tau)K_2(j(\tau))d\tau \bigg|_{t=0}^{t=1}
\end{cases}
\]

Equations (11) and (12) get a unique solution if Hypotheses (H1)-(H4) are valid. The \( j^* \in (\mathbb{B}, \mathbb{R}) \), and \( D^m \) iteration processes approach \( j^* \).
Proof. The presence of a distinctive solution from [57] \( j^* \) follows. If \( j^* \) is a fixed point of \( \exists \in (B, \mathbb{R}) \), then \( j^* \) is an answer to (11) and (12). Now, we demonstrate that the proposed iteration process converges to \( j^* \). Firstly, we prove that the self-map defined by (9) is a contraction. By using Hypotheses (H1)–(H4),

\[
|3j(t) - 3j^*(t)| \leq \frac{1}{\Gamma(m)} \int_0^t (t - b)^{m-1} |c(b)| |j(b) - j^*(b)| + \frac{1}{\Gamma(m)} \int_0^b (t - b)^{m-1} \left\{ \int_0^b |W_1(b, \tau)||K_1(j(\tau)) - K_1(j^*(\tau))|d\tau \right\} db + \int_0^b |W_2(b, \tau)||K_2(j(\tau))K_2(j^*(\tau))|d\tau \right\} db 
\leq \left( \frac{||c||_{\infty} + W_1L_{K_1} + W_2L_{K_2}}{\Gamma(m+1)} \right) |j - j^*|.
\]

Therefore,

\[
|3j(t) - 3j^*(t)| \leq \left( \frac{||c||_{\infty} + W_1L_{K_1} + W_2L_{K_2}}{\Gamma(m+1)} \right) |j - j^*|.
\] (15)

According to Hypothesis (H4), we have \( \left( \frac{||c||_{\infty} + W_1L_{K_1} + W_2L_{K_2}}{\Gamma(m+1)} \right) < 1 \).

If we take \( \lambda = \left( \frac{||c||_{\infty} + W_1L_{K_1} + W_2L_{K_2}}{\Gamma(m+1)} \right) \), then (15) can be written as

\[
|3j(t) - 3j^*(t)| \leq \lambda |j - j^*|.
\]

Hence, \( F \) is a contraction mapping. Thus, according to Theorem 1, the suggested iteration process \( j_p \) defined in (9) is strongly convergent to a unique solution of (11) and (12). \( \square \)

7. Conclusions

Fixed point theory is a mathematical discipline with wide-ranging applications across mathematics, physics, engineering, economics, and computer science. It is a robust framework for understanding the existence and characteristics of solutions to various equations and problems. Among the mathematical equations encountered in diverse applications, fractional 1-Fredholm integro-differential equations stand out. These equations involve fractional derivatives and integrals, along with a Fredholm integral operator, making them pertinent in mathematical and engineering contexts, especially when modeling phenomena characterized by memory effects and long-range interactions. In numerous instances, fixed point theory proves invaluable for the analysis and resolution of fractional 1-Fredholm integro-differential equations. In the proposed work, we introduce the \( D^{**} \) iteration method as a valuable tool for predicting fixed points within the domain of generalized \( \alpha \)-nonexpansive mappings and nearly contracted mappings. We have given a comprehensive examination of the convergence properties of the \( D^{**} \) iteration scheme when applied to the fixed points of generalized \( \alpha \)-nonexpansive mappings in uniformly convex Banach spaces. In order to underscore the effectiveness of our proposed iteration technique, we furnish detailed analytical, numerical, and graphical demonstrations. Furthermore, we explore the stability implications arising from this novel iteration strategy. As a practical application of our key findings, we utilize our approach to approximate solutions for a fractional Volterra-Fredholm integro-differential problem. Our results extend and refine several previously established conclusions, adding depth and precision to the body of knowledge in this field. In future work, anyone can develop a more generalized iterative approach that is better than our proposed iterative method for finding the fixed points.
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