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Temporal Fractal Nature of the Time-Fractional SPIDEs and Their Gradient

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Abstract: Fractional and high-order PDEs have become prominent in theory and in the modeling of many phenomena. In this article, we study the temporal fractal nature for fourth-order time-fractional stochastic partial integro-differential equations (TFSPIDEs) and their gradients, which are driven in one-to-three dimensional spaces by space–time white noise. By using the underlying explicit kernels, we prove the exact global temporal continuity moduli and temporal laws of the iterated logarithm for the TFSPIDEs and their gradients, as well as prove that the sets of temporal fast points (where the remarkable oscillation of the TFSPIDEs and their gradients happen infinitely often) are random fractals. In addition, we evaluate their Hausdorff dimensions and their hitting probabilities. It has been confirmed that these points of the TFSPIDEs and their gradients, in time, are most likely one everywhere, and are dense with the power of the continuum. Moreover, their hitting probabilities are determined by the target set B's packing dimension $\dim_p(B)$. On the one hand, this work reinforces the temporal moduli of the continuity and temporal LILs obtained in relevant literature, which were achieved by obtaining the exact values of their normalized constants; on the other hand, this work obtains the size of the set of fast points, as well as a potential theory of TFSPIDEs and their gradients.

Keywords: TFSPIDEs; Brownian-time processes; space–time white noise; temporal fractal nature; hitting probabilities; Hölder regularity

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1. Introduction

Fractional and higher-order evolution equations have been used as (stochastic) models in mathematical finance, fluid dynamics, turbulence, and mathematical physics by numerous authors in recent years (see, e.g., [1–3]). Time-fractional stochastic partial integro-differential equations (TFSPIDEs) are related to diffusion or slow diffusion in materials with memory. (For connected deterministic PDEs, see [4–6]; for connected stochastic PDEs, see [7,8]; and, for the associated stochastic integral equations (SIEs), see [9–11].)

Expanded upon by [11], Brownian-time processes (BTP) provide the foundation for the deterministic version of the TFSPIDEs. The precise dimensions and hitting probabilities for the sets of fast points, in time, for these important class of stochastic equations are obtained in this article as follows:

$$\begin{cases} C_{\partial_t^\beta U_\beta} = \dfrac{1}{2}\Delta U_\beta + I_t^{1-\beta}\Big(\dfrac{\partial^{d+1}W}{\partial t\partial x}\Big), & (t,x)\in \mathring{\mathbb{R}}_+\times\mathbb{R}^d; \\ U_\beta(0,x) = u_0(x), & x\in\mathbb{R}^d, \end{cases} \tag{1}$$

where $\partial^{d+1}W/\partial t\partial x$ is the space–time white noise corresponding to the real-valued Brownian sheet W on $\mathbb{R}_+\times\mathbb{R}^d$ ($d = 1,2,3$); Δ is the d-dimensional Laplacian operator; the time-fractional derivative of order β, $C_{\partial_t^\beta}$, is the Caputo fractional operator

$$C_{\partial_t^\beta f(t)} := \begin{cases} \dfrac{1}{\Gamma(1-\beta)}\displaystyle\int_0^t \dfrac{f'(\tau)}{(t-\tau)^\beta}d\tau, & \text{if } 0<\beta<1; \\ \dfrac{d}{dt}f(t), & \text{if } \beta = 1, \end{cases}$$

and the time-fractional integral of order α; I_t^α, is the Riemann-Liouville fractional integral

$$I_t^\alpha f := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \text{for } t > 0 \text{ and } \alpha > 0,$$

and $I_t^0 = \mathrm{I}$ is the identity operator. Here, it was assumed that the initial data u_0 are deterministic and the Borel measurable, and that there exists a constant $0 < \gamma \leq 1$ such that

$$u_0 \in C_b^{2^{k+1}-2,\gamma}(\mathbb{R}^d; \mathbb{R}), \quad \text{for } 2^{k-1} < \beta^{-1} \leq 2^k, k \in \mathbb{N}, \tag{2}$$

where $C_b^{\alpha,\gamma}(\mathbb{R}^d; \mathbb{R})$ is the set of α-continuously differentiable functions on $\mathbb{R}^d$, whose α-derivative is locally Hölder continuous with the exponent γ.

It is clear that the formal (and non-rigorous) equation is Equation (1). In this article, we work with its rigorous formulation, which is the mild form kernel SIE. Refs. [10–12] presented and addressed this SIE for the first time. We include them in Section 2 below, along with some other pertinent information.

Refs. [10,12] obtained the existence, uniqueness, sharp dimension-dependent L^p, and the Hölder regularity of the linear and non-linear noise versions of (1). The exact uniform and local continuity moduli for the TFSPIDEs in the time variable t and space variable x were separately obtained in [13]. Specifically, it was shown, in [13], that the fourth-order TFSPIDEs and their gradients have exact, spatio-temporal, dimension-dependent, uniform, and local continuity moduli. In addition to obtaining temporal central limit theorems for modifications of the quadratic variation of the solution to Equation (1) in time, it was also investigated in [14] that the solution to Equation (1) in time has infinite quadratic variation and is not a semimartingale. Ref. [15] obtained the precise, dimension-dependent, non-differentiability moduli for the TFSPIDEs and their gradients in the time variable t.

Here, we would like to mention the global temporal continuity moduli and the local temporal continuity moduli at a prescribed time $t_0 \geq 0$, as well as the laws of iterated logarithm (LILs) for $U_\beta(\cdot, x)$ and $\partial_x U_\beta(\cdot, x)$, which were obtained in [13]. These phenomena showed the existence of normalized constants for the global temporal continuity moduli and temporal LILs. But their exact values remain unknown. In this paper, we give the exact values of these normalized constants by obtaining precise estimations of the second-order increment moments. For any $d \in \mathbb{N}_+$, we define $K_{\beta,d}$ and $K_{\beta,0}$ by

$$K_{\beta,d} = \frac{4}{(2\pi)^d} \int_{\mathbb{R}} \frac{1-\cos u}{u^{2-(d\beta)/2}} du \int_{\mathbb{R}^d} \frac{1}{4 + 4|y|^2 \cos(\frac{\beta\pi}{2}) + |y|^4} dy, \tag{3}$$

and

$$K_{\beta,0} = \frac{2}{\pi} \int_{\mathbb{R}} \frac{1-\cos u}{u^{2-3\beta/2}} du \int_{\mathbb{R}} \frac{y^2}{4 + 4y^2 \cos(\frac{\beta\pi}{2}) + y^4} dy. \tag{4}$$

In this article, we obtain the following exact global temporal continuity moduli and temporal LILs for the TFSPIDE $U_\beta(t,x)$ and the gradient process $\partial_x U_\beta(t,x)$. Equations (5) and (7) below are other forms of the global temporal continuity moduli of the TFSPIDEs and their gradients, which are slightly different from those obtained in [13].

Theorem 1. *(Temporal continuity moduli) Let $\beta \in (0, 1/2]$, $x \in \mathbb{R}^d$ ($d = 1,2,3$), and $u_0 \equiv 0$ in (1) be fixed.*

(a) (Global temporal continuity modulus and temporal LIL for the TFSPIDEs) for every compact interval $I_{\text{time}} \subset \mathbb{R}_+$,

$$\mathbb{P}\Big\{ \lim_{h \to 0+} \sup_{s,t \in I_{\text{time}}, |t-s|<h} \phi_{\beta,d,h}^{-1} |U_\beta(t,x) - U_\beta(s,x)| = 1 \Big\} = 1, \tag{5}$$

where $\phi_{\beta,d,h} = h^{\frac{2-d\beta}{4}} \sqrt{2K_{\beta,d} \log(1/h)}$, *and for every fixed* $t \geq 0$

$$\mathbb{P}\Big\{ \limsup_{h\to 0+} \sup_{s,t\in I_{\text{time}},|t-s|<h} \hat{\phi}_{\beta,d,h}^{-1} |U_\beta(t,x) - U_\beta(s,x)| = 1 \Big\} = 1, \tag{6}$$

where $\hat{\phi}_{\beta,d,h} = h^{\frac{2-d\beta}{4}} \sqrt{2K_{\beta,d} \log\log(1/h)}$. *Here,* $K_{\beta,d}$ *is given in (3).*

(b) (Global temporal continuity modulus and temporal LIL for the TFSPIDE gradients.) Let $d = 1$. *For every compact interval* $I_{\text{time}} \subset \mathbb{R}_+$,

$$\mathbb{P}\Big\{ \lim_{h\to 0+} \sup_{s,t\in I_{\text{time}},|t-s|<h} \varphi_{\beta,h}^{-1} |\partial_x U_\beta(t,x) - \partial_x U_\beta(s,x)| = 1 \Big\} = 1, \tag{7}$$

where $\varphi_{\beta,h} = h^{\frac{2-3\beta}{4}} \sqrt{2K_{\beta,0} \log(1/h)}$, *and, for every fixed* $t \in \mathbb{R}_+$,

$$\mathbb{P}\Big\{ \limsup_{h\to 0+} \sup_{s,t\in I_{\text{time}},|t-s|<h} \hat{\varphi}_{\beta,h}^{-1} |\partial_x U_\beta(t,x) - \partial_x U_\beta(s,x)| = 1 \Big\} = 1, \tag{8}$$

where $\hat{\varphi}_{\beta,h} = h^{\frac{2-3\beta}{4}} \sqrt{2K_{\beta,0} \log\log(1/h)}$. *Here,* $K_{\beta,0}$ *is given in (4).*

Remark 1. *We can infer the following from the aforementioned theorem:*

- *Equations (5) and (7) are other forms of the global temporal continuity moduli of the TFSPIDEs and the TFSPIDE gradients, respectively, which are slightly different from those obtained in [13]. Equation (5) with* $k_8^{(\beta,d)} |t-s|^{\frac{2-d\beta}{4}} \sqrt{2\log(1/|t-s|)}$ *taking the place of* $\phi_{\beta,d,h}$, *and Equation (7) with* $k_{12}|t-s|^{\frac{2-3\beta}{4}} \sqrt{\log(1/|t-s|)}$ *taking the place of* $\varphi_{\beta,h}$ *were established in [13], where* $k_8^{(\beta,d)} > 0$ *and* $k_{12} > 0$ *were understood as dimension-dependent constants, i.e., independent of x (whose exact values were unknown). Here, in Equations (5) and (7), we give the exact constants for the global temporal continuity moduli of the TFSPIDEs and the TFSPIDE gradients. Moreover, by using Lemma 5 below, we can obtain* $k_8^{(\beta,d)} = \sqrt{2K_{\beta,d}}$ *and* $k_{12} = \sqrt{2K_{\beta,0}}$, *as was obtained in [13]. In this sense, the results of this paper reinforce those in [13].*
- *Equation (6) with* $k_9^{(\beta,d)} h^{\frac{2-d\beta}{4}} \sqrt{\log\log(1/h)}$ *taking the place of* $\hat{\phi}_{\beta,d,h}$, *and Equation (8) with* $k_{13} h^{\frac{2-3\beta}{4}} \sqrt{\log\log(1/h)}$ *taking the place of* $\hat{\varphi}_{\beta,h}$ *were established in [13], where* $k_9^{(\beta,d)} > 0$ *and* $k_{13} > 0$ *were understood as dimension-dependent constants, i.e., independent of x (whose exact values were unknown). Here, in Equations (6) and (8), we give the exact constants for the temporal LILs of the TFSPIDEs and the TFSPIDE gradients. Moreover, by using Lemma 5 below, we can obtain* $k_9^{(\beta,d)} = \sqrt{2K_{\beta,d}}$ *and* $k_{13} = \sqrt{2K_{\beta,0}}$, *as was obtained in [13]. In this sense, the results of this paper reinforce those in [13].*
- *Equation (5) gives the magnitude of the global maximal oscillation of the TFSPIDE solution* $U_\beta(\cdot,x)$ *over the compact rectangle* I_{time}, *which is* $\phi_{\beta,d,h}$. *Equation (7) gives the magnitude of the global maximal oscillation of the TFSPIDE gradient solution* $\partial_x U_\beta(\cdot,x)$ *over the compact rectangle* I_{time}, *which is* $\varphi_{\beta,h}$.
- *Equation (6) gives the magnitude of the local oscillation of the TFSPIDE solution* $U_\beta(\cdot,x)$ *at a prescribed time* $t_0 \geq 0$ *is* $\hat{\phi}_{\beta,d,h}$. *Equation (8) gives the magnitude of the local oscillation of the TFSPIDE gradient solution* $\partial_x U_\beta(\cdot,x)$ *at a prescribed time* $t_0 \geq 0$ *is* $\hat{\varphi}_{\beta,h}$.
- *It is interesting to compare Equations (5) and (6). The latter one states that, at some given point, the LIL of* $U_\beta(\cdot,x)$ *for any fixed x is not more than* $\hat{\phi}_{\beta,d,h}$. *On the other hand, the former tells us that the global continuity modulus of* $U_\beta(\cdot,x)$ *can be much larger, namely* $\phi_{\beta,d,h}$. *Similarly, by Equations (7) and (8), the LIL of* $\partial_x U_\beta(\cdot,x)$ *for every fixed x is less than* $\hat{\varphi}_{\beta,h}$. *On the other hand, the continuity modulus of* $\partial_x U_\beta(\cdot,x)$ *can be much larger, namely* $\varphi_{\beta,h}$.

- *With Equation (6) and Fubini's theorem, we have the random time set at*

$$S_{\beta,d,x,+} := \Big\{t \in [0,1] : \limsup_{h\to 0+} \hat{\phi}^{-1}_{\beta,d,h}|U_\beta(t+h,x) - U_\beta(t,x)| > 1\Big\},$$

which has a Lebesgue measurement of zero with a probability of one. Nevertheless, $S_{\beta,d,x,+}$ is not null. It is almost certain that the set of t that satisfies the stronger growth criterion (9) below is dense everywhere with the power of the continuum. There are similar properties for the TFSPIDE gradient $\partial_x U_\beta(\cdot, x)$.

Fix $x \in \mathbb{R}^d$. For every $\lambda \in (0,1]$, the set of temporal λ-fast points for the fourth-order TFSPIDE are defined by

$$S_{\beta,d,x}(\lambda) := \Big\{t \in [0,1] : \limsup_{h\to 0+} \phi_{\beta,d,h}|U_\beta(t+h,x) - U_\beta(t,x)| \geq \lambda\Big\}, \tag{9}$$

where $\phi_{\beta,d,h}$ is given in (5). For every $\chi \in (0,1]$, the set of the temporal χ-fast points for the fourth-order TFSPIDE gradients are defined by

$$\mathcal{S}_{\beta,x}(\chi) := \Big\{t \in [0,1] : \limsup_{h\to 0+} \varphi_{\beta,h}|\partial_x U_\beta(t+h,x) - \partial_x U_\beta(t,x)| \geq \chi\Big\}, \tag{10}$$

where $\varphi_{\beta,h}$ is given in (7).

The $S_{\beta,d,x}(\lambda)$ are the sets of t, where the temporal LIL of TFSPIDEs fail, and the $\mathcal{S}_{\beta,x}(\chi)$ are the sets of t, where the temporal LIL of TFSPIDE gradients fail. This kind of set is usually called the *fast point set* or *exceptional time set*. It is interesting to obtain information about the sizes of $S_{\beta,d,x}(\lambda)$ and $\mathcal{S}_{\beta,x}(\chi)$. We usually do this by considering their Hausdorff measures. This problem was first introduced in Orey and Taylor [16] on the fast set for Brownian motion. After this famous paper, there were several papers that studied this problem for general Gaussian processes. Among other things, the fractal nature of the fast set of empirical processes with independent increments was studied in [17]. The fractal nature of the fast point set of L^p-valued Gaussian processes was studied in [18]. The limsup fractal nature of the fast point sets of Gaussian processes was studied in [19]. The solutions and gradient solutions for TFSPIDEs are spatio-temporal Gaussian random fields. It is, therefore, natural to study this type of fractal nature (in the sense of [16,19]). This paper is devoted to establishing the fractal nature and hitting probabilities for the sets of temporal fast points for TFSPIDE $U_\beta(t,x)$ and the gradient process $\partial_x U_\beta(t,x)$.

Recall (see, e.g., [20,21]) that the Hausdorff dimension $\dim B$ of a subset B of $[0,1]$ is defined by

$$\dim(B) = \inf\{\alpha > 0 : \mu_g(B) = 0 \ \text{ for } \ g(s) = s^\alpha\}.$$

The Hausdorff g-measure of a subset B of a real line for any continuous increasing function $g : [0,1] \to [0,+\infty]$ with $g(0) = 0$ is defined as follows:

$$\mu_g(B) = \lim_{\delta\to 0}\Big[\inf_{\substack{B\subseteq \cup C_i \\ d(C_i)<\delta}} \sum g(d(C_i))\Big], \tag{11}$$

where the infimum in (11) extends over all countable covers of B by sets C_i of diameter $d(C_i) < \delta$. Keep in mind that, while $\mu_g(B)$ simplifies to an Lebesgue outer measure if $g(s) = s$, using a distinct g creates a hierarchy of measures. By being familiar with the class of measure functions g for which $\mu_g(B) = 0$, one may determine the metric features of B. The purpose of this article is to show the following two theorems. In the first one, we show that $S_{\beta,d,x}(\lambda)$ and $\mathcal{S}_{\beta,x}(\chi)$ are random fractals, and we also evaluate their Hausdorff dimensions. In the second one, we show that hitting probabilities are determined by the the target set B's packing dimension $\dim_p(B)$ rather than its Hausdorff dimension $\dim(B)$. For a definition of packing dimension, see [22].

Theorem 2. *(Fractal nature for the sets of the temporal fast points.) Let $\beta \in (0, 1/2]$, $x \in \mathbb{R}^d$ ($d = 1, 2, 3$) and $u_0 \equiv 0$ in (1) be fixed.*

(a) Suppose $d \in \{1, 2, 3\}$. For every $\lambda \in [0, 1]$ with a probability of one, we have

$$\dim(S_{\beta,d,x}(\lambda)) = 1 - \lambda^2. \tag{12}$$

(b) Suppose $d = 1$. For every $\chi \in [0, 1]$ with a probability of one, we have

$$\dim(\mathcal{S}_{\beta,x}(\chi)) = 1 - \chi^2. \tag{13}$$

The following theorem demonstrates that the appropriate index through which to determine whether sets overlap $S_{\beta,d,x}(\lambda)$ and $\mathcal{S}_{\beta,x}(\chi)$ is the packing dimension.

Theorem 3. *(Hitting probabilities for the sets of temporal fast points.) Let $\beta \in (0, 1/2]$, $x \in \mathbb{R}^d$ ($d = 1, 2, 3$) and $u_0 \equiv 0$ in (1) be fixed.*

(a) Suppose $d \in \{1, 2, 3\}$. For every $\lambda \in [0, 1]$ and every analytic set $B \subset \mathbb{R}_+$, we have

$$P\{S_{\beta,d,x}(\lambda) \cap B \neq \emptyset\} = \begin{cases} 1, & \text{if } \dim_p(B) > \lambda^2, \\ 0, & \text{if } \dim_p(B) < \lambda^2. \end{cases} \tag{14}$$

(b) Suppose $d = 1$. For every $\chi \in [0, 1]$ and every analytic set $B \subset \mathbb{R}_+$, we have

$$P\{\mathcal{S}_{\beta,x}(\chi) \cap B \neq \emptyset\} = \begin{cases} 1, & \text{if } \dim_p(B) > \chi^2, \\ 0, & \text{if } \dim_p(B) < \chi^2. \end{cases} \tag{15}$$

Remark 2. *It is easy to see that Equations (14) and (15) are respectively equivalent for every analytic set $B \subset \mathbb{R}_+$. As such, we have*

$$\mathbb{P}\Big\{\sup_{t\in B} \limsup_{h\to 0+} \phi_{\beta,d,h}|U_\beta(t+h,x) - U_\beta(t,x)| = (\dim_p(B))^{1/2}\Big\} = 1, \tag{16}$$

and

$$\mathbb{P}\Big\{\sup_{t\in B} \limsup_{h\to 0+} \varphi_{\beta,h}|\partial_x U_\beta(t+h,x) - \partial_x U_\beta(t,x)| = (\dim_p(B))^{1/2}\Big\} = 1. \tag{17}$$

Thus, in the context of TFSPIDEs and their gradients, Equations (16) and (17) can be understood as two probabilistic interpretations of the packing dimension of an analytic set $B \subset \mathbb{R}_+$.

Remark 3. *We obtain the following probabilistic interpretations of the upper and lower Minkowski dimensions of B, which are denoted by $\overline{\dim}_M(B)$ and $\underline{\dim}_M(B)$, respectively. This was achieved by reversing the order of $\sup$ and $\limsup$ in Equation (16); these definitions are provided in [22].*

$$\mathbb{P}\Big\{\limsup_{h\to 0+} \sup_{t\in B} \phi_{\beta,d,h}|U_\beta(t+h,x) - U_\beta(t,x)| = (\overline{\dim}_M(B))^{1/2}\Big\} = 1, \tag{18}$$

$$\mathbb{P}\Big\{\liminf_{h\to 0+} \sup_{t\in B} \phi_{\beta,d,h}|U_\beta(t+h,x) - U_\beta(t,x)| = (\underline{\dim}_M(B))^{1/2}\Big\} = 1. \tag{19}$$

According to Equations (18) and (19), there are also probabilistic interpretations of the upper and lower Minkowski dimensions of B.

An undefined positive, finite constant, c, will be used throughout this work; however, it might not always be the same. $c_{i,1}, c_{i,2}, \ldots$ were found to be more particularly positive and finite constants (independent of x), as shown in Section 1.

The remainder of the article is organized as follows. In Section 2, using the time-fractional SPIDEs kernel SIE formulation, the rigorous TFSPIDE kernel SIE (mild) formulation and temporal spectral density for TFSPIDEs and their gradients are discussed.

Estimations on the second-order moments of temporal increments of the fourth-order TFSPIDEs and their gradients are also obtained. In Section 3, we prove Theorem 1 and thereby establish the exact temporal continuity moduli for the TFSPIDEs and their gradients; in addition, we prove Theorem 2 and thereby obtain Hausdorff dimensions of the sets of temporal fast points for the TFSPIDEs and their gradients. Furthermore, we prove Theorem 3 and thereby obtain the hitting probabilities of the sets of temporal fast points for the TFSPIDEs and their gradients. In Section 4, the results are summarized and discussed.

2. Preliminaries

2.1. Rigorous Kernel SIE Formulations

We define the rigorous mild SIE formulations of the TFSPIDEs, as in [13], using the density of an inverse stable Lévy time Brownian motion. According to [10–12], this density is the time-fractional PDE's solution as follows:

$$\begin{cases} C_{\partial_t^\beta U_\beta} = \dfrac{1}{2}\Delta U_\beta, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\ U_\beta(0,x) = \delta(x), & x \in \mathbb{R}^d, \end{cases} \tag{20}$$

where $\delta(x)$ is the Dirac function. This solution is the transition density of a d-dimensional β-inverse-stable-Lévy-time Brownian motion (β-ISLTBM). It starts from $x \in \mathbb{R}^d$, $\mathbb{B}^x_{A_\beta} := \{B^x(A_\beta(t)), t \geq 0\}$, where the inverse stable Lévy motion A_β of index $\beta \in (0, 1/2]$ serves as the time clock for an independent d-dimensional Brownian motion B^x (see [10,23]), which is given by the following:

$$\mathbb{H}^{(\beta,d)}_{t;x,y} = \int_0^{+\infty} H^{\mathrm{BM}^d}_{s;x,y} H^{A_\beta}_{t;0,s} ds, \tag{21}$$

where $H^{\mathrm{BM}^d}_{s;x,y} = \frac{-|x-y|^2/2s}{(2\pi s)^{d/2}}$ and $H^{A_\beta}_{t;0,s} = t\beta^{-1}s^{-1-1/\beta}g_\beta(ts^{-1/\beta})$. Here, the density of a stable subordinator is denoted by $g_\beta(u)$, and its Laplace transform is e^{-s^β}. When $\beta = 1/2$, the density of the Brownian-time Brownian motion (BTBM) is represented by the kernel $\mathbb{H}^{(\beta,d)}_{t;x}$, as described in [9]; for $\beta \in \{1/2^k; k \in \mathbb{N}\}$, the density of the k-iterated BTBM is represented by the kernel $\mathbb{H}^{(\beta,d)}_{t;x}$, as explained in [10,11].

Let $b : \mathbb{R} \to \mathbb{R}$ be Borel measurable. The non-linear drift diffusion TFSPIDE is thus

$$\begin{cases} C_{\partial_t^\beta U_\beta} = \dfrac{1}{2}\Delta U_\beta + I_t^{1-\beta}\Big[b(U_\beta) + a(U_\beta)\dfrac{\partial^{d+1}W}{\partial t \partial x}\Big], & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d; \\ U_\beta(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{22}$$

Then, the rigorous TFSPIDE kernel SIE formulation is the SIE (see Equation (1.11), and Definition 1.1 in [12], as well as p. 530 in [9]), is as follows:

$$\begin{aligned} U(t,x) = &\int_{\mathbb{R}^d} \mathbb{H}^{(\beta,d)}_{t;x,y} u_0(y) dy \\ &\int_{\mathbb{R}^d}\int_0^t \mathbb{H}^{(\beta,d)}_{t-s;x,y}[b(U(s,y))dsdy + a(U(s,y))W(ds \times dy)]. \end{aligned} \tag{23}$$

Naturally, this yields the mild formulation of (1.1), which is when $a \equiv 1$ and $b \equiv 0$ are set in (22).

The spatial Fourier transform of the β-time-fractional (including the $\beta = 1/2$ BTBM example) kernels from Lemma 2.1 in [13] is cited to conclude this section.

Lemma 1 (Transforms of a spatial Fourier type)**.** *Let $0 < \beta < 1$ and $\mathbb{H}^{(\beta,d)}_{t;x,y}$ be the β-time-fractional kernel. The β-time-fractional kernel's spatial Fourier transform is provided by*

$$\hat{\mathbb{H}}^{(\beta,d)}_{t;x,\xi} = (2\pi)^{-\frac{d}{2}} E_\beta\Big(-\frac{|\xi|^2}{2}t^\beta\Big), \tag{24}$$

where

$$E_\beta(u) = \sum_{k=0}^{+\infty} \frac{u^k}{\Gamma(1+\beta k)}, \tag{25}$$

is the well-known function of Mittag–Leffler. The spatial Fourier transform in its symmetric form is applied here as follows: $\hat{f}(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(u)\, e^{-\mathrm{i}\xi\cdot u} du$.

2.2. Estimations on the Variances of Temporal Increments of TFSPIDEs and Their Gradients

For the purposes of this subsection, let $x \in \mathbb{R}^d$ be an arbitrary, fixed variable. The auxiliary Gaussian random field $\{X_\beta(t,x), t \in \mathbb{R}_+, x \in \mathbb{R}^d\}$ is defined by the following:

$$X_\beta(t,x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}} \left(\mathbb{H}^{(\beta,d)}_{(t-r)_+;x,y} - \mathbb{H}^{(\beta,d)}_{(-r)_+;x,y}\right) W(dr\times dy), \tag{26}$$

where $z_+ = \max\{z, 0\}$ for any $z \in \mathbb{R}$. Then, the TFSPIDE solution U_β has a decomposition as $U_\beta(t,x) = X_\beta(t,x) - Y_\beta(t,x)$, where

$$Y_\beta(t,x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_-} \left(\mathbb{H}^{(\beta,d)}_{(t-r)_+;x,y} - \mathbb{H}^{(\beta,d)}_{(-r)_+;x,y}\right) W(dr\times dy). \tag{27}$$

This decomposition idea was first introduced in the second-order SPDE setting in [23]. It has since been implemented in the second-order heat SPDE setting in [24,25].

Using the previously mentioned decomposition of U_β, we first calculated the exact variance for the temporal increments of the auxiliary process X_β. Then, we transferred these to our TFSPIDE solution U_β in terms of X_β and a smooth process of Y_β. The outcome that followed was crucial.

Lemma 2. *Let $\beta \in (0, 1/2]$, $x \in \mathbb{R}^d$ ($d = 1,2,3$) and $u_0 \equiv 0$ in (1) be fixed. Then, for any $s, t \in (0, T]$ such that t/s is sufficiently close to 1, we have*

$$\mathbb{E}[(U_\beta(t,x) - U_\beta(s,x))^2] = (K_{\beta,d} + o(1))|t-s|^{\frac{2-d\beta}{2}}, \tag{28}$$

where $K_{\beta,d}$ is given in (3).

Proof. With Theorem 4.1 in [13], we have

$$\mathbb{E}[|X_\beta(t,x) - X_\beta(s,x)|^2] = 2\int_{\mathbb{R}} (1-\cos((t-s)\tau)) f_\beta(\tau) d\tau, \tag{29}$$

where

$$f_\beta(\tau) = (2\pi)^{-d} \frac{1}{|\tau|^{2-((d\beta))/2}} \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2\cos(\frac{\pi\beta}{2}) + \frac{1}{4}|\xi|^4} d\xi.$$

With the change in variable $\tau \mapsto u : u = (t-s)\tau$, (29) yields

$$\mathbb{E}[|X_\beta(t,x) - X_\beta(s,x)|^2] = K_{\beta,d}|t-s|^{1-\frac{d\beta}{2}}. \tag{30}$$

Let $x \in \mathbb{R}^d$ be fixed. For each $0 < s < t$, we can obtain the following by using Parseval's identity to the integral in y:

$$\begin{aligned} \mathbb{E}[|Y_\beta(t,x) - Y_\beta(s,x)|^2] &= \int_{\mathbb{R}^d}\int_{\mathbb{R}} \left|\mathbb{H}^{(\beta,d)}_{t-r;x,y}\mathbb{I}_{\{0>r\}} - \mathbb{H}^{(\beta,d)}_{s-r;x,y}\mathbb{I}_{\{0>r\}}\right|^2 dr dy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} \left|\hat{\mathbb{H}}^{(\beta,d)}_{t-r;x,\xi}\mathbb{I}_{\{0>r\}} - \hat{\mathbb{H}}^{(\beta,d)}_{s-r;x,\xi}\mathbb{I}_{\{0>r\}}\right|^2 d\xi dr. \end{aligned} \tag{31}$$

Note that

$$\hat{\mathbb{H}}^{(\beta,d)}_{t-r;x,\xi} = (2\pi)^{-\frac{d}{2}} E_\beta\Big(-\frac{|\xi|^2(t-r)^\beta}{2}\Big). \tag{32}$$

Through the corollary on page 23 in [26], we have

$$\int_{\mathbb{R}^d} f\Big(\sum_{i=1}^{d} x_i^2\Big)dx_1\cdots dx_d = \frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^{+\infty} y^{d/2-1} f(y)dy, \tag{33}$$

Through Equations (32) and (33), Equation (31) becomes

$$\begin{aligned}
&\mathbb{E}[|Y_\beta(t,x)-Y_\beta(s,x)|^2] \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}} \frac{\Big|E_\beta\Big(-\frac{|\xi|^2}{2}(t-r)^\beta\Big)\mathbb{I}_{\{0>r\}} - E_\beta\Big(-\frac{|\xi|^2}{2}(s-r)^\beta\Big)\mathbb{I}_{\{0>r\}}\Big|^2}{(2\pi)^d} drd\xi \\
&= (2\pi)^{-d}\frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^{+\infty} y^{d/2-1}\int_{-\infty}^0 \Big|E_\beta\Big(-\frac{y}{2}(t-r)^\beta\Big) - E_\beta\Big(-\frac{y}{2}(s-r)^\beta\Big)\Big|^2 drdy.
\end{aligned} \tag{34}$$

It follows from (7.7) in [27] that

$$E_\beta(-x) = \frac{\sin(\beta\pi)}{\pi}\int_0^{+\infty} a_\beta(\zeta)e^{-\zeta x^{1/\beta}}d\zeta, \tag{35}$$

where

$$a_\beta(\zeta) = \frac{\zeta^\beta}{1+2\zeta^\beta\cos(\beta\pi)+\zeta^{2\beta}}.$$

Thus, Equation (34) yields

$$\begin{aligned}
&\mathbb{E}[|Y_\beta(t,x)-Y_\beta(s,x)|^2] \\
&= \frac{1}{2^d\Gamma(d/2)}\frac{\sin^2(\beta\pi)}{\pi^{2+d/2}}\int_0^{+\infty} y^{d/2-1}\int_{-\infty}^0\Big|\int_0^{+\infty} a_\beta(\zeta)(e^{-\zeta(\frac{y}{2})^{1/\beta}(t-r)} - e^{-\zeta(\frac{y}{2})^{1/\beta}(s-r)})d\zeta\Big|^2 drdy \\
&\le \frac{1}{2^d\Gamma(d/2)}\frac{\sin^2(\beta\pi)}{\pi^{2+d/2}}\int_0^{+\infty} y^{d/2-1}\int_{-\infty}^0\Big|\int_0^{+\infty} a_\beta(\zeta)e^{-\zeta(\frac{y}{2})^{1/\beta}(s-r)}\Big|e^{-\zeta(\frac{y}{2})^{1/\beta}(t-s)}-1\Big|d\zeta\Big|^2 drdy.
\end{aligned} \tag{36}$$

Since, for any $u \ge 0$, $|1-e^{-u}| \le 2u$, Equation (36) yields

$$\begin{aligned}
&\mathbb{E}[|Y_\beta(t,x)-Y_\beta(s,x)|^2] \\
&\le \frac{1}{2^{d+2/\beta-2}\Gamma(d/2)}\frac{\sin^2(\beta\pi)}{\pi^{d/2}}(t-s)^2\int_0^{+\infty} y^{d/2+2/\beta-1}\int_{-\infty}^0\Big|\int_0^{+\infty} a_\beta(\zeta)e^{-\zeta(\frac{y}{2})^{1/\beta}(s-r)}d\zeta\Big|^2 drdy.
\end{aligned} \tag{37}$$

By changing the variables $r \mapsto u : u = (\frac{y}{2})^{1/\beta}r$ and $y \mapsto v : v = (\frac{y}{2})^{1/\beta}s$, (37) yields

$$\begin{aligned}
&\mathbb{E}[|Y_\beta(t,x)-Y_\beta(s,x)|^2] \\
&\le \frac{2^{d/2+2}\beta}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2)}\frac{\sin^2(\beta\pi)}{\pi^2}\frac{(t-s)^2}{s^{(d\beta)/2+1}}\int_0^{+\infty} v^{(d\beta)/2}\int_0^{+\infty}\Big|\int_0^{+\infty}\zeta a_\beta(\zeta)e^{-\zeta v}e^{-\zeta u}d\zeta\Big|^2 dudv \\
&\le \frac{c_{2,1}}{s^{(d\beta)/2+1}}(t-s)^2,
\end{aligned} \tag{38}$$

since the integral above is finite for $0<\beta\le 1/2$. Furthermore, as U_β and Y_β are independent, we have

$$\mathbb{E}[|X_\beta(t,x)-X_\beta(s,x)|^2] = \mathbb{E}[|U_\beta(t,x)-U_\beta(s,x)|^2] + \mathbb{E}[|Y_\beta(t,x)-Y_\beta(s,x)|^2]. \tag{39}$$

Note that $s^{-((d\beta)/2+1)}(t-s)^2 = (t-s)^{1-(d\beta)/2}(t/s-1)^{1+(d\beta)/2}$. Combining (30), (38) and (39), we also obtain the following for any $0<s<t$:

$$|\mathbb{E}[(U_\beta(t,x)-U_\beta(s,x))^2] - K_{\beta,d}(t-s)^{1-(d\beta)/2}| \le c_{2,2}(t-s)^{1-(d\beta)/2}(t/s-1)^{1+(d\beta)/2}. \tag{40}$$

This yields (28) and completes the proof. □

We also need the following estimation on the variances of temporal increments of the TFSPIDE gradient process $\partial_x U_\beta(\cdot,x)$.

Lemma 3. *Let $\beta \in (0,1/2]$, $x \in \mathbb{R}^d$ ($d = 1,2,3$) and $u_0 \equiv 0$ in (1) be fixed. Then, for all $s,t \in (0,T]$, such that t/s is sufficiently close to 1, we have*

$$\mathbb{E}[(\partial_x U_\beta(t,x) - \partial_x U_\beta(s,x))^2] = (K_{\beta,0} + o(1))|t-s|^{\frac{2-3\beta}{2}}, \tag{41}$$

where $K_{\beta,0}$ is given in (4).

Proof. Through (4.40) in [13], we have

$$\mathbb{E}[|\partial_x X_\beta(t,x) - \partial_x X_\beta(s,x)|^2] = 2\int_{\mathbb{R}} (1-\cos((t-s)\tau)) f_\beta(\tau) d\tau, \tag{42}$$

where

$$f_\beta(\tau) = (2\pi)^{-1}|\tau|^{-2+\frac{3\beta}{2}} \int_{\mathbb{R}} \frac{\xi^2}{1+\xi^2\cos(\frac{\pi\beta}{2}) + \frac{1}{4}\xi^4} d\xi.$$

Via a change in the variables to the integral in τ, (42) yields

$$\mathbb{E}[|\partial_x X_\beta(t,x) - \partial_x X_\beta(s,x)|^2] = K_{\beta,0}|t-s|^{1-\frac{3\beta}{2}}. \tag{43}$$

Let $x \in \mathbb{R}^d$ be fixed. For each $0 < s < t$, we obtain the following by using Parseval's identity to the integral in y:

$$\begin{aligned} \mathbb{E}[|\partial_x Y_\beta(t,x) - \partial_x Y_\beta(s,x)|^2] &= \int_{\mathbb{R}^d}\int_{\mathbb{R}} \left| \partial_x \mathbb{H}^{(\beta,1)}_{t-r;x,y}\mathbb{I}_{\{0>r\}} - \partial_x \mathbb{H}^{(\beta,1)}_{s-r;x,y}\mathbb{I}_{\{0>r\}} \right|^2 drdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} \xi^2 \left| \hat{\mathbb{H}}^{(\beta,1)}_{t-r;x,\xi}\mathbb{I}_{\{0>r\}} - \hat{\mathbb{H}}^{(\beta,1)}_{s-r;x,\xi}\mathbb{I}_{\{0>r\}} \right|^2 d\xi dr. \end{aligned} \tag{44}$$

Following the same route as the proof of (38), via (44), we have

$$\mathbb{E}[|\partial_x Y_\beta(t,x) - \partial_x Y_\beta(s,x)|^2] \le \frac{c_{2,3}}{s^{3\beta/2+1}}(t-s)^2. \tag{45}$$

Thus, with (43) and (45), similar to the proof of (40), we obtain (41). This completes the proof. □

3. Results

3.1. Temporal Moduli of Continuity

We prove Theorem 1 in this subsection, thus establishing the temporal moduli of continuity for the TFSPIDEs, as well as their gradients, in the process. The following precise large deviation estimates for the TFSPIDEs and their gradients are necessary for our results.

Lemma 4. *Let $\beta \in (0,1/2]$, $x \in \mathbb{R}^d$ ($d = 1,2,3$) and $u_0 \equiv 0$ in (1) be fixed.*

(a) Suppose $d \in \{1,2,3\}$. Then, for any $t,h \in \mathring{\mathbb{R}}_+$, such that h/t is sufficiently close to 0, we have

$$\lim_{u\to+\infty} u^{-2}\log\mathbb{P}\Big(|U_\beta(t+h,x) - U_\beta(t,x)| \ge uK^{1/2}_{\beta,d}h^{\frac{2-d\beta}{4}}\Big) = -\frac{1}{2}. \tag{46}$$

(b) Suppose $d = 1$. Then, for any $t,h \in \mathring{\mathbb{R}}_+$, such that h/t is sufficiently close to 0, we have

$$\lim_{u\to+\infty} u^{-2}\log\mathbb{P}\Big(|\partial_x U_\beta(t+h,x) - \partial_x U_\beta(t,x)| \ge uK^{1/2}_{\beta,0}h^{\frac{2-3\beta}{4}}\Big) = -\frac{1}{2}. \tag{47}$$

Proof. We only show (46) because the proof of (47), which is similar to that of (46). Since h/t is sufficiently close to 0, via Lemma 2, we have $\mathbb{E}[(U_\beta(t+h,x) - U_\beta(t,x))^2] = (K_{\beta,d} + o(1))h^{1-\frac{d\beta}{2}}$. Thus, via a well-known estimation (cf., e.g., [28] (p. 23)), we have

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x} - \frac{1}{x^3}\Big)e^{-x^2/2} \le 1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}, \quad \forall x > 0, \tag{48}$$

in which we obtain (46) immediately. The proof is thus completed. □

We needed the following Fernique-type inequality for the TFSPIDEs and their gradients as it is required in the proof.

Lemma 5. *Let $\beta \in (0, 1/2]$, $x \in \mathbb{R}^d$ $(d = 1, 2, 3)$ and $u_0 \equiv 0$ in (1) be fixed.*

(a) Suppose $d \in \{1, 2, 3\}$. Then, for any $\epsilon > 0$, there exist positive and finite constants, i.e., independent of x, and $h_0 = h_0(\epsilon)$ and $c = c(\epsilon)$ are such that, for any compact interval $I_{\text{time}} \subset \mathbb{R}_+$, $0 < h < h_0$ and $u > 0$, we have

$$\mathbb{P}\Big(\sup_{s,t \in I_{\text{time}}, |t-s|<h} |U_\beta(t,x) - U_\beta(s,x)| \geq u K_{\beta,d}^{1/2} h^{\frac{2-d\beta}{4}} \Big) \leq \frac{c}{h} e^{-\frac{u^2}{2+\epsilon}}. \tag{49}$$

(b) Suppose $d = 1$. Then, for any $\epsilon > 0$, there exist positive and finite constants, i.e., independent of x, and $h_0 = h_0(\epsilon)$ and $c = c(\epsilon)$ are such that, for any compact interval $I_{\text{time}} \subset \mathbb{R}_+$, $0 < h < h_0$ and $u > 0$, we have

$$\mathbb{P}\Big(\sup_{s,t \in I_{\text{time}}, |t-s|<h} |\partial_x U_\beta(t,x) - \partial_x U_\beta(s,x)| \geq u K_{\beta,0}^{1/2} h^{\frac{2-3\beta}{4}} \Big) \leq \frac{c}{h} e^{-\frac{u^2}{2+\epsilon}}. \tag{50}$$

Proof. By using (46) and (47), as well as by following the same route as the proof of Proposition 3.3 in [29], we obtain (49) and (50), respectively. This completes the proof. □

Now, we can complete the Proof of Theorem 1.

Proof of Theorem 1. By making use of (49) and (50), as well as by following the same route as the proof of Theorems 1.4 and 1.7 in [13], we obtain (5)–(8). This completes the proof. □

3.2. Hausdorff Dimensions for the Sets of Temporal Fast Points

We prove Theorem 2 in this subsection, thus obtaining Hausdorff dimensions for the sets of temporal fast points of the TFSPIDEs, as well as their gradients, in the process.

Proof of Theorem 2. We only show Equation (11) because Equation (12) can be proved similarly. Equation (11). Via Lemma 5 and the following, i.e., the same lines in the proof of Theorem 2 of [16] (p. 180), we can show that, with a probability of one,

$$\forall \lambda \in [0, 1], \quad \dim(S_{\beta,d,x}(\lambda)) \leq 1 - \lambda^2. \tag{51}$$

That is, the upper bound of Equation (11) is validated. □

We now turn to the proof of the opposite inequality. It suffices to show that, with a probability of one,

$$\forall \lambda \in [0, 1], \quad \dim(S_{\beta,d,x}(\lambda)) \geq 1 - \lambda^2. \tag{52}$$

We follow Theorem 1.1 of [18]. Without a loss of generality, we can assume $0 < \lambda < 1$. For every fixed $0 < \lambda_0 < \lambda < 1$, we show that $S_{\beta,d,x}(\lambda)$ contains a Cantor-like subset of dimension of at least $\eta - 2\epsilon$, where $0 < \epsilon < \eta/2 < 1$ and $\eta = 1 - \lambda_0^2$. A sequence of values for λ_0 converging to λ, as well as ϵ converging to 0, was then used to determine the outcome. The focus of the proof was on creating this Cantor-like subset, which was essentially a generalized version of the reasoning presented in the proofs of [16,18].

We state the following lemma that is required in the proof (see [18]).

Lemma 6. *Suppose $g : [0, 1] \to [0, +\infty)$ is a continuous function with $g(0) = 0$. Let $F \subset [0, 1]$ be such that $F = \cap_{m=1}^{+\infty} F_m$, where $F_1 \supset \cdots \supset F_m \cdots$ for $m = 1, 2, \ldots$, and $F_m = \cup_{k=1}^{N_m} I_{m,i}$ with $\{I_{m,i} : 1 \leq i \leq N_m\}$ being, for each $m \geq 1$, a collection of disjoint closed subintervals of $[0, 1]$.*

Then, if there exist two constants $\delta > 0$ and $C > 0$, such that, for every interval $I \subset [0,1]$ with $|I| \le \delta$, there is a constant $m(I)$, such that, for all $m \ge m(I)$, we have

$$N_m(I) =: \#\{I_{m,i} \subset I; 1 \le i \le N_m\} \le Cg(|I|)N_m, \tag{53}$$

we have $\mu_g(F) > 0$.

Let $\mathcal{T}$ be the collection of intervals $[s,t] \subset [0,1]$ such that

$$U_\beta(t,x) - U_\beta(s,x) \ge \lambda\phi_{|t-s|}.$$

The modulus of continuity (5) tells us that

$$|U_\beta(t,x) - U_\beta(s,x)| \le \sqrt{2}\phi_{|t-s|}, \tag{54}$$

for all $s,t \in [0,1]$ that have a $|s-t|$ that is small enough. Thus, there is $b > 0$, which depends only on λ and λ_0 such that, for every small value, we have $I_{\text{time}} = [s,t] \subset [0,1]$,

$$U_\beta(t,x) - U_\beta(s,x) \ge \lambda_0\phi_{|t-s|}, \tag{55}$$

which implies that $[v,t] \in \mathcal{T}$ for all $v \in I_{\text{time}}(b) = [s, s+b(t-s)]$. For convenience, we assume that b is the reciprocal of an integer.

Suppose that r_m is the reciprocal of an integer, $r_{m+1} < br_m$, and br_m/r_{m+1} is an integer for $m = 1,2,\ldots$ Let δ be a positive number such that $\delta < \epsilon/16$. For every $m \ge 1$, define $\nu_m = \lfloor r_m^{-\delta} \rfloor$, $\varrho_m = \lfloor (r_m^{-1} - 1)/\nu_m \rfloor + 1$ and

$$t_{m,i} = i\nu_m r_m, \quad i = 0,1,\ldots,\varrho_m - 1,$$

$$\mathcal{J}_m = \{[t_{m,i}, t_{m,i} + r_m]; i = 0,1,\ldots,\varrho_m - 1\}.$$

For every $m \ge 1$ and $I_{\text{time}} = [t_{m,i}, t_{m,i} + r_m] \in \mathcal{J}_m$, define

$$\Lambda_{\beta,d,x}(m, I_{\text{time}}) = \gamma_{r_m}^{-1}(U_\beta(t_{m,i} + r_m, x) - U_\beta(t_{m,i}, x)),$$

where $\gamma_h = K_{\beta,d}^{1/2} h^{\frac{2-d\beta}{4}}$. Moreover, we define

$$\begin{aligned}
&\mathcal{J}_{m,+} = \{I_{\text{time}} \in \mathcal{F}_m; \Lambda_{\beta,x}(m, I_{\text{time}}) > \lambda(2\log(1/r_m))^{1/2}\},\\
&\mathcal{J}_{m,+}(b) = \{I_{\text{time}}(b) = [s, s+b(t-s)], I_{\text{time}} = [s,t] \in \mathcal{J}_{m,+}\},\\
&\rho_m(J_{\text{time}}) = \#\{I_{\text{time}} \in \mathcal{J}_{m,+}, I_{\text{time}} \subset J_{\text{time}}\}, \quad \rho_m = \rho_m([0,1]),\\
&\varrho_m(J_{\text{time}}) = \#\{I_{\text{time}} \in \mathcal{J}_m, I_{\text{time}} \subset J_{\text{time}}\}, \quad \varrho_m = \varrho_m([0,1]),\\
&r_m^{1-\eta(m)} = P(N(0,1) > \lambda(2\log(1/r_m))^{1/2}),
\end{aligned}$$

where $0 < \eta(m) \to \eta := 1 - \lambda_0^2$ as $m \to +\infty$.

From (4), we derive that, for any m large enough, $I_{\text{time}} = [s,t] \in \mathcal{J}_{m,+}$ is implied (55). Then, we have $[v,t] \in \mathcal{T}$ for any $s \in I_{\text{time}}(b) \in \mathcal{J}_{m,+}(b)$.

Lemma 7. *Let $\beta \in (0,1/2]$, $x \in \mathbb{R}^d$ ($d = 1,2,3$) and $u_0 \equiv 0$ in (1) be fixed. Then, there exists a positive, independent of x, constant $c = c(d) > 0$, such that, for all $I_{\text{time}} = [t_{m,i}, t_{m,i} + r_m] \in \mathcal{J}_m$ and $J_{\text{time}} = [t_{m,j}, t_{m,j} + r_m] \in \mathcal{J}_m$ with $I_{\text{time}} \cap J_{\text{time}} = \emptyset$, as well as all $m \ge m_0$ with some $m_0 > 0$, we have*

$$\mathbb{E}[\Lambda_{\beta,d,x}(m, I_{\text{time}})\Lambda_{\beta,d,x}(m, J_{\text{time}})] \le c\nu_m^{-1}. \tag{56}$$

Proof. For convenience, we assume that $j > i > 0$. For brevity, we define $Z_{\xi,x}(\cdot,\cdot)$ with the increments of the process $\xi(\cdot,\cdot)$ as follows:

$$Z_{\xi,x}(s,t) = \xi(t,x) - \xi(s,x), \quad t,s \in \mathbb{R}_+, x \in \mathbb{R}^d.$$

It follows from (28) that, for $j > i > 0$ and large m, we have

$$\begin{aligned}
&\mathbb{E}[Z_{X_\beta,x}(i\nu_m, i\nu_m+1)Z_{X_\beta,x}(j\nu_m, j\nu_m+1)] \\
&= \mathbb{E}[(Z_{X_\beta,x}(j\nu_m, i\nu_m+1))^2] - \mathbb{E}[(Z_{X_\beta,x}(j\nu_m, i\nu_m))^2] \\
&\quad - \mathbb{E}[(Z_{X_\beta,x}(j\nu_m+1, i\nu_m+1))^2] + \mathbb{E}[(Z_{X_\beta,x}(j\nu_m+1, i\nu_m))^2] \\
&= K_{\beta,d}(1+o(1))[(j-i+1)^{1-\frac{d\beta}{2}} - 2(j-i)^{1-\frac{d\beta}{2}} + (j-i-1)^{1-\frac{d\beta}{2}}],
\end{aligned} \tag{57}$$

where $X_\beta(\cdot,\cdot)$ is given in (26). Let $f(x) = x^{1-\frac{d\beta}{2}}$ for $x > 0$. Then, $f''(x) < 0$ for $x > 0$. Note that $f(j-i+1) - f(j-i) = \nu_m^{-1} f'(j-i+\theta_1)$ and $f(j-i) - f(j-i-1) = f'(j-i-\theta_2)$, where $\theta_1, \theta_2 \in [0,1]$. This yields the following for $j > i > 0$:

$$\begin{aligned}
&(j-i+1)^{1-\frac{d\beta}{2}} - 2(j-i)^{1-\frac{d\beta}{2}} + (j-i-1)^{1-\frac{d\beta}{2}} \\
&= f'(j-i+\theta_1) - f'(j-i-\theta_2) \\
&= (\theta_1+\theta_2) f''(j-i+\theta_3) \\
&< 0,
\end{aligned}$$

where $\theta_3 \in [-1,1]$. As such, together with (57), we have

$$\mathbb{E}[Z_{X_\beta,x}(i\nu_m, i\nu_m+1)Z_{X_\beta,x}(j\nu_m, j\nu_m+1)] < 0. \tag{58}$$

Similarly to (57),

$$\begin{aligned}
&\mathbb{E}[Z_{Y_\beta,x}(i\nu_m, i\nu_m+1)Z_{Y_\beta,x}(j\nu_m, j\nu_m+1)] \\
&= \mathbb{E}[(Z_{Y_\beta,x}(j\nu_m, i\nu_m+1))^2] - \mathbb{E}[(Z_{Y_\beta,x}(j\nu_m, i\nu_m))^2] \\
&\quad - \mathbb{E}[(Z_{Y_\beta,x}(j\nu_m+1, i\nu_m+1))^2] + \mathbb{E}[(Z_{Y_\beta,x}(j\nu_m+1, i\nu_m))^2],
\end{aligned} \tag{59}$$

where $Y_\beta(\cdot,\cdot)$ is given in (27). It follows from (36) that, for any $t, s \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\mathbb{E}[(Z_{Y_\beta,x}(t,s))^2] \\
&= \frac{1}{2^d\Gamma(d/2)} \frac{\sin^2(\beta\pi)}{\pi^{2+d/2}} \int_0^{+\infty} y^{d/2-1} \int_{-\infty}^0 \left| \int_0^{+\infty} a_\beta(\zeta)\mathbb{K}_{\beta;r,y}(\zeta,t,s)d\zeta \right|^2 drdy \\
&= \frac{1}{2^d\Gamma(d/2)} \frac{\sin^2(\beta\pi)}{\pi^{2+d/2}} \int_0^{+\infty} y^{d/2-1} \int_{-\infty}^0 \int_0^{+\infty} \int_0^{+\infty} a_\beta(\zeta_1)a_\beta(\zeta_2) \\
&\quad \times \mathbb{K}_{\beta;r,y}(\zeta_1,t,s)\mathbb{K}_{\beta;r,y}(\zeta_2,t,s)d\zeta_1 d\zeta_2 drdy,
\end{aligned} \tag{60}$$

where the following notation is used:

$$\mathbb{K}_{\beta;r,y}(\zeta,t,s) = e^{-\zeta(\frac{y}{2})^{1/\beta}(s-r)} \left(e^{-\zeta(\frac{y}{2})^{1/\beta}(t-s)} - 1 \right).$$

By some element calculations, we can conclude that, for $j > i > 0$, we have

$$\begin{aligned}
&\mathbb{K}_{\beta;r,y}(\zeta_1, j\nu_m, i\nu_m+1)\mathbb{K}_{\beta;r,y}(\zeta_2, j\nu_m, i\nu_m+1) \\
&- \mathbb{K}_{\beta;r,y}(\zeta_1, j\nu_m, i\nu_m)\mathbb{K}_{\beta;r,y}(\zeta_2, j\nu_m, i\nu_m) \\
&- \mathbb{K}_{\beta;r,y}(\zeta_1, j\nu_m+1, i\nu_m+1)\mathbb{K}_{\beta;r,y}(\zeta_2, j\nu_m+1, i\nu_m+1) \\
&+ \mathbb{K}_{\beta;r,y}(\zeta_1, j\nu_m+1, i\nu_m)\mathbb{K}_{\beta;r,y}(\zeta_2, j\nu_m+1, i\nu_m)
\end{aligned} \tag{61}$$

$$
\begin{aligned}
&= e^{r(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}}(e^{-(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}}-1)\\
&\quad\times [e^{-(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}(j\nu_m-1)}(1-e^{-\zeta_1(\frac{y}{2})^{1/\beta}})(1-e^{-\zeta_2(\frac{y}{2})^{1/\beta}})\\
&\quad+e^{-((j\nu_m-1)\zeta_1+i\nu_m\zeta_2)(\frac{y}{2})^{1/\beta}}(1-e^{-\zeta_1(\frac{y}{2})^{1/\beta}})(e^{-\zeta_2(\frac{y}{2})^{1/\beta}(j-i)\nu_m}-1)\\
&\quad+e^{-(i\nu_m\zeta_1+(j\nu_m-1)\zeta_2)(\frac{y}{2})^{1/\beta}}(1-e^{-\zeta_2(\frac{y}{2})^{1/\beta}})(e^{-\zeta_1(\frac{y}{2})^{1/\beta}(j-i)\nu_m}-1)]\\
&+e^{r(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}}\Big\{e^{-(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}(j\nu_m-1)}(1-e^{-\zeta_1(\frac{y}{2})^{1/\beta}})(1-e^{-\zeta_2(\frac{y}{2})^{1/\beta}})\\
&\quad-e^{-j(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}\nu_m}(1-e^{-\zeta_1(\frac{y}{2})^{1/\beta}})(1-e^{-\zeta_2(\frac{y}{2})^{1/\beta}})\\
&\quad+e^{-((j\nu_m-1)\zeta_1+i\nu_m\zeta_2)(\frac{y}{2})^{1/\beta}}(1-e^{-\zeta_1(\frac{y}{2})^{1/\beta}})\\
&\qquad\times[e^{-\zeta_2(\frac{y}{2})^{1/\beta}(j-i)\nu_m}(1-e^{-(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}})+(e^{-\zeta_1(\frac{y}{2})^{1/\beta}}-1)]\\
&\quad+e^{-(i\nu_m\zeta_1+(j\nu_m-1)\zeta_2)(\frac{y}{2})^{1/\beta}}(1-e^{-\zeta_2(\frac{y}{2})^{1/\beta}})\\
&\qquad\times[e^{-\zeta_1(\frac{y}{2})^{1/\beta}(j-i)\nu_m}(1-e^{-(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}})+(e^{-\zeta_2(\frac{y}{2})^{1/\beta}}-1)]\Big\}.
\end{aligned}
$$

Since for any $u\geq 0$, $|1-e^{-u}|\leq 2u$, the absolute value of the above equation is less than the following quantity, for any $j>i>0$, we have

$$48e^{r(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}}e^{-(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}(i\nu_m-1)}. \tag{62}$$

Integrating this first in r, we have $\int_{-\infty}^{0} e^{r(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}}dr=\frac{1}{(\zeta_1+\zeta_2)(\frac{y}{2})^{1/\beta}}$. Then, noting that $a_\beta(\zeta_i)\leq\frac{1}{2\cos(\beta\pi)}$ for all $\zeta_i\in\mathbb{R}_+$ and $i=1,2$, via the change in variables $\zeta_1\mapsto u_1 : u_1=(\frac{y}{2})^{1/\beta}(i\nu_m-1)\zeta_1$ and $\zeta_2\mapsto u_2 : u_2=(\frac{y}{2})^{1/\beta}(i\nu_m-1)\zeta_2$—as well as by integrating r, ζ_1 and ζ_2 in (62) separately—we can conclude this integration is less that $\frac{c}{\nu_m(\frac{y}{2})^{1/\beta}}$. Thus, together with (59)–(62), we obtain

$$|\mathbb{E}[Z_{Y_\beta,x}(i\nu_m,i\nu_m+1)Z_{Y_\beta,x}(j\nu_m,j\nu_m+1)]|\leq c\nu_m^{-1}. \tag{63}$$

Since the Gaussian process $\{U_\beta(t,x),t\geq 0\}$ is self-similar with the index $(2-\beta d)/4$ (see [13] (p. 1591)), we obtain

$$\mathbb{E}[\Lambda_{\beta,d,x}(m,I_{\text{time}})\Lambda_{\beta,d,x}(m,J_{\text{time}})]=\mathbb{E}[Z_{U_\beta,x}(i\nu_m,i\nu_m+1)Z_{U_\beta,x}(j\nu_m,j\nu_m+1)]. \tag{64}$$

Since $U_\beta(t,x) = X_\beta(t,x)-Y_\beta(t,x)$, $U_\beta(t,x)$ and $Y_\beta(t,x)$ are independent for $(t,x)\in\mathring{\mathbb{R}}_+\times\mathbb{R}^d$, Equation (64) becomes

$$
\begin{aligned}
&\mathbb{E}[\Lambda_{\beta,d,x}(m,I_{\text{time}})\Lambda_{\beta,d,x}(m,J_{\text{time}})]\\
&=\mathbb{E}[Z_{X_\beta,x}(i\nu_m,i\nu_m+1)Z_{X_\beta,x}(j\nu_m,j\nu_m+1)]-\mathbb{E}[Z_{Y_\beta,x}(i\nu_m,i\nu_m+1)Z_{Y_\beta,x}(j\nu_m,j\nu_m+1)].
\end{aligned} \tag{65}
$$

With (58), (63) and (65), we obtain (56). The proof is thus completed. □

We also need the following three lemmas.

Lemma 8. *For any $0<\zeta\leq 1/2$, there exists an integer m_0, such that*

$$\mathbb{P}(|\rho_m(J_{\text{time}})-\mathbb{E}[\rho_m(J_{\text{time}})]|\geq\lambda\mathbb{E}[\rho_m(J_{\text{time}})])\leq 2\exp(-\zeta(\lambda-2\zeta)\mathbb{E}[\rho_m(J_{\text{time}})])+r_m^5 \tag{66}$$

for all $J_{\text{time}}\subseteq[0,1]$, $m\geq m_0$ and $\lambda>0$.

Proof. We follow Lemma 2.3 of [18]. For brevity, we denote $Z_{m,i} = Z_{U_\beta,x}(t_{m,i}, t_{m,i} + r_m)$, $Y_{m,i} = \gamma_{r_m}^{-1} Z_{m,i}$, $\varrho_m = \varrho_m(J_{\text{time}})$, $\ell_m = (2\log(1/r_m))^{1/2}$ and $\delta_m = \nu_m^{-1}$. Note that

$$\rho_m(J_{\text{time}}) = \sum_{i=1}^{\varrho_m(J_{\text{time}})} \mathbb{I}(\gamma_{r_m}^{-1} Z_{m,i} > \lambda \ell_m).$$

Let $\{\xi_m, Y'_{m,i}, i = 1, \ldots, \varrho_m\}$ be independent mean zero Gaussian random variables with $\mathbb{E}[\xi_m^2] = \delta_m$ and $\mathbb{E}[(Y'_{m,i})^2] = 1 - \delta_m$. Then, $\mathbb{E}[(\xi_m + Y'_{m,i})^2] = \mathbb{E}[(Y_{m,i})^2] = 1$ and $\mathbb{E}[Y_{m,i} Y_{m,j}] \le \mathbb{E}[(\xi_m + Y'_{m,i})(\xi_m + Y'_{m,j})] = \mathbb{E}[\xi_m^2] = \delta_m$ $(i \ne j)$.

For any m large enough, define $p_{m,0} = p_m(\lambda)$, such that $q_m = \zeta(\lambda+1)\mathbb{E}[\rho_m(J_{\text{time}})] = \zeta(\lambda+1)\varrho_m p_{m,0}$, $p_{m,1} = p_m((\lambda - 3\delta_m^{1/2})/(1-\delta_m)^{1/2})$ and $p_{m,2} = p_m((\lambda + 3\delta_m^{1/2})/(1-\delta_m)^{1/2})$, where

$$p_m(z) = \mathbb{P}(N(0,1) > z\ell_m), \ \ z > 0.$$

Let $f(z) = e^z$ if $0 \le z \le q_m$, and $= e^{q_m}(z - q_m + 1)$ if $z \ge q_m$, and let $g(Y_{m,1}, \ldots, Y_{m,\varrho_m}) = f(\zeta \rho_m(J_{\text{time}}))$. Then, $g(Y_{m,1}, \ldots, Y_{m,\varrho_m}) \le e^{\zeta \rho_m(J_{\text{time}})} \vee \varrho_m e^{q_m}$. Via the well-known comparison property (cf. Theorem 3.11 of [30] (p. 74)), we have

$$\mathbb{E}[g(Y_{m,1}, \ldots, Y_{m,\varrho_m})] \le \mathbb{E}[g(\xi_m + Y'_{m,1}, \ldots, \xi_m + Y'_{m,\varrho_m})].$$

Thus, we conclude that

$$\begin{aligned}
&\mathbb{P}(\rho_m(J_{\text{time}}) - \mathbb{E}[\rho_m(J_{\text{time}})] \ge \lambda \mathbb{E}[\rho_m(J_{\text{time}})]) \\
&= \mathbb{P}(f(\zeta \rho_m(J_{\text{time}})) \ge f(q_m))) \\
&= \mathbb{P}(g(Y_{m,1}, \ldots, Y_{m,\varrho_m}) \ge e^{q_m})) \\
&\le e^{-q_m} \mathbb{E}[g(Y_{m,1}, \ldots, Y_{m,\varrho_m})] \\
&\le e^{-q_m} \mathbb{E}[g(\xi_m + Y'_{m,1}, \ldots, \xi_m + Y'_{m,\varrho_m})] \\
&\le e^{-q_m} \Big\{ \mathbb{E}[e^{\zeta \sum_{i=1}^{\varrho_m} \mathbb{I}\{\xi_m + Y'_{m,i} > \lambda \ell_m\}} \mathbb{I}(\xi_m \le 3\delta_m^{1/2} \ell_m)] + \varrho_m e^{q_m} \mathbb{P}(\xi_m > 3\delta_m^{1/2} \ell_m) \Big\} \\
&\le e^{-q_m} \mathbb{E}[e^{\zeta \sum_{i=1}^{\varrho_m} \mathbb{I}\{Y'_{m,i} > (\lambda - 3\delta_m^{1/2})\ell_m\}}] + \varrho_m \mathbb{P}(\xi_m > 3\delta_m^{1/2} \ell_m)\}.
\end{aligned}$$

Via the fact that $\{Y'_{m,i}, i = 1, \ldots, \varrho_m\}$ are independent, it is easy to see that

$$\begin{aligned}
&\mathbb{E}[e^{\zeta \sum_{i=1}^{\varrho_m} \mathbb{I}\{Y'_{m,i} > (\lambda - 3\delta_m^{1/2})\ell_m\}}] \\
&= e^{\zeta \varrho_m p_{m,1}} (\mathbb{E}[e^{\zeta(\mathbb{I}\{Y'_{m,i} > (\lambda - 3\delta_m^{1/2})\ell_m\} - p_{m,1})}])^{\varrho_m} \\
&\le e^{\zeta \varrho_m p_{m,1}} (1 + p_{m,1}(1 - p_{m,1})\zeta^2)^{\varrho_m} \\
&\le e^{\zeta \varrho_m p_{m,1} + \zeta^2 \varrho_m p_{m,1}(1 - p_{m,1})}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\mathbb{P}(\rho_m(J_{\text{time}}) - \mathbb{E}[\rho_m(J_{\text{time}})] \ge \lambda \mathbb{E}[\rho_m(J_{\text{time}})]) \\
&\le e^{-\zeta \varrho_m((\lambda+1)p_{m,0} - (1+\zeta)p_{m,1})} + \varrho_m \mathbb{P}(\xi_m > 3\delta_m^{1/2} \ell_m).
\end{aligned} \tag{67}$$

It follows from (48) that $p_{m,0} \sim p_{m,1}$ as $m \to +\infty$. This implies that $(1+\zeta)p_{m,1} \le (1 + 2\zeta)p_{m,0}$. Thus, (67) becomes

$$\begin{aligned}
&\mathbb{P}(\rho_m(J_{\text{time}}) - \mathbb{E}[\rho_m(J_{\text{time}})] \ge \lambda \mathbb{E}[\rho_m(J_{\text{time}})]) \\
&\le e^{-\zeta \varrho_m(\lambda - 2\zeta)p_{m,0}} + c r_m^{-1} r_m^9 \\
&\le e^{-\zeta \varrho_m(\lambda - 2\zeta)p_{m,0}} + r_m^5.
\end{aligned} \tag{68}$$

Similarly to (68), by choosing $q_m = \zeta((\lambda-1)\varrho_m p_{m,0} + \varrho_m)$, we have

$$\begin{aligned}&\mathbb{P}(\mathbb{E}[\rho_m(J_{\text{time}})] - \rho_m(J_{\text{time}}) \geq \lambda\mathbb{E}[\rho_m(J_{\text{time}})])\\&\leq e^{-\zeta\varrho_m(\lambda-2\zeta)p_{m,0}} + r_m^5.\end{aligned} \tag{69}$$

Thus, together with (68), (66) is yielded. The proof is thus completed. □

Lemma 9. *Given $\epsilon > 0$, $\delta > 0$, with a probability of one, there exists an integer m_0 such that*

$$|\rho_m(J_{\text{time}}) - \mathbb{E}[\rho_m(J_{\text{time}})]| \leq \epsilon\mathbb{E}[\rho_m(J_{\text{time}})] \tag{70}$$

for all $J_{\text{time}} \subseteq [0,1]$, such that $|J_{\text{time}}| \geq \delta$ and all $m \geq m_0(\epsilon,\delta)$.

Proof. It follows from (46) that $p_{m,0} = r_m^{\lambda^2(1+r_m)}$, where $r_m \to 0$ as $m \to +\infty$. This, together with Lemma 8 and the Borel–Cantelli argument, yields (70). The proof is thus completed. □

Lemma 10. *Given $\eta' < \eta = 1-\lambda^2$, there is an absolute constant c such that, with a probability of one, there exists m_1 such that*

$$\rho_m(J_{\text{time}}) \leq c|J_{\text{time}}|^{\eta'}\rho_m([0,1]), \tag{71}$$

for all $J_{\text{time}} \subseteq [0,1]$, $m \geq m_1$.

Proof. It follows from Lemma 9 that it is enough to show that

$$\rho_m(J_{\text{time}}) \leq c|J_{\text{time}}|^{\eta'}\mathbb{E}[\rho_m([0,1])] \leq c|J_{\text{time}}|^{\eta'}\varrho_m r_m^{1-\eta(m)} \tag{72}$$

for $m \geq m_1$. Note that $|J_{\text{time}}| < r_m$ implies $\rho_m(J_{\text{time}}) = 0$, $r_m \leq |J_{\text{time}}| < \nu_m r_m$, which implies $\rho_m(J_{\text{time}}) \leq 1$ and $|J_{\text{time}}|^{\eta'}\varrho_m r_m^{1-\eta(m)} \geq c r_m^{\delta+\eta'-\eta(m)} \to +\infty$. Thus, we need only to consider the case of $|J_{\text{time}}| \geq \nu_m r_m$. It is clearly sufficient to consider only the class $\mathcal{D}_m$ of intervals $[ir_m, jr_m]$, where i,j are integers and $0 \leq i < j \leq (\nu_m r_m)^{-1}$. Note that $\varrho_m \sim \nu_m^{-1} r_m^{-1} \sim r_m^{\delta-1}$ and $\varrho_m(J_{\text{time}}) = |J_{\text{time}}|\varrho_m$. We deduce from Lemma 8 that for any m large enough, we have

$$\begin{aligned}&\mathbb{P}(\rho_m(J_{\text{time}}) > c|J_{\text{time}}|^{\eta'}\varrho_m r_m^{1-\eta(m)}, J \in \mathcal{D}_m)\\&\leq r_m^{-2}\exp(-c|r_m|^{\eta'}\varrho_m r_m^{1-\eta(m)}) + r_m^3\\&\leq r_m^{-2}\exp(-c r_m^{\delta+\eta'-\eta(m)}) + r_m^3.\end{aligned}$$

Since, $\delta + \eta' - \eta(m) \to \delta + \eta' - \eta < 0$, it follows that

$$\sum_{m=1}^{+\infty}\mathbb{P}(\rho_m(J_{\text{time}}) > c|J_{\text{time}}|^{\eta'}\varrho_m r_m^{1-\eta(m)}, J \in \mathcal{D}_m) < +\infty.$$

This implies that, with a probability of one, there is a $m_1 = m_1(\eta') > 0$ such that (72) holds. The proof is thus completed. □

Next, we shall show that the existence of a sequence of sets $F_1 \supset F_2 \supset \cdots$ are such that they satisfy Lemma 2.1's presumptions and that $F = \cap_{m=1}^{+\infty} F_m \subset S_{\beta,d,x}(\lambda)$. We can assume that, for every stage of the construction that is completed in the same probability 1 set, there are only a countable number of steps required and that each step can be completed with a probability of 1. Select $\eta' = \eta - \frac{1}{4}\epsilon$ and define $m_1 =: m_1(\eta')$ such that $m \geq m_1$ and

(71) hold. Assume that the sequence of positive numbers (ϵ_k) satisfies $\sum \epsilon_k < +\infty$. In the first step, when using Lemma 9, we determine an integer $n_1 \geq m_1$ such that

$$|\rho_m - \mathbb{E}[\rho_m]| < \epsilon_1 \mathbb{E}[\rho_m] \ \ (m \geq n_1).$$

And then we shall define an increasing sequence $n_1, n_2, \ldots$ inductively, as well as define for $k \geq 1$.

$$\begin{aligned}
&\{I_{k.i}, 1 \leq i \leq Q_k\} = \{I_{\text{time}}(b) \in \mathcal{J}_{n_k,+}, I_{\text{time}}(b) \subset F_{k-1}\},\\
&F_0 = [0,1], \ \ F_k = \cup_{i=1}^{Q_k} I_{k,i},\\
&Q_k(J_{\text{time}}) = \#\{i, I_{k,i} \subset J_{\text{time}}\} \ \text{ for } J_{\text{time}} \subset [0,1], Q_k = Q_k([0,1]),\\
&\varsigma(k) = \eta(n_k), \ \ \tau(k) = 1 - \varsigma(k), \ \ R_k = |I_{k,i}| = br_{n_k}.
\end{aligned}$$

For each $k \geq 2$, suppose that n_{k-1} has been defined; as such, we can define an n_k large enough to ensure the following:

$$\begin{aligned}
&n_k \geq m_0(\epsilon, R_{k-1}^{2\tau(k-1)/\epsilon}), \ \ n_k = m_0(\epsilon_k, R_{k-1}),\\
&n_k \geq 2n_{k-1}, \ \ r_{n_k} \leq r_{n_{k-1}}^2,
\end{aligned}$$

where $m_0(\epsilon, \delta)$ is the integer determined in Lemma 9 to invalidate (70) and

$$R_k^{1/(2\epsilon)} \leq b^{2\eta} \prod_{i=1}^{k-1} R_i^{\tau(i)} b^{\varsigma(i)}. \tag{73}$$

Then, we have

$$|\rho_m(J_{\text{time}}) - \mathbb{E}\rho_m(J_{\text{time}})| \leq \epsilon_k \mathbb{E}[\rho_m(J_{\text{time}})] \tag{74}$$

for all $\subseteq [0,1]$, such that $|J_{\text{time}}| \geq R_{k-1}$ and all $m \geq n_k$.

By making use of (73), (74) and Lemmas 9 and 10, via following the same route as the proof of (2.23) in [18], we can obtain

$$Q_{k+j}(J_{\text{time}}) \leq c\Big(\prod_{i=1}^{k} \nu_{n_i}\Big) R_k^{\epsilon} |J_{\text{time}}|^{\eta - 2\epsilon} Q_{k+j} \tag{75}$$

for all $R_{k+1} < |J_{\text{time}}| \leq R_k, k \geq 1, j \geq 1$.

Noting that

$$r_{n_k}^2 \leq r_{n_k}^{1+\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}} \leq r_{n_k} r_{n_{k-1}} \cdots r_{n_1}$$

and

$$\prod_{i=1}^{k} \nu_{n_i} \leq \Big(\prod_{i=1}^{k} r_{n_i}\Big)^{-\delta},$$

via (75), we can conclude that

$$Q_{k+j}(J_{\text{time}}) \leq c b^{\epsilon} r_{n_k}^{\epsilon - 2\delta} |J_{\text{time}}|^{\eta - 2\epsilon} Q_{k+j}$$

for all $R_{k+1} < |J_{\text{time}}| \leq R_k, k \geq 1, j \geq 1$. Thus, it follows from Lemma 6, as well as from the fact that $r_{n_k}^{\epsilon-2\delta} \to 0$ $(k \to +\infty)$, with a probability of one, we have

$$\mu_{s^{\eta-2\epsilon}}(F) > 0. \tag{76}$$

Hence, we have proved (52). The proof is thus completed.

3.3. Hitting Probabilities for the Sets of Temporal Fast Points

We prove Theorem 3 in this subsection, thereby obtaining hitting probabilities for the sets of the temporal fast points of the TFSPIDEs, as well as their gradients, in the process.

Proof of Theorem 3. We only show Equation (13) because Equation (14) can be proved similarly. To prove Equation (13), via Remark 2, it is enough to show that, for every analytic set $B \subset \mathbb{R}_+$, we have

$$\mathbb{P}\Big\{ \sup_{t \in B} \limsup_{h \to 0+} \phi_{\beta,d,h} |U_\beta(t+h,x) - U_\beta(t,x)| = (\dim_p(B))^{1/2} \Big\} = 1. \tag{77}$$

□

By using (4) and Lemma 5, as well as by following the same route as the proof of the upper bound of Theorem 2.1 in [19], we obtain

$$\mathbb{P}\Big\{ \sup_{t \in B} \limsup_{h \to 0+} \phi_{\beta,d,h} |U_\beta(t+h,x) - U_\beta(t,x)| \le (\dim_p(B))^{1/2} \Big\} = 1. \tag{78}$$

We now turn to the proof of the opposite inequality. That is, it is enough to show that

$$\mathbb{P}\Big\{ \sup_{t \in B} \limsup_{h \to 0+} \phi_{\beta,d,h} |U_\beta(t+h,x) - U_\beta(t,x)| \ge (\dim_p(B))^{1/2} \Big\} = 1. \tag{79}$$

Fix ϖ such that $\dim_p(B) > \varpi$. For each integer $n \ge 1$, which are denoted by $\mathcal{Q}_n$, the set of all intervals of the form $[m2^{-n}, (m+1)2^{-n}]$, $m \in \mathbb{Z}_+$ are obtained. In words, $\mathcal{Q}_n$ denotes the totality of all intervals. For all $I_{\text{time}} \in \mathcal{Q}_n$, define $\pi_n(I_{\text{time}}) = m2^{-n}$ to be the smallest element in I_{time}. For $I_{\text{time}} \in \mathcal{Q}_n$, which is denote by $\omega_n(I_{\text{time}})$, the indicator function of the event $(\Theta_{\beta,d,x}(\pi_n(I_{\text{time}}), 2^{-n}(\log n)^{-1}) > \varpi^{1/2})$ is obtained, where the following notation is used:

$$\Theta_{\beta,d,x}(t,h) = \phi_{\beta,d,h} |U_\beta(t+h,x) - U_\beta(t,x)|. \tag{80}$$

In other words, $\omega_n(I_{\text{time}})$ is a Bernoulli random variable whose values take 1 or 0 according as to whether we have

$$\Theta_{\beta,d,x}(\pi_n(I_{\text{time}}), 2^{-n}(\log n)^{-1}) > \varpi^{1/2}.$$

Define via $D := \limsup_n D(n)$ a discrete limsup random fractal, where

$$D(n) = \bigcup_{I_{\text{time}} \in \mathcal{Q}_n : \omega_n(I_{\text{time}}) = 1} I^0_{\text{time}},$$

and where I^0_{time} denotes the interior of I_{time}. We can claim that, whenever $\dim_p(B) > \varpi$, then

$$\mathbb{P}(D \cap B \neq \varnothing) = 1. \tag{81}$$

We postpone the verification of (81) and prove (79) first, which thereby completes the proof.

Since $\dim_p(B) > \varpi$, (81), implies that there exists $t \in B$ such that there is $\Theta_{\beta,d,x}(2^{-n}[t2^n], 2^{-n}(\log)^{-1}) \ge \varpi$ for infinitely many instances of n, then, we have, in particular,

$$\sup_{t \in B} \limsup_{n \to +\infty} \Theta_{\beta,d,x}(2^{-n}[t2^n], 2^{-n}(\log)^{-1}) \ge \varpi \ \text{ a.s.}$$

Via (4), we can obtain

$$\lim_{n \to +\infty} \sup_{t \in I_{\text{time}} : I_{\text{time}} \in \mathcal{Q}_n} |\Theta_{\beta,d,x}(t, 2^{-n}(\log)^{-1}) - \Theta_{\beta,d,x}(2^{-n}[t2^n], 2^{-n}(\log)^{-1})| = 0 \ \text{ a.s.}$$

Thus, if $\dim_{\mathrm{P}}(F) > \varpi$, then (79) holds; as such, (77) also holds.

(81) remains to be verified. Fix a small $\eta > 0$ such that $\dim_p(B) > \varpi + \eta$. By [31], there is a closed $B_* \subset B$, such that, for all open sets F, (whenever $B_* \cap F \neq \varnothing$), then $\overline{\dim}_M(B_* \cap F) > \varpi + \eta$ (see [22] for the definition of an upper Minkowski dimension). It is enough to show that $D \cap B_* \neq \varnothing$ when fixing an open set F such that $F \cap B_* \neq \varnothing$. We

can claim that, with a probability of one, $D(n) \cap F \cap B_* \neq \emptyset$ is such for infinitely many n. When defined via $V(n) := \cup_{k=n}^{+\infty} D(k)$, $n \geq 1$, the open sets are obtained. As such, this claim implies that, with a probability of one, $V(n) \cap F \cap B_* \neq \emptyset$ is such for all n. Furthermore, via letting F run over a countable base for the open sets, we can obtain a $V(n) \cap B_*$ that is as dense as in (the complete metric space) B_*. Via Baire's category theorem (see [32]), we have a $B_* \cap \cap_{n=1}^{+\infty} V(n)$ that is dense in B_* and, in particular, non-empty. Since $D = \cap_{n=1}^{+\infty} V(n)$, we can conclude that $D \cap B_* \neq \emptyset$, which, in turn, means that (81) holds and its results follow.

Fix an open set F by satisfying $F \cap B_* \neq \emptyset$. This is denoted by $\mathcal{N}_n$, which are the total number of intervals $I_{\text{time}} \in \mathcal{Q}_n$ that satisfy $I_{\text{time}} \cap F \cap B_* \neq \emptyset$. Since $\overline{\dim}_M(F \cap B_*) > \varpi + \eta$, via the definition of an upper Minkowski dimension, there exists $\varpi_1 > \varpi + \eta$ such that $\mathcal{N}_n \geq 2^{n\varpi_1}$ is the case for the infinitely many integers of n. Thus, $\#(\aleph) = +\infty$, where

$$\aleph := \left\{ n \geq 1 : \mathcal{N}_n \geq 2^{n\varpi_1} \right\}. \tag{82}$$

As denote by $\Omega_n := \sum \omega_n(I_{\text{time}})$, the total number of intervals $I_{\text{time}} \in \mathcal{Q}_n$ is such that $I_{\text{time}} \cap F \cap B_* \cap D(n) \neq \emptyset$, where the sum is taken over for all $I_{\text{time}} \in \mathcal{Q}_n$ such that $I_{\text{time}} \cap B_* \cap F \neq \emptyset$;

$$\Omega_n = \#\{I_{\text{time}} \in \mathcal{Q}_n : I_{\text{time}} \cap B_* \cap F \neq \emptyset, \Theta_{\beta,d,x}(\pi_n(I_{\text{time}}), 2^{-n}(\log n)^{-1}) > \varpi^{1/2}\}.$$

In order to show that, with a probability of one, $D(n) \cap F \cap B_* \neq \emptyset$ applies for the infinitely many instances of n, it suffices to show that $\Omega_n > 0$ applies for the infinitely many instances of n. That is, it is enough to show that

$$P(\Omega_n > 0 \text{ i.o.}) = 1. \tag{83}$$

It follows from Lemma 4 that $p_n = 2^{-n(\varpi + a_n)}$, where $a_n \to 0$ is to $n \to +\infty$. Hence, $\mathbb{E}[\Omega_n] = \mathcal{N}_n p_n \geq 2^{n(\varpi_1 - \varpi - a_n)}$. Thus, it follows from Lemma 9 that, with a probability of one, $\Omega_n \geq c2^{n(\varpi_1 - \varpi - a_n)}$ applies, which implies that $\mathbb{P}(\Omega_n = 0) \to 0$ as is to $n \to +\infty$. Via Fatou's lemma, one can obtain

$$\mathbb{P}(\Omega_n > 0 \ \text{ i.o.}) \geq \limsup_{n \to +\infty} \mathbb{P}(\Omega_n > 0) = 1.$$

This yields (83). This thus completes the proof.

4. Conclusions

In this article, we established the exact, dimension-dependent temporal continuity moduli for fourth-order TFSPIDEs and their gradients. This was achieved by determining the precise values of the normalized constants, and these were supplemented by the prior efforts of Allouba and Xiao on the spatio-temporal Hölder regularity of the fourth-order TFSPIDEs and their gradients. We obtained Hausdorff dimensions and the hitting probabilities of the sets of the temporal fast points for the fourth-order TFSPIDEs and their gradients in a time variable t. It was confirmed that these points of the TFSPIDEs and their gradients, in time, have a probability of one everywhere, and that they are dense with the power of the continuum. In addition, their hitting probabilities were determined by the target set B's packing dimension $\dim_p(B)$. On the one hand, this work has reinforced the temporal continuity moduli and temporal LILs obtained in [13] by obtaining the exact values of their normalized constants; on the other hand, this work has obtained the size of the set of fast points, as well as the potential theory of TFSPIDEs and their gradients.

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Abbreviations

The following abbreviations are used in this manuscript:

TFSPIDE	Time-fractional stochastic partial integro-differential equation
ISLTBM	Inverse-stable-Lévy-time Brownian motion
PDE	Partial differential equation
BTBM	Brownian-time Brownian motion
BTP	Brownian-time process
SIE	Stochastic integral equation
LIL	Law of the iterated logarithm

References

1. Allouba, H.; Nane, E. Interacting time-fractional and Δ^{ν} PDEs systems via Brownian-time and Inverse-stable-Lévy-time Brownian sheets. *Stoch. Dynam.* **2013**, *13*, 1250012. [CrossRef]
2. Caputo, M. Linear models of dissipation whose Q is almost frequency independent. *Part II Geophys. J. Roy. Astr. Soc.* **1967**, *13*, 529–539. [CrossRef]
3. Srivastava, H.M.; Adel, W.; Izadi, M.; El-Sayed, A.A. Solving some physics problems involving fractional-order differential equations with the morgan-voyce Polynomials. *Fractal Fract.* **2023**, *7*, 301. [CrossRef]
4. Chalishajar, D.; Kasinathan, R.; Kasinathan, R. Optimal control for neutral stochastic integrodifferential equations with infinite delay driven by Poisson jumps and rosenblatt process. *Fractal Fract.* **2023**, *7*, 783. [CrossRef]
5. D'Ovidio, M.; Orsingher, E.; Toaldo, B. Time-changed processes governed by space-time fractional telegraph equations. *Stoch. Anal. Appl.* **2014**, *32*, 1009–1045. [CrossRef]
6. Garra, R.; Orsingher, E.; Polito, F. Fractional diffusions with time-varying coefficients. *J. Math. Phys.* **2015**, *56*, 093301. [CrossRef]
7. Chen, Z.-Q.; Kim, K.-H.; Kim, P. Fractional time stochastic partial differential equations. *Stoch. Proc. Appl.* **2015**, *125*, 1470–1499. [CrossRef]
8. Mijena, J.B.; Erkan, N. Space-time fractional stochastic partial differential equations. *Stoch. Proc. Appl.* **2015**, *125*, 3301–3326. [CrossRef]
9. Allouba, H. A Brownian-time excursion into fourth-order PDEs, linearized Kuramoto-Sivashinsky, and BTPSPDEs on $\mathbb{R}_+ \times \mathbb{R}^d$. *Stoch. Dynam.* **2006**, *6*, 521–534. [CrossRef]
10. Allouba, H. Time-fractional and memoryful Δ^{2^k} SIEs on $\mathbb{R}_+ \times \mathbb{R}^d$: How far can we push white noise? *Ill. J. Math.* **2013**, *57*, 919–963.
11. Allouba, H. Brownian-time Brownian motion SIEs on $\mathbb{R}_+ \times \mathbb{R}^d$: Ultra regular direct and lattice-limits solutions and fourth order SPDEs links. *Discrete Contin. Dyn. Syst.* **2013**, *33*, 413–463. [CrossRef]
12. Allouba, H. L-Kuramoto-Sivashinsky SPDEs in one-to-three dimensions: L-KS kernel, sharp Hölder regularity, and Swift-Hohenberg law equivalence. *J. Differ. Equ.* **2015**, *259*, 6851–6884. [CrossRef]
13. Allouba, H.; Xiao, Y. L-Kuramoto-Sivashinsky SPDEs v.s. time-fractional SPIDEs: Exact continuity and gradient moduli, 1/2-derivative criticality, and laws. *J. Differ. Equ.* **2017**, *263*, 15521610. [CrossRef]
14. Wang, W. Spatial moduli of non-differentiability for time-fractional SPIDEs and their gradient. *Symmetry* **2021**, *13*, 380. . [CrossRef]
15. Wang, W. Variations of the solution to a fourth order time-fractional stochastic partial integro-differential equation. *Stoch. Partial Differ.* **2022**, *10*, 582–613. [CrossRef]
16. Orey, S.; Taylor, S.T. How often on a Brownian path does the iterated logarithm fail? *P. Lond. Math. Soc.* **1974**, *28*, 174–192. [CrossRef]
17. Deheuvels, P.; Mason, P. On the fractal nature of empirical increments. *Ann. Probab.* **1995**, *23*, 355–387. [CrossRef]
18. Zhang, L.-X. On the fractal nature of increments of ℓ^p-valued Gaussian processes. *Stoch. Proc. Appl.* **1997**, *71*, 91–110. [CrossRef]
19. Khoshnevisan, D.; Peres, Y.; Xiao, Y. Limsup random fractals. *Electron J. Probab.* **2000**, *5*, 1–24. [CrossRef]
20. Falconer, K.J. *The Geometry of Fractal Sets*; Cambridge Univ. Press: Cambridge, UK, 1985.
21. Taylor, S.J. The measure theory of random fractals. *Math. Proc. Camb. Phil. Soc.* **1986**, *100*, 383–406. [CrossRef]
22. Mattila, P. *Geometry of Sets and Measures in Euclidean Spaces*; Cambridge Univ. Press: Cambridge, UK, 1995.
23. Mueller, C.; Tribe, R. Hitting probabilities of a random string, Electron. *J. Probab.* **2002**, *7*, 10–29.
24. Tudor, C.A. *Analysis of Variations for Self-Similar Processes—A Stochastic Calculus Approach*; Springer: Cham, Switzerland, 2013.
25. Tudor, C.A.; Xiao, Y. Sample path properties of the solution to the fractional-colored stochastic heat equation. *Stoch. Dynam.* **2017**, *17*, 1750004. [CrossRef]

26. Fang, K.T.; Kotz, S.; Ng, K.W. *Symmetric Multivariate and Related Distribution*; Chapman and Hall Ltd.: London, UK, 1990.
27. Haubold, H.J.; Mathai, A.M.; Saxena, R.K. Mittag-Leffler Functions and Their Applications. Hindawi Publishing Corporation. *J. Appl. Math.* **2011**, *2011*, 298628. [CrossRef]
28. Csörgő, M.; Révész, P. *Strong Approxiamtions in Probability and Statistics*; Academic Press: New York, NY, USA, 1981.
29. Meerschaert, M.M.; Wang, W.; Xiao, Y. Fernique type inequality and moduli of continuity for anisotropic Gaussian random fields. *Trans. Am. Math. Soc.* **2013**, *365*, 1081–1107. [CrossRef] [PubMed]
30. Ledoux, M.; Talagrand, M. *Probability in Banach Spaces*; Springer: Berlin, Germany, 1991.
31. Joyce, H, Preiss, D. On the existence of subsets of finite positive packing measure. *Mathematika* **1995**, *42*, 15–24. [CrossRef]
32. Munkres, J.R. *Topology: A First Course*; Prentice-Hall Inc.: Englewood Cliffs, NJ, USA, 1975.