fractal and fractional

Article

On Mixed Fractional Lifting Oscillation Spaces

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Abstract: We introduce hyperbolic oscillation spaces and mixed fractional lifting oscillation spaces expressed in terms of hyperbolic wavelet leaders of multivariate signals on $\mathbb{R}^d$, with $d \geq 2$. Contrary to Besov spaces and fractional Sobolev spaces with dominating mixed smoothness, the new spaces take into account the geometric disposition of the hyperbolic wavelet coefficients at each scale $(j_1, \cdots, j_d)$, and are therefore suitable for a multifractal analysis of rectangular regularity. We prove that hyperbolic oscillation spaces are closely related to hyperbolic variation spaces, and consequently do not almost depend on the chosen hyperbolic wavelet basis. Therefore, the so-called rectangular multifractal analysis, related to hyperbolic oscillation spaces, is somehow 'robust', i.e., does not change if the analyzing wavelets were changed. We also study optimal relationships between hyperbolic and mixed fractional lifting oscillation spaces and Besov spaces with dominating mixed smoothness. In particular, we show that, for some indices, hyperbolic and mixed fractional lifting oscillation spaces are not always sharply imbedded between Besov spaces or fractional Sobolev spaces with dominating mixed smoothness, and thus are new spaces of a really different nature.

Keywords: hyperbolic oscillation spaces; mixed fractional lifting oscillation spaces; Besov spaces and fractional Sobolev spaces with dominating mixed smoothness; hyperbolic wavelet basis; multifractal analysis of rectangular regularity

Citation: Alzughaibi, I.; Ben Slimane, M.; Algahtani, O. On Mixed Fractional Lifting Oscillation Spaces. *Fractal Fract.* **2023**, *7*, 819. https://doi.org/10.3390/fractalfract7110819

Academic Editor: Maria Rosaria Lancia

Received: 6 June 2023
Revised: 20 August 2023
Accepted: 24 August 2023
Published: 13 November 2023

1. Introduction

A classical way of describing regularity is to use pointwise Hölder classes $C^\alpha(x)$ at an arbitrary fixed point $x \in \mathbb{R}^d$ (with $\alpha > 0$), Hölder-Zygmund $C^\alpha(\mathbb{R}^d)$ spaces, classical and fractional Sobolev spaces (sometimes denoted as Bessel potential spaces), Besov spaces and Triebel–Lizorkin spaces. These spaces attracted a lot of attention theoretically and have been treated systematically with numerous applications given in many areas such as numerics, signal processing and fractal analysis, to mention only a few of them (for examples, see [1,2] and references therein).

Unfortunately, these spaces have the disadvantage of not fully capturing changes in the regularity of multivariate functions f, i.e., $d \geq 2$, that have anisotropic structures. In fact, these spaces only resolve a certain minimal smoothness for functions or signals which are rather smooth in a Cartesian axis-direction but rough in another axis-direction (such as layers in the earth, stripes on a shirt, etc.). Anisotropy naturally appears whenever physics does not act the same in different directions, e.g., geophysics, oceanography, hydrology, fluid mechanics, or medical image processing (see [3–7], among others).

Great effort has been devoted by many researchers to remedy this by generalizing the spaces in various ways. New spaces with different anisotropic or mixed degrees of smoothness along direction axes have been introduced and studied locally and globally. The function decompositions of these spaces have been investigated. Fourier analytical approaches were given. Approximations by certain sums of either anisotropic or hyperbolic tensor product blocks, wavelets or splines have been used. Many fundamental results such as imbeddings, traces theorem, etc, were obtained.

In various domains, Besov spaces constitute a natural mathematical setting to study signals, as they have a convenient wavelet characterization and fit naturally to approximation problems. They also have the advantage of being sharply imbedded with fractional Sobolev spaces.

Anisotropic Besov spaces were introduced for the study of semi-elliptic pseudo-differential operators whose symbols have different degrees of smoothness along different directions, cf., e.g., [2]; see also [8–11], and references therein, for a recent use of such spaces for optimal regularity results of the heat equation.

Let $\mathcal{D} = \{1, \cdots, d\}$. For $i \in \mathcal{D}$, let e_i be the i-th unit vector in $\mathbb{R}^d$. For $f : \mathbb{R}^d \to \mathbb{R}$, $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and $h \in \mathbb{R}$, the iterated difference of f of order $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is defined as

$$\Delta^n_{h,i} f(\mathbf{x}) = \sum_{l=0}^{n} (-1)^{l+n} \binom{n}{l} f(\mathbf{x} + lhe_i) \, .$$

Then, the anisotropic Besov space $\mathbb{A}^{\mathbf{s},q}_p(\mathbb{R}^d)$, where $1 \le p, q \le \infty$ and $\mathbf{s} = (s_1, \cdots, s_d)$ with $0 < s_i < \infty$, is defined as

$$\mathbb{A}^{\mathbf{s},q}_p(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) \ : \ \|f|\mathbb{A}^{\mathbf{s},q}_p(\mathbb{R}^d)\|_{\Delta,\mathbf{M}} < \infty\}$$

, where $\mathbf{M} = (M_1, \cdots, M_d) \in \mathbb{N}^d$ with $M_i > s_i$ and

$$\|f|\mathbb{A}^{\mathbf{s},q}_p(\mathbb{R}^d)\|_{\Delta,\mathbf{M}} = \|f|L^p(\mathbb{R}^d)\| + \sum_{i=1}^{d} \left(\int_{-1}^{1} \||t|^{-s_i} \Delta^{M_i}_{t,i} |L^p(\mathbb{R}^d)\|^q \frac{dt}{|t|} \right)^{1/q}$$

(with a standard appropriate modification to use the sup norm in the case of $q = \infty$).

As in the isotropic case, anisotropic Besov spaces encompass a large class of classical anisotropic functional spaces (see [2,12–14] for details). For example, if $1 < p < \infty$, $\mathbf{s} \in \mathbb{N}^d$ and $\mathbb{W}^{\mathbf{s}}_p(\mathbb{R}^d)$ is the classical anisotropic Sobolev space given by

$$\mathbb{W}^{\mathbf{s}}_p(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) \ : \ \|f|\mathbb{W}^{\mathbf{s}}_p(\mathbb{R}^d)\| < \infty\}$$

where

$$\|f|\mathbb{W}^{\mathbf{s}}_p(\mathbb{R}^d)\| = \|f|L^p(\mathbb{R}^d)\| + \sum_{i=1}^{d} \|\frac{\partial^{s_i} f}{\partial x_i^{s_i}} |L^p(\mathbb{R}^d)\|$$

then

$$\mathbb{A}^{\mathbf{s},2}_2(\mathbb{R}^d) = \mathbb{W}^{\mathbf{s}}_2(\mathbb{R}^d)$$

and

$$\mathbb{A}^{\mathbf{s},\min(p,2)}_p(\mathbb{R}^d) \hookrightarrow \mathbb{W}^{\mathbf{s}}_p(\mathbb{R}^d) \hookrightarrow \mathbb{A}^{\mathbf{s},\max(p,2)}_p(\mathbb{R}^d) \, .$$

The latest equality and embeddings remain true if the classical anisotropic Sobolev space is replaced by the fractional anisotropic Sobolev space $\mathbb{H}^{\mathbf{s}}_p(\mathbb{R}^d)$ for $\mathbf{s} \notin \mathbb{N}^d$. Actually, fractional anisotropic Sobolev spaces are special cases of anisotropic Lizorkin–Triebel spaces, and are therefore sharply imbedded between anisotropic Besov spaces (see [2]).

In order to describe new types of dominating mixed smoothness, mixed differences and mixed derivatives conditions have been added (see [15–17]). Let $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}_0^d$. For $\mathbf{h} = (h_1, \cdots, h_d) \in \mathbb{R}^d$, define the mixed differences of order $\mathbf{n}$ as

$$\triangle^{\mathbf{n}}_{\mathbf{h}} f(\mathbf{x}) = \triangle^{n_1}_{h_1,1} \circ \cdots \circ \triangle^{n_d}_{h_d,d} f(\mathbf{x}).$$

For $A = \{i_1, \cdots, i_k\} \subset \mathcal{D}$, set

$$\triangle^{\mathbf{n}}_{\mathbf{h},A} f = \triangle^{n_{i_1}}_{h_{i_1},i_1} \circ \cdots \circ \triangle^{n_{i_k}}_{h_{i_k},i_k} f \, .$$

Besov (or Nikol'skii) spaces with dominating mixed smoothness are defined as

$$\begin{aligned}\|f|B_p^{\mathbf{s},q}(\mathbb{R}^d)\|_{\Delta,\mathbf{M}} &= \|f|\mathbb{A}_p^{\mathbf{s},q}(\mathbb{R}^d)\|_{\Delta,\mathbf{M}} \\ &+ \sum_{k=2}^{d}\sum_{A=\{i_1,\cdots,i_k\}\subset\mathcal{D}}\left(\int_{[-1,1]^k}\left(\prod_{l=1}^{k}|t_{i_l}|^{-s_{i_k}}\|\Delta_{\mathbf{t},A}^{\mathbf{M}}f|L^p(\mathbb{R}^d)\|\right)^q\prod_{l=1}^{k}\frac{dt_{i_l}}{|t_{i_l}|}\right)^{1/q}.\end{aligned}$$

The standard appropriate modification to use the sup norm in the case of $q=\infty$ leads to the so-called Hölder spaces in L^p with dominating mixed smoothness.

Note that the above norm does not depend on the size of $\mathbf{M}$ (equivalent norms).

Write $\mathbf{x}\leq\mathbf{x}'$ (resp. $\mathbf{x}<\mathbf{x}'$ (with respect to $\mathbf{x}\geq\mathbf{x}'$)) if $x_i\leq x_i'$ (with respect to $x_i<x_i'$ (with respect to $x_i\geq x_i'$)) for all $i\in\mathcal{D}$, etc.. . . Put $\mathbf{1}=(1,\cdots,1)$ and $\mathbf{0}=(0,\cdots,0)\in\mathbb{R}^d$.

Mixed differences $\|\Delta_{\mathbf{t},A}^{\mathbf{M}}f|L^p(\mathbb{R}^d)\|$ can also be replaced by mixed moduli of continuity (or smoothness) $\sup_{-\mathbf{t}\leq\mathbf{h}\leq\mathbf{t}}\|\Delta_{\mathbf{h},A}^{\mathbf{M}}f|L^p(\mathbb{R}^d)\|$ (for example, see [15,18,19]).

Sobolev spaces with dominating mixed derivatives $W_p^{\mathbf{s}}(\mathbb{R}^d)$ were also given. They are subsets of anisotropic Sobolev spaces $W_p^{\mathbf{s}}(\mathbb{R}^d)$ with additional mixed derivative conditions:

$$W_p^{\mathbf{s}}(\mathbb{R}^d):=\{f\in L^p(\mathbb{R}^d)\ :\ \|f|W_p^{\mathbf{s}}(\mathbb{R}^d)\|<\infty\}$$

where

$$\|f|W_p^{\mathbf{s}}(\mathbb{R}^d)\|=\|f|\mathbb{W}_p^{\mathbf{s}}(\mathbb{R}^d)\|+\sum_{k=2}^{d}\sum_{A=\{i_1,\cdots,i_k\}\subset\mathcal{D}}\|\frac{\partial^{s_{i_1}+\cdots+s_{i_k}}f}{\partial x_{i_1}^{s_{i_1}}\cdots\partial x_{i_k}^{s_{i_k}}}|L^p(\mathbb{R}^d)\|\,.$$

Fractional Sobolev spaces with dominating mixed derivatives were also given and are sharply imbedded with Besov spaces with dominating mixed derivatives (for example, see [15,20]).

Many authors have performed a detailed study of the above spaces with dominating mixed derivatives, for example, see [20] and references therein. These spaces have been also studied from the viewpoint of function dyadic decomposition [21].

As in the theory of classical Sobolev spaces, alternative definitions in terms of Fourier transform may be given. They have lead to the representation theorems by Littlewood–Paley blocks [2,15,17,22]. This allows for a natural extension to parameters p and q which are less than 1 [2].

Function spaces with dominating mixed smoothness represent a suitable framework for multivariate appoximation; see [23–25]. For example, it was proved that it suffices to use entire functions whose spectrals lie in hyperbolic crosses.

The above spaces for the full range of the parameters p,q have been characterized by more or less elementary building dyadic blocks such as atoms, molecules, quarks, splines [18] and wavelet bases [15,20] (see also [17,26]). Actually, the standard isotropic dyadic blocks yield a bad decay for the coefficients and consequently do not contain the anisotropic smoothness information. Anisotropic or hyperbolic dyadic blocks are better suited in this scenario [27–29].

These developments together with the interrelations with hyperbolic crosses have numerous applications in computational mathematics, the numerical solution of partial differential equations, data analysis and signal processing [17,23,24,30–34]. In particular, ref. [35] has recently generalized the boundary crossing theorem and the zero exclusion principle for fractional systems.

Let us recall the wavelet characterization of Besov spaces with dominating mixed smoothness. Let $\psi_1=\psi$ be the univariate smooth enough and compactly supported

Daubechies mother wavelet (see [36]). Let $\psi_{-1} = \varphi$ be the corresponding father wavelet. Put $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. For $j \in \mathbb{N}_{-1}$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, write

$$[j] = \begin{cases} j & \text{if } j \in \mathbb{N}_0 \\ 0 & \text{if } j = -1 \end{cases} \quad \text{and} \quad \psi_{j,k}(x) = \begin{cases} \psi_1(2^j x - k) & \text{if } j \in \mathbb{N}_0\,, \\ \psi_{-1}(x-k) & \text{if } j = -1\,. \end{cases}$$

Then $\left(2^{[j]/2}\psi_{j,k}\right)_{j\in\mathbb{N}_{-1},k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$.

For $\mathbf{j} = (j_1, \cdots, j_d) \in \mathbb{N}^d_{-1}$, put $[\mathbf{j}] = ([j_1], \cdots, [j_d])$ and $|\mathbf{j}| = \sum_{i=1}^{d} j_i$.

For $\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^d$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, put

$$\Psi_{\mathbf{j},\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{d} \psi_{j_i,k_i}(x_i)\,.$$

Then, the collection $\{2^{|[\mathbf{j}]|/2}\ \Psi_{\mathbf{j},\mathbf{k}} \ :\ \mathbf{j} \in \mathbb{N}^d_{-1}, \mathbf{k} \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, a called hyperbolic wavelet basis [23,25,37,38]. Thus, any function $f \in L^2(\mathbb{R}^d)$ can be written as

$$f = \sum_{\mathbf{j}\in\mathbb{N}^d_{-1}} \sum_{\mathbf{k}\in\mathbb{Z}^d} C_{\mathbf{j},\mathbf{k}}\Psi_{\mathbf{j},\mathbf{k}}$$

with

$$C_{\mathbf{j},\mathbf{k}} = 2^{|[\mathbf{j}]|} \int_{\mathbb{R}^d} f(\mathbf{x})\Psi_{\mathbf{j},\mathbf{k}}(\mathbf{x})\, d\mathbf{x}$$

called a hyperbolic wavelet coefficient.

For any $\mathbf{j} \in \mathbb{N}^d_{-1}$, and $\mathbf{k} \in \mathbb{Z}^d$, let

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}(\mathbf{j}, \mathbf{k}) = \prod_{i=1}^{d} \left[k_i 2^{-[j_i]}, (k_i + 1)2^{-[j_i]}\right)\,.$$

Set

$$C_{\boldsymbol{\lambda}} = C_{\mathbf{j},\mathbf{k}}\,,\ \Psi_{\boldsymbol{\lambda}} = \Psi_{\mathbf{j},\mathbf{k}} \ \text{ and } \ \boldsymbol{\Lambda}_{\mathbf{j}} = \{\boldsymbol{\lambda}(\mathbf{j}, \mathbf{k})\ :\ \mathbf{k} \in \mathbb{Z}^d\}\,.$$

We have the following proposition.

Proposition 1 ([15,20,26])**.** *Let $0 < p, q \leq \infty$ and $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}^d$. A function f belongs to the Besov (or Nikol'skii) space $B^{\mathbf{s},q}_p$ with dominating mixed smoothness if*

$$\|f|B^{\mathbf{s},q}_p(\mathbb{R}^d)\| = \|b_{\mathbf{j}}\|_{\ell^q(\mathbb{N}^d_{-1})} < \infty$$

with

$$b_{\mathbf{j}} := 2^{(s_1-\frac{1}{p})[j_1]+\cdots+(s_d-\frac{1}{p})[j_d]} \left(\sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_{\mathbf{j}}} |C_{\boldsymbol{\lambda}}|^p\right)^{1/p}.$$

From now on, we will call it a hyperbolic Besov space. Clearly, hyperbolic Besov spaces involve simultaneous axes directions behaviors; nevertheless, they are only defined for positive p's and are invariant with respect to permutations of the locations $\mathbf{k}$'s of wavelet coefficients $C_{\boldsymbol{\lambda}}$ with $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathbf{j}}$ at each scale $\mathbf{j}$. In [39,40], it is proved that such locations affect the fractal geometry of rectangular pointwise singularities of f. Such positions are important for the computation of the local suprema of a specific family of coefficients of smaller scales and located at the same place. These suprema are called hyperbolic wavelet leaders and are given by

$$d_{\boldsymbol{\lambda}} = \sup_{\boldsymbol{\lambda}'\subset\boldsymbol{\lambda}} |C_{\boldsymbol{\lambda}'}|,$$

where $\boldsymbol{\lambda}' \subset \boldsymbol{\lambda}$ means that there exists $\mathbf{j}' \geq \mathbf{j}$ such that $\boldsymbol{\lambda}' \in \boldsymbol{\Lambda}_{\mathbf{j}'}$, and $\boldsymbol{\lambda}'$ is a subset of $\boldsymbol{\lambda}$. In general, the computation of the hyperbolic wavelet leaders is hard. Nevertheless, in [39,40], it is shown that the decay of hyperbolic wavelet leaders around each given point $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$ characterizes the rectangular regularity $\mathfrak{L}^{(\alpha_1, \cdots, \alpha_d)}(\mathbf{x})$ of the function at $\mathbf{x}$. The latest is defined through local oscillations over rectangles around $\mathbf{x}$. For $\boldsymbol{\epsilon} > \mathbf{0}$, we denote by $B(\mathbf{x}, \boldsymbol{\epsilon})$ the rectangle

$$B(\mathbf{x}, \boldsymbol{\epsilon}) = \prod_{i=1}^{d} [x_i - \epsilon_i, x_i + \epsilon_i] \ .$$

For $\mathbf{n} \in \mathbb{N}_0^d$ and $E \subset \mathbb{R}^d$, the hyperbolic $\mathbf{n}$-oscillation of f in E is defined by

$$\omega^{\mathbf{n}}(f, E) = \sup_{[\mathbf{y}, \mathbf{y}+\mathbf{nh}] \subset E} |\Delta_{\mathbf{h}}^{\mathbf{n}} f(\mathbf{y})| \tag{1}$$

where

$$\mathbf{nh} = (n_1 h_1, \cdots, n_d h_d) \ \text{ and } \ [\mathbf{y}, \mathbf{y} + \mathbf{nh}] = \prod_{i=1}^{d} [y_i, y_i + n_i h_i] \ . \tag{2}$$

Definition 1. *Let $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) > \mathbf{0}$ and $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$. We say that f is a rectangular pointwise regular $\boldsymbol{\alpha}$ at point $\mathbf{x}$, and we write $f \in \mathfrak{L}^{\boldsymbol{\alpha}}(\mathbf{x})$, if*

$$\forall\, \mathbf{n} > \boldsymbol{\alpha} \quad \exists\, C > 0 \ \ \forall\, \boldsymbol{\epsilon} > \mathbf{0} \ , \qquad \omega^{\mathbf{n}}(f, B(\mathbf{x}, \boldsymbol{\epsilon})) \leq C \prod_{i=1}^{d} \epsilon_i^{\alpha_i} \ .$$

Contrary to the classical pointwise Hölder regularity $C^\alpha(\mathbf{x})$ with $\alpha > 0$, Definition 1 involves simultaneous axes direction behaviors in a neighborhood of $\mathbf{x}$. In [40] (with respect to [39]), it is shown that rectangular regularity fluctuates widely from point to point for a large class of self-affine cascade Schauder (with respect to wavelet) series. These rectangular multifractal series are written as the superposition of similar anisotropic structures at different scales of $\mathbf{j}$, reminiscent of some possible modelization of turbulent flows or cascade models. The anisotropy corresponds to self-affine transformations of a Sierpinski carpet K (see Section 3.1). Note that fractional Brownian sheets are rectangular monofractal [18,40,41]. Note also that there has been a growing interest in the multifractal analysis of self-affine functions and measures from a different point of view (see for example [42–44]).

In [45], it is proved that the knowledge of the p-domain of f allows one to extract some relevant information concerning the fractal print dimensions of sets of level rectangular pointwise behaviors. Fractal print dimensions $print A$ distinguish between sets A that are easily covered by long thin rectangles R_n (with edge-lengths $l_1(R_n), \cdots, l_d(R_n)$) and sets which are not. Recall that

$$print A = \{\delta = (\delta_1, \cdots, \delta_d) \geq \mathbf{0} \ : \ \sup_{\varepsilon > 0} \left(\inf_{A \subset \bigcup_n R_n \,;\, l_d(R_n) \leq \cdots \leq l_1(R_n) < \varepsilon} \{ \sum_{n \in \mathbb{N}} \prod_{i=1}^{d} (l_i(R_n))^{\delta_i} \} \right) > 0\} \ . \tag{3}$$

For $\mathbf{j} \in \mathbb{N}_{-1}^d$ and $p > 0$, put the following quantity based on the wavelet leaders

$$S_{p,\mathbf{j}} = 2^{-|[\mathbf{j}]|} \left(\sum_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathbf{j}}} d_{\boldsymbol{\lambda}}^p \right) . \tag{4}$$

The p-domain of f is defined as

$$\mathcal{D}_p = \{ \mathbf{s} = (s_1, \cdots, s_d) \ : \ \exists\, C > 0 \quad \forall \mathbf{j} \ \ \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathbf{j}} \quad S_{p,\mathbf{j}} \leq C 2^{-p(s_1[j_1] + \cdots + s_d[j_d])} \} \ .$$

The results were applied for $d = 2$ for the self-affine cascade wavelet series. In particular, it was proved that the p-hyperbolic domain depends on the selected generic boxes of the Sierpinski carpet. Note that this is not the case if the $d_{\boldsymbol{\lambda}}$s in (4) are replaced by the $|C_{\boldsymbol{\lambda}}|$s. Thus, hyperbolic Besov and fractional Sobolev spaces are not convenient for the multifractal analysis of rectangular regularity (see Section 3.1).

The p-domain of f has a functional interpretation.

Definition 2. *Let* $\mathbf{s} = (s_1, \cdots, s_d)$ *and* $p > 0$*. A function* f *belongs to the hyperbolic oscillation space* $O_p^{\mathbf{s}}$ *if*

$$\|f|O_p^{\mathbf{s}}(\mathbb{R}^d)\| = \|\sigma_{\mathbf{j}}\|_{\ell^\infty(\mathbb{N}_{-1}^d)} < \infty$$

with

$$\sigma_{\mathbf{j}} = \sigma_{\mathbf{j}}(\mathbf{s}, p) := 2^{(s_1-\frac{1}{p})[j_1]+\cdots+(s_d-\frac{1}{p})[j_d]} \left(\sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_{\mathbf{j}}} d_{\boldsymbol{\lambda}}^p \right)^{1/p} \tag{5}$$

Clearly

$$\mathcal{D}_p = \{\mathbf{s} \ : \ f \in O_p^{\mathbf{s}}(\mathbb{R}^d)\} \ . \tag{6}$$

Note that

$$O_p^{\mathbf{s}}(\mathbb{R}^d) \hookrightarrow B_p^{\mathbf{s},\infty}(\mathbb{R}^d) \ . \tag{7}$$

Contrary to Jaffard's isotropic oscillation spaces [1,46], hyperbolic oscillation spaces capture axes direction behaviors. Moreover, the p-domain can be extended for $p < 0$; in this case, we require that $f \in B_\infty^{(0,\cdots,0),\infty}$ and $\mathcal{D}_p$ is the set of all $\mathbf{s}$ such that for all $\varepsilon > 0$ there exists $C > 0$, such that $2^{(s_1p-1-\varepsilon)[j_1]+\cdots+(s_dp-1-\varepsilon)[j_d]} \sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_{\mathbf{j}}} \left(\sup_{\boldsymbol{\lambda}'\subset\prod_{i=1}^d[(k_i-1)2^{-[j_i]},(k_i+2)2^{-[j_i]})} |C_{\boldsymbol{\lambda}'}| \right)^p \leq C$ for $\mathbf{j}$ large enough. Hyperbolic oscillation spaces yield good interaction with rectangular multifractal analysis.

Furthermore, hyperbolic oscillation spaces can be extended via mixed fractional derivatives or primitives. Let $\mathbf{s}' = (s'_1, \cdots, s'_d) \in \mathbb{R}^d$. In [20] Definition 1.8, the mixed fractional lifting operator $I_{\mathbf{s}'}$ of derivatives or primitives is defined by the Fourier transform property

$$\widehat{I_{\mathbf{s}'}f}(\xi) = \hat{f}(\xi) \prod_{i=1}^d (1 + |\xi_i|^2)^{s'_i/2} \ .$$

Remark 1. *In [20], it is shown that* $I_{\mathbf{s}'}$ *maps* $B_p^{\mathbf{s},q}$ *isomorphically onto* $B_p^{\mathbf{s}-\mathbf{s}',q}$*.*

Definition 3. *Let* $p > 0$, $\mathbf{s} \in \mathbb{R}^d$ *and* $\mathbf{s}' \in \mathbb{R}^d$*. We say that* f *belongs to the mixed fractional lifting oscillation space* $O_p^{\mathbf{s},\mathbf{s}'}$ *if* $I_{\mathbf{s}'} f \in O_p^{\mathbf{s}-\mathbf{s}'}$*.*

Hyperbolic and mixed fractional lifting oscillation spaces are defined through wavelet leaders. The independence from the chosen hyperbolic wavelet basis is a natural requirement. This will be demonstrated in the next section; we will prove that hyperbolic oscillation spaces are closely related to hyperbolic variation spaces. Therefore, the rectangular multifractal analysis, related to hyperbolic oscillation spaces, is somehow 'robust', i.e., does not change if the analyzing wavelets were changed.

In the third section, we study optimal relationships between hyperbolic and mixed fractional lifting oscillation spaces and hyperbolic Besov spaces. In particular, we will prove that, for some indices, hyperbolic and mixed fractional lifting oscillation spaces are not always sharply imbedded between hyperbolic Besov spaces, and thus are new spaces of a really different nature.

The results and proofs are full hyperbolic counterparts of isotropic techniques conducted in [1] Theorem 3 and [46] Proposition 2. This requires many technical efforts to overcome a lot of troubles caused by the different dilation factors $2^{[j_1]}, \cdots, 2^{[j_d]}$ in coordinate axes and will be demonstrated in detail.

2. Independence from the Chosen Hyperbolic Wavelet Basis

We will prove that hyperbolic and mixed fractional lifting oscillation spaces are almost independent from the chosen hyperbolic wavelet basis. For that purpose, it suffices to show that hyperbolic oscillation spaces are closely related to the following hyperbolic variation spaces.

Definition 4. *We say that f belongs to the hyperbolic variation space $V^{\mathbf{s},p}$ if $f \in L^\infty$ and $\forall \mathbf{n} > \mathbf{s}$*

$$\sup_{\mathbf{j}\in\mathbb{N}^d_{-1}} 2^{(s_1-\frac{1}{p})[j_1]+\cdots+(s_d-\frac{1}{p})[j_d]} \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \left(\omega^{\mathbf{n}}(f, 2\overline{\lambda})\right)^p \right)^{1/p} < \infty,$$

where

$$c\overline{\lambda} := \prod_{i=1}^{d} [(k_i - c)2^{-[j_i]}, (k_i + c)2^{-[j_i]}].$$

and $\omega^{\mathbf{n}}(f, 2\overline{\lambda})$ is as in (1) and (2).

Theorem 1. *The following imbeddings hold*

$$\forall \varepsilon > 0, \forall \eta > 0 \quad B^{\varepsilon,\infty}_{\infty} \bigcap O^{\mathbf{s}+\boldsymbol{\eta}}_{p} \hookrightarrow V^{\mathbf{s},p} \hookrightarrow O^{\mathbf{s}}_{p} \, .$$

Proof. It suffices to consider f compactly supported. In fact, if f has no compact support, then using a partition of unity, we can write $f = \sum_{\mathbf{k}\in\mathbb{Z}^d} f(.)\Theta(. - \mathbf{k})$, and Θ is a C^∞ compactly supported function such that $\sum_{\mathbf{k}\in\mathbb{Z}^d} \Theta(. - \mathbf{k}) = 1$. Moreover, we use the fact that

$$\|f\|_{O^{\mathbf{s}}_p} \sim \left(\sum_{\mathbf{k}} \|f(.)\Theta(. - \mathbf{k})\|^p_{O^{\mathbf{s}}_p} \right)^{1/p} \quad \text{and} \quad \|f\|_{V^{\mathbf{s},p}} \sim \left(\sum_{\mathbf{k}} \|f(.)\Theta(. - \mathbf{k})\|^p_{V^{\mathbf{s},p}} \right)^{1/p} .$$

Without any loss of generality, we consider $d = 2$. Let $A > 0$ be large enough so that the support of f is included in $[-A, A]^2$ and the supports of ψ and φ are included in $[-A, A]$.

- We will show the first imbedding.

  Let $f \in B^{\varepsilon,\infty}_{\infty} \bigcap O^{\mathbf{s}+\boldsymbol{\eta}}_{p}$. Write

  $$f = \sum_{\mathbf{j}'\in\mathbb{N}^d_{-1}} f_{\mathbf{j}'}$$

  with

  $$f_{\mathbf{j}'} = \sum_{\boldsymbol{\lambda}'\in\boldsymbol{\Lambda}_{\mathbf{j}'}} C_{\boldsymbol{\lambda}'}\Psi_{\boldsymbol{\lambda}'}.$$

  Clearly

$$
\begin{aligned}
\omega^{\mathbf{n}}(f, A\overline{\lambda}) \quad &\leq \sum_{\mathbf{j}' \in \mathbb{N}^d_{-1}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \\
&\leq \sum_{\mathbf{j}' \leq \mathbf{j}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \\
&\quad + \sum_{\mathbf{j} \leq \mathbf{j}' \leq (j_1^2, j_2^2)} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \\
&\quad + \sum_{\substack{j_1' \geq j_1^2 \\ j_2 \leq j_2' \leq j_2^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) + \sum_{\substack{j_2' \geq j_2^2 \\ j_1 \leq j_1' \leq j_1^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \\
&\quad + \sum_{\substack{j_1' \leq j_1 \\ j_2 \leq j_2' \leq j_2^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) + \sum_{\substack{j_1' \leq j_1 \\ j_2' \geq j_2^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \\
&\quad + \sum_{\substack{j_2' \leq j_2 \\ j_1 \leq j_1' \leq j_1^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) + \sum_{\substack{j_2' \leq j_2 \\ j_1' \geq j_1^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \\
&\quad + \sum_{\mathbf{j}' \geq (j_1^2, j_2^2)} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \,.
\end{aligned}
$$

We have

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C \parallel f_{\mathbf{j}'} \parallel_{L^\infty(A\overline{\lambda})} \,.$$

Using the localization of the wavelets

$$\sup_{\mathbf{x}} \sum_{\lambda' \in \Lambda_{\mathbf{j}'}} |\Psi_{\lambda'}(\mathbf{x})| < \infty \,. \tag{8}$$

Since the support of $\Psi_{\lambda'}$ is included in $A\overline{\lambda'}$, it follows that

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C \sup_{A\overline{\lambda'} \bigcap A\overline{\lambda} \neq \emptyset} |C_{\lambda'}|. \tag{9}$$

In particular, since $f \in B^{\varepsilon,\infty}_{\infty}$, then

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C 2^{-\varepsilon_1 [j_1'] - \varepsilon_2 [j_2']} \,. \tag{10}$$

It follows that

$$\sum_{\mathbf{j}' \geq (j_1^2, j_2^2)} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C 2^{-\varepsilon_1 j_1^2 - \varepsilon_2 j_2^2}. \tag{11}$$

Let

$$\tilde{d}_{\lambda} = \sup_{A\overline{R} \bigcap A\overline{\lambda} \neq \emptyset} d_R \,. \tag{12}$$

If $\mathbf{j}' \geq \mathbf{j}$ then by (9)

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C \tilde{d}_{\lambda}. \tag{13}$$

Therefore

$$\sum_{\mathbf{j} \leq \mathbf{j}' \leq (j_1^2, j_2^2)} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C j_1^2 j_2^2 \tilde{d}_{\lambda}. \tag{14}$$

Properties (10) and (13) imply that for all $0 \le \alpha \le 1$

$$\begin{aligned} \sum_{\substack{j_1' \ge j_1^2 \\ j_2 \le j_2' \le j_2^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) &\le C\tilde{d}_{\lambda}^{\alpha} \sum_{\substack{j_1' \ge j_1^2 \\ j_2 \le j_2' \le j_2^2}} 2^{-(\varepsilon_1 [j_1'] + \varepsilon_2 [j_2'])(1-\alpha)} \\ &\le C 2^{-(1-\alpha)\varepsilon_1 j_1^2} j_2^2 \tilde{d}_{\lambda}^{\alpha} \,. \end{aligned} \tag{15}$$

Similarly

$$\sum_{\substack{j_2' \ge j_2^2 \\ j_1 \le j_1' \le j_1^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \le C 2^{-(1-\alpha)\varepsilon_2 j_2^2} j_1^2 \tilde{d}_{\lambda}^{\alpha} \,. \tag{16}$$

On the other hand

$$\triangle_{\mathbf{h}}^{\mathbf{n}} f_{\mathbf{j}'}(\mathbf{y}) = \sum_{\lambda' \in \Lambda_{\mathbf{j}'}} C_{\lambda'} \prod_{i=1}^{2} \triangle_{h_i}^{n_i} \psi_{j_i', k_i'}(y_i).$$

Let $\mathbf{j}' \le \mathbf{j}$. Let $\mathbf{y}$ and $\mathbf{y} + \mathbf{nh}$ be in $A\overline{\lambda}$. We apply the Taylor's formula to each $\triangle_{h_i}^{n_i} \psi_{j_i', k_i'}$ and we use the localization of wavelets (similar to (8) for the derivatives). We obtain

$$|\triangle_{\mathbf{h}}^{\mathbf{n}} f_{\mathbf{j}'}(\mathbf{y})| \le C \left(\prod_{i=1}^{2} 2^{n_i [j_i']} |h_i|^{n_i} \right) \sup_{A\overline{\lambda'} \cap A\overline{\lambda} \ne \emptyset} |C_{\lambda'}|.$$

Clearly, $|h_i| \le A 2^{-[j_i]}$ for all i, and consequently

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \le C \left(\prod_{i=1}^{2} 2^{n_i ([j_i'] - [j_i])} \right) \sup_{A\overline{\lambda'} \cap A\overline{\lambda} \ne \emptyset} |C_{\lambda'}|.$$

Therefore, if $\lambda^{\mathbf{j}'}$ denotes the dyadic rectangle of scale $\mathbf{j}'$ which includes λ, then

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \le C \left(\prod_{i=1}^{2} 2^{n_i ([j_i'] - [j_i])} \right) \tilde{d}_{\lambda^{\mathbf{j}'}} \,.$$

Thus

$$\sum_{\mathbf{j}' \le \mathbf{j}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \le C \sum_{\mathbf{j}' \le \mathbf{j}} \prod_{i=1}^{2} 2^{n_i ([j_i'] - [j_i])} \tilde{d}_{\lambda^{\mathbf{j}'}} \,. \tag{17}$$

If $j_1' \le j_1$ and $j_2' \ge j_2$, we apply the Taylor's formula to only $\triangle_{h_1}^{n_1} \psi_{j_1', k_1'}$ and we use the localization of wavelets. We obtain

$$|\triangle_{\mathbf{h}}^{\mathbf{n}} f_{\mathbf{j}'}(\mathbf{y})| \le C 2^{n_1 [j_1']} |h_1|^{n_1} \sup_{A\overline{\lambda'} \cap A\overline{\lambda} \ne \emptyset} |C_{\lambda'}| \le C 2^{n_1 ([j_1'] - [j_1])} \sup_{A\overline{\lambda'} \cap A\overline{\lambda} \ne \emptyset} |C_{\lambda'}|.$$

Therefore

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \le C 2^{n_1 ([j_1'] - [j_1])} 2^{-\varepsilon_1 [j_1'] - \varepsilon_2 [j_2']} \tag{18}$$

(thanks to (10)) and

$$\omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \le C 2^{n_1 ([j_1'] - [j_1])} \tilde{d}_{\lambda^{(j_1', j_2)}} \tag{19}$$

with $\lambda^{(j_1', j_2)}$ being the dyadic rectangle of scale (j_1', j_2), which includes λ.
Thus

$$\sum_{\substack{j_1' \le j_1 \\ j_2 \le j_2' \le j_2^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \le C j_2^2 \sum_{j_1' \le j_1} 2^{n_1 ([j_1'] - [j_1])} \tilde{d}_{\lambda^{(j_1', j_2)}} \,. \tag{20}$$

Properties (18) and (19) imply that for all $0 \leq \alpha \leq 1$

$$\begin{aligned}
\sum_{\substack{j_1' \leq j_1 \\ j_2' \geq j_2^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) &\leq C \sum_{\substack{j_1' \leq j_1 \\ j_2' \geq j_2^2}} 2^{n_1([j_1']-[j_1])} \tilde{d}^{\alpha}_{\lambda^{(j_1',j_2)}} 2^{-(\varepsilon_1[j_1']+\varepsilon_2[j_2'])(1-\alpha)} \\
&\leq C 2^{-(1-\alpha)\varepsilon_2 j_2^2} \sum_{j_1' \leq j_1} 2^{n_1([j_1']-[j_1])} \tilde{d}^{\alpha}_{\lambda^{(j_1',j_2)}} 2^{-(1-\alpha)\varepsilon_1[j_1']} . \qquad (21)
\end{aligned}$$

Similarly

$$\sum_{\substack{j_2' \leq j_2 \\ j_1 \leq j_1' \leq j_1^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C j_1^2 \sum_{j_2' \leq j_2} 2^{n_2([j_2']-[j_2])} \tilde{d}_{\lambda^{(j_1,j_2')}} \qquad (22)$$

and

$$\sum_{\substack{j_2' \leq j_2 \\ j_1' \geq j_1^2}} \omega^{\mathbf{n}}(f_{\mathbf{j}'}, A\overline{\lambda}) \leq C 2^{-(1-\alpha)\varepsilon_1 j_1^2} \sum_{j_2' \leq j_2} C 2^{n_2([j_2']-[j_2])} \tilde{d}^{\alpha}_{\lambda^{(j_1,j_2')}} 2^{-(1-\alpha)\varepsilon_2[j_2']} , \qquad (23)$$

where $\lambda^{(j_1,j_2')}$ denotes the dyadic rectangle of scale (j_1, j_2') which includes λ. Inequality

$$N^{-p}\left(\sum_{i=1}^{N} |a_i|\right)^p \leq \sum_{i=1}^{N} |a_i|^p ,$$

together with (8), (11), (14)–(17), (20)–(23) imply that

$$\begin{aligned}
& C(([j_1]+1)([j_2]+1))^{-p}\left(\omega^{\mathbf{n}}(f, A\overline{\lambda})\right)^p \\
& \leq \sum_{\mathbf{j}' \leq \mathbf{j}} \left(\prod_{i=1}^{2} 2^{n_i([j_i']-[j_i])} \tilde{d}_{\lambda^{\mathbf{j}'}}\right)^p \\
& +(j_1^2 j_2^2 \tilde{d}_{\lambda})^p \\
& +(2^{-(1-\alpha)\varepsilon_1 j_1^2} j_2^2 \tilde{d}^{\alpha}_{\lambda})^p + (2^{-(1-\alpha)\varepsilon_2 j_2^2} j_1^2 \tilde{d}^{\alpha}_{\lambda})^p \\
& + j_2^2 \sum_{j_1' \leq j_1} \left(2^{n_1([j_1']-[j_1])} \tilde{d}_{\lambda^{(j_1',j_2)}}\right)^p \\
& + j_1^2 \sum_{j_2' \leq j_2} \left(2^{n_2([j_2']-[j_2])} \tilde{d}_{\lambda^{(j_1,j_2')}}\right)^p \\
& + 2^{-\varepsilon_2 j_2^2(1-\alpha)p} \sum_{j_1' \leq j_1} \left(2^{n_1([j_1']-[j_1])} 2^{-\varepsilon_1[j_1'](1-\alpha)} \tilde{d}^{\alpha}_{\lambda^{(j_1',j_2)}}\right)^p \\
& + 2^{-\varepsilon_1 j_1^2(1-\alpha)p} \sum_{j_2' \leq j_2} \left(2^{n_2([j_2']-[j_2])} 2^{-\varepsilon_2[j_2'](1-\alpha)} \tilde{d}^{\alpha}_{\lambda^{(j_1,j_2')}}\right)^p \\
& + 2^{(-\varepsilon_1 j_1^2 - \varepsilon_2 j_2^2)p} .
\end{aligned}$$

When we sum on $\lambda \in \Lambda_{\mathbf{j}}$ each $\tilde{d}_{\lambda^{\mathbf{j}'}}$ is repeated $C\prod_{i=1}^{2} 2^{[j_i]-[j_i']}$ times in $\sum_{\mathbf{j}' \leq \mathbf{j}}$. So that

$$
\begin{aligned}
&C(([j_1]+1)([j_2]+1))^{-p} \sum_{\lambda\in\Lambda_{\mathbf{j}}} \left(\omega^{\mathbf{n}}(f, A\overline{\lambda})\right)^p \\
&\leq \sum_{\mathbf{j}'\leq\mathbf{j}} \left(\prod_{i=1}^{2} 2^{(n_i p-1)([j_i']-[j_i])} \sum_{\lambda'\in\Lambda_{\mathbf{j}'}} \tilde{d}^p_{\lambda^{\mathbf{j}'}}\right) \\
&+j_1^{2p} j_2^{2p} \sum_{\lambda\in\Lambda_{\mathbf{j}}} \tilde{d}^p_{\lambda} \\
&+\left(2^{-p(1-\alpha)\varepsilon_1 j_1^2} j_2^{2p} + 2^{-p(1-\alpha)\varepsilon_2 j_2^2} j_1^{2p}\right) \sum_{\lambda\in\Lambda_{\mathbf{j}}} \tilde{d}^{\alpha p}_{\lambda} \\
&+j_2^{2p} \sum_{j_1'\leq j_1} \left(2^{(n_1 p-1)([j_1']-[j_1])} \sum_{\lambda^{(j_1',j_2)}\in\Lambda_{(j_1',j_2)}} \tilde{d}^p_{\lambda^{(j_1',j_2)}}\right) \\
&+j_1^{2p} \sum_{j_2'\leq j_2} \left(2^{(n_2 p-1)([j_2']-[j_2])} \sum_{\lambda^{(j_1,j_2')}\in\Lambda_{(j_1,j_2')}} \tilde{d}^p_{\lambda^{(j_1,j_2')}}\right) \\
&+2^{-\varepsilon_2 j_2{}^2(1-\alpha)p} \sum_{j_1'\leq j_1} \left(2^{(n_1 p-1)([j_1']-[j_1])} 2^{-p\varepsilon_1 [j_1'](1-\alpha)} \sum_{\lambda^{(j_1',j_2)}\in\Lambda_{(j_1',j_2)}} \tilde{d}^{\alpha p}_{\lambda^{(j_1',j_2)}}\right) \\
&+2^{-\varepsilon_1 j_1^2(1-\alpha)p} \sum_{j_2'\leq j_2} \left(2^{(n_2 p-1)([j_2']-[j_2])} \sum_{\lambda^{(j_1,j_2')}\in\Lambda_{(j_1,j_2')}} \tilde{d}^{\alpha p}_{\lambda^{(j_1,j_2')}}\right) \\
&+2^{[j_1]+[j_2]} 2^{-(\varepsilon_1 j_1^2+\varepsilon_2 j_2^2)p} .
\end{aligned}
$$

Note that $2^{[j_1]+[j_2]}$ in the last term follows from the fact that f is compactly supported. From (12), $\tilde{d}_{\lambda}$ is a supremum of d_{λ} on at most CA^d rectangles, then we can replace $\tilde{d}_{\lambda}$ by d_{λ} in the previous $\sum_{\lambda^{\cdots}\in\Lambda_{\ldots}}$.

Since $f \in O_p^{\mathbf{s}+\boldsymbol{\eta}}$, then

$$\sum_{\lambda'\in\Lambda_{\mathbf{j}'}} d^p_{\lambda'} \leq C2^{[j_1']+[j_2']-(s_1+\eta_1)p[j_1']-(s_2+\eta_2)p[j_2']} \qquad \forall \mathbf{j}' .$$

Since $f \in O_p^{\mathbf{s}+\boldsymbol{\eta}}$ and f is compactly supported, then, using the third result in Proposition 3, $f \in O_{\alpha p}^{\mathbf{s}+\boldsymbol{\eta}}$ for all $\alpha \in [0,1]$. It follows that

$$\sum_{\lambda'\in\Lambda_{\mathbf{j}'}} d^{\alpha p}_{\lambda'} \leq C2^{[j_1']+[j_2']-(s_1+\eta_1)\alpha p[j_1']-(s_2+\eta_2)\alpha p[j_2']} \qquad \forall \mathbf{j}' .$$

Thus

$$C(([j_1]+1)([j_2]+1))^{-3p}\sum_{\lambda\in\Lambda_{\mathbf{j}}}\left(\omega^{\mathbf{n}}(f,A\overline{\lambda})\right)^p$$

$$\leq \sum_{\mathbf{j}'\leq\mathbf{j}} 2^{(n_1p-1)([j_1']-[j_1])+(n_2p-1)([j_2']-[j_2])}2^{[j_1']+[j_2']-(s_1+\eta_1)p[j_1']-(s_2+\eta_2)p[j_2']} \tag{24}$$

$$+2^{[j_1]+[j_2]-(s_1+\eta_1)p[j_1]-(s_2+\eta_2)p[j_2]}$$

$$+\left(2^{-p(1-\alpha)\varepsilon_1 j_1^2}+2^{-p(1-\alpha)\varepsilon_2 j_2^2}\right)2^{[j_1]+[j_2]-(s_1+\eta_1)\alpha p[j_1]-(s_2+\eta_2)\alpha p[j_2]} \tag{25}$$

$$+\sum_{j_1'\leq j_1} 2^{(n_1p-1)([j_1']-[j_1])}2^{[j_1']+[j_2]-(s_1+\eta_1)p[j_1']-(s_2+\eta_2)p[j_2]} \tag{26}$$

$$+\sum_{j_2'\leq j_2} 2^{(n_2p-1)([j_2']-[j_2])}2^{[j_1]+[j_2']-(s_1+\eta_1)p[j_1]-(s_2+\eta_2)p[j_2']} \tag{27}$$

$$+2^{-\varepsilon_2 j_2^2(1-\alpha)p}\sum_{j_1\leq j_1} 2^{(n_1p-1)([j_1']-[j_1])}2^{-p\varepsilon_1[j_1'](1-\alpha)}2^{[j_1']+[j_2]-(s_1+\eta_1)\alpha p[j_1']-(s_2+\eta_2)\alpha p[j_2]} \tag{28}$$

$$+2^{-\varepsilon_1 j_1^2(1-\alpha)p}\sum_{j_2'\leq j_2} 2^{(n_2p-1)([j_2']-[j_2])}2^{-p\varepsilon_2[j_2'](1-\alpha)}2^{[j_1]+[j_2']-(s_1+\eta_1)\alpha p[j_1]-(s_2+\eta_2)\alpha p[j_2']} \tag{29}$$

$$+2^{[j_1]+[j_2]}2^{-(\varepsilon_1 j_1^2+\varepsilon_2 j_2^2)p}\,. \tag{30}$$

The coefficient of $[j_i']$ in (24) is $p(n_i-(s_i+\eta_i)) > 0$ if $n_i > s_i$, thus (24)$\sim 2^{[j_1]+[j_2]-(s_1+\eta_1)p[j_1]-(s_2+\eta_2)p[j_2]}$.
The coefficient of $[j_1']$ in (26) is $p(n_1-(s_1+\eta_1)) > 0$ if $n_1 > s_1$, so (26) $\sim 2^{[j_1]+[j_2]-(s_1+\eta_1)p[j_1]-(s_2+\eta_2)p[j_2]}$.
Similarly (27) $\sim 2^{[j_1]+[j_2]-(s_1+\eta_1)p[j_1]-(s_2+\eta_2)p[j_2]}$ if $n_2 > s_2$.
The coefficient of $[j_1']$ in (28) is $p(n_1-\varepsilon_1(1-\alpha)-\alpha(s_1+\eta_1)) > 0$ if $n_1 > s_1$; thus, (28) $\sim 2^{-\varepsilon_2[j_2]^2(1-\alpha)p}2^{-\varepsilon_1[j_1](1-\alpha)p}2^{[j_1]+[j_2]-(s_1+\eta_1)\alpha p[j_1]-(s_2+\eta_2)\alpha p[j_2]}$.
Similarly, (29) $\sim 2^{-\varepsilon_1[j_1]^2(1-\alpha)p}2^{-\varepsilon_2[j_2](1-\alpha)p}2^{[j_1]+[j_2]-(s_1+\eta_1)\alpha p[j_1]-(s_2+\eta_2)\alpha p[j_2]}$.
If $\frac{s_i}{s_i+\eta_i} < \alpha < 1$ then $s_ip < (s_i+\eta_i)\alpha p$; thus, (25), together with (28) and (29) are $\leq C2^{[j_1]+[j_2]-s_1p[j_1]-s_2p[j_2]}$.
Finally, it is clear that (30) $\leq C2^{[j_1]+[j_2]-s_1p[j_1]-s_2p[j_2]}$.
We conclude that $f\in V^{\mathbf{s},p}$.

- Let us now show the second imbedding. Let $f\in V^{\mathbf{s},p}$.
Assume first that $\mathbf{j}\geq\mathbf{0}$. By a change of variable

$$C_\lambda = C_{\mathbf{j},\mathbf{k}} = 2^{j_1+j_2}\int f(\mathbf{x})\Psi_\lambda(\mathbf{x})d\mathbf{x} = \int \tilde{f}_\lambda(\mathbf{y})\psi(y_1)\psi(y_2)d\mathbf{y}$$

with $\tilde{f}_\lambda(\mathbf{y}) = f\left(\frac{y_1+k_1}{2^{j_1}},\frac{y_2+k_2}{2^{j_2}}\right)$. If $\psi\in C^r(\mathbb{R})$ and $r > n\geq 1$, then thanks to the scaling properties of a multiresolution analysis, there exists a compactly supported function θ_n such that $\psi = \triangle^n_{\frac{1}{2}}\theta_n$ (see Lemma 2 in [1]), so that

$$C_\lambda = \int \tilde{f}_\lambda(\mathbf{y})\prod_{i=1}^{2}\triangle^{n_i}_{\frac{1}{2}}\theta_{n_i}(y_i)d\mathbf{y} = (-1)^{n_1+n_2}\int \triangle^{\mathbf{n}}_{-\frac{1}{2}\mathbf{1}}\tilde{f}_\lambda(\mathbf{y})\prod_{i=1}^{2}\theta_{n_i}(y_i)d\mathbf{y}\,,$$

where $\mathbf{1} = (1,\cdots,1)$. We can assume that the supports of the θ_{n_i}s are included in $[-A,A]$. Then

$$|C_\lambda| \leq \left(\sup_{\mathbf{y}\in[-A,A]^2}|\triangle^{\mathbf{n}}_{-\frac{1}{2}\mathbf{1}}\tilde{f}_\lambda(\mathbf{y})|\right)\int|\prod_{i=1}^{2}\theta_{n_i}(y_i)|d\mathbf{y}.$$

If $\mathbf{y} \in [-A, A]^2$ and $\mathbf{h} = -\frac{1}{2}\mathbf{1} = (-\frac{1}{2}, -\frac{1}{2})$, then $\mathbf{y} + \mathbf{nh} := (y_1 - \frac{n_1}{2}, y_2 - \frac{n_2}{2})$ and $\tilde{F}_\lambda(\mathbf{y} + \mathbf{nh}) = F\left(\frac{y_1 - \frac{n_1}{2} + k_1}{2^{j_1}}, \frac{y_2 - \frac{n_2}{2} + k_2}{2^{j_2}}\right)$. Since $-A \le y_i \le A$, then

$$\frac{y_i - \frac{n_i}{2} + k_i}{2^{j_i}} \in k_i 2^{-j_i} + 2^{-j_i}\left[-(A + \frac{N}{2}), (A + \frac{N}{2})\right]$$

with $N = \max\{n_1, n_2\}$. Thus

$$|C_\lambda| \le C\omega^{\mathbf{n}}\left(f, (A + \frac{N}{2})\overline{\boldsymbol{\lambda}}\right)$$

If $\boldsymbol{\lambda}' \subset \boldsymbol{\lambda}$, then

$$|C_{\lambda'}| \le C\omega^{\mathbf{n}}(f, (A + \frac{N}{2})\overline{\boldsymbol{\lambda}}).$$

So that

$$\forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathbf{j}} \qquad d_\lambda \le C\omega^{\mathbf{n}}\left(f, (A + \frac{N}{2})\overline{\boldsymbol{\lambda}}\right).$$

And

$$2^{p(s_1 j_1 + s_2 j_2)} 2^{-j_1 - j_2} \sum_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathbf{j}}} d_\lambda^p \le C 2^{p(s_1 j_1 + s_2 j_2)} 2^{-j_1 - j_2} \sum_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathbf{j}}} \left(\omega^{\mathbf{n}}\left(f, (A + \frac{N}{2})\overline{\boldsymbol{\lambda}}\right)\right)^p.$$

If $j_1 = -1$ and $j_2 \ge 0$, then

$$C_\lambda = 2^{j_2} \int f(\mathbf{x}) \Psi_\lambda(\mathbf{x}) d\mathbf{x} = \int \tilde{f}_\lambda(\mathbf{y}) \varphi(y_1) \psi(y_2) d\mathbf{y}$$

with $\tilde{f}_\lambda(\mathbf{y}) = f\left(y_1 + k_1, \frac{y_2 + k_2}{2^{j_2}}\right)$. If the support of φ is included in $[-A, A]$, then $\varphi(y_1) = (-1)^{n_1} \triangle_A^{n_1} \varphi(y_1)$, so that

$$C_\lambda = (-1)^{n_1} \int \tilde{f}_\lambda(\mathbf{y}) \triangle_A^{n_1} \varphi(y_1) \triangle_{\frac{1}{2}}^{n_2} \theta_{n_2}(y_2) d\mathbf{y} = (-1)^{n_1 + n_2} \int \triangle^{\mathbf{n}}_{(-A, -\frac{1}{2})} \tilde{f}_\lambda(\mathbf{y}) \varphi(y_1) \theta_{n_2}(y_2) d\mathbf{y}\,,$$

where θ_n is as above
Therefore

$$|C_\lambda| \le \left(\sup_{\mathbf{y} \in [-A, A]^2} |\triangle^{\mathbf{n}}_{(-A, -\frac{1}{2})} \tilde{f}_\lambda(\mathbf{y})|\right) \int |\varphi(y_1) \theta_{n_2}(y_2)| d\mathbf{y}.$$

If $\mathbf{y} \in [-A, A]^2$ and $\mathbf{h} = (-A, -\frac{1}{2})$, then $\tilde{f}_\lambda(\mathbf{y} + \mathbf{nh}) = f\left(y_1 - An_1 + k_1, \frac{y_2 - \frac{n_2}{2} + k_2}{2^{j_2}}\right)$. Since the support of f is included in $[-A, A]^2$, then

$$|C_\lambda| \le C\omega^{\mathbf{n}}\left(f, (A + \frac{N}{2})\overline{\boldsymbol{\lambda}}\right).$$

If $\boldsymbol{\lambda}' = \boldsymbol{\lambda}'((-1, j_2'), \mathbf{k}') \subset \boldsymbol{\lambda}$ then

$$|C_{\lambda'}| \le C\omega^{\mathbf{n}}(f, (A + \frac{N}{2})\overline{\boldsymbol{\lambda}}).$$

So that

$$\forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}_{\mathbf{j}} \qquad d_\lambda \le C\omega^{\mathbf{n}}\left(f, (A + \frac{N}{2})\overline{\boldsymbol{\lambda}}\right).$$

And
$$2^{ps_2j_2}2^{-j_2}\sum_{\lambda\in\Lambda_{\mathbf{j}}} d_\lambda^p \le C2^{ps_2j_2}2^{-j_2}\sum_{\lambda\in\Lambda_{\mathbf{j}}}\left(\omega^{\mathbf{n}}\left(F,(A+\frac{N}{2})\overline{\lambda}\right)\right)^p .$$

Analogously, if $j_1 \geq 0$ and $j_2 = -1$, then
$$\forall\lambda\in\Lambda_{\mathbf{j}}\qquad d_\lambda \le C\omega^{\mathbf{n}}\left(f,(A+\frac{N}{2})\overline{\lambda}\right).$$

And
$$2^{ps_1j_1}2^{-j_1}\sum_{\lambda\in\Lambda_{\mathbf{j}}} d_\lambda^p \le C2^{ps_1j_1}2^{-j_1}\sum_{\lambda\in\Lambda_{\mathbf{j}}}\left(\omega^{\mathbf{n}}\left(f,(A+\frac{N}{2})\overline{\lambda}\right)\right)^p .$$

Finally, if $\mathbf{j} = -\mathbf{1}$, then using the fact that $\varphi(y_i) = (-1)^{n_i}\triangle_A^{n_i}\varphi(y_i)$, for $i = 1,2$, we also obtain
$$\sum_{\lambda\in\Lambda_{\mathbf{j}}} d_\lambda^p \le C\sum_{\lambda\in\Lambda_{\mathbf{j}}}\left(\omega^{\mathbf{n}}\left(f,(A+\frac{N}{2})\overline{\lambda}\right)\right)^p .$$

In all cases
$$\forall\ \mathbf{j}\ge-\mathbf{1}\qquad \sigma_{\mathbf{j}}^p(\mathbf{s},p)\le C2^{p(s_1[j_1]+s_2[j_2])}2^{-[j_1]-[j_2]}\sum_{\lambda\in\Lambda_{\mathbf{j}}}\left(\omega^{\mathbf{n}}\left(f,(A+\frac{N}{2})\overline{\lambda}\right)\right)^p . \tag{31}$$

Now, let $\mathbf{j}\ge-\mathbf{1}$. Let l be such that $2^{l-1} < A+\frac{N}{2} < 2^l$, then
$$\sum_{\lambda\in\Lambda_{\mathbf{j}}}\left(\omega^{\mathbf{n}}(f,(A+\frac{N}{2})\overline{\lambda})\right)^p \le C2^{2l}\sum_{\lambda'\in\Lambda_{\mathbf{j}-l\mathbf{1}}}\left(\omega^{\mathbf{n}}(f,2\lambda')\right)^p$$

because each $\lambda\in\Lambda_{\mathbf{j}}$ is contained in a rectangle $\lambda'\in\Lambda_{\mathbf{j}-l\mathbf{1}}$, and each $\lambda'\in\Lambda_{\mathbf{j}-l\mathbf{1}}$ contains at most $C2^{2l}$ rectangles $\lambda\in\Lambda_{\mathbf{j}}$.
Since $f\in V^{\mathbf{s},p}$, then
$$\sum_{\lambda'\in\Lambda_{\mathbf{j}-l\mathbf{1}}}\left(\omega^{\mathbf{n}}(f,2\lambda')\right)^p \le C2^{-(s_1[j_1-l]+s_2[j_2-l])p}2^{[j_1-l]+[j_2-l]} .$$

Then
$$\sum_{\lambda\in\Lambda_{\mathbf{j}}}\left(\omega^{\mathbf{n}}(f,(A+\frac{N}{2})\overline{\lambda})\right)^p \le C2^{-(s_1[j_1]+s_2[j_2])p}2^{[j_1]+[j_2]} . \tag{32}$$

Both (31) and (32) yield $f\in O_p^{\mathbf{s}}$.
□

3. Optimal Relationships with Hyperbolic Besov Spaces

3.1. *The Advantage of the Hyperbolic Oscillation Approach Self-Affine Cascade Functions*

We will first explain the advantage of the hyperbolic oscillation approach for self-affine cascade functions. Let N_1 and N_2 be two integers with $N_1 < N_2$. We divide the unit square $\mathfrak{R} = [0,1]^2$ into a uniform grid of $2^{N_1+N_2}$ rectangles of sides 2^{-N_1} and 2^{-N_2}. Choose $A\subset\{0,1,\ldots,2^{N_1}-1\}\times\{0,1,\ldots,2^{N_2}-1\}$. For $\omega=(a,b)\in A$, the contraction $S_\omega(x_1,x_2)=\left(2^{-N_1}(x_1+a),2^{-N_2}(x_2+b)\right)$ maps the unit square $\mathfrak{R}$ into the rectangle
$$\mathfrak{R}_\omega = [2^{-N_1}a, 2^{-N_1}(1+a)]\times[2^{-N_2}b, 2^{-N_2}(1+b)]. \tag{33}$$

If G is a subset of $\mathbb{R}^2$, we define the mapping S by
$$S(G)=\bigcup_{\omega\in A} S_\omega(G) .$$

The Sierpinski carpet K (see, for example, [43,44]) is given by

$$K = \bigcap_{n \in \mathbb{N}} S^n(\mathfrak{R}) \,. \tag{34}$$

Let $(\gamma_\omega)_{\omega \in A}$ be scalars with $0 < |\gamma_\omega| < 1$. The self-affine cascade function F adapted to the subdivision A satisfies the self-affine equation

$$F = \Psi \,+\, \sum_{\omega \in A} \gamma_\omega \, F \circ S_\omega^{-1} \,, \tag{35}$$

where $\Psi = \Psi_{\mathbf{0},\mathbf{0}}$.

The unique solution in $L^1(\mathfrak{R})$ is given by the hyperbolic wavelet series

$$F(\mathbf{x}) = \Psi(\mathbf{x}) + \sum_{n=1}^{\infty} \sum_{(\omega_1,\ldots,\omega_n) \in A^n} \gamma_{\omega_1} \cdots \gamma_{\omega_n} \, \Psi\Big(S_{\omega_n}^{-1} \cdots S_{\omega_1}^{-1}(\mathbf{x})\Big) \,. \tag{36}$$

Define

$$|\gamma|_{\max} = \max_{\omega \in A} |\gamma_\omega| \quad \text{and} \quad H_{\min} = -\frac{\log|\gamma|_{\max}}{N_2 \log 2}.$$

If $|\gamma|_{\max} > 2^{-N_2}$ then F is uniformly Lipschitz. For $p > 0$, set

$$\tau(p) = -\frac{\log(\sum_{\omega \in A} |\gamma_\omega|^p)}{N_1 \log 2}.$$

Let $\sigma = \dfrac{N_1}{N_2}$. It is easy to show that the self-affine cascade function F adapted to the subdivision A belongs to the hyperbolic Besov space $B_p^{(s_1,s_2),q}$ iff

$$\sigma s_1 + s_2 \leq \frac{1 + \sigma(1 + \tau(p))}{p} \,. \tag{37}$$

The following result (which can be proved as in [45]) shows that, on the contrary, the p-domain $\mathcal{D}_p$ of F defined in (6) depends on the geometric disposition of the elements of A. We consider the following two geometric choices:

$$\text{Each row and column of the grid contains at most one chosen} \quad \mathfrak{R}_\omega \tag{38}$$

and

$$\text{Only one column contains all boxes} \quad (\mathfrak{R}_\omega)_{\omega \in A}. \tag{39}$$

Proposition 2. *1. Assume that (38) is satisfied. Then, (s_1, s_2) belongs to $\mathcal{D}_p$ if (37) together with*

$$s_1 \leq \frac{1 + \tau(p)}{p} \tag{40}$$

and

$$s_2 \leq \frac{1 + \sigma\tau(p)}{p} \tag{41}$$

hold.

2. Assume that (39) is satisfied. Then, (s_1, s_2) belongs to $\mathcal{D}_p$ if (37) together with (41) and

$$s_1 \leq \frac{1}{p} + \frac{H_{\min}}{\sigma} \tag{42}$$

hold.

Thus, as mentioned in the introduction, contrary to hyperbolic Besov and fractional Sobolev spaces, the new hyperbolic oscillation spaces are convenient for the multifractal analysis of rectangular regularity.

3.2. Optimal General Relationships

We will now study general optimal relationships between hyperbolic and mixed fractional lifting oscillation spaces and hyperbolic Besov spaces. In particular, we will prove that, for $\mathbf{0} < \mathbf{s} < \frac{1}{p}\mathbf{1}$ (with respect to $\mathbf{s} - \frac{1}{p}\mathbf{1} < \mathbf{s}' < \mathbf{s}$), hyperbolic (with respect to mixed fractional lifting) oscillation spaces are not always sharply imbedded between hyperbolic Besov spaces, and thus are new spaces of a really different nature.

Proposition 3. *Let $p > 0$. The following embeddings between hyperbolic oscillation spaces hold.*

1. *If $\mathbf{s} \leq \mathbf{s}^*$ then $O_p^{\mathbf{s}^*} \hookrightarrow O_p^{\mathbf{s}}$.*
2. *If $q \geq p$. Then $O_p^{\mathbf{s}} \hookrightarrow O_q^{\mathbf{s} - \frac{1}{p}\mathbf{1} + \frac{1}{q}\mathbf{1}}$.*
3. *If f is compactly supported and $f \in O_p^{\mathbf{s}}$ then $f \in O_q^{\mathbf{s}}$ for all $q < p$.*

Proposition 4. *Let $p > 0$. The following sharp relationships between hyperbolic oscillation spaces and hyperbolic Besov spaces hold.*

1. *If $\mathbf{s} > \frac{1}{p}\mathbf{1}$, then $O_p^{\mathbf{s}} = B_p^{\mathbf{s},\infty}$.*
2. *$B_p^{\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\frac{1}{p}\mathbf{1}} \hookrightarrow B_p^{\frac{1}{p}\mathbf{1},\infty}$.*
3. *If $\mathbf{s} < \frac{1}{p}\mathbf{1}$, then $B_p^{\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\mathbf{s}}$.*
4. *If $\mathbf{s} > \mathbf{0}$ then $O_p^{\mathbf{s}} \hookrightarrow B_\infty^{0,\infty}$ and $B_\infty^{\mathbf{s},\infty} \hookrightarrow (O_p^{\mathbf{s}})_{loc}$, where $(O_p^{\mathbf{s}})_{loc}$ denotes the spaces of functions that locally belong to $O_p^{\mathbf{s}}$.*
5. *If $\mathbf{s} \leq \mathbf{0}$, then $B_\infty^{\mathbf{0},\infty} \hookrightarrow (O_p^{\mathbf{s}})_{loc}$.*

It follows that, for $\mathbf{0} < \mathbf{s} < \frac{1}{p}\mathbf{1}$, hyperbolic oscillation spaces are not sharply imbedded between hyperbolic Besov spaces, and thus are new spaces of a really different nature.

Remark 2. *If $f = \sum_{\mathbf{j} \in \mathbb{N}_{-1}^d} \sum_{\mathbf{k} \in \mathbb{Z}^d} C_{\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}}$, then we can replace $I_{\mathbf{s}'} f$ by the mixed fractional lifting fractional hyperbolic wavelet series*

$$\tilde{f}_{\mathbf{s}'} = \sum_{\mathbf{j} \in \mathbb{N}_{-1}^d} \sum_{\mathbf{k} \in \mathbb{Z}^d} \tilde{C}_{\mathbf{s}',\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}}$$

with

$$\tilde{C}_{\mathbf{s}',\mathbf{j},\mathbf{k}} = 2^{[j_1]s_1' + \cdots + [j_d]s_d'} C_{\mathbf{j},\mathbf{k}} \ .$$

Using Remark 1, we therefore deduce the following results.

Corollary 1. *Let $p > 0$. The following embeddings between mixed fractional lifting hyperbolic oscillation spaces hold.*

1. *If $\mathbf{s} \leq \mathbf{s}^*$ and $\mathbf{s}' \leq \mathbf{s}'^*$ then $O_p^{\mathbf{s}^*,\mathbf{s}'^*} \hookrightarrow O_p^{\mathbf{s},\mathbf{s}'}$, where $\mathbf{s}'^* = (s_1'^*, s_2'^*)$.*
2. *If $q \geq p$ then $O_p^{\mathbf{s},\mathbf{s}'} \hookrightarrow O_q^{\mathbf{s} - \frac{1}{p}\mathbf{1} + \frac{1}{q}\mathbf{1},\mathbf{s}'}$.*

Corollary 2. *Let $p > 0$. The following sharp relationships between mixed fractional lifting hyperbolic oscillation spaces and hyperbolic Besov spaces hold.*

1. *If $\mathbf{s}' < \mathbf{s} - \frac{1}{p}\mathbf{1}$, then $O_p^{\mathbf{s},\mathbf{s}'} = B_p^{\mathbf{s},\infty}$.*
2. *If $\mathbf{s}' = \mathbf{s} - \frac{1}{p}\mathbf{1}$. Then $B_p^{\mathbf{s},p} \hookrightarrow O_p^{\mathbf{s},\mathbf{s}'} \hookrightarrow B_p^{\mathbf{s},\infty}$.*

3. *If* $\mathbf{s} - \frac{1}{p}\mathbf{1} < \mathbf{s}'$. *Then* $B_p^{\mathbf{s}'+\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\mathbf{s},\mathbf{s}'}$.
4. *If* $\mathbf{s} > \mathbf{s}'$. *Then* $O_p^{\mathbf{s},\mathbf{s}'} \hookrightarrow B_\infty^{\mathbf{s}',\infty}$ *and* $B_\infty^{\mathbf{s},\infty} \hookrightarrow (O_p^{\mathbf{s},\mathbf{s}'})_{loc}$.
5. *If* $\mathbf{s}' \geq \mathbf{s}$, *then* $B_\infty^{\mathbf{s}',\infty} \hookrightarrow (O_p^{\mathbf{s},\mathbf{s}'})_{loc}$.

It follows that, for $\mathbf{s} - \frac{1}{p}\mathbf{1} < \mathbf{s}' < \mathbf{s}$, mixed fractional lifting hyperbolic oscillation spaces are not always sharply imbedded between hyperbolic Besov spaces, and thus are new spaces of a really different nature.

Proof of Proposition 3. 1. The first imbedding follows from the fact that $\sigma_{\mathbf{j}}(\mathbf{s},p) \leq \sigma_{\mathbf{j}}(\mathbf{s}^*,p)$, where $\sigma_{\mathbf{j}}(\mathbf{s},p)$ was given in (5).

2. Let $f \in O_p^{\mathbf{s}}$. Clearly

$$\sigma_{\mathbf{j}}(\mathbf{s} - \frac{1}{p}\mathbf{1} + \frac{1}{q}\mathbf{1}, q) = 2^{(s_1-\frac{1}{p})[j_1]+\cdots+(s_d-\frac{1}{p})[j_d]} \|d_{\boldsymbol{\lambda}}\|_{\ell_q(\boldsymbol{\Lambda}_{\mathbf{j}})} .$$

Since $q \geq p$ then $\ell_p \hookrightarrow \ell_q$. Thus, the second imbedding holds.

3. Let f be compactly supported. Assume that $f \in O_p^{\mathbf{s}}$. Since the wavelets are compactly supported, then the sum in $\sigma_{\mathbf{j}}(\mathbf{s},p)$ bears on at most $C2^{[j_1]+\cdots+[j_d]}$ dyadic rectangles at scale $\mathbf{j}$. Let $q < p$. Applying the Hölder inequality

$$\sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_{\mathbf{j}}} d_{\boldsymbol{\lambda}}^q \leq (C2^{[j_1]+\cdots+[j_d]})^{1-\frac{q}{p}} \left(\sum_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}_{\mathbf{j}}} d_{\boldsymbol{\lambda}}^p\right)^{q/p} .$$

Therefore, $\sigma_{\mathbf{j}}(\mathbf{s},q) \leq C(\sigma_{\mathbf{j}}(\mathbf{s},p))^{q/p}$. Hence, $f \in O_q^{\mathbf{s}}$ for all $q < p$.
□

Proof of Proposition 4. 1. We have already observed (7). Now, we will show that $B_p^{\mathbf{s},\infty} \hookrightarrow O_p^{\mathbf{s}}$ if $\mathbf{s} > \frac{1}{p}\mathbf{1}$.

$$\begin{aligned}\sigma_{\mathbf{j}}^p(\mathbf{s},p) &\leq 2^{p(s_1[j_1]+\cdots+s_d[j_d])}2^{-([j_1]+\cdots+[j_d])}\sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{j}'\geq\mathbf{j}}\sum_{\boldsymbol{\lambda}'\subset\boldsymbol{\lambda}}|C_{\boldsymbol{\lambda}'}|^p \\ &\leq \sum_{\mathbf{j}'\geq\mathbf{j}}\sum_{\mathbf{k}'\in\mathbb{Z}^d}|C_{\boldsymbol{\lambda}'}|^p 2^{(ps_1-1)[j_1']+\cdots+(ps_d-1)[j_d']}2^{-(ps_1-1)([j_1']-[j_1])}\cdots 2^{-(ps_d-1)([j_d']-[j_d])} \\ &\leq \sum_{\mathbf{j}'\geq\mathbf{j}} b_{\mathbf{j}'}^p(\mathbf{s},p)2^{-(ps_1-1)([j_1']-[j_1])}\cdots 2^{-(ps_d-1)([j_d']-[j_d])} .\end{aligned} \tag{43}$$

If $f \in B_p^{\mathbf{s},\infty}$, it follows that

$$\begin{aligned}\sigma_{\mathbf{j}}^p(\mathbf{s},p) &\leq \|f\|_{B_p^{\mathbf{s},\infty}}^p \sum_{\mathbf{j}'\geq\mathbf{j}} 2^{-(ps_1-1)([j_1']-[j_1])}\cdots 2^{-(ps_d-1)([j_d']-[j_d])} \\ &\leq C \qquad (\text{because } \mathbf{s} > \tfrac{1}{p}\mathbf{1}) .\end{aligned}$$

Thus, $f \in O_p^{\mathbf{s}}$.

2. We have already observed (7). In order to prove the optimality of the embedding $O_p^{\frac{1}{p}\mathbf{1}} \hookrightarrow B_p^{\frac{1}{p}\mathbf{1},\infty}$, it is easy to show that if

$$f = \sum_{\mathbf{j}\in\mathbb{N}_{-1}^2}\sum_{k_1=0}^{2^{[j_1]}-1}\sum_{k_2=0}^{2^{[j_2]}-1} 2^{-\frac{[j_1]+[j_2]}{p}}\Psi_{\mathbf{j},\mathbf{k}} \tag{44}$$

then $f \in O_p^{\frac{1}{p}\mathbf{1}}$ but $f \notin B_p^{\frac{1}{p}\mathbf{1},q}$ for all $q < \infty$.

Let us now prove that $B_p^{\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\frac{1}{p}\mathbf{1}}$. If $f \in B_p^{\frac{1}{p}\mathbf{1},p}$, then by (43)

$$\sigma_{\mathbf{j}}(\frac{1}{p}\mathbf{1},p) \le \left(\sum_{\mathbf{j}' \ge \mathbf{j}} b_{\mathbf{j}'}^p(\mathbf{s},p)\right)^{1/p} \le \|f\|^p_{B_p^{\frac{1}{p}\mathbf{1},1/p}} .$$

So $f \in O_p^{\frac{1}{p}\mathbf{1}}$.

In order to prove the optimality of the embedding $B_p^{\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\frac{1}{p}\mathbf{1}}$, we take a $q > p$ and construct a function F in $B_p^{\frac{1}{p}\mathbf{1},q}$, such that $F \notin O_p^{\frac{1}{p}\mathbf{1}}$.

Let $j_1 = 2$ and $j_{n+1} = 2^{j_n}$ for all $n \in \mathbb{N}$. Consider

$$F = \sum_{n \ge 1} C_{(j_{n+1}-1,j_{n+1}-1),\mathbf{0}} \Psi_{(j_{n+1}-1,j_{n+1}-1),\mathbf{0}} .$$

where

$$C_{(j_{n+1}-1,j_{n+1}-1),\mathbf{0}} = \frac{1}{\left((j_{n+1}-1)(\log(j_{n+1}-1))^2\right)^{\frac{2}{q}}} .$$

Clearly $b_{\mathbf{j}} = \dfrac{1}{\left((j_{n+1}-1)(\log(j_{n+1}-1))^2\right)^{\frac{2}{q}}}$ if $\mathbf{j} = (j_{n+1}-1, j_{n+1}-1)$, and $b_{\mathbf{j}} = 0$ elsewhere. Since the series $\sum_{n \ge 2} \dfrac{1}{n(\log n)^2}$ converges, then $F \in B_p^{\mathbf{s},q}$.

On the other hand, if $\mathbf{j} = \mathbf{j}_n$, then

$$\sigma_{\mathbf{j}} \ge \frac{2^{2j_n}}{\left((j_{n+1}-1)(\log(j_{n+1}-1))^2\right)^{\frac{2p}{q}}} .$$

It follows that

$$\lim_{n \to \infty} \sigma_{\mathbf{j}} = \infty ,$$

therefore $F \notin O_p^{\frac{1}{p}\mathbf{1}}$.

3. Let us now show that $B_p^{\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\mathbf{s}}$ if $\mathbf{s} < \frac{1}{p}\mathbf{1}$. Clearly, from above

$$B_p^{\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\frac{1}{p}\mathbf{1}} \hookrightarrow O_p^{\mathbf{s}}.$$

The embedding $B_p^{\frac{1}{p}\mathbf{1},p} \hookrightarrow O_p^{\mathbf{s}}$ is optimal since it improves the Sobolev type embedding obtained by the combination of a sharp result from Proposition 5.6 p. 188 in [17] with Theorem 1.9 in [20].

4. Let $\mathbf{s} > \mathbf{0}$ and $f \in O_p^{\mathbf{s}}$. For $[\mathbf{j}] = \mathbf{0}$, since

$$\sigma_{\mathbf{j}}(\mathbf{s},p) = \|d_{\boldsymbol{\lambda}}\|_{\ell_p(\boldsymbol{\Lambda}_{\mathbf{j}})} \le C$$

then

$$\forall\ \boldsymbol{\lambda}' \qquad |C_{\boldsymbol{\lambda}'}| \le C .$$

Therefore $f \in B_\infty^{\mathbf{0},\infty}$. The embedding $O_p^{\mathbf{s}} \hookrightarrow B_\infty^{\mathbf{0},\infty}$ is optimal since it improves the Sobolev type embedding, see [17] Proposition 5.6 p. 188, which is sharp.

Let us now show that $B^{\mathbf{s},\infty}_{\infty} \hookrightarrow (O^{\mathbf{s}}_p)_{loc}$ if $\mathbf{s} > \mathbf{0}$. Without any loss of generality, we focus only on functions f supported on the unit square of $\mathbb{R}^2$.
Let $A > 0$ be such that ψ and φ are supported in $[-A, A]$. Then

$$f = \sum_{\mathbf{j}\in\mathbb{N}^2_{-1}} \sum_{\substack{|k_1|\le A+2^{[j_1]}\\ |k_2|\le A+2^{[j_2]}}} C_{\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}}\,. \tag{45}$$

Assume that $f \in B^{\mathbf{s},\infty}_{\infty}$. We will prove that

$$2^{p(s_1[j_1]+s_2[j_2])}2^{-([j_1]+[j_2])} \sum_{\substack{|k_1|\le A+2^{[j_1]}\\ |k_2|\le A+2^{[j_2]}}} d^p_{\boldsymbol{\lambda}} \le C\,. \tag{46}$$

For $\mathbf{j}' \ge \mathbf{j}$, set

$$\boldsymbol{\Lambda}_{\mathbf{j}'}(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}' \in \boldsymbol{\Lambda}_{\mathbf{j}'} \ : \ \boldsymbol{\lambda}' \subset \boldsymbol{\lambda}\}\,.$$

Write

$$d^p_{\boldsymbol{\lambda}} \le \sup_{\mathbf{j}'\ge\mathbf{j}} \sup_{\boldsymbol{\lambda}'\in\boldsymbol{\Lambda}_{\mathbf{j}'}(\boldsymbol{\lambda})} |C_{\boldsymbol{\lambda}'}|^p\,.$$

Then

$$\sum_{\substack{|k_1|\le A+2^{[j_1]}\\ |k_2|\le A+2^{[j_2]}}} d^p_{\boldsymbol{\lambda}} \le C2^{[j_1]+[j_2]} \sup_{\mathbf{j}'\ge\mathbf{j}} \sup_{\boldsymbol{\lambda}'\in\boldsymbol{\Lambda}_{\mathbf{j}'}(\boldsymbol{\lambda})} |C_{\boldsymbol{\lambda}'}|^p\,.$$

Since $f \in B^{\mathbf{s},\infty}_{\infty}$, then

$$\sum_{\substack{|k_1|\le A+2^{[j_1]}\\ |k_2|\le A+2^{[j_2]}}} d^p_{\boldsymbol{\lambda}} \le C2^{[j_1]+[j_2]} \sup_{\mathbf{j}'\ge\mathbf{j}} \sup_{\boldsymbol{\lambda}'\in\boldsymbol{\Lambda}_{\mathbf{j}'}(\boldsymbol{\lambda})} 2^{-p(s_1[j_1']+s_2[j_2'])} \le C2^{[j_1]+[j_2]}2^{-p(s_1[j_1]+s_2[j_2])}\,. \tag{47}$$

Hence (46) holds.
The optimality is a consequence of the optimality of $B^{\mathbf{s},\infty}_{\infty} \hookrightarrow B^{\mathbf{s},\infty}_{p}$.

5. Let us now show that $B^{\mathbf{0},\infty}_{\infty} \hookrightarrow (O^{\mathbf{s}}_p)_{loc}$ if $\mathbf{s} \le \mathbf{0}$. Without any loss of generality, we focus only on functions f supported on the unit square.
Let $A > 0$ be such that ψ and φ are supported in $[-A, A]$. As in (47)

$$\sum_{\substack{|k_1|\le A+2^{[j_1]}\\ |k_2|\le A+2^{[j_2]}}} d^p_{\boldsymbol{\lambda}} \le C2^{[j_1]+[j_2]}\,.$$

Since $\mathbf{s} \le \mathbf{0}$, then (46) holds.

□

4. Conclusions

Besov spaces and fractional Sobolev spaces with dominating mixed smoothness are invariant under permutations of the wavelet coefficients at the same scale. Such permutations allow one to modify the rectangular singularities and therefore affect the multifractal analysis of rectangular regularity. This leads us to new functional spaces that we studied and related to Besov spaces with dominating mixed smoothness. We proved that hyperbolic oscillation spaces are closely related to hyperbolic variation spaces, and consequently do not almost depend on the chosen hyperbolic wavelet basis. Therefore, the so-called rectangular multifractal analysis, related to hyperbolic oscillation spaces is somehow 'robust', i.e., does not change if the analyzing wavelets were changed. We also showed that, for $\mathbf{0} < \mathbf{s} < \frac{1}{p}\mathbf{1}$ (with respect to $\mathbf{s} - \frac{1}{p}\mathbf{1} < \mathbf{s}' < \mathbf{s}$), hyperbolic oscillation spaces (with respect to mixed fractional lifting oscillation spaces) are not sharply imbedded between fractional

Sobolev spaces with dominating mixed smoothness, and thus are new spaces of a really different nature.

Author Contributions: Conceptualization, I.A., M.B.S. and O.A.; formal analysis, I.A., M.B.S. and O.A.; investigation, I.A., M.B.S. and O.A.; methodology, I.A., M.B.S. and O.A.; supervision, M.B.S. and O.A.; writing—original draft, I.A., M.B.S. and O.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research work is funded by the Researchers Supporting Project number (RSP2023R447), King Saud University, Riyadh, Saudi Arabia.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

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