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Article

Systems of Hilfer–Hadamard Fractional Differential Equations with Nonlocal Coupled Boundary Conditions

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Abstract: We study the existence and uniqueness of solutions for a system of Hilfer–Hadamard fractional differential equations. These equations are subject to coupled nonlocal boundary conditions that incorporate Riemann–Stieltjes integrals and a range of Hadamard fractional derivatives. To establish our key findings, we apply various fixed point theorems, notably including the Banach contraction mapping principle, the Krasnosel'skii fixed point theorem applied to the sum of two operators, the Schaefer fixed point theorem, and the Leray–Schauder nonlinear alternative.

Keywords: Hilfer–Hadamard fractional differential equations; nonlocal coupled boundary conditions; existence; uniqueness

MSC: 34A08; 34B10; 34B15

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1. Introduction

We examine the system of fractional differential equations

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}u(t) = f(t,u(t),v(t)), \quad t \in [1,T], \\ {}^{HH}D_1^{\gamma,\delta}v(t) = g(t,u(t),v(t)), \quad t \in [1,T], \end{cases} \tag{1}$$

subject to the nonlocal coupled boundary conditions

$$\begin{cases} u(1) = 0, \ \ {}^{H}D_1^{\varsigma}u(T) = \displaystyle\sum_{i=1}^{m}\int_1^T {}^{H}D_1^{\varrho_i}u(s)\,dH_i(s) + \sum_{i=1}^{n}\int_1^T {}^{H}D_1^{\sigma_i}v(s)\,dK_i(s), \\ v(1) = 0, \ \ {}^{H}D_1^{\vartheta}v(T) = \displaystyle\sum_{i=1}^{p}\int_1^T {}^{H}D_1^{\eta_i}u(s)\,dP_i(s) + \sum_{i=1}^{q}\int_1^T {}^{H}D_1^{\theta_i}v(s)\,dQ_i(s), \end{cases} \tag{2}$$

where $T > 1$, $\alpha, \gamma \in (1,2]$, $\beta, \delta \in [0,1]$, $m,n,p,q \in \mathbb{N}$, $\varsigma, \vartheta, \varrho_i, \sigma_i, \eta_i, \theta_i \in [0,1]$, ${}^{HH}D_1^{\kappa_1,\kappa_2}$ denotes the Hilfer–Hadamard fractional derivative of order κ_1 and type κ_2 (for $\kappa_1 = \alpha, \gamma$ and $\kappa_2 = \beta, \delta$), ${}^{H}D_1^{\kappa}$ represents the Hadamard fractional derivative of order κ (for $\kappa = \varsigma, \vartheta, \varrho_i, \sigma_j, \eta_k, \theta_\iota$, $i = 1,\dots,m$, $j = 1,\dots,n$, $k = 1,\dots,p$ and $\iota = 1,\dots,q$), the continuous functions f and g are defined on $[1,T] \times \mathbb{R}^2$, and the integrals from the boundary conditions (2) are Riemann–Stieltjes integrals with H_i, K_j, P_k, Q_ι, $i = 1,\dots,m$, $j = 1,\dots,n$, $k = 1,\dots,p$ and $\iota = 1,\dots,q$ functions of bounded variation.

In this paper, we present a variety of conditions for the functions f and g such that problem (1) and (2) has at least one solution. We will write our problem as an equivalent system of integral equations, and then we will associate it with an operator whose fixed points are our solutions. The proof of our primary outcomes involves the utilization of diverse fixed point theorems. Noteworthy among these theorems are the Banach contraction mapping principle, the Krasnosel'skii fixed point theorem applied to the sum of two operators, the Schaefer fixed point theorem, and the Leray–Schauder nonlinear alternative. The nonlocal boundary conditions (2) are general ones, and they include different particular cases. For example, if $\kappa = 0$, for $\kappa = \varsigma, \vartheta, \varrho_i, \sigma_j, \eta_k, \theta_\iota$, $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, p$ and $\iota = 1, \dots, q$, then the Hadamard derivative ${}^{H}D_1^{\kappa} z(t)$ coincides with $z(t)$. If one of the order of the Hadamard derivatives from the right-hand side of the relations from (2) is zero (for example, if ϱ_1 is zero), then the term $\int_1^T {}^{H}D_1^{\varrho_1} u(s)\, dH_1(s)$ becomes $\int_1^T u(s)\, dH_1(s)$, which contains the cases of the multi-point boundary conditions for the function u (if H_1 is a step function); a classical integral condition; a combination of them; or even a Hadamard fractional integral for a special form of H_1 (as we mentioned in [1]). If $\varrho_1 \in (0, 1]$ and H_1 is a step function, then $\int_1^T {}^{H}D_1^{\varrho_1} u(s)\, dH_1(s) = \sum_{i=1}^{n_0} {}^{H}D_1^{\varrho_1} u(\xi_i)$, which is a combination of the Hadamard fractional derivatives of function u in various points. If all functions K_i, $i = 1, \dots, n$ and P_j, $j = 1, \dots, p$ are constant functions, then the boundary conditions become uncoupled boundary conditions (where the Hadamard derivative of order ς of the function u in the point T is dependent only of the derivatives ${}^{H}D_1^{\varrho_i}$, $i = 1, \dots, m$ of the function u, and the Hadamard derivative of order ϑ of the function v in the point T is dependent only of the derivatives ${}^{H}D_1^{\theta_i}$, $i = 1, \dots, q$ of function v), and if H_i, $i = 1, \dots, m$ and Q_j, $j = 1, \dots, q$ are constant functions, then the boundary conditions become purely coupled boundary conditions (in which the Hadamard derivative of order ς of the function u in T is dependent only of the derivatives ${}^{H}D_1^{\sigma_i}$, $i = 1, \dots, n$ of the function v, and the Hadamard derivative of order ϑ of the function v in T is dependent only of the derivatives ${}^{H}D_1^{\eta_i}$, $i = 1, \dots, p$ of the function u).

Next, we will introduce some papers that are relevant to the issue posed by Equations (1) and (2). In [2], the authors investigated the existence and uniqueness of solutions for the Hilfer–Hadamard fractional differential equation with nonlocal boundary conditions

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta} x(t) = f(t, x(t)), \ t \in [1, T], \\ x(1) = 0, \ x(T) = \displaystyle\sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i \, {}^{H}I_1^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k \, {}^{H}D_1^{\omega_k} x(\mu_k), \end{cases} \tag{3}$$

where $\alpha \in (1, 2]$, $\beta \in [0, 1]$, $\eta_j, \zeta_i, \lambda_k \in \mathbb{R}$, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, ${}^{H}I^{\phi_i}$ is the Hadamard fractional integral operator of order $\phi_i > 0$, and $\xi_j, \theta_i, \mu_k \in (1, T)$ for $j = 1, \dots, m$, $i = 1, \dots, n$, $k = 1, \dots, r$. The multi-valued version of problem (3) is also studied. For the proof of the main results, they used differing fixed point theorems. In [3], the authors proved the existence of solutions for the system of sequential Hilfer–Hadamard fractional differential equations supplemented with boundary conditions

$$\begin{cases} ({}^{HH}D_1^{\alpha_1,\beta_1} + \lambda_1 \, {}^{HH}D_1^{\alpha_1 - 1,\beta_1}) u(t) = f(t, u(t), v(t)), \ t \in [1, e], \\ ({}^{HH}D_1^{\alpha_2,\beta_2} + \lambda_2 \, {}^{HH}D_1^{\alpha_2 - 1,\beta_2}) v(t) = g(t, u(t), v(t)), \ t \in [1, e], \\ u(1) = 0, \ u(e) = A_1, \ v(1) = 0, \ v(e) = A_2, \end{cases} \tag{4}$$

where $\alpha_1, \alpha_2 \in (1, 2]$, $A_1, A_2, \lambda_1, \lambda_2 \in \mathbb{R}_+$, and $f, g : [1, e] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions.

In paper [4], Hadamard defined a fractional derivative with a kernel involving a logarithmic function with an arbitrary exponent. In [5], Hilfer introduced a new fractional derivative (known as the Hilfer fractional derivative), which is a generalization of the Riemann-Liouville fractional derivative and the Caputo fractional derivative. Some applications of this new fractional derivative are presented in papers [6,7]. The Hilfer–Hadamard fractional derivative is an interpolation of the Hadamard fractional derivative, and it covers the cases of the Riemann–Liouville–Hadamard and Caputo–Hadamard fractional derivatives (see the definition in Section 2). The distinctive aspects of our presented challenge, (1) and (2), emerge from the exploration of a set of Hilfer–Hadamard fractional differential equations encompassing diverse orders and types. Additionally, the introduction of general nonlocal boundary conditions (2) contributes novelty, extending beyond numerous specific instances as previously observed. To the best of our knowledge, this issue represented by Equations (1) and (2) is a novel problem in the literature. Our theorems represent original contributions and make substantial advancements in the realm of coupled systems involving Hilfer–Hadamard fractional derivatives. Although the techniques employed in demonstrating our primary findings in Section 3 are conventional, their adaptation to address our problem (1) and (2) is innovative. For more recent investigations concerning Hadamard, Hilfer, and Hilfer–Hadamard fractional differential equations and their applications, we recommend the monograph [8] and the following papers: [1,9–28].

The structure of the paper unfolds as follows: In Section 2, we offer definitions and properties related to fractional derivatives, along with a result regarding the existence of solutions for the linear boundary value problem linked to Equations (1) and (2). Moving on, Section 3 is dedicated to the core findings concerning the existence and uniqueness of solutions for problem (1) and (2). Subsequently, in Section 4, we provide illustrative examples that demonstrate the practical application of our theorems. Lastly, concluding insights for this paper can be found in Section 5.

2. Auxiliary Results

In this section, we present some definitions and properties of fractional derivatives and an existence result for the linear boundary value problem associated with (1) and (2).

Definition 1 (Hadamard fractional integral [29])**.** *For a function $z : [a,\infty) \to \mathbb{R}$, ($a \geq 0$), the Hadamard fractional integral of order $p > 0$ is defined by*

$$({}^{H}I_a^{p} z)(x) = \frac{1}{\Gamma(p)} \int_a^x \left(\ln \frac{x}{t}\right)^{p-1} \frac{z(t)}{t}\, dt, \quad x > a, \tag{5}$$

and $({}^{H}I_a^{0} z)(x) = z(x), \ x > a$.

Definition 2 (Hadamard fractional derivative [29])**.** *For a function $z : [a,\infty) \to \mathbb{R}$, ($a \geq 0$), the Hadamard fractional derivative of order $p > 0$ is defined by*

$${}^{H}D_a^{p} z(x) = \left(x\frac{d}{dx}\right)^n {}^{H}I_a^{n-p} z(x) = \frac{1}{\Gamma(n-p)} \left(x\frac{d}{dx}\right)^n \int_a^x \left(\ln \frac{x}{t}\right)^{n-p-1} \frac{z(t)}{t}\, dt, \tag{6}$$

where $n-1 < p < n$, ($n \in \mathbb{N}$). For $p = m \in \mathbb{N}$, ${}^{H}D_a^{m} z(x) = (\delta^m z)(x), \ x > a$, where $\delta = x\frac{d}{dx}$ is the δ-derivative, and, for $p = 0$, ${}^{H}D_a^{0} z(x) = z(x)$.

Lemma 1 ([29])**.** *If $\alpha,\ \beta > 0$, and $a > 0$, then*

$$\begin{aligned} \left({}^{H}I_a^{\alpha} \left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \left(\ln \frac{x}{a}\right)^{\beta+\alpha-1}, \\ \left({}^{H}D_a^{\alpha} \left(\ln \frac{t}{a}\right)^{\beta-1}\right)(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{x}{a}\right)^{\beta-\alpha-1}. \end{aligned} \tag{7}$$

Definition 3 (Hilfer–Hadamard fractional derivative [2,7])**.** *Let $z \in L^1(a,b)$ and $n-1 < \alpha \leq n$, ($n \in \mathbb{N}$), $0 \leq \beta \leq 1$. The Hilfer–Hadamard fractional derivative of order α and type β for the function z is defined by*

$$\begin{aligned} &\left({}^{HH}D_a^{\alpha,\beta} z\right)(t) = \left({}^{H}I_a^{\beta(n-\alpha)} \delta^n \, {}^{H}I_a^{(n-\alpha)(1-\beta)} z\right)(t) \\ &= \left({}^{H}I_a^{\beta(n-\alpha)} \delta^n \, {}^{H}I_a^{(n-\gamma)} z\right)(t) = \left({}^{H}I_a^{\beta(n-\alpha)} \, {}^{H}D_a^{\gamma} z\right)(t), \end{aligned} \tag{8}$$

where $\gamma = \alpha + n\beta - \alpha\beta$.

If $\beta = 0$, the Hilfer–Hadamard fractional derivative ${}^{HH}D_a^{\alpha,\beta}z$ coincides with the Hadamard fractional derivative ${}^{H}D_a^{\alpha}z$. If $\beta = 1$, the fractional derivative ${}^{HH}D_a^{\alpha,\beta}z$ coincides with the Caputo–Hadamard derivative, given by ${}^{CH}D_a^{\alpha}z(t) = ({}^{H}I_a^{n-\alpha}\delta^n z)(t)$.

Theorem 1 ([2]). *Let $\alpha > 0$, $n-1 < \alpha \le n$, $(n \in \mathbb{N})$, $\beta \in [0,1]$, $\gamma = \alpha + n\beta - \alpha\beta$, and $0 < a < b < \infty$. If $z \in L^1(a,b)$ and $({}^{H}I_a^{n-\gamma}z)(t) \in AC_\delta^n[a,b]$, then the following relation holds*

$$ {}^{H}I_a^{\alpha}\left({}^{HH}D_a^{\alpha,\beta}z\right)(t) = {}^{H}I_a^{\gamma}\left({}^{H}D_a^{\gamma}z\right)(t) = z(t) - \sum_{i=0}^{n-1}\frac{(\delta^{(n-i-1)}({}^{H}I_a^{n-\gamma}z))(a)}{\Gamma(\gamma-i)}\left(\ln\frac{t}{a}\right)^{\gamma-i-1}. \quad (9)$$

We consider now the system of linear fractional differential equations

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}u(t) = h(t), & t \in [1,T], \\ {}^{HH}D_1^{\gamma,\delta}v(t) = k(t), & t \in [1,T], \end{cases} \quad (10)$$

subject to the boundary conditions (2), where $h,k \in C([1,T],\mathbb{R})$. We denote by $\lambda = \alpha + (2-\alpha)\beta$, $\mu = \gamma + (2-\gamma)\delta$, and

$$\begin{aligned} a &= \frac{\Gamma(\lambda)}{\Gamma(\lambda-\varsigma)}(\ln T)^{\lambda-\varsigma-1} - \sum_{i=1}^{m}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\varrho_i)}\int_1^T(\ln s)^{\lambda-\varrho_i-1}\,dH_i(s), \\ b &= \sum_{i=1}^{n}\frac{\Gamma(\mu)}{\Gamma(\mu-\sigma_i)}\int_1^T(\ln s)^{\mu-\sigma_i-1}\,dK_i(s), \\ c &= \sum_{i=1}^{p}\frac{\Gamma(\lambda)}{\Gamma(\lambda-\eta_i)}\int_1^T(\ln s)^{\lambda-\eta_i-1}\,dP_i(s), \\ d &= \frac{\Gamma(\mu)}{\Gamma(\mu-\vartheta)}(\ln T)^{\mu-\vartheta-1} - \sum_{i=1}^{q}\frac{\Gamma(\mu)}{\Gamma(\mu-\theta_i)}\int_1^T(\ln s)^{\mu-\theta_i-1}\,dQ_i(s), \\ \Delta &= ad - bc. \end{aligned} \quad (11)$$

Lemma 2. *We suppose that $a,b,c,d \in \mathbb{R}$, $\Delta \neq 0$, and $h,k \in C([1,T],\mathbb{R})$. Then, the solution of problem (10) and (2) is given by*

$$\begin{aligned} u(t) &= {}^{H}I_1^{\alpha}h(t) + \frac{(\ln t)^{\lambda-1}}{\Delta}\Big[-d\,{}^{H}I_1^{\alpha-\varsigma}h(T) + d(A_1(h)+A_2(k)) \\ &\quad -b\,{}^{H}I_1^{\gamma-\vartheta}k(T) + b(A_3(h)+A_4(k))\Big], \quad t \in [1,T], \\ v(t) &= {}^{H}I_1^{\gamma}k(t) + \frac{(\ln t)^{\mu-1}}{\Delta}\Big[-a\,{}^{H}I_1^{\gamma-\vartheta}k(T) + a(A_3(h)+A_4(k)) \\ &\quad -c\,{}^{H}I_1^{\alpha-\varsigma}h(T) + c(A_1(h)+A_2(k))\Big], \quad t \in [1,T], \end{aligned} \quad (12)$$

where operators $A_i : C([1,T],\mathbb{R}) \to \mathbb{R},\ i = 1,\ldots,4$ are defined by

$$\begin{aligned} A_1(h) &= \sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}\frac{h(\tau)}{\tau}d\tau\right)dH_i(s), \\ A_2(k) &= \sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}\frac{k(\tau)}{\tau}\,d\tau\right)dK_i(s), \\ A_3(h) &= \sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}\frac{h(\tau)}{\tau}\,d\tau\right)dP_i(s), \\ A_4(k) &= \sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}\frac{k(\tau)}{\tau}\,d\tau\right)dQ_i(s). \end{aligned} \quad (13)$$

Proof. We apply the integral operators ${}^HI_1^\alpha$ and ${}^HI_1^\gamma$, respectively, to equations of system (10). Then, the solutions of system (10) are given by

$$\begin{cases} u(t) = a_1(\ln t)^{\lambda-1} + a_2(\ln t)^{\lambda-2} + {}^H I_1^\alpha h(t), \ \ t \in [1,T], \\ v(t) = b_1(\ln t)^{\mu-1} + b_2(\ln t)^{\mu-2} + {}^H I_1^\gamma k(t), \ \ t \in [1,T], \end{cases} \tag{14}$$

where $a_i, b_i \in \mathbb{R}, \ i = 1,2$. Because $u(1) = v(1) = 0$, we deduce that $a_2 = b_2 = 0$. So, we obtain, for the solutions of (10), the formulas

$$\begin{cases} u(t) = a_1(\ln t)^{\lambda-1} + {}^H I_1^\alpha h(t), \ \ t \in [1,T], \\ v(t) = b_1(\ln t)^{\mu-1} + {}^H I_1^\gamma k(t), \ \ t \in [1,T]. \end{cases} \tag{15}$$

For $\kappa = \varsigma, \varrho_i, \eta_j, i = 1,\dots,m, \ j = 1,\dots,p$, we find

$$\begin{aligned} {}^HD_1^\kappa u(t) &= a_1\, {}^HD_1^\kappa(\ln t)^{\lambda-1} + {}^HD_1^\kappa\, {}^HI_1^\alpha h(t) = a_1 \frac{\Gamma(\lambda)}{\Gamma(\lambda-\kappa)}(\ln t)^{\lambda-\kappa-1} + {}^HI_1^{\alpha-\kappa} h(t) \\ &= a_1 \frac{\Gamma(\lambda)}{\Gamma(\lambda-\kappa)}(\ln t)^{\lambda-\kappa-1} + \frac{1}{\Gamma(\alpha-\kappa)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-\kappa-1} \frac{h(s)}{s}\, ds, \end{aligned} \tag{16}$$

and for $\widetilde{\kappa} = \vartheta, \sigma_i, \theta_j, i = 1,\dots,n, \ j = 1,\dots,q$, we obtain

$$\begin{aligned} {}^HD_1^{\widetilde{\kappa}} v(t) &= b_1\, {}^HD_1^{\widetilde{\kappa}}(\ln t)^{\mu-1} + {}^HD_1^{\widetilde{\kappa}}\, {}^HI_1^\gamma k(t) = b_1 \frac{\Gamma(\mu)}{\Gamma(\mu-\widetilde{\kappa})}(\ln t)^{\mu-\widetilde{\kappa}-1} + {}^HI_1^{\gamma-\widetilde{\kappa}} k(t) \\ &= b_1 \frac{\Gamma(\mu)}{\Gamma(\mu-\widetilde{\kappa})}(\ln t)^{\mu-\widetilde{\kappa}-1} + \frac{1}{\Gamma(\gamma-\widetilde{\kappa})} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-\widetilde{\kappa}-1} \frac{k(s)}{s}\, ds. \end{aligned} \tag{17}$$

By applying the conditions ${}^HD_1^\varsigma u(T) = \sum_{i=1}^m \int_1^T {}^HD_1^{\varrho_i} u(s)\, dH_i(s) + \sum_{i=1}^n \int_1^T {}^HD_1^{\sigma_i} v(s)\, dK_i(s)$, and ${}^HD_1^\vartheta v(T) = \sum_{i=1}^p \int_1^T {}^HD_1^{\eta_i} u(s)\, dP_i(s) + \sum_{i=1}^q \int_1^T {}^HD_1^{\theta_i} v(s)\, dQ_i(s)$, we deduce

$$\begin{aligned} & a_1 \frac{\Gamma(\lambda)}{\Gamma(\lambda-\varsigma)}(\ln T)^{\lambda-\varsigma-1} + \frac{1}{\Gamma(\alpha-\varsigma)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-\varsigma-1} \frac{h(s)}{s}\, ds \\ & \quad = \sum_{i=1}^m \int_1^T \left[a_1 \frac{\Gamma(\lambda)}{\Gamma(\lambda-\varrho_i)}(\ln s)^{\lambda-\varrho_i-1} + \frac{1}{\Gamma(\alpha-\varrho_i)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\varrho_i-1} \frac{h(\tau)}{\tau}\, d\tau \right] dH_i(s) \\ & \quad + \sum_{i=1}^n \int_1^T \left[b_1 \frac{\Gamma(\mu)}{\Gamma(\mu-\sigma_i)}(\ln s)^{\mu-\sigma_i-1} + \frac{1}{\Gamma(\gamma-\sigma_i)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\sigma_i-1} \frac{k(\tau)}{\tau}\, d\tau \right] dK_i(s), \\ & b_1 \frac{\Gamma(\mu)}{\Gamma(\mu-\vartheta)}(\ln T)^{\mu-\vartheta-1} + \frac{1}{\Gamma(\gamma-\vartheta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\gamma-\vartheta-1} \frac{k(s)}{s}\, ds \\ & \quad = \sum_{i=1}^p \int_1^T \left[a_1 \frac{\Gamma(\lambda)}{\Gamma(\lambda-\eta_i)}(\ln s)^{\lambda-\eta_i-1} + \frac{1}{\Gamma(\alpha-\eta_i)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\eta_i-1} \frac{h(\tau)}{\tau}\, d\tau \right] dP_i(s) \\ & \quad + \sum_{i=1}^q \int_1^T \left[b_1 \frac{\Gamma(\mu)}{\Gamma(\mu-\theta_i)}(\ln s)^{\mu-\theta_i-1} + \frac{1}{\Gamma(\gamma-\theta_i)} \int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\theta_i-1} \frac{k(\tau)}{\tau}\, d\tau \right] dQ_i(s), \end{aligned} \tag{18}$$

or

$$\begin{aligned} & a_1 \left[\frac{\Gamma(\lambda)}{\Gamma(\lambda-\varsigma)}(\ln T)^{\lambda-\varsigma-1} - \sum_{i=1}^m \frac{\Gamma(\lambda)}{\Gamma(\lambda-\varrho_i)} \int_1^T (\ln s)^{\lambda-\varrho_i-1}\, dH_i(s) \right] \\ & \quad - b_1 \sum_{i=1}^n \frac{\Gamma(\mu)}{\Gamma(\mu-\sigma_i)} \int_1^T (\ln s)^{\mu-\sigma_i-1}\, dK_i(s) \\ & \quad = -{}^HI_1^{\alpha-\varsigma} h(T) + A_1(h) + A_2(k), \\ & -a_1 \sum_{i=1}^p \frac{\Gamma(\lambda)}{\Gamma(\lambda-\eta_i)} \int_1^T (\ln s)^{\lambda-\eta_i-1}\, dP_i(s) \\ & \quad + b_1 \left[\frac{\Gamma(\mu)}{\Gamma(\mu-\vartheta)}(\ln T)^{\mu-\vartheta-1} - \sum_{i=1}^q \frac{\Gamma(\mu)}{\Gamma(\mu-\theta_i)} \int_1^T (\ln s)^{\mu-\theta_i-1}\, dQ_i(s) \right] \\ & \quad = -{}^HI_1^{\gamma-\vartheta} k(T) + A_3(h) + A_4(k). \end{aligned} \tag{19}$$

The determinant of system (19) in the unknowns a_1 and b_1 is

$$\begin{vmatrix} a & -b \\ -c & d \end{vmatrix} = ad - bc, \tag{20}$$

that is, Δ, given by (11), which is different than zero by the assumptions of this lemma. So, the solution of system (19) is unique, namely

$$\begin{aligned} a_1 &= -\frac{d}{\Delta}\,{}^H I_1^{\alpha-\varsigma} h(T) + \frac{d}{\Delta}(A_1(h) + A_2(k)) - \frac{b}{\Delta}\,{}^H I_1^{\gamma-\vartheta} k(T) + \frac{b}{\Delta}(A_3(h) + A_4(k)), \\ b_1 &= -\frac{a}{\Delta}\,{}^H I_1^{\gamma-\vartheta} k(T) + \frac{a}{\Delta}(A_3(h) + A_4(k)) - \frac{c}{\Delta}\,{}^H I_1^{\alpha-\varsigma} h(T) + \frac{c}{\Delta}(A_1(h) + A_2(k)). \end{aligned} \tag{21}$$

By replacing the above formulas for a_1 and b_1 in (15), we obtain the solution of problems (10) and (2) given by (12). $\square$

3. Existence Results

In this section, we will give the main existence and uniqueness theorems for the solutions of problem (1) and (2). By using Lemma 2, our problem (1) and (2) can be equivalently written as the following system of integral equations

$$\begin{cases} u(t) = {}^H I_1^{\alpha} F_{uv}(t) + \dfrac{(\ln t)^{\lambda-1}}{\Delta}\Big[-d\,{}^H I_1^{\alpha-\varsigma} F_{uv}(T) + d(A_1(F_{uv}) + A_2(G_{uv})) \\ \quad -b\,{}^H I_1^{\gamma-\vartheta} G_{uv}(T) + b(A_3(F_{uv}) + A_4(G_{uv}))\Big], \ \ t \in [1,T], \\ v(t) = {}^H I_1^{\gamma} G_{uv}(t) + \dfrac{(\ln t)^{\mu-1}}{\Delta}\Big[-a\,{}^H I_1^{\gamma-\vartheta} G_{uv}(T) + a(A_3(F_{uv}) + A_4(G_{uv})) \\ \quad -c\,{}^H I_1^{\alpha-\varsigma} F_{uv}(T) + c(A_1(F_{uv}) + A_2(G_{uv}))\Big], \ \ t \in [1,T], \end{cases} \tag{22}$$

where $F_{uv}(\tau) = f(\tau, u(\tau), v(\tau)),\ G_{uv}(\tau) = g(\tau, u(\tau), v(\tau)), \tau \in [1,T]$.

We consider the Banach $\mathcal{X} = C([1,T], \mathbb{R})$ with the supremum norm $\|u\| = \sup_{t\in[1,T]} |u(t)|$ and the Banach space $\mathcal{Y} = \mathcal{X} \times \mathcal{X}$ with the norm $\|(u,v)\|_{\mathcal{Y}} = \|u\| + \|v\|$. We define the operator $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{A}(u,v) = (\mathcal{A}_1(u,v), \mathcal{A}_2(u,v))$, with $\mathcal{A}_1$, $\mathcal{A}_2 : \mathcal{Y} \to \mathcal{X}$ given by

$$\begin{aligned} \mathcal{A}_1(u,v)(t) &= {}^H I_1^{\alpha} F_{uv}(t) + \frac{(\ln t)^{\lambda-1}}{\Delta}\Big[-d\,{}^H I_1^{\alpha-\varsigma} F_{uv}(T) + d(A_1(F_{uv}) + A_2(G_{uv})) \\ &\quad -b\,{}^H I_1^{\gamma-\vartheta} G_{uv}(T) + b(A_3(F_{uv}) + A_4(G_{uv}))\Big], \\ \mathcal{A}_2(u,v)(t) &= {}^H I_1^{\gamma} G_{uv}(t) + \frac{(\ln t)^{\mu-1}}{\Delta}\Big[-a\,{}^H I_1^{\gamma-\vartheta} G_{uv}(T) + a(A_3(F_{uv}) + A_4(G_{uv})) \\ &\quad -c\,{}^H I_1^{\alpha-\varsigma} F_{uv}(T) + c(A_1(F_{uv}) + A_2(G_{uv}))\Big], \end{aligned} \tag{23}$$

for all $t \in [1,T]$ and $(u,v) \in \mathcal{Y}$. We see that the solutions of problem (1) and (2) (or system (22)) are the fixed points of operator $\mathcal{A}$. So, next, we will investigate the existence of the fixed points of this operator $\mathcal{A}$ in the space $\mathcal{Y}$.

We present now the basic assumptions that we will use in the next results.

$(H1)$ $\alpha, \gamma \in (1,2]$; $\beta, \delta \in [0,1]$; $m,n,p,q \in \mathbb{N}$; $\varsigma, \vartheta, \varrho_i, \sigma_j, \eta_k, \theta_\iota \in [0,1]$; H_i, K_j, P_k, Q_ι are bounded variation functions, for all $i = 1,\ldots,m$, $j = 1,\ldots,n$, $k = 1,\ldots,p$, $\iota = 1,\ldots,q$; $a,b,c,d \in \mathbb{R}$, and $\Delta \neq 0$ (given by (11)).

We also introduce the constants

$$\begin{aligned} \Xi_1 &= \frac{1}{\Gamma(\alpha+1)}(\ln T)^{\alpha} + \frac{(\ln T)^{\lambda-1}}{|\Delta|}\Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma+1)}(\ln T)^{\alpha-\varsigma} \\ &\quad + |d|\sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i+1)}\left|\int_1^T (\ln s)^{\alpha-\varrho_i}\, dH_i(s)\right| \\ &\quad + |b|\sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i+1)}\left|\int_1^T (\ln s)^{\alpha-\eta_i}\, dP_i(s)\right|\Bigg], \end{aligned}$$

$$
\begin{aligned}
\Xi_2 = {} & \frac{(\ln T)^{\lambda-1}}{|\Delta|}\left[|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i+1)}\left|\int_1^T(\ln s)^{\gamma-\sigma_i}\,dK_i(s)\right|\right.\\
& \left.+\frac{|b|}{\Gamma(\gamma-\vartheta+1)}(\ln T)^{\gamma-\vartheta}+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i+1)}\left|\int_1^T(\ln s)^{\gamma-\theta_i}\,dQ_i(s)\right|\right],\\
\Xi_3 = {} & \frac{(\ln T)^{\mu-1}}{|\Delta|}\left[|a|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i+1)}\left|\int_1^T(\ln s)^{\alpha-\eta_i}\,dP_i(s)\right|\right.\\
& \left.+\frac{|c|}{\Gamma(\alpha-\varsigma+1)}(\ln T)^{\alpha-\varsigma}+|c|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i+1)}\left|\int_1^T(\ln s)^{\alpha-\varrho_i}\,dH_i(s)\right|\right],\\
\Xi_4 = {} & \frac{1}{\Gamma(\gamma+1)}(\ln T)^{\gamma}+\frac{(\ln T)^{\mu-1}}{|\Delta|}\left[\frac{|a|}{\Gamma(\gamma-\vartheta+1)}(\ln T)^{\gamma-\vartheta}\right.\\
& +|a|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i+1)}\left|\int_1^T(\ln s)^{\gamma-\theta_i}\,dQ_i(s)\right|\\
& \left.+|c|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i+1)}\left|\int_1^T(\ln s)^{\gamma-\sigma_i}\,dK_i(s)\right|\right].
\end{aligned}
\tag{24}
$$

Our first existence and uniqueness theorem for problem (1) and (2) is the following one, which is based on the Banach contraction mapping principle (see [30]).

Theorem 2. *We assume that assumption* $(H1)$ *holds. In addition, we suppose that the functions* $f,g:[1,T]\times\mathbb{R}^2\to\mathbb{R}$ *are continuous and satisfy the condition*

$(H2)$ *There exist* $l_i>0$, $i=1,\ldots,4$ *such that*

$$
\begin{aligned}
|f(t,x_1,y_1)-f(t,x_2,y_2)| &\le l_1|x_1-x_2|+l_2|y_1-y_2|,\\
|g(t,x_1,y_1)-g(t,x_2,y_2)| &\le l_3|x_1-x_2|+l_4|y_1-y_2|,
\end{aligned}
\tag{25}
$$

for all $t\in[1,T]$ *and* $x_i,y_i\in\mathbb{R}$, $i=1,2$.

If

$$
l_5(\Xi_1+\Xi_3)+l_6(\Xi_2+\Xi_4)<1, \tag{26}
$$

where $l_5=\max\{l_1,l_2\}$, $l_6=\max\{l_3,l_4\}$, *then the boundary value problem (1) and (2) has a unique solution* $(u(t),v(t))$, $t\in[1,T]$.

Proof. We will verify that operator $\mathcal{A}$ is a contraction in the space $\mathcal{Y}$. We denote this by $\Lambda_1=\sup_{t\in[1,T]}|f(t,0,0)|$ and $\Lambda_2=\sup_{t\in[1,T]}|g(t,0,0)|$. By using $(H2)$, we find

$$
\begin{aligned}
|F_{uv}(t)| &= |f(t,u(t),v(t))|\le|f(t,u(t),v(t))-f(t,0,0)|+|f(t,0,0)|\\
&\le l_1|u(t)|+l_2|v(t)|+\Lambda_1\le l_5(|u(t)|+|v(t)|)+\Lambda_1,\\
|G_{uv}(t)| &= |g(t,u(t),v(t))|\le|g(t,u(t),v(t))-g(t,0,0)|+|g(t,0,0)|\\
&\le l_3|u(t)|+l_4|v(t)|+\Lambda_2\le l_6(|u(t)|+|v(t)|)+\Lambda_2,
\end{aligned}
\tag{27}
$$

for all $t\in[1,T]$ and $(u,v)\in\mathcal{Y}$. We consider now the positive number

$$
R\ge\frac{\Lambda_1(\Xi_1+\Xi_3)+\Lambda_2(\Xi_2+\Xi_4)}{1-l_5(\Xi_1+\Xi_3)-l_6(\Xi_2+\Xi_4)}, \tag{28}
$$

and let the set $B_R=\{(u,v)\in\mathcal{Y},\ \|(u,v)\|_{\mathcal{Y}}\le R\}$.

We will show firstly that $\mathcal{A}(B_R)\subset B_R$. Indeed, for this, let $(u,v)\in B_R$. Then, we obtain

$$
\begin{aligned}
|\mathcal{A}_1(u,v)(t)| \le {} & \frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}|F_{uv}(s)|\,\frac{ds}{s}\\
& +\frac{(\ln t)^{\lambda-1}}{|\Delta|}\left[\frac{|d|}{\Gamma(\alpha-\varsigma)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1}|F_{uv}(s)|\,\frac{ds}{s}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+|d|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}|F_{uv}(\tau)|\,\frac{d\tau}{\tau}\right)dH_i(s)\right|\\
&+|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}|G_{uv}(\tau)|\,\frac{d\tau}{\tau}\right)dK_i(s)\right|\\
&+\frac{|b|}{\Gamma(\gamma-\vartheta)}\int_1^T\left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1}|G_{uv}(s)|\,\frac{ds}{s}\\
&+|b|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}|F_{uv}(\tau)|\,\frac{d\tau}{\tau}\right)dP_i(s)\right|\\
&+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}|G_{uv}(\tau)|\,\frac{d\tau}{\tau}\right)dQ_i(s)\right|\Bigg]\\
\le{}&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}[l_5(|u(s)|+|v(s)|)+\Lambda_1]\,\frac{ds}{s}\\
&+\frac{(\ln t)^{\lambda-1}}{|\Delta|}\Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1}[l_5(|u(s)|+|v(s)|)+\Lambda_1]\,\frac{ds}{s}\\
&+|d|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}[l_5(|u(\tau)+|v(\tau)|)+\Lambda_1]\,\frac{d\tau}{\tau}\right)dH_i(s)\right|\\
&+|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}[l_6(|u(\tau)|+|v(\tau)|)+\Lambda_2]\,\frac{d\tau}{\tau}\right)dK_i(s)\right|\\
&+\frac{|b|}{\Gamma(\gamma-\vartheta)}\int_1^T\left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1}[l_6(|u(s)|+|v(s)|)+\Lambda_2]\,\frac{ds}{s}\\
&+|b|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}[l_5(|u(\tau)|+|v(\tau)|)+\Lambda_1]\,\frac{d\tau}{\tau}\right)dP_i(s)\right|\\
&+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}[l_6(|u(\tau)|+|v(\tau)|)+\Lambda_2]\,\frac{d\tau}{\tau}\right)dQ_i(s)\right|\Bigg].
\end{aligned}
\tag{29}
$$

So, we find

$$
\begin{aligned}
|\mathcal{A}_1(u,v)(t)|\le{}&(l_5\|(u,v)\|_{\mathcal{Y}}+\Lambda_1)\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
&+\frac{(\ln t)^{\lambda-1}}{|\Delta|}\Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1}\frac{ds}{s}\\
&+|d|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}\frac{d\tau}{\tau}\right)dH_i(s)\right|\\
&+|b|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}\frac{d\tau}{\tau}\right)dP_i(s)\right|\Bigg]\Bigg\}\\
&+(l_6\|(u,v)\|_{\mathcal{Y}}+\Lambda_2)\frac{(\ln t)^{\lambda-1}}{|\Delta|}\Bigg[|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}\frac{d\tau}{\tau}\right)dK_i(s)\right|\\
&+\frac{|b|}{\Gamma(\gamma-\vartheta)}\int_1^T\left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1}\frac{ds}{s}\\
&+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}\frac{d\tau}{\tau}\right)dQ_i(s)\right|\Bigg]\\
={}&(l_5\|(u,v)\|_{\mathcal{Y}}+\Lambda_1)\Bigg\{\frac{(\ln t)^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\ln t)^{\lambda-1}}{|\Delta|}\Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma+1)}(\ln T)^{\alpha-\varsigma}\\
&+|d|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i+1)}\left|\int_1^T(\ln s)^{\alpha-\varrho_i}\,dH_i(s)\right|\\
&+|b|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i+1)}\left|\int_1^T(\ln s)^{\alpha-\eta_i}\,dP_i(s)\right|\Bigg]\Bigg\}
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&+(l_6\|(u,v)\|_{\mathcal{Y}}+\Lambda_2)\frac{(\ln t)^{\lambda-1}}{|\Delta|}\left[|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i+1)}\left|\int_1^T(\ln s)^{\gamma-\sigma_i}\,dK_i(s)\right|\right.\\
&\left.+\frac{|b|}{\Gamma(\gamma-\vartheta+1)}(\ln T)^{\gamma-\vartheta}+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i+1)}\left|\int_1^T(\ln s)^{\gamma-\theta_i}\,dQ_i(s)\right|\right]\\
\le\ &(l_5\|(u,v)\|_{\mathcal{Y}}+\Lambda_1)\Xi_1+(l_6\|(u,v)\|_{\mathcal{Y}}+\Lambda_2)\Xi_2\\
\le\ &R(l_5\Xi_1+l_6\Xi_2)+\Lambda_1\Xi_1+\Lambda_2\Xi_2,\ \ \forall t\in[1,T].
\end{aligned}
$$

In a similar manner, we obtain

$$
\begin{aligned}
&|\mathcal{A}_2(u,v)(t)|\le(l_5\|(u,v)\|_{\mathcal{Y}}+\Lambda_1)\Xi_3+(l_6\|(u,v)\|_{\mathcal{Y}}+\Lambda_2)\Xi_4\\
&\le R(l_5\Xi_3+l_6\Xi_4)+\Lambda_1\Xi_3+\Lambda_2\Xi_4,\ \ \forall t\in[1,T].
\end{aligned}
\tag{31}
$$

Then, by condition (26) and relations (30) and (31), we deduce

$$
\begin{aligned}
&\|\mathcal{A}(u,v)\|_{\mathcal{Y}}=\|\mathcal{A}_1(u,v)\|+\|\mathcal{A}_2(u,v)\|\le R[l_5(\Xi_1+\Xi_3)+l_6(\Xi_2+\Xi_4)]\\
&\quad+\Lambda_1(\Xi_1+\Xi_3)+\Lambda_2(\Xi_2+\Xi_4)\le R.
\end{aligned}
\tag{32}
$$

So, $\mathcal{A}(B_R)\subset B_R$.

Next, we will prove that operator $\mathcal{A}$ is a contraction. For this, let (u_1,v_1), $(u_2,v_2)\in\mathcal{Y}$. Then, for any $t\in[1,T]$ we obtain

$$
\begin{aligned}
&|\mathcal{A}_1(u_1,v_1)(t)-\mathcal{A}_1(u_2,v_2)(t)|\\
\le\ &\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}|F_{u_1v_1}(s)-F_{u_2v_2}(s)|\,\frac{ds}{s}\\
&+\frac{(\ln t)^{\lambda-1}}{|\Delta|}\left[\frac{|d|}{\Gamma(\alpha-\varsigma)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1}|F_{u_1v_1}(s)-F_{u_2v_2}(s)|\frac{ds}{s}\right.\\
&+|d|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}|F_{u_1v_1}(\tau)-F_{u_2v_2}(\tau)|\,\frac{d\tau}{\tau}\right)dH_i(s)\right|\\
&+|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}|G_{u_1v_1}(\tau)-G_{u_2v_2}(\tau)|\,\frac{d\tau}{\tau}\right)dK_i(s)\right|\\
&+\frac{|b|}{\Gamma(\gamma-\vartheta)}\int_1^T\left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1}|G_{u_1v_1}(s)-G_{u_2v_2}(s)|\,\frac{ds}{s}\\
&+|b|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}|F_{u_1v_1}(\tau)-F_{u_2v_2}(\tau)|\,\frac{d\tau}{\tau}\right)dP_i(s)\right|\\
&\left.+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}|G_{u_1v_1}(\tau)-G_{u_2v_2}(\tau)|\,\frac{d\tau}{\tau}\right)dQ_i(s)\right|\right]\\
\le\ &l_5\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\
&+\frac{(\ln t)^{\lambda-1}}{|\Delta|}\left[\frac{|d|}{\Gamma(\alpha-\varsigma)}l_5\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1}\frac{ds}{s}\right.\\
&+|d|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}l_5\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}\frac{d\tau}{\tau}\right)dH_i(s)\right|\\
&+|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}l_6\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}\frac{d\tau}{\tau}\right)dK_i(s)\right|\\
&+\frac{|b|}{\Gamma(\gamma-\vartheta)}l_6\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\int_1^T\left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1}\frac{ds}{s}\\
&+|b|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}l_5\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}\frac{d\tau}{\tau}\right)dP_i(s)\right|\\
&\left.+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}l_6\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}\frac{d\tau}{\tau}\right)dQ_i(s)\right|\right].
\end{aligned}
\tag{33}
$$

Therefore, we find

$$\begin{aligned}
&|\mathcal{A}_1(u_1,v_1)(t)-\mathcal{A}_1(u_2,v_2)(t)| \\
&\leq \|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\Bigg\{l_5\Bigg\{\frac{1}{\Gamma(\alpha+1)}(\ln T)^{\alpha}+\frac{(\ln T)^{\lambda-1}}{|\Delta|}\Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma+1)}(\ln T)^{\alpha-\varsigma} \\
&\quad+|d|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i+1)}\left|\int_1^T(\ln s)^{\alpha-\varrho_i}\,dH_i(s)\right| \\
&\quad+|b|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i+1)}\left|\int_1^T(\ln s)^{\alpha-\eta_i}\,dP_i(s)\right|\Bigg]\Bigg\} \\
&\quad+l_6\frac{(\ln T)^{\lambda-1}}{|\Delta|}\Bigg[|d|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i+1)}\left|\int_1^T(\ln s)^{\gamma-\sigma_i}\,dK_i(s)\right| \\
&\quad+\frac{|b|}{\Gamma(\gamma-\vartheta+1)}(\ln T)^{\gamma-\vartheta}+|b|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i+1)}\left|\int_1^T(\ln s)^{\gamma-\theta_i}\,dQ_i(s)\right|\Bigg]\Bigg\} \\
&= \|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}(l_5\Xi_1+l_6\Xi_2).
\end{aligned} \tag{34}$$

In a similar manner, we obtain

$$|\mathcal{A}_2(u_1,v_1)(t)-\mathcal{A}_2(u_2,v_2)(t)| \leq \|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}(l_5\Xi_3+l_6\Xi_4),\ \ \forall\, t\in[1,T]. \tag{35}$$

Then, by relations (34) and (35), we deduce

$$\begin{aligned}
&\|\mathcal{A}(u_1,v_1)-\mathcal{A}(u_2,v_2)\|_{\mathcal{Y}} = \|\mathcal{A}_1(u_1,v_1)-\mathcal{A}_1(u_2,v_2)\|+\|\mathcal{A}_2(u_1,v_1)-\mathcal{A}_2(u_2,v_2)\| \\
&\leq [l_5(\Xi_1+\Xi_3)+l_6(\Xi_2+\Xi_4)]\|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}.
\end{aligned} \tag{36}$$

By (26), we conclude that operator $\mathcal{A}$ is a contraction. Therefore, operator $\mathcal{A}$ has a unique fixed point by the Banach contraction mapping principle. Hence, problem (1) and (2) has a unique solution $(u(t),v(t)),\ t\in[1,T]$. □

The next two results for the existence of solutions of problem (1) and (2) are based on the Krasnosel'skii fixed point theorem for the sum of two operators (see [31]).

Theorem 3. *We suppose that assumptions $(H1)$ and $(H2)$ hold. In addition, we assume that the functions $f,g:[1,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous and satisfy the following condition:*

$(H3)$ *There exist the continuous functions $\phi,\psi\in C([1,T],\mathbb{R}_+)$, ($\mathbb{R}_+=[0,\infty)$) such that*

$$|f(t,x,y)|\leq\phi(t),\ \ |g(t,x,y)|\leq\psi(t),\ \ \forall\,(t,x,y)\in[1,T]\times\mathbb{R}^2. \tag{37}$$

If

$$l_5\left[\Xi_1+\Xi_3-\frac{1}{\Gamma(\alpha+1)}(\ln T)^{\alpha}\right]+l_6\left[\Xi_2+\Xi_4-\frac{1}{\Gamma(\gamma+1)}(\ln T)^{\gamma}\right]<1, \tag{38}$$

then problem (1) and (2) has at least one solution $(u(t),v(t)),\ t\in[1,T]$.

Proof. We consider the number $r>0$ satisfying the condition

$$r\geq(\Xi_1+\Xi_3)\|\phi\|+(\Xi_2+\Xi_4)\|\psi\|, \tag{39}$$

and the closed ball $B_r=\|(u,v)\in\mathcal{Y},\ \ \|(u,v)\|_{\mathcal{Y}}\leq r\}$. We will verify the assumptions of the Krasnosel'skii fixed point theorem for the sum of two operators. We split operator $\mathcal{A}$, defined on B_r, as $\mathcal{A}=\mathcal{B}+\mathcal{C}$, $\mathcal{B}=(\mathcal{B}_1,\mathcal{B}_2)$, $\mathcal{C}=(\mathcal{C}_1,\mathcal{C}_2)$, where $\mathcal{B}_i,\mathcal{C}_i,\ i=1,2$ are defined by

$$
\begin{aligned}
&\mathcal{B}_1(u,v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} F_{uv}(s)\, \frac{ds}{s}, \\
&\mathcal{C}_1(u,v)(t) = \mathcal{A}_1(u,v)(t) - \mathcal{B}_1(u,v)(t), \\
&\mathcal{B}_2(u,v)(t) = \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} G_{uv}(s)\, \frac{ds}{s}, \\
&\mathcal{C}_2(u,v)(t) = \mathcal{A}_2(u,v)(t) - \mathcal{B}_2(u,v)(t),
\end{aligned} \tag{40}
$$

for all $t \in [1,T]$ and $(u,v) \in B_r$.

We will prove firstly that $\mathcal{B}(u_1,v_1) + \mathcal{C}(u_2,v_2) \in B_r$ for all (u_1,v_1), $(u_2,v_2) \in B_r$. For this, let (u_1,v_1), $(u_2,v_2) \in B_r$. Then, we obtain

$$
\begin{aligned}
&|\mathcal{B}_1(u_1,v_1)(t) + \mathcal{C}_1(u_2,v_2)(t)| \le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} |F_{u_1v_1}(s)|\, \frac{ds}{s} \\
&\quad + \frac{(\ln t)^{\lambda-1}}{|\Delta|} \left[\frac{|d|}{\Gamma(\alpha-\varsigma)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-\varsigma-1} |F_{u_2v_2}(s)|\, \frac{ds}{s} \right. \\
&\quad + |d| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\varrho_i-1} |F_{u_2v_2}(\tau)|\, \frac{d\tau}{\tau} \right) dH_i(s) \right| \\
&\quad + |d| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\sigma_i-1} |G_{u_2v_2}(\tau)|\, \frac{d\tau}{\tau} \right) dK_i(s) \right| \\
&\quad + \frac{|b|}{\Gamma(\gamma-\vartheta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\gamma-\vartheta-1} |G_{u_2v_2}(s)|\, \frac{ds}{s} \\
&\quad + |b| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\eta_i-1} |F_{u_2v_2}(\tau)|\, \frac{d\tau}{\tau} \right) dP_i(s) \right| \\
&\quad \left. + |b| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\theta_i-1} |G_{u_2v_2}(\tau)|\, \frac{d\tau}{\tau} \right) dQ_i(s) \right| \right] \\
&\le \|\phi\| \left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \right. \\
&\quad + \frac{(\ln t)^{\lambda-1}}{|\Delta|} \left[\frac{|d|}{\Gamma(\alpha-\varsigma)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-\varsigma-1} \frac{ds}{s} \right. \\
&\quad + |d| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\varrho_i-1} \frac{d\tau}{\tau} \right) dH_i(s) \right| \\
&\quad \left.\left. + |b| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\eta_i-1} \frac{d\tau}{\tau} \right) dP_i(s) \right| \right] \right\} \\
&\quad + \|\psi\| \frac{(\ln t)^{\lambda-1}}{|\Delta|} \left[|d| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\sigma_i-1} \frac{d\tau}{\tau} \right) dK_i(s) \right| \right. \\
&\quad + \frac{|b|}{\Gamma(\gamma-\vartheta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\gamma-\vartheta-1} \frac{ds}{s} \\
&\quad \left. + |b| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\theta_i-1} \frac{d\tau}{\tau} \right) dQ_i(s) \right| \right] \\
&\le \Xi_1 \|\phi\| + \Xi_2 \|\psi\|, \quad \forall\, t \in [1,T],
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
&|\mathcal{B}_2(u_1,v_1)(t)+\mathcal{C}_2(u_2,v_2)(t)| \leq \frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}|G_{u_1v_1}(s)|\frac{ds}{s}\\
&\quad+\frac{(\ln t)^{\mu-1}}{|\Delta|}\Bigg[\frac{|a|}{\Gamma(\gamma-\vartheta)}\int_1^T\left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1}|G_{u_2v_2}(s)|\frac{ds}{s}\\
&\quad+|a|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}|F_{u_2v_2}(\tau)|\frac{d\tau}{\tau}\right)dP_i(s)\right|\\
&\quad+|a|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}|G_{u_2v_2}(\tau)|\frac{d\tau}{\tau}\right)dQ_i(s)\right|\\
&\quad+\frac{|c|}{\Gamma(\alpha-\varsigma)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1}|F_{u_2v_2}(s)|\frac{ds}{s}\\
&\quad+|c|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}|F_{u_2v_2}(\tau)|\frac{d\tau}{\tau}\right)dH_i(s)\right|\\
&\quad+|c|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}|G_{u_2v_2}(\tau)|\frac{d\tau}{\tau}\right)dK_i(s)\right|\Bigg]\\
&\leq \|\phi\|\frac{(\ln t)^{\mu-1}}{|\Delta|}\Bigg[|a|\sum_{i=1}^{p}\frac{1}{\Gamma(\alpha-\eta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1}\frac{d\tau}{\tau}\right)dP_i(s)\right|\\
&\quad+\frac{|c|}{\Gamma(\alpha-\varsigma)}\int_1^T\left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1}\frac{ds}{s}\\
&\quad+|c|\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha-\varrho_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1}\frac{d\tau}{\tau}\right)dH_i(s)\right|\Bigg]\\
&\quad+\|\psi\|\Bigg\{\frac{1}{\Gamma(\gamma)}\int_1^t\left(\ln\frac{t}{s}\right)^{\gamma-1}\frac{ds}{s}\\
&\quad+\frac{(\ln t)^{\mu-1}}{|\Delta|}\Bigg[\frac{|a|}{\Gamma(\gamma-\vartheta)}\int_1^T\left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1}\frac{ds}{s}\\
&\quad+|a|\sum_{i=1}^{q}\frac{1}{\Gamma(\gamma-\theta_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1}\frac{d\tau}{\tau}\right)dQ_i(s)\right|\\
&\quad+|c|\sum_{i=1}^{n}\frac{1}{\Gamma(\gamma-\sigma_i)}\left|\int_1^T\left(\int_1^s\left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1}\frac{d\tau}{\tau}\right)dK_i(s)\right|\Bigg]\Bigg\}\\
&\leq \Xi_3\|\phi\|+\Xi_4\|\psi\|,\ \ \forall t\in[1,T].
\end{aligned} \tag{42}
$$

Then, by (41) and (42), we deduce

$$
\begin{aligned}
&\|\mathcal{B}(u_1,v_1)+\mathcal{C}(u_2,v_2)\|_{\mathcal{Y}} = \|\mathcal{B}_1(u_1,v_1)+\mathcal{C}_1(u_2,v_2)\|+\|\mathcal{B}_2(u_1,v_1)+\mathcal{C}_2(u_2,v_2)\|\\
&\leq (\Xi_1+\Xi_3)\|\phi\|+(\Xi_2+\Xi_4)\|\psi\|\leq r.
\end{aligned} \tag{43}
$$

Next, we will prove that operator $\mathcal{C}$ is a contraction mapping. Indeed, for all (u_1,v_1), $(u_2,v_2)\in B_r$, we find

$$
\begin{aligned}
&|\mathcal{C}_1(u_1,v_1)(t)-\mathcal{C}_1(u_2,v_2)(t)|\leq \|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\\
&\quad\times\left\{l_5\left[\Xi_1-\frac{1}{\Gamma(\alpha+1)}(\ln T)^{\alpha}\right]+l_6\Xi_2\right\},\ \ \forall t\in[1,T],\\
&|\mathcal{C}_2(u_1,v_1)(t)-\mathcal{C}_2(u_2,v_2)(t)\|\leq \|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\\
&\quad\times\left\{l_5\Xi_3+l_6\left[\Xi_4-\frac{1}{\Gamma(\gamma+1)}(\ln T)^{\gamma}\right]\right\},\ \ \forall t\in[1,T].
\end{aligned} \tag{44}
$$

Therefore, by (44), we obtain

$$
\begin{aligned}
&\|\mathcal{C}(u_1,v_1)-\mathcal{C}(u_2,v_2)\|_{\mathcal{Y}}\leq \|(u_1,v_1)-(u_2,v_2)\|_{\mathcal{Y}}\\
&\times\left\{l_5\left[\Xi_1+\Xi_3-\frac{1}{\Gamma(\alpha+1)}(\ln T)^{\alpha}\right]+l_6\left[\Xi_2+\Xi_4-\frac{1}{\Gamma(\gamma+1)}(\ln T)^{\gamma}\right]\right\}.
\end{aligned} \tag{45}
$$

By condition (38), we conclude that operator $\mathcal{C}$ is a contraction.

Operators $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}$ are continuous by the continuity of functions f and g. In addition, $\mathcal{B}$ is uniformly bounded on B_r because

$$\|\mathcal{B}_1(u,v)\| \leq \frac{(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\|\phi\|, \ \ \|\mathcal{B}_2(u,v)\| \leq \frac{(\ln T)^{\gamma}}{\Gamma(\gamma+1)}\|\psi\|, \ \ \forall\,(u,v) \in B_r, \tag{46}$$

and then

$$\|\mathcal{B}(u,v)\| \leq \frac{(\ln T)^{\alpha}}{\Gamma(\alpha+1)}\|\phi\| + \frac{(\ln T)^{\gamma}}{\Gamma(\gamma+1)}\|\psi\|, \ \ \forall\,(u,v) \in B_r. \tag{47}$$

We finally prove that operator $\mathcal{B}$ is compact. Let $t_1, t_2 \in [1,T]$, $t_1 < t_2$. Then, for all $(u,v) \in B_r$, we find

$$\begin{aligned}
&|\mathcal{B}_1(u,v)(t_2) - \mathcal{B}_1(u,v)(t_1)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} F_{uv}(s)\,\frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\ln\frac{t_1}{s}\right)^{\alpha-1} F_{uv}(s)\,\frac{ds}{s} \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \right] |F_{uv}(s)|\,\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} |F_{uv}(s)|\,\frac{ds}{s} \\
&\leq \|\phi\| \left\{ \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\ln\frac{t_2}{s}\right)^{\alpha-1} - \left(\ln\frac{t_1}{s}\right)^{\alpha-1} \right] \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s} \right\} \\
&= \|\phi\| \left\{ \frac{1}{\Gamma(\alpha+1)} \left[(\ln t_2)^{\alpha} - \left(\ln\frac{t_2}{t_1}\right)^{\alpha} - (\ln t_1)^{\alpha} \right] + \frac{1}{\Gamma(\alpha+1)} \left(\ln\frac{t_2}{t_1}\right)^{\alpha} \right\} \\
&= \|\phi\| \frac{1}{\Gamma(\alpha+1)} [(\ln t_2)^{\alpha} - (\ln t_1)^{\alpha}],
\end{aligned} \tag{48}$$

which tends to zero as $t_2 \to t_1$, independently of $(u,v) \in B_r$. We also have

$$\begin{aligned}
&|\mathcal{B}_2(u,v)(t_2) - \mathcal{B}_2(u,v)(t_1)| \\
&= \left| \frac{1}{\Gamma(\gamma)} \int_1^{t_2} \left(\ln\frac{t_2}{s}\right)^{\gamma-1} G_{uv}(s)\,\frac{ds}{s} - \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left(\ln\frac{t_1}{s}\right)^{\gamma-1} G_{uv}(s)\,\frac{ds}{s} \right| \\
&\leq \frac{1}{\Gamma(\gamma)} \int_1^{t_1} \left[\left(\ln\frac{t_2}{s}\right)^{\gamma-1} - \left(\ln\frac{t_1}{s}\right)^{\gamma-1} \right] |G_{uv}(s)|\,\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_1}^{t_2} \left(\ln\frac{t_2}{s}\right)^{\gamma-1} |G_{uv}(s)|\,\frac{ds}{s} \\
&\leq \|\psi\| \frac{1}{\Gamma(\gamma+1)} [(\ln t_2)^{\gamma} - (\ln t_1)^{\gamma}],
\end{aligned} \tag{49}$$

which tends to zero as $t_2 \to t_1$, independently of $(u,v) \in B_r$.

Hence, by using (48) and (49), we obtain that operators $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}$ are equicontinuous. By the Arzela–Ascoli theorem, we deduce that $\mathcal{B}$ is compact on B_r. Therefore, by the Krasnosel'skii fixed point theorem ([31]), we conclude that problem (1) and (2) has at least one solution $(u(t), v(t)),\ t \in [1,T]$. $\square$

Theorem 4. *We suppose that assumption* $(H1)$ *holds and the functions* $f, g : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous and satisfy assumptions* $(H2)$ *and* $(H3)$. *If*

$$l_5 \frac{1}{\Gamma(\alpha+1)} (\ln T)^{\alpha} + l_6 \frac{1}{\Gamma(\gamma+1)} (\ln T)^{\gamma} < 1, \tag{50}$$

then problem (1) *and* (2) *has at least one solution* $(u(t), v(t)),\ t \in [1,T]$.

Proof. As in the proof of Theorem 3, we consider the positive number $r \geq (\Xi_1 + \Xi_3)\|\phi\| + (\Xi_2 + \Xi_4)\|\psi\|$ and the closed ball B_r. We also split operator $\mathcal{A}$, defined on B_r, as $\mathcal{A} = \mathcal{B} + \mathcal{C}$, $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$, $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$, where $\mathcal{B}_i$, $\mathcal{C}_i$, $i = 1, 2$ are defined by (40).

For (u_1, v_1), $(u_2, v_2) \in B_r$, we obtain as in the proof of Theorem 3, that

$$\|\mathcal{B}(u_1, v_1) + \mathcal{C}(u_2, v_2)\|_{\mathcal{Y}} \leq r. \tag{51}$$

We will prove next that the operator $\mathcal{B}$ is a contraction. Indeed, we find

$$\begin{aligned}
&|\mathcal{B}_1(u_1,v_1)(t) - \mathcal{B}_1(u_2,v_2)(t)\| \leq \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{Y}} l_5 \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&= l_5 \frac{1}{\Gamma(\alpha+1)} (\ln t)^{\alpha} \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{Y}} \\
&\leq l_5 \frac{1}{\Gamma(\alpha+1)} (\ln T)^{\alpha} \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{Y}}, \ \ \forall\, t \in [1,T], \\
&|\mathcal{B}_2(u_1,v_1)(t) - \mathcal{B}_2(u_2,v_2)(t)\| \leq \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{Y}} l_6 \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} \frac{ds}{s} \\
&= l_6 \frac{1}{\Gamma(\gamma+1)} (\ln t)^{\gamma} \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{Y}} \\
&\leq l_6 \frac{1}{\Gamma(\gamma+1)} (\ln T)^{\gamma} \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{Y}}, \ \ \forall\, t \in [1,T].
\end{aligned} \tag{52}$$

Then, we deduce

$$\begin{aligned}
\|\mathcal{B}(u_1,v_1) - \mathcal{B}(u_2,v_2)\|_{\mathcal{Y}} &\leq \left[l_5 \frac{1}{\Gamma(\alpha+1)} (\ln T)^{\alpha} + l_6 \frac{1}{\Gamma(\gamma+1)} (\ln T)^{\gamma} \right] \\
&\quad \times \|(u_1,v_1) - (u_2,v_2)\|_{\mathcal{Y}},
\end{aligned} \tag{53}$$

that is, by (50), operator $\mathcal{B}$ is a contraction.

Operators $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}$ are continuous by the continuity of the functions f and g. Moreover, $\mathcal{C}$ is uniformly bounded on B_r because we have

$$\begin{aligned}
|\mathcal{C}_1(u,v)(t)| &\leq \frac{(\ln T)^{\lambda-1}}{|\Delta|} \Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-\varsigma-1} \|\phi\| \frac{ds}{s} \\
&\quad + |d| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\varrho_i-1} \|\phi\| \frac{d\tau}{\tau} \right) dH_i(s) \right| \\
&\quad + |d| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\sigma_i-1} \|\psi\| \frac{d\tau}{\tau} \right) dK_i(s) \right| \\
&\quad + \frac{|b|}{\Gamma(\gamma-\vartheta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\gamma-\vartheta-1} \|\psi\| \frac{ds}{s} \\
&\quad + |b| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\eta_i-1} \|\phi\| \frac{d\tau}{\tau} \right) dP_i(s) \right| \\
&\quad + |b| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\theta_i-1} \|\psi\| \frac{d\tau}{\tau} \right) dQ_i(s) \right| \Bigg] \\
&= \|\phi\| \frac{(\ln T)^{\lambda-1}}{|\Delta|} \Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma+1)} (\ln T)^{\alpha-\varsigma} \\
&\quad + |d| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i+1)} \left| \int_1^T (\ln s)^{\alpha-\varrho_i} dH_i(s) \right| \\
&\quad + |b| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i+1)} \left| \int_1^T (\ln s)^{\alpha-\eta_i} dP_i(s) \right| \Bigg] \\
&\quad + \|\psi\| \frac{(\ln T)^{\lambda-1}}{|\Delta|} \Bigg[|d| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i+1)} \left| \int_1^T (\ln s)^{\gamma-\sigma_i} dK_i(s) \right| \\
&\quad + \frac{|b|}{\Gamma(\gamma-\vartheta+1)} (\ln T)^{\gamma-\vartheta} + |b| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i+1)} \left| \int_1^T (\ln s)^{\gamma-\theta_i} dQ_i(s) \right| \Bigg]
\end{aligned} \tag{54}$$

$$= \|\phi\| \left[\Xi_1 - \frac{1}{\Gamma(\alpha+1)} (\ln T)^{\alpha} \right] + \|\psi\| \Xi_2, \ \ \forall\, t \in [1,T], \ (u,v) \in B_r,$$

and

$$\begin{aligned}
|\mathcal{C}_2(u,v)(t)| \le{}& \frac{(\ln T)^{\mu-1}}{|\Delta|} \left[\frac{|a|}{\Gamma(\gamma-\vartheta)} \int_1^T \left(\ln\frac{T}{s}\right)^{\gamma-\vartheta-1} \|\psi\| \frac{ds}{s} \right.\\
&+|a| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i)} \left| \int_1^T \left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-\eta_i-1} \|\phi\| \frac{d\tau}{\tau} \right) dP_i(s) \right|\\
&+|a| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i)} \left| \int_1^T \left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\gamma-\theta_i-1} \|\psi\| \frac{d\tau}{\tau} \right) dQ_i(s) \right|\\
&+\frac{|c|}{\Gamma(\alpha-\varsigma)} \int_1^T \left(\ln\frac{T}{s}\right)^{\alpha-\varsigma-1} \|\phi\| \frac{ds}{s}\\
&+|c| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i)} \left| \int_1^T \left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\alpha-\varrho_i-1} \|\phi\| \frac{d\tau}{\tau} \right) dH_i(s) \right|\\
&\left.+|c| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i)} \left| \int_1^T \left(\int_1^s \left(\ln\frac{s}{\tau}\right)^{\gamma-\sigma_i-1} \|\psi\| \frac{d\tau}{\tau} \right) dK_i(s) \right| \right]\\
={}& \|\phi\| \frac{(\ln T)^{\mu-1}}{|\Delta|} \left[|a| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i+1)} \left| \int_1^T (\ln s)^{\alpha-\eta_i} dP_i(s) \right| \right.\\
&\left.+\frac{|c|}{\Gamma(\alpha-\varsigma+1)} (\ln T)^{\alpha-\varsigma} + |c| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i+1)} \left| \int_1^T (\ln s)^{\alpha-\varrho_i} dH_i(s) \right| \right]\\
&+\|\psi\| \frac{(\ln T)^{\mu-1}}{|\Delta|} \left[\frac{|a|}{\Gamma(\gamma-\vartheta+1)} (\ln T)^{\gamma-\vartheta} \right.\\
&+|a| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i+1)} \left| \int_1^T (\ln s)^{\gamma-\theta_i} dQ_i(s) \right|\\
&\left.+|c| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i+1)} \left| \int_1^T (\ln s)^{\gamma-\sigma_i} dK_i(s) \right| \right]\\
={}& \|\phi\| \Xi_3 + \|\psi\| \left[\Xi_4 - \frac{1}{\Gamma(\gamma+1)} (\ln T)^{\gamma} \right], \ \ \forall\, t \in [1,T], \ (u,v) \in B_r.
\end{aligned} \tag{55}$$

Therefore, by (54) and (55), we obtain

$$\begin{aligned}
\|\mathcal{C}_1(u,v)\| &\le \|\phi\| \left[\Xi_1 - \frac{1}{\Gamma(\alpha+1)} (\ln T)^{\alpha} \right] + \|\psi\| \Xi_2,\\
\|\mathcal{C}_2(u,v)\| &\le \|\phi\| \Xi_3 + \|\psi\| \left[\Xi_4 - \frac{1}{\Gamma(\gamma+1)} (\ln T)^{\gamma} \right],
\end{aligned} \tag{56}$$

and then

$$\begin{aligned}
\|\mathcal{C}(u,v)\| \le{}& \|\phi\| \left[\Xi_1 + \Xi_3 - \frac{1}{\Gamma(\alpha+1)} (\ln T)^{\alpha} \right]\\
&+\|\psi\| \left[\Xi_2 + \Xi_4 - \frac{1}{\Gamma(\gamma+1)} (\ln T)^{\gamma} \right], \ \ \forall\, (u,v) \in B_r.
\end{aligned} \tag{57}$$

We finally prove that operator $\mathcal{C}$ is compact. Let $t_1,\, t_2 \in [1,T]$, $t_1 < t_2$. Then, for all $(u,v) \in B_r$, we find

$$\begin{aligned}
&|\mathcal{C}_1(u,v)(t_2) - \mathcal{C}_1(u,v)(t_1)|\\
&\le \left\{ \|\phi\| \left[\Xi_1 - \frac{1}{\Gamma(\alpha+1)} (\ln T)^{\alpha} \right] + \|\psi\| \Xi_2 \right\} \frac{1}{(\ln T)^{\lambda-1}} \left[(\ln t_2)^{\lambda-1} - (\ln t_1)^{\lambda-1} \right],
\end{aligned} \tag{58}$$

which tends to zero as $t_2 \to t_1$, independently of $(u,v) \in B_r$. We also obtain

$$\begin{aligned}
&|\mathcal{C}_2(u,v)(t_2) - \mathcal{C}_2(u,v)(t_1)|\\
&\le \left\{ \|\phi\| \Xi_3 + \|\psi\| \left[\Xi_4 - \frac{1}{\Gamma(\gamma+1)} (\ln T)^{\gamma} \right] \right\} \frac{1}{(\ln T)^{\mu-1}} \left[(\ln t_2)^{\mu-1} - (\ln t_1)^{\mu-1} \right],
\end{aligned} \tag{59}$$

which tends to zero as $t_2 \to t_1$, independently of $(u, v) \in B_r$.

So, by using inequalities (58) and (59), we obtain that operators $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}$ are equicontinuous. By the Arzela–Ascoli theorem, we conclude that $\mathcal{C}$ is compact on B_r. Then, by applying the Krasnosel'skii fixed point theorem (see [31]), we deduce that problem (1) and (2) has at least one solution $(u(t), v(t)),\ t \in [1, T]$. $\square$

Our next result is based on the Schaefer fixed point theorem (see [32]).

Theorem 5. *We assume that assumption $(H1)$ holds. In addition, we suppose that the functions $f, g : [1, T] \times (\mathbb{R})^2 \to \mathbb{R}$ are continuous and satisfy the following condition:*
$(H4)$ There exist positive constants M_1, M_2 such that

$$|f(t, x, y)| \leq M_1, \ \ |g(t, x, y)| \leq M_2, \ \ \forall\, t \in [1, T], \ \ x, y \in \mathbb{R}. \tag{60}$$

Then, there exists at least one solution $(u(t), v(t)),\ t \in [1, T]$ for problem (1) and (2).

Proof. Firstly, we show that $\mathcal{A}$ is completely continuous. Operator $\mathcal{A}$ is continuous. Indeed, let $(u_n, v_n) \in \mathcal{Y}$, $n \in \mathbb{N}$, $(u_n, v_n) \to (u, v)$, as $n \to \infty$ in $\mathcal{Y}$. Then, for each $t \in [1, T]$, we obtain

$$\begin{aligned}
&|\mathcal{A}_1(u_n, v_n)(t) - \mathcal{A}_1(u, v)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} |F_{u_n v_n}(s) - F_{uv}(s)| \frac{ds}{s} \\
&\quad + \frac{(\ln t)^{\lambda-1}}{|\Delta|} \Bigg[\frac{|d|}{\Gamma(\alpha-\varsigma)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-\varsigma-1} |F_{u_n v_n}(s) - F_{uv}(s)| \frac{ds}{s} \\
&\quad + |d| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\varrho_i-1} |F_{u_n v_n}(\tau) - F_{uv}(\tau)| \frac{d\tau}{\tau} \right) dH_i(s) \right| \\
&\quad + |d| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\sigma_i-1} |G_{u_n v_n}(\tau) - G_{uv}(\tau)| \frac{d\tau}{\tau} \right) dK_i(s) \right| \\
&\quad + \frac{|b|}{\Gamma(\gamma-\vartheta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\gamma-\vartheta-1} |G_{u_n v_n}(s) - G_{uv}(s)| \frac{ds}{s} \\
&\quad + |b| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\eta_i-1} |F_{u_n v_n}(\tau) - F_{uv}(\tau)| \frac{d\tau}{\tau} \right) dP_i(s) \right| \\
&\quad + |b| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\theta_i-1} |G_{u_n v_n}(\tau) - G_{uv}(\tau)| \frac{d\tau}{\tau} \right) dQ_i(s) \right| \Bigg],
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
&|\mathcal{A}_2(u_n, v_n)(t) - \mathcal{A}_2(u, v)(t)| \\
&\leq \frac{1}{\Gamma(\gamma)} \int_1^t \left(\ln \frac{t}{s}\right)^{\gamma-1} |G_{u_n v_n}(s) - G_{uv}(s)| \frac{ds}{s} \\
&\quad + \frac{(\ln t)^{\mu-1}}{|\Delta|} \Bigg[\frac{|a|}{\Gamma(\gamma-\vartheta)} \int_1^T \left(\ln \frac{T}{s}\right)^{\gamma-\vartheta-1} |G_{u_n v_n}(s) - G_{uv}(s)| \frac{ds}{s} \\
&\quad + |a| \sum_{i=1}^{p} \frac{1}{\Gamma(\alpha-\eta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\eta_i-1} |F_{u_n v_n}(\tau) - F_{uv}(\tau)| \frac{d\tau}{\tau} \right) dP_i(s) \right| \\
&\quad + |a| \sum_{i=1}^{q} \frac{1}{\Gamma(\gamma-\theta_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\theta_i-1} |G_{u_n v_n}(\tau) - G_{uv}(\tau)| \frac{d\tau}{\tau} \right) dQ_i(s) \right| \\
&\quad + \frac{|c|}{\Gamma(\alpha-\varsigma)} \int_1^T \left(\ln \frac{T}{s}\right)^{\alpha-\varsigma-1} |F_{u_n v_n}(s) - F_{uv}(s)| \frac{ds}{s} \\
&\quad + |c| \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha-\varrho_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\alpha-\varrho_i-1} |F_{u_n v_n}(\tau) - F_{uv}(\tau)| \frac{d\tau}{\tau} \right) dH_i(s) \right| \\
&\quad + |c| \sum_{i=1}^{n} \frac{1}{\Gamma(\gamma-\sigma_i)} \left| \int_1^T \left(\int_1^s \left(\ln \frac{s}{\tau}\right)^{\gamma-\sigma_i-1} |G_{u_n v_n}(\tau) - G_{uv}(\tau)| \frac{d\tau}{\tau} \right) dK_i(s) \right| \Bigg].
\end{aligned} \tag{62}$$

Because f and g are continuous, we find

$$\begin{aligned} &|F_{u_n v_n}(s) - F_{uv}(s)| = |f(s, u_n(s), v_n(s)) - f(s, u(s), v(s))| \to 0, \text{ and} \\ &|G_{u_n v_n}(s) - G_{uv}(s)| = |g(s, u_n(s), v_n(s)) - g(s, u(s), v(s))| \to 0, \end{aligned} \tag{63}$$

as $n \to \infty$, for all $s \in [1, T]$. So, by relations (61)–(63), we deduce

$$\|\mathcal{A}_1(u_n, v_n) - \mathcal{A}_1(u, v)\| \to 0, \ \|\mathcal{A}_2(u_n, v_n) - \mathcal{A}_2(u, v)\| \to 0, \text{ as } n \to \infty, \tag{64}$$

and then $\|\mathcal{A}(u_n, v_n) - \mathcal{A}(u, v)\|_{\mathcal{Y}} \to 0$, as $n \to \infty$; i.e., $\mathcal{A}$ is a continuous operator.

We prove now that $\mathcal{A}$ maps bounded sets into bounded sets in $\mathcal{Y}$. For $R > 0$, let $B_R = \{(u, v) \in \mathcal{Y}, \ \|(u, v)\|_{\mathcal{Y}} \le R\}$. Then, by using (60) and similar computations to those in the first part of the proof of Theorem 2, we obtain

$$|\mathcal{A}_1(u, v)(t)| \le M_1 \Xi_1 + M_2 \Xi_2, \ |\mathcal{A}_2(u, v)(t)| \le M_1 \Xi_3 + M_2 \Xi_4, \tag{65}$$

for all $t \in [1, T]$ and $(u, , v) \in B_R$. Then, by (65), we conclude

$$\|\mathcal{A}(u, v)\|_{\mathcal{Y}} \le M_1(\Xi_1 + \Xi_3) + M_2(\Xi_2 + \Xi_4), \ \forall (u, v) \in B_R; \tag{66}$$

i.e., $\mathcal{A}(B_R)$ is bounded.

In the following, we will prove that $\mathcal{A}$ maps bounded sets into equicontinuous sets. For this, let $t_1, t_2 \in [1, T]$, $t_1 < t_2$, and $(u, v) \in B_R$. Then, by using similar computations to those in the proofs of Theorems 3 and 4, we find

$$\begin{aligned} &|\mathcal{A}_1(u, v)(t_2) - \mathcal{A}_1(u, v)(t_1)| \\ &\le |\mathcal{B}_1(u, v)(t_2) - \mathcal{B}_1(u, v)(t_1)| + |\mathcal{C}_1(u, v)(t_2) - \mathcal{C}_1(u, v)(t_1)| \\ &\le M_1 \frac{1}{\Gamma(\alpha + 1)} [(\ln t_2)^\alpha - (\ln t_1)^\alpha] \\ &\quad + \left\{ M_1 \left[\Xi_1 - \frac{1}{\Gamma(\alpha + 1)} (\ln T)^\alpha \right] + M_2 \Xi_2 \right\} \\ &\quad \times \frac{1}{(\ln T)^{\lambda - 1}} \left[(\ln t_2)^{\lambda - 1} - (\ln t_1)^{\lambda - 1} \right] \to 0, \text{ as } t_2 \to t_1, \end{aligned} \tag{67}$$

independently of $(u, v) \in B_R$, and

$$\begin{aligned} &|\mathcal{A}_2(u, v)(t_2) - \mathcal{A}_2(u, v)(t_1)| \\ &\le |\mathcal{B}_2(u, v)(t_2) - \mathcal{B}_2(u, v)(t_1)| + |\mathcal{C}_2(u, v)(t_2) - \mathcal{C}_2(u, v)(t_1)| \\ &\le M_2 \frac{1}{\Gamma(\gamma + 1)} [(\ln t_2)^\gamma - (\ln t_1)^\gamma] \\ &\quad + \left\{ M_1 \Xi_3 + M_2 \left[\Xi_4 - \frac{1}{\Gamma(\gamma + 1)} (\ln T)^\gamma \right] \right\} \\ &\quad \times \frac{1}{(\ln T)^{\mu - 1}} \left[(\ln t_2)^{\mu - 1} - (\ln t_1)^{\mu - 1} \right] \to 0, \text{ as } t_2 \to t_1, \end{aligned} \tag{68}$$

independently of $(u, v) \in B_R$.

Therefore, by using relations (67) and (68), we obtain that operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are equicontinuous, and so, $\mathcal{A}$ is equicontinuous. So, the operator $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$ is completely continuous, by using the Arzela–Ascoli theorem.

Finally, we show that set $\mathcal{U} = \{(u, v) \in \mathcal{Y}, \ (u, v) = \omega \mathcal{A}(u, v), \ 0 \le \omega \le 1\}$ is bounded. Let $(u, v) \in \mathcal{U}$, i.e., there exists $\omega \in [0, 1]$ such that $(u, v) = \omega \mathcal{A}(u, v)$ or $u(t) = \omega \mathcal{A}_1(u, v)(t)$ and $v(t) = \omega \mathcal{A}_2(u, v)(t)$ for all $t \in [1, T]$. Then, by $(H4)$, we obtain in a similar manner as that used in the first part of this proof that

$$\begin{aligned} &|u(t)| = \omega |\mathcal{A}_1(u, v)(t)| \le |\mathcal{A}_1(u, v)(t)| \le M_1 \Xi_1 + M_2 \Xi_2, \ \forall t \in [1, T], \\ &|v(t)| = \omega |\mathcal{A}_2(u, v)(t)| \le |\mathcal{A}_2(u, v)(t)| \le M_1 \Xi_3 + M_2 \Xi_4, \ \forall t \in [1, T], \end{aligned} \tag{69}$$

and then
$$\|(u,v)\|_{\mathcal{Y}} = \|u\| + \|v\| \leq M_1(\Xi_1 + \Xi_3) + M_2(\Xi_2 + \Xi_4). \tag{70}$$

This shows that the set $\mathcal{U}$ is bounded. Therefore, by the Schaefer fixed point theorem (see [32]), we deduce that operator $\mathcal{A}$ has at least one fixed point. Hence, problem (1) and (2) has at least one solution. □

In our last existence result, we will use the Leray–Schauder nonlinear alternative (see [33]).

Theorem 6. *We suppose that assumption $(H1)$ holds. Moreover, we assume that the functions $f, g : [1,T] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, and the following conditions are satisfied:*

$(H5)$ *There exist the functions $p_1, p_2 \in C([1,T], \mathbb{R}_+)$ and the functions $\varphi_1, \varphi_2 \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ nondecreasing in each of both variables such that*

$$|f(t,x,y)| \leq p_1(t)\varphi_1(|x|,|y|), \quad |g(t,x,y)| \leq p_2(t)\varphi_2(|x|,|y|), \quad \forall\, t \in [1,T], \ x,y \in \mathbb{R}. \tag{71}$$

$(H6)$ *There exists a positive constant L such that*

$$\frac{L}{\|p_1\|\varphi_1(L,L)(\Xi_1+\Xi_3) + \|p_2\|\varphi_2(L,L)(\Xi_2+\Xi_4)} > 1. \tag{72}$$

Then, the fractional boundary value problem (1) and (2) has at least one solution $(u(t), v(t))$, $t \in [1,T]$.

Proof. We define the set $\mathcal{V} = \{(u,v) \in \mathcal{Y}, \ \|(u,v)\|_{\mathcal{Y}} < L\}$, where L is the constant given by (72). The operator $\mathcal{A} : \overline{\mathcal{V}} \to \mathcal{Y}$ is completely continuous.

We assume that there exist $(u,v) \in \partial\mathcal{V}$ such that $(u,v) = \nu\mathcal{A}(u,v)$ for some $\nu \in (0,1)$. Then, we find

$$\begin{aligned} |u(t)| &= \nu|\mathcal{A}_1(u,v)(t)| \leq |\mathcal{A}_1(u,v)(t)| \leq \|p_1\|\varphi_1(\|u\|,\|v\|)\Xi_1 + \|p_2\|\varphi_2(\|u\|,\|v\|)\Xi_2, \\ |v(t)| &= \nu|\mathcal{A}_2(u,v)(t)| \leq |\mathcal{A}_2(u,v)(t)| \leq \|p_1\|\varphi_1(\|u\|,\|v\|)\Xi_3 + \|p_2\|\varphi_2(\|u\|,\|v\|)\Xi_4, \end{aligned} \tag{73}$$

for all $t \in [1,T]$, and, therefore,

$$\begin{aligned} &\|(u,v)\|_{\mathcal{Y}} = \|u\| + \|v\| \leq \|p_1\|\varphi_1(\|u\|,\|v\|)(\Xi_1+\Xi_3) + \|p_2\|\varphi_2(\|u\|,\|v\|)(\Xi_2+\Xi_4) \\ &\leq \|p_1\|\varphi_1(\|(u,v)\|_{\mathcal{Y}},\|(u,v)\|_{\mathcal{Y}})(\Xi_1+\Xi_3) + \|p_2\|\varphi_2(\|(u,v)\|_{\mathcal{Y}},\|(u,v)\|_{\mathcal{Y}})(\Xi_2+\Xi_4). \end{aligned} \tag{74}$$

So, we obtain

$$\frac{L}{\|p_1\|\varphi_1(L,L)(\Xi_1+\Xi_3) + \|p_2\|\varphi_2(L,L)(\Xi_2+\Xi_4)} \leq 1, \tag{75}$$

which, based on (72), is a contradiction.

We deduce that there is no $(u,v) \in \partial\mathcal{V}$ such that $(u,v) = \nu\mathcal{A}(u,v)$ for some $\nu \in (0,1)$. Therefore, by the Leray–Schauder nonlinear alternative (see [33]), we conclude that $\mathcal{A}$ has a fixed point $(u,v) \in \overline{\mathcal{V}}$, which is a solution of problem (1) and (2). □

4. Examples

In this section, we will present some examples that illustrate our theorems obtained in Section 3.

We consider $T = 9$, $\alpha = \frac{3}{2}$, $\gamma = \frac{4}{3}$, $\beta = \frac{2}{5}$, $\delta = \frac{1}{7}$, $\varsigma = \frac{1}{4}$, $\vartheta = 0$, $m = 2$, $n = 2$, $p = 2$, $q = 2$, $\varrho_1 = 0$, $\varrho_2 = \frac{2}{11}$, $\sigma_1 = \frac{3}{8}$, $\sigma_2 = \frac{5}{12}$, $\eta_1 = 0$, $\eta_2 = \frac{1}{2}$, $\theta_1 = 0$, and $\theta_2 = \frac{4}{9}$.

In addition, we introduce the following functions

$$
\begin{aligned}
H_1(s) &= \begin{cases} -\dfrac{513}{22}, \ s \in \left[1, \dfrac{19}{6}\right); \\ \dfrac{15}{22}, \ s \in \left[\dfrac{19}{6}, \dfrac{11}{2}\right); \\ \dfrac{8}{27}(s-2)^{9/2} + \dfrac{15}{22} - \dfrac{7^{9/2}}{27 \cdot 2^{3/2}}, \ s \in \left[\dfrac{11}{2}, \dfrac{25}{3}\right); \\ \dfrac{8 \cdot 19^{9/2}}{3^{15/2}} + \dfrac{15}{22} - \dfrac{7^{9/2}}{27 \cdot 2^{3/2}}, \ s \in \left[\dfrac{25}{3}, 9\right]; \end{cases} \\
H_2(s) &= \begin{cases} \dfrac{35}{12}, \ s \in \left[1, \dfrac{54}{13}\right); \\ \dfrac{8}{3}, \ s \in \left[\dfrac{54}{13}, 9\right]; \end{cases} \\
K_1(s) &= -\dfrac{67}{72}(s-1)^3, \ s \in [1, 9]; \\
K_2(s) &= \begin{cases} 2, \ s \in [1, 7); \\ \dfrac{97}{38}, \ s \in [7, 9]; \end{cases} \\
P_1(s) &= \begin{cases} 2s + 15, \ s \in \left[1, \dfrac{72}{11}\right); \\ 2s + \dfrac{49}{6}, \ s \in \left[\dfrac{72}{11}, 9\right]; \end{cases} \\
P_2(s) &= \begin{cases} \dfrac{1}{2}, \ s \in \left[1, \dfrac{33}{8}\right); \\ \dfrac{23}{26}, \ s \in \left[\dfrac{33}{8}, 9\right]; \end{cases} \\
Q_1(s) &= \begin{cases} 17, \ s \in [1, 2); \\ 14, \ s \in [2, 7); \\ \dfrac{5}{68}(s-3)^{17/5} + 14 - \dfrac{5}{17} \cdot 2^{24/5}, \ s \in [7, 9]; \end{cases} \\
Q_2(s) &= \begin{cases} \dfrac{102}{25}\left(s - \dfrac{1}{2}\right)^{-25/6}, \ s \in \left[1, \dfrac{53}{8}\right); \\ \dfrac{51 \cdot 2^{27/2}}{25 \cdot 49^{25/6}}, \ s \in \left[\dfrac{53}{8}, 9\right]. \end{cases}
\end{aligned} \tag{76}
$$

We consider the system of fractional differential equations

$$
\begin{cases} {}^{HH}D_1^{\frac{3}{2}, \frac{2}{5}} u(t) = f(t, u(t), v(t)), \ \ t \in [1, T], \\ {}^{HH}D_1^{\frac{4}{3}, \frac{1}{7}} v(t) = g(t, u(t), v(t)), \ \ t \in [1, T], \end{cases} \tag{77}
$$

subject to the nonlocal coupled boundary conditions

$$
\begin{cases} u(1) = 0, \ \ {}^{H}D_1^{\frac{1}{4}} u(9) = 24\, u\left(\dfrac{19}{6}\right) + \dfrac{4}{3} \displaystyle\int_{\frac{11}{2}}^{\frac{25}{3}} (s-2)^{\frac{7}{2}} u(s)\, ds - \dfrac{1}{4}\, {}^{H}D_1^{\frac{2}{11}} u\left(\dfrac{54}{13}\right) \\ \quad -\dfrac{67}{24} \displaystyle\int_1^9 (s-1)^2\, {}^{H}D_1^{\frac{3}{8}} v(s)\, ds + \dfrac{21}{38}\, {}^{H}D_1^{\frac{5}{12}} v\left(\dfrac{7}{3}\right), \\ v(1) = 0, \ \ v(9) = 2 \displaystyle\int_1^9 u(s)\, ds - \dfrac{41}{6}\, u\left(\dfrac{72}{11}\right) + \dfrac{5}{13}\, {}^{H}D_1^{1/2} u\left(\dfrac{33}{8}\right) \\ \quad -3\, v(2) + \dfrac{1}{4} \displaystyle\int_7^9 (s-3)^{\frac{12}{5}} v(s)\, ds - 17 \displaystyle\int_1^{\frac{53}{8}} \left(s - \dfrac{1}{2}\right)^{-\frac{31}{6}} {}^{H}D_1^{\frac{4}{9}} v(s)\, ds. \end{cases} \tag{78}
$$

After some computations, using the Mathematica program, we obtain $\lambda = \frac{17}{10}$, $\mu = \frac{10}{7}$, $a \approx -1831.69757626$, $b \approx -449.04583604$, $c \approx 10.3109466$, $d \approx 37.64671349$, and $\Delta \approx -64327.3062105 \neq 0$. So, assumption $(H1)$ is satisfied.

In addition, we find $\Xi_1 \approx 4.99459858$, $\Xi_2 \approx 1.73975237$, $\Xi_3 \approx 0.93885308$, and $\Xi_4 \approx 5.31130101$.

Example 1. We consider the functions

$$\begin{aligned} f(t,x,y) &= \frac{1}{27} e^{-(t-1)^3} \sqrt{x^2+1} - \frac{2}{21} e^{-3t^2} \arctan y + \frac{\cos(4t+1)}{\sqrt[3]{t^2+1}}, \\ g(t,x,y) &= \frac{1}{12(t^2+1)} \cdot \frac{|x|}{1+|x|} + \frac{3}{41(t+2)} \cos^2 y - t^3 + 7, \end{aligned} \tag{79}$$

for all $t \in [1,9]$ and $x, y \in \mathbb{R}$.

We have

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &\leq \frac{1}{27}|x_1 - x_2| + \frac{2}{21}|y_1 - y_2|, \\ |g(t,x_1,y_1) - g(t,x_2,y_2)| &\leq \frac{1}{24}|x_1 - x_2| + \frac{2}{41}|y_1 - y_2|, \end{aligned} \tag{80}$$

for all $t \in [1,9]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$. So, we find $l_1 = \frac{1}{27}$, $l_2 = \frac{2}{21}$, $l_3 = \frac{1}{24}$, $l_4 = \frac{2}{41}$ (from assumption $(H2)$), and then $l_5 = \frac{2}{21}$ and $l_6 = \frac{2}{41}$. Because $l_5(\Xi_1 + \Xi_3) + l_6(\Xi_2 + \Xi_4) \approx 0.909 < 1$, then condition (26) is satisfied. Therefore, by Theorem 2, we conclude that the boundary value problem (77) and (78) with the nonlinearities (79) has a unique solution $(u(t), v(t))$, $t \in [1,9]$.

Example 2. We consider the functions

$$\begin{aligned} f(t,x,y) &= \frac{1}{t^2+12} \cdot \frac{x}{x^2+1} + \frac{(t^7+1)e^{-2t+3}}{15(t^3+31)} \sin y - \frac{\cos(4t^2+9)}{t+6}, \\ g(t,x,y) &= \frac{2t+1}{7\sqrt{t^4+18}} e^{-x^2} - \frac{4t^2+1}{3(t^5+42)} \cdot \frac{2y^4+1}{y^4+3} + \frac{e^{-t+6}}{5t+4}, \end{aligned} \tag{81}$$

for all $t \in [1,9]$ and $x, y \in \mathbb{R}$.

We have the following inequalities

$$\begin{aligned} |f(t,x,y)| &\leq \frac{1}{2(t^2+12)} + \frac{(t^7+1)e^{-2t+3}}{15(t^3+31)} + \frac{1}{t+6} =: \phi(t), \\ |g(t,x,y)| &\leq \frac{2t+1}{7\sqrt{t^4+18}} + \frac{2(4t^2+1)}{3(t^5+42)} + \frac{e^{-t+6}}{5t+4} =: \psi(t), \end{aligned} \tag{82}$$

for all $t \in [1,9]$ and $x, y \in \mathbb{R}$, and

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &\leq 0.077|x_1 - x_2| + 0.1265|y_1 - y_2|, \\ |g(t,x_1,y_1) - g(t,x_2,y_2)| &\leq 0.1057|x_1 - x_2| + 0.1041|y_1 - y_2|, \end{aligned} \tag{83}$$

for all $t \in [1,9]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$. So $l_1 = 0.077$, $l_2 = 0.1265$, $l_3 = 0.1057$, $l_4 = 0.1041$, and so $l_5 = 0.1265$ and $l_6 = 0.1057$. So, assumptions $(H2)$ and $(H3)$ are satisfied. In addition, we obtain $z_1 := \Xi_1 + \Xi_3 - \frac{1}{\Gamma(5/2)}(\ln 9)^{3/2} \approx 3.48339869$ and $z_2 := \Xi_2 + \Xi_4 - \frac{1}{\Gamma(7/3)}(\ln 9)^{4/3} \approx 4.65193119$. Then, we find $l_5 z_1 + l_6 z_2 \approx 0.932359 < 1$, i.e., condition (38) is also satisfied. By Theorem 3, we deduce that problem (77) and (78) with the nonlinearities (81) has at least one solution $(u(t), v(t))$, $t \in [1,9]$.

Example 3. We consider the functions

$$\begin{aligned} f(t,x,y) &= \frac{e^{-3t+1}}{\sqrt[3]{t^4+2}}\sin(x+y)+\frac{1}{5}\sqrt{t^2+13}, \\ g(t,x,y) &= \frac{t}{t^2+1}\cdot\frac{3x^2+1}{x^2+4}-(t^3-2t)e^{-y^2}+\arctan\frac{t^2}{t^6+2}, \end{aligned} \tag{84}$$

for all $t \in [1,9]$ and $x, y \in \mathbb{R}$. We obtain $|f(t,x,y)| \le 2$ and $|g(t,x,y)| \le 712.5+\frac{\pi}{2}$, for all $t \in [1,9]$ and $x, y \in \mathbb{R}$. So, $M_1 = 2$ and $M_2 = 712.5+\frac{\pi}{2}$ (from assumption $(H4)$). By Theorem 5, we conclude that the boundary value problem (77) and (78) with the nonlinearities (84) has at least one solution $(u(t), v(t)),\ t \in [1,9]$.

Example 4. We consider the functions

$$\begin{aligned} f(t,x,y) &= \frac{3}{t^2+81}\left(\frac{x}{28}\cos(x^3+y)-\frac{y}{50}\arctan\frac{x}{y^2+1}+\frac{1}{2}\right), \\ g(t,x,y) &= \frac{2t}{t^3+93}\left(\frac{5x}{23}\sin(x^2-4y)+\frac{y}{34}\arctan\frac{x+1}{y^4+2}-\frac{7}{8}\right), \end{aligned} \tag{85}$$

for all $t \in [1,9]$ and $x, y \in \mathbb{R}$. We have the inequalities

$$|f(t,x,y)| \le \frac{3}{t^2+81}\left(\frac{|x|}{28}+\frac{\pi|y|}{100}+\frac{1}{2}\right),\ \ |g(t,x,y)| \le \frac{2t}{t^3+93}\left(\frac{5|x|}{23}+\frac{\pi|y|}{34}+\frac{7}{8}\right), \tag{86}$$

for all $t \in [1,9]$ and $x, y \in \mathbb{R}$. So, we find $p_1(t) = \frac{3}{t^2+81}$, $p_2(t) = \frac{2t}{t^3+93}$, $\varphi_1(u,v) = \frac{u}{28}+\frac{\pi v}{100}+\frac{1}{2}$, $\varphi_2(u,v) = \frac{5u}{23}+\frac{\pi v}{34}+\frac{7}{8}$, for all $t \in [1,9]$ and $u, v \in \mathbb{R}_+$, and then assumption $(H5)$ is satisfied. In addition, we obtain $\|p_1\| \approx 0.03658537$ and $\|p_2\| \approx 0.05155531$. The condition from assumption $(H6)$ becomes

$$L > \frac{\frac{1}{2}(\Xi_1+\Xi_3)\|p_1\|+\frac{7}{8}(\Xi_3+\Xi_4)\|p_2\|}{1-(\frac{1}{28}+\frac{\pi}{100})(\Xi_1+\Xi_3)\|p_1\|-(\frac{5}{23}+\frac{\pi}{34})(\Xi_2+\Xi_4)\|p_2\|} \approx 0.48878552, \tag{87}$$

So, if $L \ge 0.4888$, then assumption $(H6)$ is also satisfied. Therefore, by Theorem 6, we deduce the existence of at least one solution $(u(t), v(t)),\ t \in [1,9]$ for problem (77) and (78) with the nonlinearities (85).

5. Conclusions

In this paper, we investigated the existence and uniqueness of solutions for a system of fractional differential equations denoted as (1). These equations are subject to nonlocal boundary conditions as specified in (2). System (1) encompasses Hilfer–Hadamard fractional derivatives that vary in orders and types, while the conditions (2) are nonlocal, featuring a combination of Riemann–Stieltjes integrals and Hadamard derivatives with varying orders. It is worth noting that these conditions are general ones, encompassing scenarios that range from uncoupled boundary conditions (in the event that all functions K_i for $i = 1, \ldots, n$ and P_j for $j = 1, \ldots, p$ are constants) to more complex cases that generalize multi-point boundary conditions, classical integral conditions, and various combinations thereof. In Section 2, we have provided an existence theorem for the linear fractional differential problem associated with (1) and (2). In Section 3, we have presented our primary findings, supported by rigorous proofs in which we have employed various fixed point theorems. These theorems include the Banach contraction mapping principle (applied to prove Theorem 2), the Krasnosel'skii fixed point theorem for the sum of two operators (utilized in proving Theorems 3 and 4), the Schaefer fixed point theorem (employed for Theorem 5), and the Leray–Schauder nonlinear alternative (used to establish Theorem 6). Finally, in Section 4, we have provided several illustrative examples to elucidate the implications of our main existence results. Going forward, our aim is to investigate different sets of fractional equations, which include fractional derivatives of various kinds and are subject to diverse boundary conditions.

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