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Asymptotic and Mittag–Leffler Synchronization of Fractional-Order Octonion-Valued Neural Networks with Neutral-Type and Mixed Delays

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Abstract: Very recently, a different generalization of real-valued neural networks (RVNNs) to multidimensional domains beside the complex-valued neural networks (CVNNs), quaternion-valued neural networks (QVNNs), and Clifford-valued neural networks (ClVNNs) has appeared, namely octonion-valued neural networks (OVNNs), which are not a subset of ClVNNs. They are defined on the octonion algebra, which is an 8D algebra over the reals, and is also the only other normed division algebra that can be defined over the reals beside the complex and quaternion algebras. On the other hand, fractional-order neural networks (FONNs) have also been very intensively researched in the recent past. Thus, the present work combines FONNs and OVNNs and puts forward a fractional-order octonion-valued neural network (FOOVNN) with neutral-type, time-varying, and distributed delays, a very general model not yet discussed in the literature, to our awareness. Sufficient criteria expressed as linear matrix inequalities (LMIs) and algebraic inequalities are deduced, which ensure the asymptotic and Mittag–Leffler synchronization properties of the proposed model by decomposing the OVNN system of equations into a real-valued one, in order to avoid the non-associativity problem of the octonion algebra. To accomplish synchronization, we use two different state feedback controllers, two different types of Lyapunov-like functionals in conjunction with two Halanay-type lemmas for FONNs, the free-weighting matrix method, a classical lemma, and Young's inequality. The four theorems presented in the paper are each illustrated by a numerical example.

Keywords: fractional-order neural networks (FONNs); Mittag–Leffler function; synchronization analysis; octonion-valued neural networks (OVNNs); time delays

Citation: Popa, C.-A. Asymptotic and Mittag–Leffler Synchronization of Fractional-Order Octonion-Valued Neural Networks with Neutral-Type and Mixed Delays. *Fractal Fract.* **2023**, 7, 830. https://doi.org/10.3390/fractalfract7110830

Academic Editors: António Lopes, Xujun Yang and Xiaofeng Chen

Received: 23 October 2023
Revised: 10 November 2023
Accepted: 14 November 2023
Published: 20 November 2023

1. Introduction

In the last few years, there has been an increasing interest in neural networks with values in multidimensional domains. Thus, the first type of such neural networks that have appeared are CVNNs, which are the 2D generalization of the classical RVNNs, defined on the complex number algebra. Afterwards, QVNNs appeared, which are defined on the 4D algebra of quaternions. Both types of networks are a part of the larger family of ClVNNs, which can be defined on any Clifford algebra of dimension 2^n, where $n \geq 1$.

Recently, a different type of generalization of CVNNs and QVNNs was proposed, namely OVNNs, which are defined on the 8D algebra of octonions. It is easy to see that the algebra of octonions is not a particular case of Clifford algebra because all of the Clifford algebras are associative and the algebra of octonions is not associative. Nonetheless, the octonion algebra possesses the property of being a normed division algebra, which, together with the complex and quaternion algebras, are the only ones that can be defined over the reals. It is this interesting property that gave rise to the idea of defining OVNNs, first in their feed forward variant, in [1]. Additionally, octonions have interesting applications in signal processing [2–4], salient object detection [5,6], and hyperspectral fluorescence data fusion [7], but also in physics, for example, in electrodynamics [8], fluid dynamics

[9], and gravitational field equations [10]. Taking these applications into account, it was natural to extend neural networks to the octonion domain in order to increase the expressiveness of the networks and their representational power. The dynamics of recurrent neural networks defined on the algebra of octonions was firstly studied in [11,12]. More recently, other papers have appeared that discuss dynamic properties of OVNNs, for example, [13–16]. We can observe an increasing interest in studying OVNNs in the last two years, which means that the domain of OVNNs is captivating the attention of the research community and that more papers will appear in the very near future.

On the other hand, fractional calculus discusses different ways to define real or complex orders in the context of differential and integral operators. Even though it appeared long time ago, fractional calculus did not have any important applications until recently, when it was proved that some natural phenomena can be better expressed using fractional derivatives or integrals [17,18]. Thus, fractional calculus has proved more effective in describing systems in fields such as physics, electromagnetics, acoustics, mechanics, heat transfer, biology, chemistry, economy, and finance.

It was also proved that fractional-order systems have the interesting property of infinite memory. This means that the use of a fractional derivative or integral could also provide an enhancement for neural networks. Thus, FONNs were defined for the first time in [19]. Over the years, many papers studying different dynamic properties of FONNs have appeared. Recently, the asymptotic stability and synchronization properties of FONNs were discussed, for example, in [20–27]. Then, the equivalent of the exponential stability and synchronization from integer-order neural networks, namely Mittag–Leffler stability and synchronization, were discussed in [28–33]. Finite-time stability and synchronization have been the focus of the papers [34–39]. Other dynamic properties were also studied, for example, dissipativity, in [40–45], etc. As such, it can be said that FONNs constitute a domain of research in their own right, with recently gained popularity.

Combining the two directions discussed above, the current paper aims to study the synchronization property of FONNs with values in the octonion algebra, namely FOOVNNs. To the best of our knowledge, this type of model has not been studied before in the existing literature. There exist, however, papers discussing fractional-order CVNNs (FOCVNNs), for example, [32,37,46–49], and also papers dealing with different dynamic properties of fractional-order QVNNs (FOQVNNs), for example, [29,35,42–44,50]. As such, the current research presents a model that generalizes the models proposed in these papers.

When defining the recurrent neural network model to be studied, it is essential to also add different types of delays to the model, because delays appear naturally when implementing neural networks in practice. The most classical types of delays are the time-varying delays. Then, distribution delays exist, which are determined by the dispersion of conduction velocities along the neural network's paths. When they appear together with time-varying delays, distributed delays are called mixed delays. FONNs with this type of delays have been discussed in the recent literature, for example, in [21,26,31,39,47,51–56].

A different and equally interesting family of delays are the neutral-type delays, which actually dramatically change the dynamics of the system, giving rise to neutral-type systems. In these systems, it is assumed that past derivative information affects the current state. It is natural that this type of delay was also added to neural network models, because they have been seen to appear when VLSI circuits are used for implementing neural networks. FONNs with neutral-type delays were the focus of the following recent papers: [20,21,26,57–59].

Taking into account all the above considerations, the most important highlights of the paper are as follows:

1. For the first time in the literature, to our knowledge, an FOOVNN with time-varying, distributed, and neutral-type delays is put forward, combining fractional calculus with octonion algebra.
2. The asymptotic and Mittag–Leffler synchronization properties are studied for the proposed model, and sufficient criteria are given both in terms of LMIs and of algebraic

inequalities by decomposing the OVNN system of equations into a real-valued one, in order to avoid the non-associativity problem of octonion algebra.

3. To accomplish synchronization, we use two different types of state feedback controllers, two different types of Lyapunov-type functionals in conjunction with two Halanay-type lemmas specific to FONNs, as well as the free weighting matrix method, a classical lemma, and Young's inequality.
4. Each of the four theorems formulated in the paper is illustrated by one numerical example.
5. The model is general, and can be particularized for fractional-order CVNNs (FOCVNNs) and fractional-order QVNNs (FOQVNNs), for which the corresponding results do not exist in the literature, to our knowledge, for such general models.

Thus, the rest of the paper is organized as follows: Section 2 is dedicated to presenting the algebra of octonions, the definitions related to fractional calculus, the proposed model, its decomposition into a real-valued system, and the assumption and lemmas that are used in the proofs of the paper. Then, Section 3 contains the main results of the research: four theorems expressed in the form of LMIs and algebraic inequalities for the asymptotic and Mittag–Leffler synchronization of the proposed FOOVNN model using two types of state feedback controllers and two types of Lyapunov-like functions. Each of the four theorems is illustrated by a numerical simulation in Section 4. Finally, the conclusions of the paper are drawn in Section 5.

Notations: $\mathbb{R}$—real numbers, $\mathbb{R}^+$—positive real numbers, $\mathbb{O}$—octonion numbers, $||\cdot||$—L_2 norm, $|\cdot|$—L_1 norm, $\mathbb{R}^N$ ($\mathbb{O}^N$)—real-valued (octonion-valued) N-dimensional vectors, $\mathbb{R}^{N\times N}$ ($\mathbb{O}^{N\times N}$)—real-valued (octonion-valued) matrices of dimensions $N \times N$, $A < 0$—matrix A is negative definite, A^T—transpose of matrix A, and $\lambda_{\min}(A)$—smallest eigenvalue of matrix A.

2. Preliminaries

We start by giving details about the algebra of octonions, mainly based on [11,12]. The set of octonions is defined as:

$$\mathbb{O} = \left\{ o = \sum_{q=0}^{7} o^q e_q \,\middle|\, o^q \in \mathbb{R},\ \forall 0 \leq q \leq 7 \right\},$$

where e_q represent the unit octonions, $\forall 0 \leq q \leq 7$. On this set, we define the octonion addition by $o + p := \sum_{q=0}^{7}(o^q + p^q)e_q$ and the scalar multiplication by $\alpha o := \sum_{q=0}^{7}(\alpha o^q)e_q$. The multiplication of octonions is defined by the multiplication table of the octonion units:

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

For each octonion $o \in \mathbb{O}$, its conjugate is defined as $\bar{o} := o^0 e_0 - \sum_{q=1}^{7} o^q e_q$. Then, we define the norm of octonion o as $|o| := \sqrt{\bar{o}o} = \sqrt{\sum_{q=0}^{7}(o^q)^2}$ and its inverse as $o^{-1} := \frac{\bar{o}}{|o|^2}$. With all these operations, it can be proved that $\mathbb{O}$ is a normed division algebra. Actually, there exists a famous result by Hurwitz, who showed that the real, complex, quaternion, and octonion number sets with their respective operations are the only normed division algebras over the real numbers.

As can be seen from the multiplication table, we have that $e_q e_l = -e_l e_q \neq e_l e_q$ if $q \neq l$, $q \neq 0$, $l \neq 0$, which means that, like the algebra of quaternions, the above-defined algebra of octonions is not commutative. Moreover, we can observe that $(e_q e_l)e_m = -e_q(e_l e_m) \neq e_q(e_l e_m)$ if q, l, m are non-zero, different, and $e_q e_l \neq \pm e_m$. This tells us that, unlike the algebra of quaternions and other Clifford algebras in general, the algebra of octonions is also not associative.

On the other hand, we introduce the basics of the calculus with fractional order.

Definition 1 ([60])**.** *The fractional integral of order α for an integrable function $x : [t_0, \infty) \to \mathbb{R}$ is defined as:*

$$I_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} x(s) ds,$$

where $t \geq t_0$, $\alpha > 0$, and $\Gamma(\cdot)$ is the gamma function, defined by:

$$\Gamma(\tau) = \int_0^{\infty} t^{\tau-1} e^{-t} dt,$$

for $Re(\tau) > 0$, where $Re(\cdot)$ represents the real part.

Definition 2 ([60])**.** *The fractional Caputo derivative of order α for a function $x \in C^n([t_0, \infty), \mathbb{R})$ is defined by:*

$$D_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{x^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $t \geq t_0$ and n is a positive integer, with $n-1 < \alpha < n$. Moreover, when $0 < \alpha < 1$, we have that:

$$D_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\dot{x}(s)}{(t-s)^{\alpha}} ds.$$

Definition 3 ([60])**.** *The Mittag–Leffler function is defined by:*

$$E_{\alpha}(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(p\alpha+1)},$$

where $\alpha > 0$ and $z \in \mathbb{C}$. When $\alpha = 1$, we have that $E_1(z) = e^z$.

Now, we are ready to introduce the model that will represent the drive system. Assume we have the following OVNN with neutral-type, time-varying, and distributed delays:

$$D_0^{\alpha} o_i(t) = -c_i o_i(t) + \sum_{j=1}^{N} a_{ij} f_j(o_j(t)) + \sum_{j=1}^{N} b_{ij} f_j(o_j(t-\pi(t))) + \sum_{j=1}^{N} g_{ij} \int_{t-\varepsilon}^{t} f_j(o_j(s)) ds + h_i D_{-\gamma}^{\alpha} o_i(t-\gamma) + I_i, \quad (1)$$

$\forall i \in \{1, \ldots, N\}$, $\forall t \in [0, +\infty)$, where $o(t) = (o_1(t), \ldots, o_N(t))^T \in \mathbb{O}^N$ represents the vector of states at $t \in [0, +\infty)$, $C = \text{diag}(c_1, \ldots, c_N) \in \mathbb{R}^{N \times N}$ represents the self-feedback weight matrix, $A = (a_{ij})_{1 \leq i,j \leq N} \in \mathbb{O}^{N \times N}$ is the weight matrix without delay, $B = (b_{ij})_{1 \leq i,j \leq N} \in \mathbb{O}^{N \times N}$ is the weight matrix with delay, $G = (g_{ij})_{1 \leq i,j \leq N} \in \mathbb{O}^{N \times N}$ is the distributed delay weight matrix, $H = \text{diag}(h_1, \ldots, h_N) \in \mathbb{O}^{N \times N}$ is the neutral-type weight matrix, $f_j : \mathbb{O} \to \mathbb{O}$ represent the activation functions, $\forall j \in \{1, \ldots, N\}$, and $I = (I_1, \ldots, I_N)^T \in \mathbb{O}^N$ is the external input vector. The time-varying delays are $\pi : \mathbb{R}^+ \to \mathbb{R}^+$, and we assume the existence of $\pi > 0$ with $\pi(t) \leq \pi$, $\forall t \in [0, +\infty)$; the distributed delay is $\varepsilon > 0$, and the neutral-type delay is $\gamma > 0$. We denote $\varsigma := \max\{\pi, \varepsilon, \gamma\}$. Additionally, suppose that the functions f_j can be written in the form $f_j(o) = \sum_{q=0}^{7} f_j^q(o) e_q$, $\forall o \in \mathbb{O}$, where $f_j^q : \mathbb{O} \to \mathbb{R}$, $\forall j \in \{1, \ldots, N\}$, $\forall 0 \leq q \leq 7$.

For System (1), the initial condition is given as:

$$o_i(t) = \nu_i(t), \quad \forall t \in [-\varsigma, 0],$$

where $\nu_i \in \mathcal{C}([-\varsigma, 0], \mathbb{O})$, $\forall i \in \{1, \dots, N\}$, and the norm on $\mathcal{C}([-\varsigma, 0], \mathbb{O}^N)$ has the definition $||\nu|| := \sum_{i=1}^{N} \sup_{[-\varsigma,0]} |\nu_i(t)|$.

On the other hand, in order to study synchronization, we will correspondingly define the response system as:

$$D_0^\alpha p_i(t) = -c_i p_i(t) + \sum_{j=1}^{N} a_{ij} f_j(p_j(t)) + \sum_{j=1}^{N} b_{ij} f_j(p_j(t - \pi(t))) + \sum_{j=1}^{N} g_{ij} \int_{t-\varepsilon}^{t} f_j(p_j(s)) ds + h_i D_{-\gamma}^{\alpha} p_i(t - \gamma) + I_i - u_i(t), \quad (2)$$

$\forall i \in \{1, \dots, N\}$, $\forall t \in [0, +\infty)$, and $p(t) = (p_1(t), \dots, p_N(t))^T \in \mathbb{O}^N$ represents the vector of states at $t \in [0, +\infty)$, and $u(t) = (u_1(t), \dots, u_N(t))^T \in \mathbb{O}^N$ represents the control input vector at $t \in [0, +\infty)$.

For System (2), the initial condition is given by:

$$p_i(t) = v_i(t), \quad \forall t \in [-\varsigma, 0],$$

where $v_i \in \mathcal{C}([-\varsigma, 0], \mathbb{O})$, $\forall i \in \{1, \dots, N\}$.

Now, by denoting $\mathfrak{r}_i(t) = p_i(t) - o_i(t)$, $\forall i \in \{1, \dots, N\}$, $\forall t \in [0, +\infty)$ and also considering Relations (1) and (2), the expression of the error system is given as:

$$D_0^\alpha \mathfrak{r}_i(t) = -c_i \mathfrak{r}_i(t) + \sum_{j=1}^{N} a_{ij} \tilde{f}_j(\mathfrak{r}_j(t)) + \sum_{j=1}^{N} b_{ij} \tilde{f}_j(\mathfrak{r}_j(t - \pi(t))) + \sum_{j=1}^{N} g_{ij} \int_{t-\varepsilon}^{t} \tilde{f}_j(\mathfrak{r}_j(s)) ds + h_i D_{-\gamma}^{\alpha} \mathfrak{r}_i(t - \gamma) - u_i(t), \quad (3)$$

$\forall i \in \{1, \dots, N\}$ and $\forall t \in [0, +\infty)$, where $\tilde{f}_j(\mathfrak{r}_j(t)) = f_j(\mathfrak{r}_j(t) + o_j(t)) - f_j(o_j(t))$, $\forall t \in [0, +\infty)$, $\forall j \in \{1, \dots, N\}$.

The initial conditions of System (3) now have the following form:

$$\mathfrak{r}_i(t) = \psi_i(t) = v_i(t) - \nu_i(t), \quad \forall t \in [-\varsigma, 0],$$

where $\psi_i \in \mathcal{C}([-\varsigma, 0], \mathbb{O})$, $\forall i \in \{1, \dots, N\}$.

At this point, we will transform the System of Equations (3) into 8 real-valued systems. In order to do this, each equation in (3) can be written as the following 8 equations:

$$\begin{aligned} D_0^\alpha \mathfrak{r}_i^q(t) \quad = \quad & -c_i \mathfrak{r}_i^q(t) + \sum_{j=1}^{N} \sum_{l=0}^{7} a_{ij}^{ql} \tilde{f}_j^l(\mathfrak{r}_j(t)) + \sum_{j=1}^{N} \sum_{l=0}^{7} b_{ij}^{ql} \tilde{f}_j^l(\mathfrak{r}_j(t - \pi(t))) + \sum_{j=1}^{N} \sum_{l=0}^{7} g_{ij}^{ql} \int_{t-\varepsilon}^{t} \tilde{f}_j^l(\mathfrak{r}_j(s)) ds \\ & + \sum_{l=0}^{7} h_i^{ql} D_{-\gamma}^{\alpha} \mathfrak{r}_i^l(t - \gamma) - u_i^q(t), \end{aligned}$$

$\forall 0 \le q \le 7$, $\forall i \in \{1, \dots, N\}$, and o^{ql}, $\forall 0 \le q, l \le 7$ represents an element of matrix $\mathrm{mat}(o)$, defined by:

$$\mathrm{mat}(o) := \begin{bmatrix} o^0 & -o^1 & -o^2 & -o^3 & -o^4 & -o^5 & -o^6 & -o^7 \\ o^1 & o^0 & -o^3 & o^2 & -o^5 & o^4 & o^7 & -o^6 \\ o^2 & o^3 & o^0 & -o^1 & -o^6 & -o^7 & o^4 & o^5 \\ o^3 & -o^2 & o^1 & o^0 & -o^7 & o^6 & -o^5 & -o^4 \\ o^4 & o^5 & o^6 & o^7 & o^0 & -o^1 & -o^2 & -o^3 \\ o^5 & -o^4 & o^7 & -o^6 & o^1 & o^0 & o^3 & -o^2 \\ o^6 & -o^7 & -o^4 & o^5 & o^2 & -o^3 & o^0 & o^1 \\ o^7 & o^6 & -o^5 & -o^4 & o^3 & o^2 & -o^1 & o^0 \end{bmatrix}.$$

If we now denote $\mathrm{vec}(o) := (o^0, o^1, \ldots, o^7)^T$, System (3) will have the following expression:

$$\begin{aligned}\mathrm{vec}(D_0^\alpha \mathfrak{r}_i(t)) \;=\;& -c_i \mathrm{vec}(\mathfrak{r}_i(t)) + \sum_{j=1}^{N} \mathrm{mat}(a_{ij}) \mathrm{vec}(\tilde{f}_j(\mathfrak{r}_j(t))) + \sum_{j=1}^{N} \mathrm{mat}(b_{ij}) \mathrm{vec}(\tilde{f}_j(\mathfrak{r}_j(t - \pi(t)))) \\ & + \sum_{j=1}^{N} \mathrm{mat}(g_{ij}) \int_{t-\varepsilon}^{t} \mathrm{vec}(\tilde{f}_j(\mathfrak{r}_j(s)))ds + \mathrm{mat}(h_i) \mathrm{vec}(D_{-\gamma}^\alpha \mathfrak{r}_i(t-\gamma)) - \mathrm{vec}(u_i(t)),\end{aligned}$$

$\forall t \in [0, +\infty)$ and $\forall i \in \{1, \ldots, N\}$.

Finally, by denoting:

$$\check{\mathfrak{r}}(t) := (\mathrm{vec}(\mathfrak{r}_1(t))^T, \ldots, \mathrm{vec}(\mathfrak{r}_N(t))^T)^T \in \mathbb{R}^{8N},\ \check{f}(\check{\mathfrak{r}}(t)) := (\mathrm{vec}(\tilde{f}_1(\mathfrak{r}_1(t)))^T, \ldots, \mathrm{vec}(\tilde{f}_N(\mathfrak{r}_N(t)))^T)^T \in \mathbb{R}^{8N},$$

$$\check{C} := \mathrm{diag}(c_1 I_8, c_2 I_8, \ldots, c_N I_8) \in \mathbb{R}^{8N\times 8N},\ \check{A} := \left(\mathrm{mat}(a_{ij})\right)_{1\le i,j\le N} \in \mathbb{R}^{8N\times 8N},$$

$$\check{B} := \left(\mathrm{mat}(b_{ij})\right)_{1\le i,j\le N} \in \mathbb{R}^{8N\times 8N},\ \check{G} := \left(\mathrm{mat}(g_{ij})\right)_{1\le i,j\le N} \in \mathbb{R}^{8N\times 8N},$$

$$\check{H} := \mathrm{diag}(\mathrm{mat}(h_1), \ldots, \mathrm{mat}(h_N)) \in \mathbb{R}^{8N\times 8N},\ \check{u}(t) := (\mathrm{vec}(u_1(t))^T, \ldots, \mathrm{vec}(u_N(t))^T)^T \in \mathbb{R}^{8N},$$

System (3) becomes:

$$D_0^\alpha \check{\mathfrak{r}}(t) = -\check{C}\check{\mathfrak{r}}(t) + \check{A}\check{f}(\check{\mathfrak{r}}(t)) + \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) + \check{G}\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds + \check{H}D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma) - \check{u}, \quad \forall t \in [0, +\infty). \tag{4}$$

The following assumption regarding the activation functions has to be made:

Assumption 1. *The activation functions f_j satisfy, $\forall o,\ o' \in \mathbb{O}$, the following Lipschitz conditions:*

$$||f_j^q(o) - f_j^q(o')|| \le l_j^q ||o - o'||,$$

$\forall 0 \le q \le 7$ and $\forall j \in \{1, \ldots, N\}$, where $l_j^q > 0$ represent the Lipschitz constants. We denote $\check{L} := diag(l_1^0, \ldots, l_1^7, \ldots, l_N^0, \ldots, l_N^7) \in \mathbb{R}^{8N\times 8N}$.

In order to conduct our proofs, the following lemmas will also be needed:

Lemma 1 ([61]). *If $x \in \mathcal{C}^1([t_0, \infty), \mathbb{R}^N)$ and $P \in \mathbb{R}^{N\times N}$ is a positive definite matrix, then*

$$D_{t_0}^\alpha (x^T(t)Px(t)) \le x^T(t)PD_{t_0}^\alpha x(t) + D_{t_0}^\alpha x(t)^T Px(t),\ \forall t \ge t_0,$$

where $0 < \alpha < 1$.

Lemma 2 ([53]). *Let $V : [t_0 - \rho, \infty) \to \mathbb{R}^+$ be bounded on $[t_0 - \rho, t_0]$ and continuous on $[t_0, \infty)$. If there exist ϕ, v_h, $\forall h = 1, \ldots, m$ such that*

$$D_{t_0}^\alpha V(t) \le -\phi V(t) + \sum_{h=1}^{m} v_h \sup_{-\rho_h \le \omega \le 0} V(t+\omega),$$

where $0 < \alpha < 1$, $v_h > 0$, $\phi > \sum_{h=1}^{m} v_h$, $\rho = \max\{\rho_1, \ldots, \rho_m\}$, then $\lim_{t\to\infty} V(t) = 0$.

Lemma 3 ([62]). *For any vectors $X, Y \in \mathbb{R}^N$ and any positive definite matrix $Q \in \mathbb{R}^{N\times N}$, the following inequality holds:*

$$X^T Y + Y^T X \le X^T QX + Y^T Q^{-1} Y.$$

Lemma 4 ([36])**.** *If $x \in \mathcal{C}^1([t_0, \infty), \mathbb{R})$ and $p \geq 1$, then*

$$D^{\alpha}_{t_0}|x(t)|^p \leq p|x(t)|^{p-1} sign(x(t)) D^{\alpha}_{t_0} x(t),\ \forall t \geq t_0,$$

where $0 < \alpha < 1$.

Lemma 5 (Young's Inequality)**.** *Let $u > 0$, $v > 0$, $r > 1$, $s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, then the inequality*

$$uv \leq \frac{1}{r}u^r + \frac{1}{s}v^s,$$

holds, with equality if $u^r = v^s$.

Lemma 6 ([63])**.** *Let $V \in \mathcal{C}([t_0, \infty), \mathbb{R})$, which satisfies*

$$D^{\alpha}_{t_0} V(t) \leq -\phi V(t),\ \forall t \geq t_0,$$

where $0 < \alpha < 1$ and $\phi > 0$. Then,

$$V(t) \leq V(t_0) E_{\alpha}(-\phi(t - t_0)^{\alpha}),\ \forall t \geq t_0.$$

3. Main Results

The assumption that $0 < \alpha < 1$ will be made in the remaining part of the paper.

The state feedback control technique will be used to study the synchronization between Drive System (1) and Response System (2). The control input is given, in this case, by the following expression:

$$u_i(t) = k_{i1}\mathfrak{r}_i(t) + k_{i2}\mathfrak{r}_i(t - \pi(t)) + k_{i3}\int_{t-\varepsilon}^{t} \mathfrak{r}_i(s)ds + k_{i4} D^{\alpha}_{-\gamma}\mathfrak{r}_i(t - \gamma), \tag{5}$$

and $k_{i1},\ k_{i2},\ k_{i3},\ k_{i4} \in \mathbb{R}^+, \forall i \in \{1, \ldots, N\}$ represent the control gain parameters. We can now write System (3) as:

$$\begin{aligned} D^{\alpha}_0 \mathfrak{r}_i(t) \;=\; & -(c_i + k_{i1})\mathfrak{r}_i(t) - k_{i2}\mathfrak{r}_i(t - \pi(t)) - k_{i3}\int_{t-\varepsilon}^{t} \mathfrak{r}_i(s)ds + \sum_{j=1}^{N} a_{ij}\tilde{f}_j(\mathfrak{r}_j(t)) \\ & + \sum_{j=1}^{N} b_{ij}\tilde{f}_j(\mathfrak{r}_j(t - \pi(t))) + \sum_{j=1}^{N} g_{ij}\int_{t-\varepsilon}^{t} \tilde{f}_j(\mathfrak{r}_j(s))ds + (h_i - k_{i4}) D^{\alpha}_{-\gamma}\mathfrak{r}_i(t - \gamma), \end{aligned} \tag{6}$$

$\forall i \in \{1, \ldots, N\}$.

In matrix form, we can write System (6) as:

$$\begin{aligned} D^{\alpha}_0 \check{\mathfrak{r}}(t) \;=\; & -(\check{C} + \check{K}_1)\check{\mathfrak{r}}(t) - \check{K}_2\check{\mathfrak{r}}(t - \pi(t)) - \check{K}_3\int_{t-\varepsilon}^{t} \check{\mathfrak{r}}(s)ds + \check{A}\check{f}(\check{\mathfrak{r}}(t)) \\ & + \check{B}\check{f}(\check{\mathfrak{r}}(t - \pi(t))) + \check{G}\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds + (\check{H} - \check{K}_4) D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t - \gamma), \end{aligned} \tag{7}$$

where $\check{K}_1 := \mathrm{diag}(k_{11}I_8, k_{21}I_8, \ldots, k_{N1}I_8) \in \mathbb{R}^{8N \times 8N}$, $\check{K}_2 := \mathrm{diag}(k_{12}I_8, k_{22}I_8, \ldots, k_{N2}I_8) \in \mathbb{R}^{8N \times 8N}$,

$\check{K}_3 := \mathrm{diag}(k_{13}I_8, k_{23}I_8, \ldots, k_{N3}I_8) \in \mathbb{R}^{8N \times 8N}$, $\check{K}_4 := \mathrm{diag}(k_{14}I_8, k_{24}I_8, \ldots, k_{N4}I_8) \in \mathbb{R}^{8N \times 8N}$.

Theorem 1. *System (1) and System (2) are asymptotically synchronized under Controller (5) if the subsequent LMI is true:*

$$\Omega := (\Omega_{i,j})_{1 \leq i,j \leq 10} < 0, \tag{8}$$

where $\Omega_{1,1} = (2 + \phi)P + \check{L}^T R_1 \check{L} - P(\check{C} + \check{K}_1) - (\check{C} + \check{K}_1)P$, $\Omega_{1,2} = -(\check{C} + \check{K}_1)N_1^T$, $\Omega_{1,3} = -P\check{K}_2 - (\check{C} + \check{K}_1)N_2^T$, $\Omega_{1,6} = P\check{A} + (\check{C} + \check{K}_1)N_3^T$, $\Omega_{1,7} = P\check{B} + (\check{C} + \check{K}_1)N_4^T$, $\Omega_{1,8} = -P\check{K}_3 -$

$(\check{C}+\check{K}_1)N_5^T$, $\Omega_{1,9} = (\check{C}+\check{K}_1)N_6^T$, $\Omega_{1,10} = (\check{C}+\check{K}_1)N_7^T$, $\Omega_{2,2} = -N_1 - N_1^T$, $\Omega_{2,3} = -N_1\check{K}_2 - N_2^T$, $\Omega_{2,6} = N_1\check{A} + N_3^T$, $\Omega_{2,7} = N_1\check{B} + N_4^T$, $\Omega_{2,8} = -N_1\check{K}_3 - N_5^T$, $\Omega_{2,9} = N_1\check{G} + N_6^T$, $\Omega_{2,10} = N_1(\check{H}-\check{K}_4) + N_7^T$, $\Omega_{3,3} = -v_1P + \check{L}^T R_2 \check{L} - N_2\check{K}_3 - \check{K}_3 N_2^T$, $\Omega_{3,6} = N_2\check{A} + \check{K}_2 N_3^T$, $\Omega_{3,7} = N_2\check{B} + \check{K}_2 N_4^T$, $\Omega_{3,8} = -N_2\check{K}_3 - \check{K}_2 N_5^T$, $\Omega_{3,9} = N_2\check{G} + \check{K}_2 N_6^T$, $\Omega_{3,10} = N_2(\check{H}-\check{K}_4) + \check{K}_2 N_7^T$, $\Omega_{4,4} = -v_2 P$, $\Omega_{5,5} = -v_3 P$, $\Omega_{6,6} = -R_1 - N_3\check{A} - \check{A}^T N_3^T$, $\Omega_{6,7} = -N_3\check{B} - \check{A}^T N_4^T$, $\Omega_{6,8} = N_3\check{K}_3 + \check{A}^T N_5^T$, $\Omega_{6,9} = -N_3\check{G} - \check{A}^T N_6^T$, $\Omega_{6,10} = -N_3(\check{H}-\check{K}_4) - \check{A}^T N_7^T$, $\Omega_{7,7} = -R_2 - N_4\check{B} - \check{B}^T N_4^T$, $\Omega_{7,8} = N_4\check{K}_3 + \check{B}^T N_5^T$, $\Omega_{7,9} = -N_4\check{G} - \check{B}^T N_6^T$, $\Omega_{7,10} = -N_4(\check{H}-\check{K}_4) - \check{B}^T N_7^T$, $\Omega_{8,8} = -N_5\check{K}_3 - \check{K}_3 N_5^T$, $\Omega_{8,9} = N_5\check{G} + \check{K}_3 N_6^T$, $\Omega_{8,10} = N_5(\check{H}-\check{K}_4) + \check{K}_3 N_7^T$, $\Omega_{9,9} = \check{G}^T P \check{G} - N_6\check{G} - \check{G}^T N_6^T$, $\Omega_{9,10} = -N_6(\check{H}-\check{K}_4) - \check{G}^T N_7^T$, $\Omega_{10,10} = (\check{H}-\check{K}_4)^T P (\check{H}-\check{K}_4) - N_7(\check{H}-\check{K}_4) - (\check{H}-\check{K}_4)^T N_7^T$, *and* $P \in \mathbb{R}^{8N\times 8N}$ *is a positive definite matrix,* $R_1,\ R_2 \in \mathbb{R}^{8N\times 8N}$ *are diagonal positive definite matrices,* $N_1,\ldots,N_7 \in \mathbb{R}^{8N\times 8N}$ *are any matrices, and* $\phi,\ v_1,\ v_2,\ v_3$ *are positive real numbers.*

Proof. Define the following Lyapunov-like functional:

$$V(t) = \check{\mathfrak{r}}(t)^T P \check{\mathfrak{r}}(t).$$

Using Lemma 1 and taking the fractional-order derivative of the function defined above, along the trajectories of System (7), we obtain:

$$D_0^\alpha V(t) + \phi V(t) - \sum_{h=1}^{m} v_h \sup_{-\rho_h \le \varsigma \le 0} V(t+\varsigma)$$

$$\begin{aligned}
\le\ & D_0^\alpha V(t) + \phi V(t) - v_1 V(t-\pi(t)) - v_2 V(t-\varepsilon) - v_3 V(t-\gamma) \\
\le\ & \check{\mathfrak{r}}(t)^T P D_0^\alpha \check{\mathfrak{r}}(t) + D_0^\alpha \check{\mathfrak{r}}(t)^T P \check{\mathfrak{r}}(t) + \phi \check{\mathfrak{r}}(t)^T P \check{\mathfrak{r}}(t) - v_1 \check{\mathfrak{r}}(t-\pi(t))^T P \check{\mathfrak{r}}(t-\pi(t)) \\
& - v_2 \check{\mathfrak{r}}(t-\varepsilon)^T P \check{\mathfrak{r}}(t-\varepsilon) - v_3 \check{\mathfrak{r}}(t-\gamma)^T P \check{\mathfrak{r}}(t-\gamma) \\
=\ & \check{\mathfrak{r}}(t)^T P \Big[-(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t) - \check{K}_2 \check{\mathfrak{r}}(t-\pi(t)) - \check{K}_3 \int_{t-\varepsilon}^{t} \check{\mathfrak{r}}(s)ds + \check{A}\check{f}(\check{\mathfrak{r}}(t)) \\
& + \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) + \check{G}\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds + (\check{H}-\check{K}_4) D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma) \Big] \\
& + \Big[-(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t) - \check{K}_2 \check{\mathfrak{r}}(t-\pi(t)) - \check{K}_3 \int_{t-\varepsilon}^{t} \check{\mathfrak{r}}(s)ds + \check{A}\check{f}(\check{\mathfrak{r}}(t)) \\
& + \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) + \check{G}\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds + (\check{H}-\check{K}_4) D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma) \Big]^T P \check{\mathfrak{r}}(t) \\
& + \phi \check{\mathfrak{r}}(t)^T P \check{\mathfrak{r}}(t) - v_1 \check{\mathfrak{r}}(t-\pi(t))^T P \check{\mathfrak{r}}(t-\pi(t)) - v_2 \check{\mathfrak{r}}(t-\varepsilon)^T P \check{\mathfrak{r}}(t-\varepsilon) - v_3 \check{\mathfrak{r}}(t-\gamma)^T P \check{\mathfrak{r}}(t-\gamma) \\
=\ & -\check{\mathfrak{r}}(t)^T P (\check{C}+\check{K}_1)\check{\mathfrak{r}}(t) - \check{\mathfrak{r}}(t)^T P \check{K}_2 \check{\mathfrak{r}}(t-\pi(t)) - \check{\mathfrak{r}}(t)^T P \check{K}_3 \int_{t-\varepsilon}^{t} \check{\mathfrak{r}}(s)ds + \check{\mathfrak{r}}(t)^T P \check{A}\check{f}(\check{\mathfrak{r}}(t)) \\
& + \check{\mathfrak{r}}(t)^T P \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) + \check{\mathfrak{r}}(t)^T P \check{G}\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds + \check{\mathfrak{r}}(t)^T P (\check{H}-\check{K}_4) D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma) \\
& - \check{\mathfrak{r}}(t)^T (\check{C}+\check{K}_1) P \check{\mathfrak{r}}(t) - \check{\mathfrak{r}}(t-\pi(t))^T \check{K}_2 P \check{\mathfrak{r}}(t) - \left(\int_{t-\varepsilon}^{t} \check{\mathfrak{r}}(s)ds\right)^T \check{K}_3 P \check{\mathfrak{r}}(t) + \check{f}(\check{\mathfrak{r}}(t))^T \check{A}^T P \check{\mathfrak{r}}(t) \\
& + \check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T \check{B}^T P \check{\mathfrak{r}}(t) + \left(\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds\right)^T \check{G}^T P \check{\mathfrak{r}}(t) + D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma)^T (\check{H}-\check{K}_4) P \check{\mathfrak{r}}(t) \\
& + \phi \check{\mathfrak{r}}(t)^T P \check{\mathfrak{r}}(t) - v_1 \check{\mathfrak{r}}(t-\pi(t))^T P \check{\mathfrak{r}}(t-\pi(t)) - v_2 \check{\mathfrak{r}}(t-\varepsilon)^T P \check{\mathfrak{r}}(t-\varepsilon) - v_3 \check{\mathfrak{r}}(t-\gamma)^T P \check{\mathfrak{r}}(t-\gamma).
\end{aligned} \tag{9}$$

If we take $Q = P^{-1}$ in Lemma 3, we obtain:

$$\begin{aligned}
& \check{\mathfrak{r}}(t)^T P \check{G}\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds + \left(\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds\right)^T \check{G}^T P \check{\mathfrak{r}}(t) \\
\le\ & \check{\mathfrak{r}}(t)^T P P^{-1} P \check{\mathfrak{r}}(t) + \left(\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds\right)^T \check{G}^T P \check{G} \left(\int_{t-\varepsilon}^{t} \check{f}(\check{\mathfrak{r}}(s))ds\right),
\end{aligned} \tag{10}$$

$$\check{\mathfrak{r}}(t)^T P(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)+D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T(\check{H}-\check{K}_4)P\check{\mathfrak{r}}(t)$$
$$\leq \check{\mathfrak{r}}(t)^T PP^{-1}P\check{\mathfrak{r}}(t)+D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T(\check{H}-\check{K}_4)^T P(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma). \quad (11)$$

Now, Inequality (9) becomes:

$$D^{\alpha}_0 V(t)+\phi V(t)-\sum_{h=1}^{m} v_h \sup_{-\rho_h\leq\varsigma\leq 0} V(t+\varsigma)$$
$$\begin{aligned}
\leq\ & -\check{\mathfrak{r}}(t)^T P(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t)-\check{\mathfrak{r}}(t)^T(\check{C}+\check{K}_1)P\check{\mathfrak{r}}(t)-\check{\mathfrak{r}}(t)^T P\check{K}_2\check{\mathfrak{r}}(t-\pi(t))-\check{\mathfrak{r}}(t-\pi(t))^T\check{K}_2 P\check{\mathfrak{r}}(t)\\
& -\check{\mathfrak{r}}(t)^T P\check{K}_3\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds-\left(\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds\right)^T\check{K}_3 P\check{\mathfrak{r}}(t)+\check{\mathfrak{r}}(t)^T P\check{A}\check{f}(\check{\mathfrak{r}}(t))+\check{f}(\check{\mathfrak{r}}(t))^T\check{A}^T P\check{\mathfrak{r}}(t)\\
& +\check{\mathfrak{r}}(t)^T P\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t)))+\check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T\check{B}^T P\check{\mathfrak{r}}(t)\\
& +\check{\mathfrak{r}}(t)^T P\check{\mathfrak{r}}(t)+\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)^T\check{G}^T P\check{G}\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)\\
& +\check{\mathfrak{r}}(t)^T P\check{\mathfrak{r}}(t)+D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T(\check{H}-\check{K}_4)^T P(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)\\
& +\phi\check{\mathfrak{r}}(t)^T P\check{\mathfrak{r}}(t)-v_1\check{\mathfrak{r}}(t-\pi(t))^T P\check{\mathfrak{r}}(t-\pi(t))-v_2\check{\mathfrak{r}}(t-\varepsilon)^T P\check{\mathfrak{r}}(t-\varepsilon)-v_3\check{\mathfrak{r}}(t-\gamma)^T P\check{\mathfrak{r}}(t-\gamma). \quad (12)
\end{aligned}$$

Assumption 1 allows us to ascertain the existence of positive definite diagonal matrices R_1 and R_2 for which the following inequalities are valid $\forall t\in[0,+\infty)$:

$$0\leq\check{\mathfrak{r}}(t)^T\check{L}^T R_1\check{L}\check{\mathfrak{r}}(t)-\check{f}(\check{\mathfrak{r}}(t))^T R_1\check{f}(\check{\mathfrak{r}}(t)), \quad (13)$$
$$0\leq\check{\mathfrak{r}}(t-\pi(t))^T\check{L}^T R_2\check{L}\check{\mathfrak{r}}(t-\pi(t))-\check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T R_2\check{f}(\check{\mathfrak{r}}(t-\pi(t))). \quad (14)$$

On the other hand, there exist any matrices $N_1,\ N_2,\ N_3,\ N_4,\ N_5,\ N_6,\ N_7$ such that the subsequent identity is true:

$$\begin{aligned}
0\ =\ & \Bigg[D^{\alpha}_0\check{\mathfrak{r}}(t)^T N_1+\check{\mathfrak{r}}(t-\pi(t))^T N_2-\check{f}(\check{\mathfrak{r}}(t))^T N_3-\check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T N_4+\left(\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds\right)^T N_5\\
& -\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)^T N_6-D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T N_7\Bigg]\times\Big[-D^{\alpha}_0\check{\mathfrak{r}}(t)-(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t)\\
& -\check{K}_2\check{\mathfrak{r}}(t-\pi(t))-\check{K}_3\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds+\check{A}\check{f}(\check{\mathfrak{r}}(t))+\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t)))+\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\\
& +(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)\Big]. \quad (15)
\end{aligned}$$

By taking the transpose of Identity (15) and adding them both, as well as Inequalities (13)–(14) to (12), we have that:

$$D^{\alpha}_0 V(t)+\phi V(t)-\sum_{h=1}^{m} v_h \sup_{-\rho_h\leq\varsigma\leq 0} V(t+\varsigma)\leq\zeta^T(t)\Omega\zeta(t),$$

where Ω is given in (8), and

$$\begin{aligned}
\zeta(t)\ =\ & \Big[\ \check{\mathfrak{r}}(t)^T\quad D^{\alpha}_0\check{\mathfrak{r}}(t)^T\quad \check{\mathfrak{r}}(t-\pi(t))^T\quad \check{\mathfrak{r}}(t-\varepsilon)^T\quad \check{\mathfrak{r}}(t-\gamma)^T\quad \check{f}(\check{\mathfrak{r}}(t))^T\\
& \check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T\quad \left(\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds\right)^T\quad \left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)^T\quad D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T\ \Big]^T.
\end{aligned}$$

Condition (8) tells us that $\Omega<0$, so we can conclude that:

$$D^{\alpha}_0 V(t)+\phi V(t)-\sum_{h=1}^{m} v_h \sup_{-\rho_h\leq\varsigma\leq 0} V(t+\varsigma)\leq 0.$$

The conclusion $\lim_{t\to\infty} V(t) = 0$ is drawn after applying Lemma 2. From $\lambda_{\min}(P)||\check{\mathfrak{r}}(t)||^2 \leq \check{\mathfrak{r}}(t)^T P\check{\mathfrak{r}}(t) = V(t)$, we deduce that Systems (1) and (2) are asymptotically synchronized using Control Scheme (5), which is what we needed to prove. □

For the next theorem, no neutral-type delay will be considered for Systems (1) and (2), meaning that they will only possess mixed delays. Consequently, the neutral-type term will also not be present in the expression of the controller.

Theorem 2. *System* (1) *and System* (2) *are asymptotically synchronized under Controller* (5) *if the following algebraic inequalities are true,* $\forall i \in \{1,\ldots,8N\}$*:*

$$\begin{aligned} -\omega_i\rho(\check{c}_i+\check{k}_{i1})+\omega_i(\rho-1)\check{k}_{i2}+\omega_i(\rho-1)\check{k}_{i3}\varepsilon+\omega_i(\rho-1)\sum_{j=1}^{8N}|\check{a}_{ij}|\check{l}_j+\check{l}_i\sum_{j=1}^{8N}|\check{a}_{ji}|\omega_j& \\ +\omega_i(\rho-1)\sum_{j=1}^{8N}|\check{b}_{ij}|\check{l}_j+\omega_i(\rho-1)\sum_{j=1}^{8N}|\breve{g}_{ij}|\check{l}_j\varepsilon+\phi\omega_i &< 0 \\ \omega_i\check{k}_{i2}+\check{l}_i\sum_{j=1}^{8N}|\check{b}_{ji}|\omega_j-v_1\omega_i &< 0 \\ \omega_i\check{k}_{i3}\varepsilon+\check{l}_i\sum_{j=1}^{8N}|\breve{g}_{ji}|\omega_j\varepsilon-v_2\omega_i &< 0, \end{aligned} \quad (16)$$

where ρ, ω_i, $\forall i \in \{1,\ldots,8N\}$, ϕ, v_1, v_2 *are positive real numbers with* $\rho > 1$*.*

Proof. Define the following Lyapunov-like functional:

$$V(t) = \sum_{i=1}^{8N}\omega_i|\check{\mathfrak{r}}_i(t)|^\rho.$$

Using Lemma 4 and Assumption 1, and taking the fractional-order derivative of the function defined above, along the trajectories of System (7), we obtain:

$$D_0^\alpha V(t)+\phi V(t)-\sum_{h=1}^{m}v_h\sup_{-\rho_h\leq\varsigma\leq0}V(t+\varsigma)$$

$$\begin{aligned} &\leq \sum_{i=1}^{8N}\Big(\omega_i\rho|\check{\mathfrak{r}}_i(t)|^{\rho-1}\mathrm{sign}(\check{\mathfrak{r}}_i(t))D_0^\alpha\check{\mathfrak{r}}_i(t)+\phi\omega_i|\check{\mathfrak{r}}_i(t)|^\rho-v_1\omega_i|\check{\mathfrak{r}}_i(t-\pi(t))|^\rho-v_2\omega_i|\check{\mathfrak{r}}_i(t-\varepsilon)|^\rho\Big) \\ &\leq \sum_{i=1}^{8N}\Bigg(\omega_i\rho|\check{\mathfrak{r}}_i(t)|^{\rho-1}\mathrm{sign}(\check{\mathfrak{r}}_i(t))\Bigg[-(\check{c}_i+\check{k}_{i1})\check{\mathfrak{r}}_i(t)-\check{k}_{i2}\check{\mathfrak{r}}_i(t-\pi(t))-\check{k}_{i3}\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}_i(s)ds+\sum_{j=1}^{8N}\check{a}_{ij}\check{f}_j(\check{\mathfrak{r}}_j(t)) \\ &\quad+\sum_{j=1}^{8N}\check{b}_{ij}\check{f}_j(\check{\mathfrak{r}}_j(t-\pi(t)))+\sum_{j=1}^{8N}\breve{g}_{ij}\int_{t-\varepsilon}^{t}\check{f}_j(\check{\mathfrak{r}}_j(s))ds\Bigg] \\ &\quad+\phi\omega_i|\check{\mathfrak{r}}_i(t)|^\rho-v_1\omega_i|\check{\mathfrak{r}}_i(t-\pi(t))|^\rho-v_2\omega_i\sup_{-\varepsilon\leq\varsigma\leq0}|\check{\mathfrak{r}}_i(t+\varsigma)|^\rho\Bigg) \\ &\leq \sum_{i=1}^{8N}\Bigg(-\omega_i\rho|\check{\mathfrak{r}}_i(t)|^\rho(\check{c}_i+\check{k}_{i1})+\omega_i\rho|\check{\mathfrak{r}}_i(t)|^{\rho-1}\check{k}_{i2}|\check{\mathfrak{r}}_i(t-\pi(t))|+\omega_i\rho|\check{\mathfrak{r}}_i(t)|^{\rho-1}\check{k}_{i3}\int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}_i(s)|ds \end{aligned}$$

$$
\begin{aligned}
&+\omega_i\rho|\check{\mathfrak{r}}_i(t)|^{\rho-1}\sum_{j=1}^{8N}|\check{a}_{ij}|\check{l}_j|\check{\mathfrak{r}}_j(t)|+\omega_i\rho|\check{\mathfrak{r}}_i(t)|^{\rho-1}\sum_{j=1}^{8N}|\check{b}_{ij}|\check{l}_j|\check{\mathfrak{r}}_j(t-\pi(t))| \\
&+\omega_i\rho|\check{\mathfrak{r}}_i(t)|^{\rho-1}\sum_{j=1}^{8N}|\check{g}_{ij}|\check{l}_j\int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}_j(s)|ds+\phi\omega_i|\check{\mathfrak{r}}_i(t)|^{\rho}-v_1\omega_i|\check{\mathfrak{r}}_i(t-\pi(t))|^{\rho}-v_2\omega_i\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_i(t+\varsigma)|^{\rho}\Bigg) \\
\le\ &\sum_{i=1}^{8N}\Bigg(-\omega_i\rho(\check{c}_i+\check{k}_{i1})|\check{\mathfrak{r}}_i(t)|^{\rho}+\omega_i\rho\check{k}_{i2}|\check{\mathfrak{r}}_i(t)|^{\rho-1}|\check{\mathfrak{r}}_i(t-\pi(t))|+\omega_i\rho\check{k}_{i3}\varepsilon|\check{\mathfrak{r}}_i(t)|^{\rho-1}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_i(t+\varsigma)| \\
&+\omega_i\rho\sum_{j=1}^{8N}|\check{a}_{ij}|\check{l}_j|\check{\mathfrak{r}}_i(t)|^{\rho-1}|\check{\mathfrak{r}}_j(t)|+\omega_i\rho\sum_{j=1}^{8N}|\check{b}_{ij}|\check{l}_j|\check{\mathfrak{r}}_i(t)|^{\rho-1}|\check{\mathfrak{r}}_j(t-\pi(t))| \\
&+\omega_i\rho\sum_{j=1}^{8N}|\check{g}_{ij}|\check{l}_j\varepsilon|\check{\mathfrak{r}}_i(t)|^{\rho-1}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_j(t+\varsigma)|^{\rho} \\
&+\phi\omega_i|\check{\mathfrak{r}}_i(t)|^{\rho}-v_1\omega_i|\check{\mathfrak{r}}_i(t-\pi(t))|^{\rho}-v_2\omega_i\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_i(t+\varsigma)|^{\rho}\Bigg). \qquad (17)
\end{aligned}
$$

The application of Lemma 5 yields that:

$$
\begin{aligned}
|\check{\mathfrak{r}}_i(t)|^{\rho-1}|\check{\mathfrak{r}}_i(t-\pi(t))| &\le \frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}|\check{\mathfrak{r}}_i(t-\pi(t))|^{\rho}, \\
|\check{\mathfrak{r}}_i(t)|^{\rho-1}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_i(t+\varsigma)| &\le \frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_i(t+\varsigma)|^{\rho}, \\
|\check{\mathfrak{r}}_i(t)|^{\rho-1}|\check{\mathfrak{r}}_j(t)| &\le \frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}|\check{\mathfrak{r}}_j(t)|^{\rho}, \\
|\check{\mathfrak{r}}_i(t)|^{\rho-1}|\check{\mathfrak{r}}_j(t-\pi(t))| &\le \frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}|\check{\mathfrak{r}}_j(t-\pi(t))|^{\rho}, \\
|\check{\mathfrak{r}}_i(t)|^{\rho-1}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_j(t+\varsigma)| &\le \frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_j(t+\varsigma)|^{\rho},
\end{aligned}
$$

$\forall i,j\in\{1,\ldots,8N\}, \forall t\in[0,+\infty)$.

With these inequalities, (17) has the following form:

$$
D_0^{\alpha}V(t)+\phi V(t)-\sum_{h=1}^{m}v_h\sup_{-\rho_h\le\varsigma\le 0}V(t+\varsigma)
$$

$$
\begin{aligned}
\le\ &\sum_{i=1}^{8N}\Bigg(-\omega_i\rho(\check{c}_i+\check{k}_{i1})|\check{\mathfrak{r}}_i(t)|^{\rho}+\omega_i\rho\check{k}_{i2}\left[\frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}|\check{\mathfrak{r}}_i(t-\pi(t))|^{\rho}\right] \\
&+\omega_i\rho\check{k}_{i3}\varepsilon\left[\frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_i(t+\varsigma)|^{\rho}\right]+\omega_i\rho\sum_{j=1}^{8N}|\check{a}_{ij}|\check{l}_j\left[\frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}|\check{\mathfrak{r}}_j(t)|^{\rho}\right] \\
&+\omega_i\rho\sum_{j=1}^{8N}|\check{b}_{ij}|\check{l}_j\left[\frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}|\check{\mathfrak{r}}_j(t-\pi(t))|^{\rho}\right]+\omega_i\rho\sum_{j=1}^{8N}|\check{g}_{ij}|\check{l}_j\varepsilon\left[\frac{\rho-1}{\rho}|\check{\mathfrak{r}}_i(t)|^{\rho}+\frac{1}{\rho}\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_j(t+\varsigma)|^{\rho}\right] \\
&+\phi\omega_i|\check{\mathfrak{r}}_i(t)|^{\rho}-v_1\omega_i|\check{\mathfrak{r}}_i(t-\pi(t))|^{\rho}-v_2\omega_i\sup_{-\varepsilon\le\varsigma\le 0}|\check{\mathfrak{r}}_i(t+\varsigma)|^{\rho}\Bigg) \\
=\ &\sum_{i=1}^{8N}\Bigg(\Bigg[-\omega_i\rho(\check{c}_i+\check{k}_{i1})+\omega_i(\rho-1)\check{k}_{i2}+\omega_i(\rho-1)\check{k}_{i3}\varepsilon+\omega_i(\rho-1)\sum_{j=1}^{8N}|\check{a}_{ij}|\check{l}_j+\check{l}_i\sum_{j=1}^{8N}|\check{a}_{ji}|\omega_j \\
&+\omega_i(\rho-1)\sum_{j=1}^{8N}|\check{b}_{ij}|\check{l}_j+\omega_i(\rho-1)\sum_{j=1}^{8N}|\check{g}_{ij}|\check{l}_j\varepsilon+\phi\omega_i\Bigg]|\check{\mathfrak{r}}_i(t)|^{\rho}
\end{aligned}
$$

$$+\left[\omega_i\check{k}_{i2}+\check{l}_i\sum_{j=1}^{8N}|\check{b}_{ji}|\omega_j-v_1\omega_i\right]|\check{\mathfrak{r}}_i(t-\pi(t))|^{\rho}+\left[\omega_i\check{k}_{i3}\varepsilon+\check{l}_i\sum_{j=1}^{8N}|\check{g}_{ji}|\omega_j\varepsilon-v_2\omega_i\right]\sup_{-\varepsilon\leq\varsigma\leq 0}|\check{\mathfrak{r}}_i(t+\varsigma)|^{\rho}\Bigg).$$

Conditions (16) allow us to conclude that:

$$D_0^{\alpha}V(t)+\phi V(t)-\sum_{h=1}^{m}v_h\sup_{-\rho_h\leq\varsigma\leq 0}V(t+\varsigma)\leq 0.$$

The conclusion $\lim_{t\to\infty}V(t)=0$ is drawn by applying Lemma 2. From $(\min_i\omega_i)\sum_{i=1}^{8N}|\check{\mathfrak{r}}_i(t)|^{\rho}\leq\sum_{i=1}^{8N}\omega_i|\check{\mathfrak{r}}_i(t)|^{\rho}=V(t)$, we deduce that Systems (1) and (2) are asymptotically synchronized on the basis of Control Scheme (5), which is what we needed to prove. □

Remark 1. *Theorems 1 and 2 give sufficient conditions expressed in terms of LMIs and algebraic inequalities, respectively, for the asymptotic synchronization of FOOVNNs with neutral-type, time-varying, and distributed delays. The asymptotic synchronization of FOCVNNs with different types of delays was discussed in the literature, for example, in [37,46,47,49,64,65]. The same property was studied for FOQVNNs with different types of delays, for example, in [44,50,66]. The asymptotic synchronization of delayed FOOVNNs was never before discussed in the existing literature, to the best of our knowledge. As such, our model is more general than the ones presented in the existing research, and thus, the results obtained are not directly comparable with the available results. However, our obtained results can be particularized for delayed FOCVNNs or FOQVNNs.*

Theorem 3. *Systems (1) and (2) are Mittag–Leffler synchronized under Control Scheme (5) if the subsequent LMI is true:*

$$\Omega:=\left(\Omega_{i,j}\right)_{1\leq i,j\leq 8}<0,\tag{18}$$

where $\Omega_{1,1}=(2+\phi)P+\check{L}^TR_1\check{L}-P(\check{C}+\check{K}_1)-(\check{C}+\check{K}_1)P$, $\Omega_{1,2}=-(\check{C}+\check{K}_1)N_1^T$, $\Omega_{1,3}=-P\check{K}_2-(\check{C}+\check{K}_1)N_2^T$, $\Omega_{1,4}=P\check{A}+(\check{C}+\check{K}_1)N_3^T$, $\Omega_{1,5}=P\check{B}+(\check{C}+\check{K}_1)N_4^T$, $\Omega_{1,6}=-P\check{K}_3-(\check{C}+\check{K}_1)N_5^T$, $\Omega_{1,7}=(\check{C}+\check{K}_1)N_6^T$, $\Omega_{1,8}=(\check{C}+\check{K}_1)N_7^T$, $\Omega_{2,2}=-N_1-N_1^T$, $\Omega_{2,3}=-N_1\check{K}_2-N_2^T$, $\Omega_{2,4}=N_1\check{A}+N_3^T$, $\Omega_{2,5}=N_1\check{B}+N_4^T$, $\Omega_{2,6}=-N_1\check{K}_3-N_5^T$, $\Omega_{2,7}=N_1\check{G}+N_6^T$, $\Omega_{2,8}=N_1(\check{H}-\check{K}_4)+N_7^T$, $\Omega_{3,3}=-v_1P+\check{L}^TR_2\check{L}-N_2\check{K}_3-\check{K}_3N_2^T$, $\Omega_{3,4}=N_2\check{A}+\check{K}_2N_3^T$, $\Omega_{3,5}=N_2\check{B}+\check{K}_2N_4^T$, $\Omega_{3,6}=-N_2\check{K}_3-\check{K}_2N_5^T$, $\Omega_{3,7}=N_2\check{G}+\check{K}_2N_6^T$, $\Omega_{3,8}=N_2(\check{H}-\check{K}_4)+\check{K}_2N_7^T$, $\Omega_{4,4}=-R_1-N_3\check{A}-\check{A}^TN_3^T$, $\Omega_{4,5}=-N_3\check{B}-\check{A}^TN_4^T$, $\Omega_{4,6}=N_3\check{K}_3+\check{A}^TN_5^T$, $\Omega_{4,7}=-N_3\check{G}-\check{A}^TN_6^T$, $\Omega_{4,8}=-N_3(\check{H}-\check{K}_4)-\check{A}^TN_7^T$, $\Omega_{5,5}=-R_2-N_4\check{B}-\check{B}^TN_4^T$, $\Omega_{5,6}=N_4\check{K}_3+\check{B}^TN_5^T$, $\Omega_{5,7}=-N_4\check{G}-\check{B}^TN_6^T$, $\Omega_{5,8}=-N_4(\check{H}-\check{K}_4)-\check{B}^TN_7^T$, $\Omega_{6,6}=-N_5\check{K}_3-\check{K}_3N_5^T$, $\Omega_{6,7}=N_5\check{G}+\check{K}_3N_6^T$, $\Omega_{6,8}=N_5(\check{H}-\check{K}_4)+\check{K}_3N_7^T$, $\Omega_{7,7}=\check{G}^TP\check{G}-N_6\check{G}-\check{G}^TN_6^T$, $\Omega_{7,8}=-N_6(\check{H}-\check{K}_4)-\check{G}^TN_7^T$, $\Omega_{8,8}=(\check{H}-\check{K}_4)^TP(\check{H}-\check{K}_4)-N_7(\check{H}-\check{K}_4)-(\check{H}-\check{K}_4)^TN_7^T$, *and* $P\in\mathbb{R}^{8N\times 8N}$ *is a positive definite matrix,* R_1, $R_2\in\mathbb{R}^{8N\times 8N}$ *are diagonal positive definite matrices,* $N_1,\ldots,N_7\in\mathbb{R}^{8N\times 8N}$ *are any matrices, and* ϕ *is a positive real number.*

Proof. Define the following Lyapunov-like functional:

$$V(t)=\check{\mathfrak{r}}^T(t)P\check{\mathfrak{r}}(t).$$

Using Lemma 1 and taking the fractional-order derivative of the function defined above, along the trajectories of System (7), we obtain:

$$\begin{aligned}
&D_0^{\alpha}V(t)+\phi V(t)\\
\leq\;&\check{\mathfrak{r}}(t)^TPD_0^{\alpha}\check{\mathfrak{r}}(t)+D_0^{\alpha}\check{\mathfrak{r}}(t)^TP\check{\mathfrak{r}}(t)+\phi\check{\mathfrak{r}}(t)^TP\check{\mathfrak{r}}(t)\\
=\;&\check{\mathfrak{r}}(t)^TP\Bigg[-(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t)-\check{K}_2\check{\mathfrak{r}}(t-\pi(t))-\check{K}_3\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds+\check{A}\check{f}(\check{\mathfrak{r}}(t))\\
&+\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t)))+\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds+(\check{H}-\check{K}_4)D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+\Big[-(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t)-\check{K}_2\check{\mathfrak{r}}(t-\pi(t))-\check{K}_3\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds+\check{A}\check{f}(\check{\mathfrak{r}}(t))\\
&+\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t)))+\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds+(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)\Big]^T P\check{\mathfrak{r}}(t)\\
&+\phi\check{\mathfrak{r}}(t)^T P\check{\mathfrak{r}}(t)\\
= \;&-\check{\mathfrak{r}}(t)^T P(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t)-\check{\mathfrak{r}}(t)^T P\check{K}_2\check{\mathfrak{r}}(t-\pi(t))-\check{\mathfrak{r}}(t)^T P\check{K}_3\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds+\check{\mathfrak{r}}(t)^T P\check{A}\check{f}(\check{\mathfrak{r}}(t))\\
&+\check{\mathfrak{r}}(t)^T P\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t)))+\check{\mathfrak{r}}(t)^T P\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds+\check{\mathfrak{r}}(t)^T P(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)\\
&-\check{\mathfrak{r}}(t)^T(\check{C}+\check{K}_1)P\check{\mathfrak{r}}(t)-\check{\mathfrak{r}}(t-\pi(t))^T\check{K}_2P\check{\mathfrak{r}}(t)-\left(\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds\right)^T\check{K}_3P\check{\mathfrak{r}}(t)+\check{f}(\check{\mathfrak{r}}(t))^T\check{A}^TP\check{\mathfrak{r}}(t)\\
&+\check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T\check{B}^TP\check{\mathfrak{r}}(t)+\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)^T\check{G}^TP\check{\mathfrak{r}}(t)+D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T(\check{H}-\check{K}_4)P\check{\mathfrak{r}}(t)\\
&+\phi\check{\mathfrak{r}}(t)^TP\check{\mathfrak{r}}(t).
\end{aligned}
\tag{19}
$$

If we take $Q=P^{-1}$ in Lemma 3, we obtain that:

$$
\begin{aligned}
&\check{\mathfrak{r}}(t)^TP\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds+\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)^T\check{G}^TP\check{\mathfrak{r}}(t)\\
\leq\;&\check{\mathfrak{r}}(t)^TPP^{-1}P\check{\mathfrak{r}}(t)+\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)^T\check{G}^TP\check{G}\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\check{\mathfrak{r}}(t)^TP(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)+D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T(\check{H}-\check{K}_4)P\check{\mathfrak{r}}(t)\\
\leq\;&\check{\mathfrak{r}}(t)^TPP^{-1}P\check{\mathfrak{r}}(t)+D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T(\check{H}-\check{K}_4)^TP(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma).
\end{aligned}
\tag{21}
$$

Now, Inequality (19) becomes:

$$
\begin{aligned}
&D^{\alpha}_{0}V(t)+\phi V(t)\\
\leq\;&-\check{\mathfrak{r}}(t)^TP(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t)-\check{\mathfrak{r}}(t)^T(\check{C}+\check{K}_1)P\check{\mathfrak{r}}(t)-\check{\mathfrak{r}}(t)^TP\check{K}_2\check{\mathfrak{r}}(t-\pi(t))-\check{\mathfrak{r}}(t-\pi(t))^T\check{K}_2P\check{\mathfrak{r}}(t)\\
&-\check{\mathfrak{r}}(t)^TP\check{K}_3\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds-\left(\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds\right)^T\check{K}_3P\check{\mathfrak{r}}(t)+\check{\mathfrak{r}}(t)^TP\check{A}\check{f}(\check{\mathfrak{r}}(t))+\check{f}(\check{\mathfrak{r}}(t))^T\check{A}^TP\check{\mathfrak{r}}(t)\\
&+\check{\mathfrak{r}}(t)^TP\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t)))+\check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T\check{B}^TP\check{\mathfrak{r}}(t)\\
&+\check{\mathfrak{r}}(t)^TP\check{\mathfrak{r}}(t)+\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)^T\check{G}^TP\check{G}\left(\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds\right)\\
&+\check{\mathfrak{r}}(t)^TP\check{\mathfrak{r}}(t)+D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)^T(\check{H}-\check{K}_4)^TP(\check{H}-\check{K}_4)D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)\\
&+\phi\check{\mathfrak{r}}(t)^TP\check{\mathfrak{r}}(t).
\end{aligned}
\tag{22}
$$

Assumption 1 allows us to ascertain the existence of positive definite diagonal matrices R_1 and R_2 for which the following inequalities are valid $\forall t\in[0,+\infty)$:

$$
0\leq\check{\mathfrak{r}}(t)^T\check{L}^TR_1\check{L}\check{\mathfrak{r}}(t)-\check{f}(\check{\mathfrak{r}}(t))^TR_1\check{f}(\check{\mathfrak{r}}(t)),
\tag{23}
$$

$$
0\leq\check{\mathfrak{r}}(t-\pi(t))^T\check{L}^TR_2\check{L}\check{\mathfrak{r}}(t-\pi(t))-\check{f}(\check{\mathfrak{r}}(t-\pi(t)))^TR_2\check{f}(\check{\mathfrak{r}}(t-\pi(t))).
\tag{24}
$$

On the other hand, there exist any matrices $N_1,\ N_2,\ N_3,\ N_4,\ N_5,\ N_6,\ N_7$ such that the subsequent identity is true:

$$
\begin{aligned}
0 \;=\;& \Bigg[D_0^\alpha \check{\mathfrak{r}}(t)^T N_1 + \check{\mathfrak{r}}(t-\pi(t))^T N_2 - \check{f}(\check{\mathfrak{r}}(t))^T N_3 - \check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T N_4 + \left(\int_{t-\varepsilon}^t \check{\mathfrak{r}}(s)ds\right)^T N_5 \\
& - \left(\int_{t-\varepsilon}^t \check{f}(\check{\mathfrak{r}}(s))ds\right)^T N_6 - D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma)^T N_7\Bigg] \times \Big[-D_0^\alpha \check{\mathfrak{r}}(t) - (\check{C}+\check{K}_1)\check{\mathfrak{r}}(t) \\
& - \check{K}_2\check{\mathfrak{r}}(t-\pi(t)) - \check{K}_3\int_{t-\varepsilon}^t \check{\mathfrak{r}}(s)ds + \check{A}\check{f}(\check{\mathfrak{r}}(t)) + \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) + \check{G}\int_{t-\varepsilon}^t \check{f}(\check{\mathfrak{r}}(s))ds \\
& + (\check{H}-\check{K}_4)D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma)\Big].
\end{aligned} \tag{25}
$$

By taking the transpose of Identity (25) and adding them both, as well as Inequalities (23)–(24) to (22), we have that:

$$D_0^\alpha V(t) + \phi V(t) \le \zeta^T(t)\Omega\zeta(t),$$

where Ω is given in (18), and

$$
\begin{aligned}
\zeta(t) \;=\;& \Big[\; \check{\mathfrak{r}}(t)^T \quad D_0^\alpha \check{\mathfrak{r}}(t)^T \quad \check{\mathfrak{r}}(t-\pi(t))^T \quad \check{f}(\check{\mathfrak{r}}(t))^T \quad \check{f}(\check{\mathfrak{r}}(t-\pi(t)))^T \\
& \left(\int_{t-\varepsilon}^t \check{\mathfrak{r}}(s)ds\right)^T \quad \left(\int_{t-\varepsilon}^t \check{f}(\check{\mathfrak{r}}(s))ds\right)^T \quad D_{-\gamma}^\alpha \check{\mathfrak{r}}(t-\gamma)^T \;\Big]^T.
\end{aligned}
$$

Condition (18) tells us that $\Omega < 0$, from which we obtain that:

$$D_0^\alpha V(t) + \phi V(t) \le 0.$$

From Lemma 6 we obtain that:

$$V(t) \le V(0)E_\alpha(-\phi t^\alpha),$$

or, equivalently,

$$
\begin{aligned}
\lambda_{\min}(P)||\check{\mathfrak{r}}(t)||^2 &\le \check{\mathfrak{r}}(t)^T P \check{\mathfrak{r}}(t) \\
&\le \check{\mathfrak{r}}(0)^T P \check{\mathfrak{r}}(0) E_\alpha(-\phi t^\alpha) \\
&\le \lambda_{\max}(P)||\check{\mathfrak{r}}(0)||^2 E_\alpha(-\phi t^\alpha) \\
&\le \lambda_{\max}(P)||\check{\psi}||^2 E_\alpha(-\phi t^\alpha),
\end{aligned}
$$

and, finally,

$$||\check{\mathfrak{r}}(t)|| \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}||\check{\psi}||(E_\alpha(-\phi t^\alpha))^{\frac{1}{2}}.$$

This means that Systems (1) and (2) are Mittag–Leffler synchronized on the basis of Control Scheme (5), which is what we needed to prove. □

We will now employ a different state feedback controller to obtain another condition for the Mittag–Leffler synchronization between Drive System (1) and Response System (2), for which the controller will be given by:

$$u_i(t) = k_{i1}\mathfrak{r}_i(t) + k_{i2}\mathrm{sign}(\mathfrak{r}_i(t))|\mathfrak{r}_i(t-\pi(t))| + k_{i3}\mathrm{sign}(\mathfrak{r}_i(t))\int_{t-\varepsilon}^t |\mathfrak{r}_i(s)|ds + k_{i4}\mathrm{sign}(\mathfrak{r}_i(t))\left|D_{-\gamma}^\alpha \mathfrak{r}_i(t-\gamma)\right|, \tag{26}$$

where $k_{i1},\ k_{i2},\ k_{i3},\ k_{i4} \in \mathbb{R}^+, \forall i \in \{1,\dots,N\}$, represent the control gain parameters. In this case, System (3) will have the form:

$$
\begin{aligned}
D_0^\alpha \mathfrak{r}_i(t) \ = \ & -(c_i + k_{i1})\mathfrak{r}_i(t) - k_{i2}\text{sign}(\mathfrak{r}_i(t))|\mathfrak{r}_i(t-\pi(t))| - k_{i3}\text{sign}(\mathfrak{r}_i(t))\int_{t-\varepsilon}^{t}|\mathfrak{r}_i(s)|ds \\
& -k_{i4}\text{sign}(\mathfrak{r}_i(t))\left|D_{-\gamma}^{\alpha}\mathfrak{r}_i(t-\gamma)\right| + \sum_{j=1}^{N} a_{ij}\tilde{f}_j(\mathfrak{r}_j(t)) + \sum_{j=1}^{N} b_{ij}\tilde{f}_j(\mathfrak{r}_j(t-\pi(t))) \\
& + \sum_{j=1}^{N} g_{ij}\int_{t-\varepsilon}^{t}\tilde{f}_j(\mathfrak{r}_j(s))ds + h_i D_{-\gamma}^{\alpha}\mathfrak{r}_i(t-\gamma),
\end{aligned} \tag{27}
$$

$\forall i \in \{1,\ldots,N\}$.

In matrix form, we can write System (27) as:

$$
\begin{aligned}
D_0^\alpha \check{\mathfrak{r}}(t) \ = \ & -(\check{C} + \check{K}_1)\check{\mathfrak{r}}(t) - \check{K}_2\text{sign}(\check{\mathfrak{r}}(t)) \odot |\check{\mathfrak{r}}(t-\pi(t))| - \check{K}_3\text{sign}(\check{\mathfrak{r}}(t)) \odot \int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}(s)|ds \\
& -\check{K}_4\text{sign}(\check{\mathfrak{r}}(t)) \odot \left|D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma)\right| + \check{A}\check{f}(\check{\mathfrak{r}}(t)) + \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) \\
& +\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds + \check{H}D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma),
\end{aligned} \tag{28}
$$

where $\check{K}_1 := \text{diag}(k_{11}I_8, k_{21}I_8, \ldots, k_{N1}I_8) \in \mathbb{R}^{8N\times 8N}$, $\check{K}_2 := \text{diag}(k_{12}I_8, k_{22}I_8, \ldots, k_{N2}I_8) \in \mathbb{R}^{8N\times 8N}$,

$\check{K}_3 := \text{diag}(k_{13}I_8, k_{23}I_8, \ldots, k_{N3}I_8) \in \mathbb{R}^{8N\times 8N}$, $\check{K}_4 := \text{diag}(k_{14}I_8, k_{24}I_8, \ldots, k_{N4}I_8) \in \mathbb{R}^{8N\times 8N}$, and $\odot$ represents the elementwise vector product.

Theorem 4. *Systems (1) and (2) are Mittag–Leffler synchronized under Control Scheme (27) if the subsequent LMIs are true:*

$$
\check{C} + \check{K}_1 - |\check{A}|\check{L} > 0,\ \check{K}_2 - |\check{B}|\check{L} > 0,\ \check{K}_3 - |\check{G}|\check{L} > 0,\ \check{K}_4 - |\check{H}| > 0. \tag{29}
$$

Proof. Define the following Lyapunov-like functional:

$$
V(t) = \frac{1}{2}\check{\mathfrak{r}}(t)^T\check{\mathfrak{r}}(t).
$$

Using Lemma 1 and taking the fractional-order derivative of the function defined above, along the trajectories of System (28), we obtain:

$$
\begin{aligned}
D_0^\alpha V(t) \ = \ & D_0^\alpha\left(\frac{1}{2}\check{\mathfrak{r}}(t)^T\check{\mathfrak{r}}(t)\right) \\
\leq \ & \check{\mathfrak{r}}(t)^T D_0^\alpha \check{\mathfrak{r}}(t) \\
= \ & \check{\mathfrak{r}}(t)^T\Big[-(\check{C} + \check{K}_1)\check{\mathfrak{r}}(t) - \check{K}_2\text{sign}(\check{\mathfrak{r}}(t)) \odot |\check{\mathfrak{r}}(t-\pi(t))| - \check{K}_3\text{sign}(\check{\mathfrak{r}}(t)) \odot \int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}(s)|ds \\
& -\check{K}_4\text{sign}(\check{\mathfrak{r}}(t)) \odot \left|D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma)\right| + \check{A}\check{f}(\check{\mathfrak{r}}(t)) + \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) \\
& +\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds + \check{H}D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma)\Big] \\
= \ & -\check{\mathfrak{r}}(t)^T(\check{C} + \check{K}_1)\check{\mathfrak{r}}(t) - |\check{\mathfrak{r}}(t)|^T\check{K}_2|\check{\mathfrak{r}}(t-\pi(t))| - |\check{\mathfrak{r}}(t)|^T\check{K}_3\int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}(s)|ds \\
& -|\check{\mathfrak{r}}(t)|^T\check{K}_4\left|D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma)\right| + \check{\mathfrak{r}}(t)^T\check{A}\check{f}(\check{\mathfrak{r}}(t)) + \check{\mathfrak{r}}(t)^T\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) + \check{\mathfrak{r}}(t)^T\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds \\
& +\check{\mathfrak{r}}(t)^T\check{H}D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma) \\
\leq \ & -|\check{\mathfrak{r}}(t)|^T(\check{C} + \check{K}_1)|\check{\mathfrak{r}}(t)| - |\check{\mathfrak{r}}(t)|^T\check{K}_2|\check{\mathfrak{r}}(t-\pi(t))| - |\check{\mathfrak{r}}(t)|^T\check{K}_3\int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}(s)|ds \\
& -|\check{\mathfrak{r}}(t)|^T\check{K}_4\left|D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma)\right| + |\check{\mathfrak{r}}(t)|^T|\check{A}|\check{L}|\check{\mathfrak{r}}(t)| + |\check{\mathfrak{r}}(t)|^T|\check{B}|\check{L}|\check{\mathfrak{r}}(t-\pi(t))|
\end{aligned}
$$

$$
\begin{aligned}
& +|\check{\mathfrak{r}}(t)|^T|\check{G}|\check{L}\int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}(s)|ds+|\check{\mathfrak{r}}(t)|^T|\check{H}|\left|D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)\right| \\
= \;& |\check{\mathfrak{r}}(t)|^T(-(\check{C}+\check{K}_1)+|\check{A}|\check{L})|\check{\mathfrak{r}}(t)|+|\check{\mathfrak{r}}(t)|^T(-\check{K}_2+|\check{B}|\check{L})|\check{\mathfrak{r}}(t-\pi(t))| \\
& +|\check{\mathfrak{r}}(t)|^T(-\check{K}_3+|\check{G}|\check{L})\int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}(s)|ds+|\check{\mathfrak{r}}(t)|^T(-\check{K}_4+|\check{H}|)\left|D^{\alpha}_{-\gamma}\check{\mathfrak{r}}(t-\gamma)\right|.
\end{aligned} \tag{30}
$$

Using Conditions (29), Inequality (30) becomes:

$$
\begin{aligned}
D_0^{\alpha}V(t) &\leq -|\check{\mathfrak{r}}(t)|^T(\check{C}+\check{K}_1-|\check{A}|L)|\check{\mathfrak{r}}(t)| \\
&\leq -\lambda V(t),
\end{aligned}
$$

where $\lambda = 2\lambda_{\min}(\check{C}+\check{K}_1-|\check{A}|L)$. From Lemma 6, we obtain:

$$
V(t) \leq V(0)E_{\alpha}(-\lambda t^{\alpha}),\ \forall t \in [0,+\infty),
$$

which is equivalent with:

$$
\begin{aligned}
\frac{1}{2}||\check{\mathfrak{r}}(t)||^2 &\leq \frac{1}{2}||\check{\mathfrak{r}}(0)||^2E_{\alpha}(-\lambda t^{\alpha}) \\
&\leq \frac{1}{2}||\check{\psi}||^2E_{\alpha}(-\lambda t^{\alpha}),\ \forall t \in [0,+\infty),
\end{aligned}
$$

yielding

$$
||\check{\mathfrak{r}}(t)|| \leq ||\check{\psi}||[E_{\alpha}(-\lambda t^{\alpha})]^{\frac{1}{2}},\ \forall t \in [0,+\infty).
$$

This means that Systems (1) and (2) are Mittag–Leffler synchronized using Control Scheme (27), which is what we needed to prove. □

Remark 2. *Theorems 3 and 4 give sufficient conditions expressed in terms of LMIs using two types of state feedback controllers for the Mittag–Leffler synchronization of FOOVNNs with neutral-type, time-varying, and distributed delays. The Mittag–Leffler synchronization of delayed FOCVNNs was discussed in the literature, for example, in [32,67]. The same property was studied for FOQVNNs with different types of delays, for example, in [29,35,68]. The Mittag–Leffler synchronization of delayed FOOVNNs has never been presented in the literature, to our knowledge. Again, our model is more general than the ones discussed in the existing research, and thus, the results obtained are not directly comparable with the available results, but they can be particularized for FOCVNNs or FOQVNNs with neutral-type, time-varying, and/or distributed delays.*

4. Numerical Examples

In all of the experiments, we take $\alpha = 0.75$.

Example 1. *Define the following two-neuron FOOVNN with neutral-type, time-varying, and distributed delays:*

$$
D_0^{\alpha}o_i(t) = -c_io_i(t)+\sum_{j=1}^{2}a_{ij}f_j(o_j(t))+\sum_{j=1}^{2}b_{ij}f_j(o_j(t-\pi(t)))+\sum_{j=1}^{2}g_{ij}\int_{t-\varepsilon}^{t}f_j(o_j(s))ds+h_iD^{\alpha}_{-\gamma}o_i(t-\gamma)+I_i, \tag{31}
$$

$\forall i \in \{1,2\}$ *and* $\forall t \in [0,+\infty)$.

System (31) will be taken as the drive system, and the response system will be the following system:

$$
D_0^{\alpha}p_i(t) = -c_ip_i(t)+\sum_{j=1}^{2}a_{ij}f_j(p_j(t))+\sum_{j=1}^{2}b_{ij}f_j(p_j(t-\pi(t)))+\sum_{j=1}^{2}g_{ij}\int_{t-\varepsilon}^{t}f_j(p_j(s))ds+h_iD^{\alpha}_{-\gamma}p_i(t-\gamma)+I_i-u_i(t), \tag{32}
$$

$\forall i \in \{1,2\}$ *and* $\forall t \in [0,+\infty)$. *If we denote* $\mathfrak{r}_i(t) = p_i(t)-o_i(t)$, *taking (31) and (32) into account, the expression of the error system is given as:*

$$D_0^\alpha \mathfrak{r}_i(t) = -c_i\mathfrak{r}_i(t) + \sum_{j=1}^{2} a_{ij}\tilde{f}_j(\mathfrak{r}_j(t)) + \sum_{j=1}^{2} b_{ij}\tilde{f}_j(\mathfrak{r}_j(t-\pi(t))) + \sum_{j=1}^{2} g_{ij}\int_{t-\varepsilon}^{t}\tilde{f}_j(\mathfrak{r}_j(s))ds + h_i D_{-\gamma}^\alpha \mathfrak{r}_i(t-\gamma) - u_i(t), \quad (33)$$

$\forall i \in \{1,2\}$ and $\forall t \in [0,+\infty)$, where $\tilde{f}_j(\mathfrak{r}_j(t)) = f_j(\mathfrak{r}_j(t) + o_j(t)) - f_j(o_j(t))$, $\forall t \in [0,+\infty)$ and $\forall j \in \{1,2\}$.

In order to realize synchronization between Systems (31) and (32), we will use the following state feedback controller:

$$u_i(t) = k_{i1}\mathfrak{r}_i(t) + k_{i2}\mathfrak{r}_i(t-\pi(t)) + k_{i3}\int_{t-\varepsilon}^{t}\mathfrak{r}_i(s)ds + k_{i4}D_{-\gamma}^\alpha \mathfrak{r}_i(t-\gamma), \quad (34)$$

and $k_{i1},\ k_{i2},\ k_{i3},\ k_{i4} \in \mathbb{R}^+$, $\forall i \in \{1,2\}$ represent the control gain parameters.

With this controller, System (33) can be written in matrix form as:

$$\begin{aligned} D_0^\alpha \check{\mathfrak{r}}(t) \quad = \quad & -(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t) - \check{K}_2\check{\mathfrak{r}}(t-\pi(t)) - \check{K}_3\int_{t-\varepsilon}^{t}\check{\mathfrak{r}}(s)ds + \check{A}\check{f}(\check{\mathfrak{r}}(t)) \\ & +\check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) + \check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds + (\check{H}-\check{K}_4)D_{-\gamma}^\alpha\check{\mathfrak{r}}(t-\gamma). \end{aligned} \quad (35)$$

The choice of parameters is the following:

$$C = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$a_{11} = -0.7e_0 + 0.9e_1 - 0.2e_2 + 0.4e_3 + 0.2e_4 + 0.8e_5 + 0.3e_6 + 0.9e_7,$$
$$a_{12} = 0.3e_0 + 0.9e_1 - 0.2e_2 - 0.2e_3 + 0.5e_4 + 0.8e_5 + 0.8e_6 - 0.9e_7,$$
$$a_{21} = -0.2e_0 - 0.4e_1 + 0.2e_2 - 0.2e_3 + 0.3e_4 + 0.2e_5 - 0.5e_6 + 0.2e_7,$$
$$a_{22} = 0.4e_0 + 0.3e_1 + 0.1e_2 + 0.4e_3 - 0.2e_4 - 0.8e_5 + 0.8e_6 + 0.9e_7,$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$b_{11} = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$
$$b_{12} = 0.8e_0 + 0.5e_1 + 0.3e_2 - 0.5e_3 + 0.8e_4 + 0.9e_5 - 0.9e_6 + 0.8e_7,$$
$$b_{21} = 0.3e_0 + 0.2e_1 - 0.2e_2 + 0.1e_3 + 0.8e_4 + 0.9e_5 + 0.7e_6 + 0.9e_7,$$
$$b_{22} = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.8e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

$$g_{11} = -0.2e_0 + 0.4e_1 + 0.5e_2 + 0.3e_3 - 0.6e_4 + 0.2e_5 - 0.4e_6 + 0.5e_7,$$
$$g_{12} = 0.3e_0 + 0.5e_1 + 0.2e_2 - 0.5e_3 + 0.3e_4 + 0.1e_5 - 0.2e_6 + 0.3e_7,$$
$$g_{21} = 0.1e_0 + 0.3e_1 - 0.2e_2 + 0.1e_3 + 0.2e_4 + 0.1e_5 + 0.3e_6 + 0.4e_7,$$
$$g_{22} = -0.3e_0 + 0.1e_1 + 0.2e_2 + 0.5e_3 + 0.4e_4 - 0.2e_5 - 0.1e_6 + 0.5e_7,$$

$$H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix},$$

$$h_1 = 0.1e_0 + 0.4e_1 + 0.3e_2 - 0.5e_3 + 0.2e_4 + 0.3e_5 - 0.4e_6 + 0.5e_7,$$
$$h_2 = -0.2e_0 + 0.1e_1 + 0.2e_2 + 0.3e_3 + 0.5e_4 - 0.3e_5 - 0.4e_6 + 0.2e_7,$$
$$f_j(o) = \frac{1}{20\sqrt{2}} \sum_{q=0}^{7} f_j^q(o)e_q = \frac{1}{20\sqrt{2}} \sum_{q=0}^{7} \frac{1}{1+\exp(-o^q)} e_q,\ \forall o \in \mathbb{O},\ \forall j \in \{1,2\},$$

from which we deduce that the activation functions satisfy Assumption 1, and $\check{L} = \begin{bmatrix} 0.025I_8 & 0 \\ 0 & 0.025I_8 \end{bmatrix}$. *The control gain matrices are the following:*

$$K_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},\ K_3 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix},\ K_4 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Additionally, $\pi(t) = 0.4|\cos t|$, $\varepsilon = 0.3$, $\gamma = 0.01$, *hence,* $\pi = 0.4$, $\varsigma = \max\{\pi, \varepsilon, \gamma\} = 0.4$, $\phi = 4$, $v_1 = 2$, $v_2 = 2$, *and* $v_3 = 2$. *Based on the above parameters, we conclude that the hypotheses of Theorem 1 are satisfied, and the LMI is solved to give* $R_1 = diag(1.196I_8, 2.3816I_8)$ *and* $R_2 = diag(2.5154I_8, 2.6213I_8)$ *(in order not to clutter the paper, the values of the other matrices are not provided). Thus, Systems (31) and (32) are asymptotically synchronized using Control Scheme (34).*

Figures 1 and 2 depict the trajectories of octonion states $\check{\mathfrak{r}}_1$ *and* $\check{\mathfrak{r}}_2$ *of System (35), with 8 initial points.*

Example 2. *In the second example, we realize the synchronization between Systems (31) and (32) based on State Feedback Controller (34), but this time with the subsequent parameters:*

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix},$$
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
$$a_{11} = -0.7e_0 + 0.9e_1 - 0.2e_2 + 0.4e_3 + 0.2e_4 + 0.8e_5 + 0.3e_6 + 0.9e_7,$$
$$a_{12} = 0.3e_0 + 0.9e_1 - 0.2e_2 - 0.2e_3 + 0.5e_4 + 0.8e_5 + 0.8e_6 - 0.9e_7,$$
$$a_{21} = -0.2e_0 - 0.4e_1 + 0.2e_2 - 0.2e_3 + 0.3e_4 + 0.2e_5 - 0.5e_6 + 0.2e_7,$$
$$a_{22} = 0.4e_0 + 0.3e_1 + 0.1e_2 + 0.4e_3 - 0.2e_4 - 0.8e_5 + 0.8e_6 + 0.9e_7,$$
$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$
$$b_{11} = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$
$$b_{12} = 0.8e_0 + 0.5e_1 + 0.3e_2 - 0.5e_3 + 0.8e_4 + 0.9e_5 - 0.9e_6 + 0.8e_7,$$
$$b_{21} = 0.3e_0 + 0.2e_1 - 0.2e_2 + 0.1e_3 + 0.8e_4 + 0.9e_5 + 0.7e_6 + 0.9e_7,$$
$$b_{22} = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.8e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$
$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$
$$g_{11} = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$
$$g_{12} = 0.9e_0 + 0.5e_1 + 0.3e_2 - 0.5e_3 + 0.8e_4 + 0.9e_5 - 0.9e_6 + 0.7e_7,$$
$$g_{21} = 0.3e_0 + 0.2e_1 - 0.2e_2 + 0.1e_3 + 0.8e_4 + 0.9e_5 + 0.8e_6 + 0.9e_7,$$
$$g_{22} = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.9e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

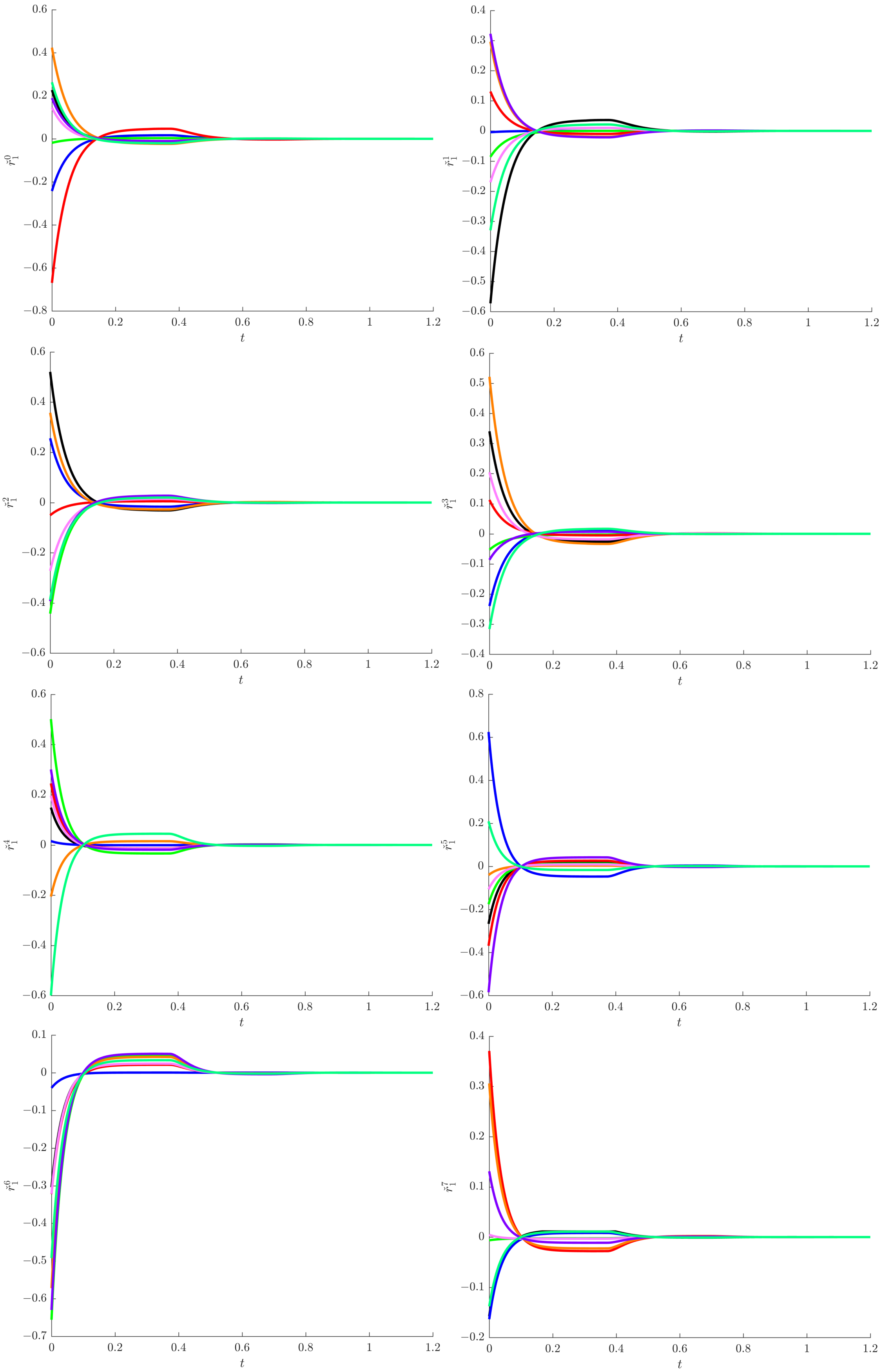

Figure 1. State trajectories of octonion components of $\check{\mathfrak{r}}_1$ in Example 1. Different colors are used for the 8 initial points. The eight graphs depict the components $\check{\mathfrak{r}}_1^q$, $0 \leq q \leq 7$, with respect to time.

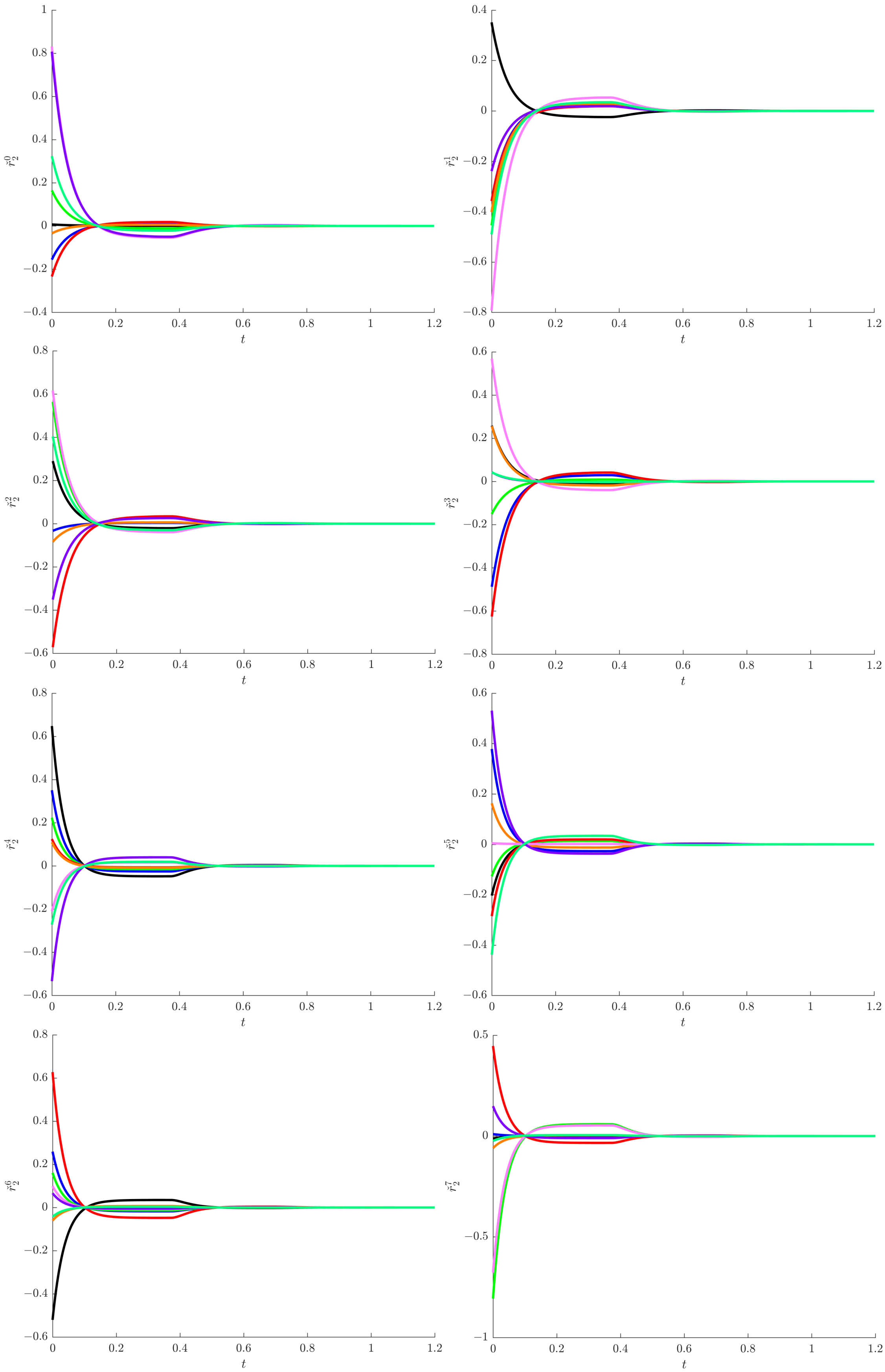

Figure 2. State trajectories of octonion components of $\check{\mathfrak{r}}_2$ in Example 1. Different colors are used for the 8 initial points. The eight graphs depict the components $\check{\mathfrak{r}}_2^q$, $0 \leq q \leq 7$, with respect to time.

$$H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix},$$

$$h_1 = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$

$$h_2 = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.9e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$f_j(o) = \frac{1}{2\sqrt{2}} \sum_{q=0}^{7} f_j^q(o) e_q = \frac{1}{2\sqrt{2}} \sum_{q=0}^{7} \frac{1}{1+\exp(-o^q)} e_q, \ \forall o \in \mathbb{O}, \ \forall j \in \{1,2\},$$

from which we deduce that the activation functions satisfy Assumption 1 and $\check{L} = \begin{bmatrix} 0.25I_8 & 0 \\ 0 & 0.25I_8 \end{bmatrix}$. *The control gain matrices are the following:*

$$K_1 = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}, \ K_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \ K_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Now, we take $\pi(t) = 0.3|\sin t|$, $\varepsilon = 0.05$, $\gamma = 0$ *(no neutral-type delay), from which we have* $\pi = 0.3$, $\varsigma = \max\{\pi, \varepsilon, \gamma\} = 0.3$, $\omega_i = 1$, $\forall i \in \{1, \dots, 16\}$, $\rho = 1.5 > 1$, $\phi = 1.6$, $v_1 = 3.5$, *and* $v_2 = 4.5$. *All the hypotheses of Theorem 2 are now satisfied, thus allowing us to conclude that Drive System (31) and Response System (32) are asymptotically synchronized using Control Scheme (34).*

Example 3. *For the next example, we again study the synchronization between Systems (31) and (32) based on State Feedback Controller (34) with the following parameters:*

$$C = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$a_{11} = -0.7e_0 + 0.9e_1 - 0.2e_2 + 0.4e_3 + 0.2e_4 + 0.8e_5 + 0.3e_6 + 0.9e_7,$$

$$a_{12} = 0.3e_0 + 0.9e_1 - 0.2e_2 - 0.2e_3 + 0.5e_4 + 0.8e_5 + 0.8e_6 - 0.9e_7,$$

$$a_{21} = -0.2e_0 - 0.4e_1 + 0.2e_2 - 0.2e_3 + 0.3e_4 + 0.2e_5 - 0.5e_6 + 0.2e_7,$$

$$a_{22} = 0.4e_0 + 0.3e_1 + 0.1e_2 + 0.4e_3 - 0.2e_4 - 0.8e_5 + 0.8e_6 + 0.9e_7,$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$b_{11} = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$

$$b_{12} = 0.8e_0 + 0.5e_1 + 0.3e_2 - 0.5e_3 + 0.8e_4 + 0.9e_5 - 0.9e_6 + 0.8e_7,$$

$$b_{21} = 0.3e_0 + 0.2e_1 - 0.2e_2 + 0.1e_3 + 0.8e_4 + 0.9e_5 + 0.7e_6 + 0.9e_7,$$

$$b_{22} = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.8e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

$$g_{11} = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$

$$g_{12} = 0.9e_0 + 0.5e_1 + 0.3e_2 - 0.5e_3 + 0.8e_4 + 0.9e_5 - 0.9e_6 + 0.7e_7,$$

$$g_{21} = 0.3e_0 + 0.2e_1 - 0.2e_2 + 0.1e_3 + 0.8e_4 + 0.9e_5 + 0.8e_6 + 0.9e_7,$$

$$g_{22} = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.9e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix},$$

$$h_1 = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$

$$h_2 = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.9e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$f_j(o) = \frac{1}{20\sqrt{2}} \sum_{q=0}^{7} f_j^q(o)e_q = \frac{1}{20\sqrt{2}} \sum_{q=0}^{7} \frac{1}{1+\exp(-o^q)} e_q, \ \forall o \in \mathbb{O}, \ \forall j \in \{1,2\},$$

from which we deduce that the activation functions satisfy Assumption 1 and $\check{L} = \begin{bmatrix} 0.025I_8 & 0 \\ 0 & 0.025I_8 \end{bmatrix}$. *The control gain matrices are the following:*

$$K_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, K_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, K_3 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix}, K_4 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

The delays are taken as $\pi(t) = 0.4|\cos t|$, $\varepsilon = 0.5$, *and* $\gamma = 0.03$, *from which we have* $\pi = 0.4$, $\varsigma = \max\{\pi, \varepsilon, \gamma\} = 0.5$, $\phi = 4.5$, $v_1 = 2.7$, $v_2 = 2.2$, *and* $v_3 = 2.3$. *All the hypotheses of Theorem 3 are now satisfied, and the LMI is solved to give* $R_1 = diag(1.5005I_8, 1.6806I_8)$ *and* $R_2 = diag(1.7819I_8, 1.8129I_8)$ *(in order not to clutter the paper, the values of the other matrices are not provided), thus allowing us to conclude that Drive System (31) and Response System (32) are Mittag–Leffler synchronized using Control Scheme (34).*

Example 4. *For the last example, the same Systems (31) and (32) are considered as drive and response systems, respectively, but this time, a different feedback controller will be used to achieve synchronization:*

$$u_i(t) = k_{i1}\mathfrak{r}_i(t) + k_{i2}sign(\mathfrak{r}_i(t))|\mathfrak{r}_i(t-\pi(t))| + k_{i3}sign(\mathfrak{r}_i(t))\int_{t-\varepsilon}^{t}|\mathfrak{r}_i(s)|ds + k_{i4}sign(\mathfrak{r}_i(t))\left|D_{-\gamma}^{\alpha}\mathfrak{r}_i(t-\gamma)\right|, \quad (36)$$

where $k_{i1}, k_{i2}, k_{i3}, k_{i4} \in \mathbb{R}^+$, $\forall i \in \{1,2\}$ *represent the control gain parameters. With this controller, System (33) can be written in matrix form as:*

$$\begin{aligned} D_0^{\alpha}\check{\mathfrak{r}}(t) \quad = \quad & -(\check{C}+\check{K}_1)\check{\mathfrak{r}}(t) - \check{K}_2 sign(\check{\mathfrak{r}}(t)) \odot |\check{\mathfrak{r}}(t-\pi(t))| - \check{K}_3 sign(\check{\mathfrak{r}}(t)) \odot \int_{t-\varepsilon}^{t}|\check{\mathfrak{r}}(s)|ds \\ & -\check{K}_4 sign(\check{\mathfrak{r}}(t)) \odot \left|D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma)\right| + \check{A}\check{f}(\check{\mathfrak{r}}(t)) + \check{B}\check{f}(\check{\mathfrak{r}}(t-\pi(t))) \\ & +\check{G}\int_{t-\varepsilon}^{t}\check{f}(\check{\mathfrak{r}}(s))ds + \check{H}D_{-\gamma}^{\alpha}\check{\mathfrak{r}}(t-\gamma). \end{aligned} \quad (37)$$

The parameters will now be the following:

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$a_{11} = -0.7e_0 + 0.9e_1 - 0.2e_2 + 0.4e_3 + 0.2e_4 + 0.8e_5 + 0.3e_6 + 0.9e_7,$$

$$a_{12} = 0.3e_0 + 0.9e_1 - 0.2e_2 - 0.2e_3 + 0.5e_4 + 0.8e_5 + 0.8e_6 - 0.9e_7,$$

$$a_{21} = -0.2e_0 - 0.4e_1 + 0.2e_2 - 0.2e_3 + 0.3e_4 + 0.2e_5 - 0.5e_6 + 0.2e_7,$$

$$a_{22} = 0.4e_0 + 0.3e_1 + 0.1e_2 + 0.4e_3 - 0.2e_4 - 0.8e_5 + 0.8e_6 + 0.9e_7,$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$b_{11} = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$

$$b_{12} = 0.8e_0 + 0.5e_1 + 0.3e_2 - 0.5e_3 + 0.8e_4 + 0.9e_5 - 0.9e_6 + 0.8e_7,$$

$$b_{21} = 0.3e_0 + 0.2e_1 - 0.2e_2 + 0.1e_3 + 0.8e_4 + 0.9e_5 + 0.7e_6 + 0.9e_7,$$

$$b_{22} = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.8e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

$$g_{11} = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$

$$g_{12} = 0.9e_0 + 0.5e_1 + 0.3e_2 - 0.5e_3 + 0.8e_4 + 0.9e_5 - 0.9e_6 + 0.7e_7,$$

$$g_{21} = 0.3e_0 + 0.2e_1 - 0.2e_2 + 0.1e_3 + 0.8e_4 + 0.9e_5 + 0.8e_6 + 0.9e_7,$$

$$g_{22} = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.9e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix},$$

$$h_1 = -0.4e_0 + 0.7e_1 + 0.2e_2 + 0.5e_3 - 0.9e_4 + 0.9e_5 - 0.8e_6 + 0.9e_7,$$

$$h_2 = -0.5e_0 + 0.5e_1 + 0.2e_2 + 0.4e_3 + 0.9e_4 - 0.9e_5 - 0.8e_6 + 0.7e_7,$$

$$f_j(o) = \frac{1}{2\sqrt{2}} \sum_{q=0}^{7} f_j^q(o) e_q = \frac{1}{2\sqrt{2}} \sum_{q=0}^{7} \frac{1}{1 + \exp(-o^q)} e_q,\ \forall o \in \mathbb{O},\ \forall j \in \{1, 2\},$$

from which we deduce that the activation functions satisfy Assumption 1 and $\check{L} = \begin{bmatrix} 0.25I_8 & 0 \\ 0 & 0.25I_8 \end{bmatrix}$. *The control gain matrices are the following:*

$$K_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},\ K_2 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix},\ K_3 = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix},\ K_4 = \begin{bmatrix} 7 & 0 \\ 0 & 5 \end{bmatrix}.$$

The delays are $\pi(t) = 0.3|\sin t|$, $\varepsilon = 0.4$, $\gamma = 0.02$, *so* $\pi = 0.3$ *and* $\varsigma = \max\{\pi, \varepsilon, \gamma\} = 0.4$. *The LMI hypotheses of Theorem 4 are easily verified to be true; thus, we can conclude that Systems (31) and (32) are Mittag–Leffler synchronized using Control Scheme (36).*

Figures 3 and 4 depict the trajectories of octonion states $\check{\mathfrak{r}}_1$ *and* $\check{\mathfrak{r}}_2$ *of System (37), with 8 initial points.*

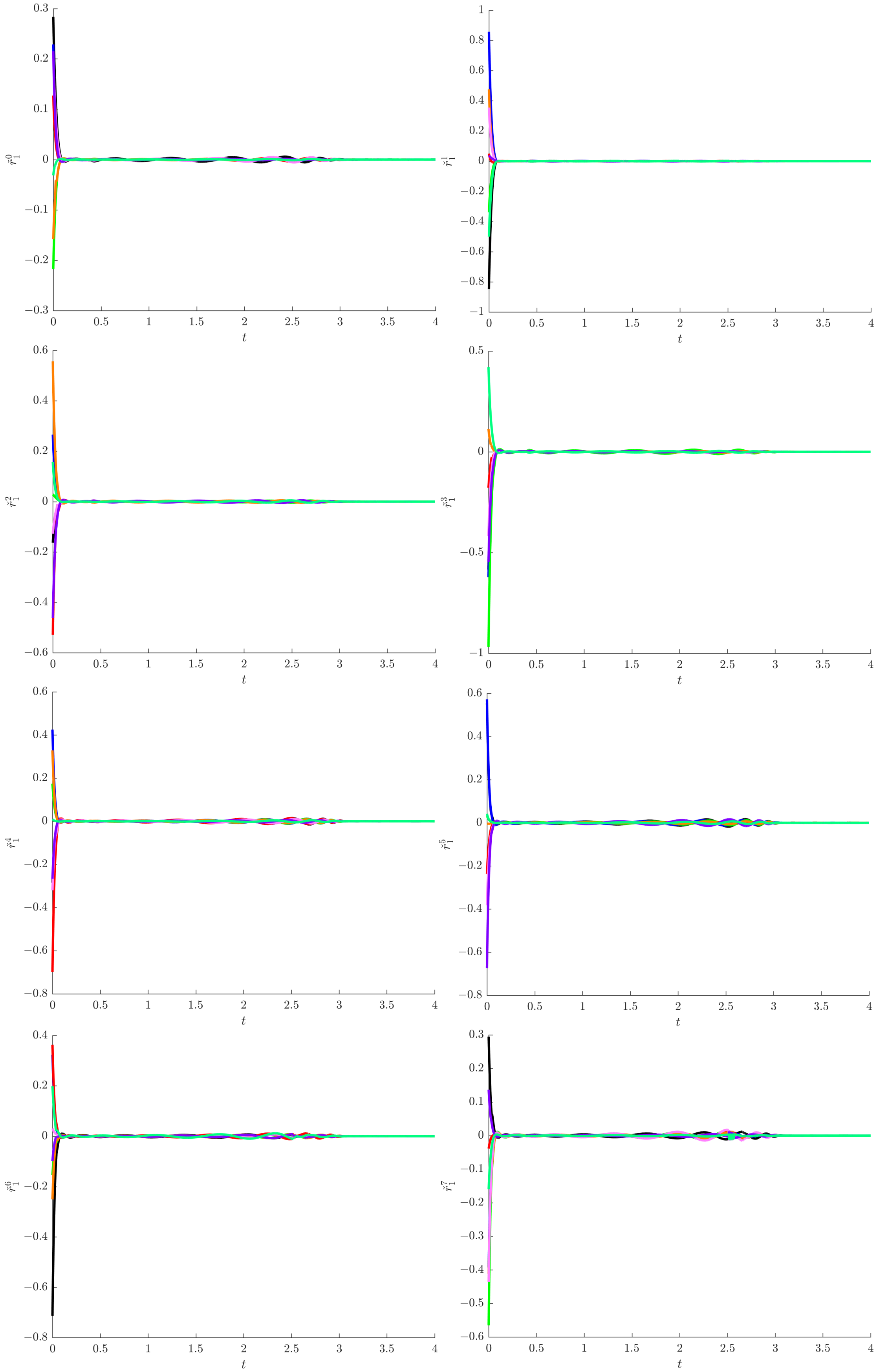

Figure 3. State trajectories of octonion components of $\check{\mathfrak{r}}_1$ in Example 4. Different colors are used for the 8 initial points. The eight graphs depict the components $\check{\mathfrak{r}}_1^q$, $0 \le q \le 7$, with respect to time.

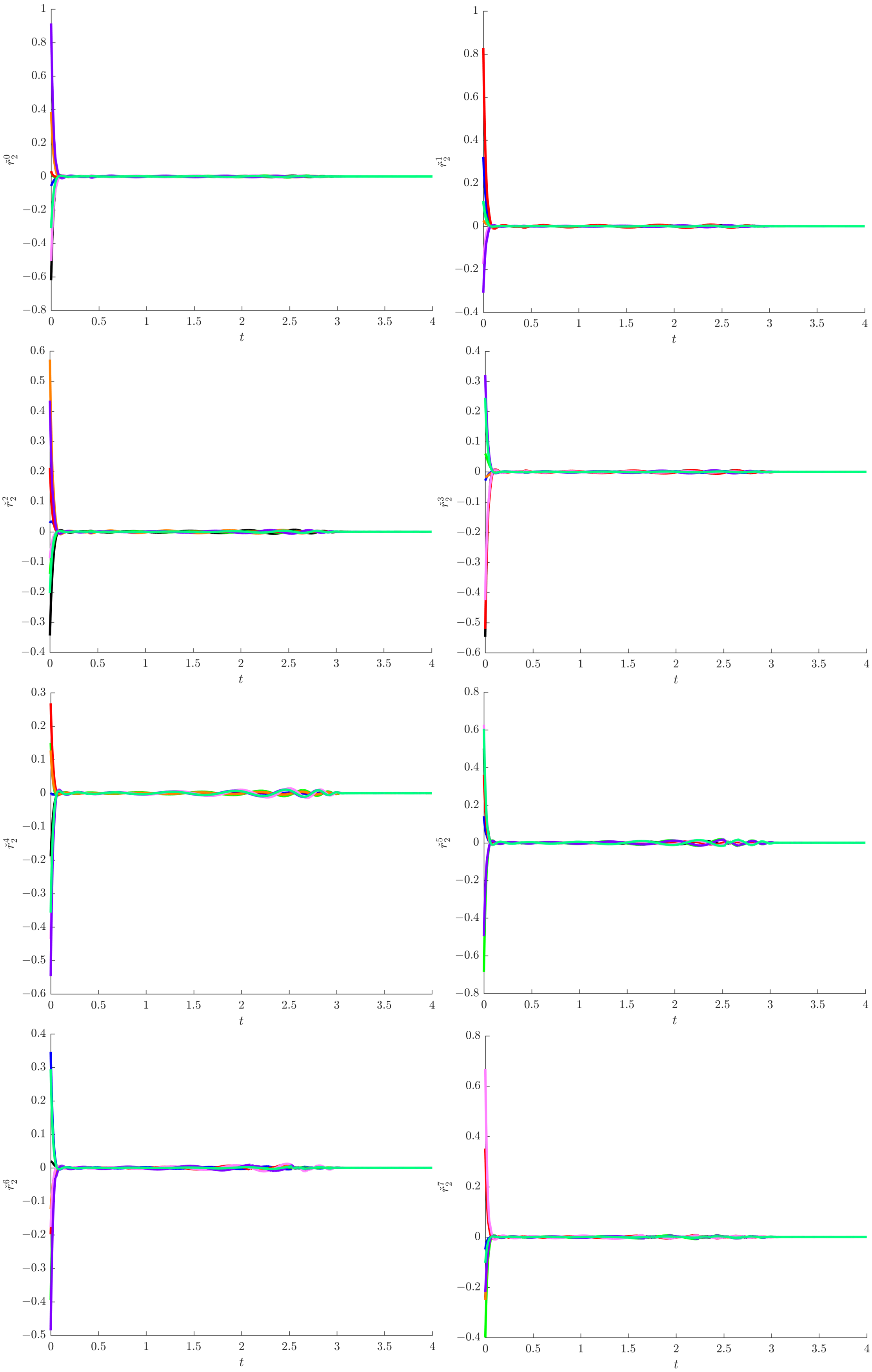

Figure 4. State trajectories of octonion components of $\check{\mathfrak{r}}_2$ in Example 4. Different colors are used for the 8 initial points. The eight graphs depict the components $\check{\mathfrak{r}}_2^q$, $0 \le q \le 7$, with respect to time.

5. Conclusions

The very general FOOVNN model with neutral-type, time-varying, and distributed delays was discussed in this paper, to our awareness, for the first time in the literature. Sufficient criteria expressed by LMIs and algebraic inequalities were deduced, guaranteeing the asymptotic and Mittag–Leffler synchronization of these FOOVNNs by means of two different state feedback control schemes. The octonion-valued system of equations was transformed into a real-valued one, in order to avoid the non-associativity problem of the octonion algebra. Two types of Lyapunov-like functions were used in conjunction with two Halanay-type lemmas designed specifically for FONNs. Additionally, in order to reduce the conservativeness of the obtained conditions, the free-weighting matrix method, a classical lemma, and Young's inequality were employed. Each of the four theorems formulated in the paper was illustrated using a numerical simulation.

The methods used in the present work are general enough that they can be applied to study other types of dynamic properties for other types of FOOVNNs, which constitute promising future work directions. Additionally, the theorems put forward in the study can be particularized for FOCVNNs and FOQVNNs, for which the respective results do not exist in the literature for models with so many types of delays.

Funding: Funding was provided by a grant of the Ministry of Research, Innovation and Digitization, project number PN-III-P1-1.1-PD-2021-0345.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: This work was supported by a grant of the Ministry of Research, Innovation and Digitization, CNCS—UEFISCDI, project number PN-III-P1-1.1-PD-2021-0345, within PNCDI III.

Conflicts of Interest: The author declares no conflict of interest.

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