On the Existence and Ulam Stability of BVP within Kernel Fractional Time

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Abstract: This manuscript, we establish novel findings regarding the existence of solutions for second-order fractional differential equations employing $\Psi$-Caputo fractional derivatives. The application of Banach’s fixed-point theorem (BFPT) ensures the uniqueness of the solutions, while Schauder’s fixed-point theorem (SFPT) is instrumental in determining the existence of these solutions. Furthermore, we assess the stability of the proposed equation using the Ulam–Hyers stability criterion. To illustrate our results, we provide a concrete example showcasing their practical implications.

Keywords: existence and uniqueness; $\Psi$-Caputo fractional derivative; fixed-point theorems

MSC: 26A33; 34A08; 34A12; 34B27; 34D20

1. Introduction

Today, fractional calculus is a well-established branch of mathematics with a rich history and a wide array of applications. It has become an indispensable tool for modeling and understanding complex physical and engineering systems that exhibit non-integer-order behaviors [1–6]. Undoubtedly, the theory of existence holds a paramount position within the realm of fractional calculus. Researchers have also come up with many results on the solutions of existence and uniqueness to the initial and boundary value problems (BVPs) of FDEs in the sense of Riemann–Liouville and Caputo fractional derivatives, see ([7–21]).

Researchers have also shown keen interest in fractional differential equations of the Hadamard-type. In 1928 [22], Hadamard discovered a derivative of a specific type of fractional derivatives. We now list a generalization of fractional derivatives, which is for $\Psi$-Caputo, which are for Caputo fractional derivatives and also Riemann–Liouville derivatives. The study conducted by reference [23] explores the existence and uniqueness of mild solutions of BVPs associated with Caputo–Hadamard fractional differential equations, incorporating integral and anti-periodic conditions.

In their respective studies, Rezapour et al. [24] dedicated their research efforts to exploring the existence of solutions within a recently defined category of fractional BVPs situated within the framework of Caputo–Hadamard calculus. In a parallel investigation, Abbas et al. [25] established the existence of solutions for a specific group of Caputo–Hadamard fractional differential equations. Their approach leaned heavily on the application of Mönch’s fixed-point theorem in conjunction with the utilization of the measure of non-compactness technique.

In recent times, there has been a growing trend among researchers to explore the practical applications of Ulam–Hyers stability, as evidenced by references ([26–33]).

Murad et al. [34] conducted a comprehensive examination in which they delved into multiple facets of a differential equation involving a combination of Caputo and Riemann fractional derivatives. Their investigation encompassed the analysis of the existence of
solutions as well as the exploration of Ulam–Hyers and Ulam–Hyers–Rassias theorems in this context.

Patil et al. [35] treated the existence and uniqueness of positive solutions to the fractional differential equation
\[ C D_{0+}^\alpha z(t) + f(t, z(t)) = 0, \quad 0 < t < 1, \]
with nonlocal integral boundary conditions
\[ \begin{align*}
  z(0) &= z'(0) + g(z), \\
  z(1) &= \int_0^1 z(s)ds,
\end{align*} \]
where \(1 < \alpha \leq 2, C D_{0+}^\alpha \) is the Caputo fractional derivative of order \(\alpha\), \(f : [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+\), and \(g : C[0, 1] \to \mathbb{R}^+\).

Motivated by the above work and the research conducted to advance further in this goal, in this research, we treat the existence and uniqueness of the solution of a \(\Psi\)-Caputo fractional differential equation
\[ D_{a}^{\Psi, \alpha} \varphi(\xi) = F(\xi, \varphi(\xi)), \quad \xi \in J = [a, 3], \tag{1} \]
with the boundary condition
\[ \begin{align*}
  \varphi(a) &= \varphi'(a), \\
  \varphi(3) &= \int_a^3 \varphi(\zeta)\Psi'(\zeta)d\zeta
\end{align*} \tag{2} \]
where \(D_{a}^{\Psi, \alpha}\) is the \(\Psi\)-Caputo fractional derivative, with \(1 < \alpha \leq 2\), and \(F : [a, 3] \times \mathbb{R} \to \mathbb{R}\) is continuous function (CF). Let \(\Psi : [a_1, a_2] \to \mathbb{R}\) be increasing via \(\Psi'(\xi) \neq 0, \forall \xi\). The symbol \(C(J, \mathbb{R})\) represents the Banach space of CFs \(\varphi : J \to [\xi] : \xi \in J\). We utilize SFPT and BFPT to establish both the existence and uniqueness of solutions for Equations (1) and (2), subject to specific conditions. Additionally, we demonstrate the validity of a stability theorem, namely the Ulam–Hyers stability theorem. To illustrate our main findings, we include an example as an application.

2. Essential Preliminaries

In this research, we require a set of fundamental definitions and lemmas that will be essential in various aspects of our study.

**Definition 1 ([1])**. Let \(\varphi : (0, \infty) \to \mathbb{R}\) be a CF. Hence, the Riemann–Liouville fractional derivative (RLFI) of order \(\alpha > 0, n = [\alpha] + 1\) ([\(\alpha\] denotes the integer part of the real number \(\alpha\)), defined as
\[ RL D_{0+}^\alpha \varphi(\xi) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{d\xi} \right)^n \int_0^\xi (\xi-s)^{n-\alpha-1} \varphi(s)ds, \]
where \(n-1 < \alpha < n\).

**Definition 2 ([1])**. Let \(\varphi : (0, \infty) \to \mathbb{R}\) be a CF. Hence, a Caputo fractional derivative (CFD) of order \(\alpha > 0, n = [\alpha] + 1\) can be defined as
\[ C D_{0+}^\alpha \varphi(\xi) = \frac{1}{\Gamma(n-\alpha)} \int_0^\xi (\xi-s)^{n-\alpha-1} \varphi^{(n)}(s)ds, \]
where \(n-1 < \alpha < n\).
Definition 3 ([1]). The $Ψ$-Riemann–Liouville fractional integral ($Ψ$-RLFI) of order $α > 0$ for a CF $ϕ: [a, 3] \rightarrow \mathbb{R}$ is referred to as

$$T_α^{Ψ} \phi(ζ) = \frac{1}{Γ(α)} \int_a^ζ (Ψ(ζ) - Ψ(s))^{α-1} Ψ'(s)ϕ(s)ds.$$ 

Definition 4 ([1]). The Caputo fractional derivative (CFD) of order $α > 0$ for a $ϕ: [0, +∞) \rightarrow \mathbb{R}$ is intended by

$$D_α^{ϕ}(ζ) = \frac{1}{Γ(n-α)} \int_a^ζ (ζ-s)^{n-α-1} \phi^{(n)}(s)ds, \quad α \in (n-1, n), \quad n \in \mathbb{N}.$$ 

Definition 5 ([1]). The $Ψ$-Caputo fractional derivative ($Ψ$-CFD) of order $α > 0$ for a CF $ϕ: [a, 3] \rightarrow \mathbb{R}$ is the aim of

$$D_α^{Ψ}(ζ) = \frac{1}{Γ(n-α)} \int_a^ζ (Ψ(ζ) - Ψ(s))^{n-α-1} Ψ'(s)\partial_α^Ψ(ζ)ds, \quad ζ > a, \quad α \in (n-1, n),$$

where $\partial_α^Ψ = \left( \frac{1}{Ψ(ζ)} \frac{d}{s} \right)^n, n \in \mathbb{N}$.

Lemma 1 ([1]). Let $q, φ > 0$, and $κ ∈ C([a, b], \mathbb{R})$. Hence, $κ ∈ [a, b]$ and by assumption $F_κ(z) = Ψ(z) - Ψ(a)$, we have:

1. $T_a^Ψ T_a^Ψ κ(ζ) = T_a^{q+φ_Ψ} κ(ζ);$  
2. $D_a^Ψ D_a^Ψ κ(ζ) = κ(ζ);$  
3. $D_a^Ψ (F_κ(z))^{q-1} = \frac{Γ(q)}{Γ(q-φ)} (F_κ(z))^{q-φ-1};$  
4. $D_a^Ψ (F_κ(z))^{q-1} = \frac{Γ(q)}{Γ(q-φ)} (F_κ(z))^{q-φ-1};$  
5. $D_a^Ψ (F_κ(z))^{q-1} = 0, \quad k \in \{0, \ldots, n-1\}, \quad n \in \mathbb{N}, \quad q \in (n-1, n)$.

Lemma 2 ([1]). Let $n - 1 < a_1 ≤ a_2 > 0$, $a > 0$, $κ ∈ L(a, T)$, $D_{a_1}^{Ψ} κ ∈ L(a, T)$. Then, the differential equation

$$D_{a_1}^{Ψ} κ = 0$$

has the unique solution

$$κ(ζ) = χ_0 + χ_1(Ψ(ζ) - Ψ(a)) + χ_2(Ψ(ζ) - Ψ(a))^2 + \cdots + χ_{n-1}(Ψ(ζ) - Ψ(a))^{n-1},$$

and

$$T_{a_1}^{Ψ} D_{a_1}^{Ψ} κ(ζ) = κ(ζ) + χ_0 + χ_1(Ψ(ζ) - Ψ(a)) + χ_2(Ψ(ζ) - Ψ(a))^2 + \cdots + χ_{n-1}(Ψ(ζ) - Ψ(a))^{n-1},$$

with $χ_κ ∈ R, \quad φ = 0, 1, \ldots, n - 1$.

Furthermore,

$$D_{a_1}^{Ψ} T_{a_1}^{Ψ} κ(ζ) = κ(ζ),$$

and

$$T_{a_1}^{Ψ} T_{a_1}^{Ψ} κ(ζ) = T_{a_1}^{Ψ+φ_Ψ} κ(ζ).$$

Definition 6 ([36]). The Equation (1) is Ulam–Hyers stable if there exists a real number $C_F > 0$ such that for each $ε > 0$ and for each solution $z ∈ C^1(J, \mathbb{R})$ of the inequality

$$|D_{a_1}^{Ψ} z(ζ) - F_κ(z(ζ))| ≤ ε, \quad ζ ∈ J,$$
there exists a solution \( \varphi \in C^1(J, \mathbb{R}) \) of Equation (1) with

\[
|\varphi(\xi) - \varphi(\xi)| \leq c \varphi(\xi), \xi \in J.
\]

**Theorem 1** ([37] (BFPT)). Consider a Banach space denoted as \( H \). If there exists an operator \( Z \) from \( H \) to itself, which satisfies the property of being a contraction, then \( Z \) possesses a unique fixed point within the confines of \( H \), and this fixed point is uniquely determined.

**Theorem 2** ([37] (SFPT)). Suppose \( H \) is a closed, bounded, and convex subset of the Banach space \( X \), and there exists a continuous mapping \( Z \) from \( H \) to \( H \) such that the set \( \{ Z_x : x \in H \} \) is relatively compact. In that case, \( Z \) possesses at least one fixed point within the set \( H \).

Note that \( \beta \neq 0 \), given by

\[
\beta = 1 - (\Psi(3) - \Psi(a))\Psi'(a) + \Psi(3) - \Psi(a) - \frac{(\Psi(3) - \Psi(a))^2}{2}.
\]

**Lemma 3.** For every \( \varphi(\xi) \in C(J, \mathbb{R}) \), \( 1 < \alpha \leq 2 \), and thus the BVP (1) and (2) has a solution

\[
\varphi(\xi) = \frac{\Psi'(a) + \Psi(\xi) - \Psi(a)}{\beta \Gamma(\alpha + 1)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{\alpha-1}(\Psi(3) - \Psi(s) - a)F(s, \varphi(s))\Psi'(s)ds
\]

\[
+ \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{\alpha-1}F(s, \varphi(s))\Psi'(s)ds.
\]

**Proof.** Referring to Lemma 2, we can streamline the problem defined in Equations (1) and (2) into an equivalent integral equation

\[
\varphi(\xi) = c_0 + c_1(\Psi(\xi) - \Psi(a)) + \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{\alpha-1}F(s, \varphi(s))\Psi'(s)ds,
\]

to find \( c_0 \) and \( c_1 \). From the first boundary condition \( \varphi(a) = \varphi'(a) \), we have \( c_1\Psi'(a) = c_0 \); hence,

\[
\varphi(\xi) = (\Psi'(a) + \Psi(\xi) - \Psi(a))c_1 + \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{\alpha-1}F(s, \varphi(s))\Psi'(s)ds.
\]

By using the condition \( \varphi(3) = \int_a^3 \varphi(\xi)\Psi'(\xi)d\xi \), the result is

\[
\varphi(\xi) = (\Psi'(a) + \Psi(\xi) - \Psi(a))c_1 + \frac{1}{\Gamma(a)} \int_a^3 (\Psi(3) - \Psi(s))^{\alpha-1}F(s, \varphi(s))\Psi'(s)ds.
\]

By applying Fubini’s theorem, we can derive the following result

\[
\int_a^3 \varphi(\xi)\Psi'(\xi)d\xi = \int_a^3 [(\Psi'(a) + \Psi(\xi) - \Psi(a))c_1
\]
\[
+ \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{\alpha-1}F(s, \varphi(s))\Psi'(s)ds]\Psi'(\xi)d\xi,
\]

and

\[
c_0 = \frac{\Psi'(a)}{\beta \Gamma(\alpha + 1)} \int_a^3 (\Psi(3) - \Psi(s))^{\alpha-1}(\Psi(3) - \Psi(s) - a)F(s, \varphi(s))\Psi'(s)ds,
\]

\[
c_1 = \frac{1}{\beta \Gamma(\alpha + 1)} \int_a^3 (\Psi(3) - \Psi(s))^{\alpha-1}(\Psi(3) - \Psi(s) - a)F(s, \varphi(s))\Psi'(s)ds.
\]
This implies that
\[
\kappa(\varsigma) = \frac{\Psi'(a) + \Psi(\varsigma) - \Psi(a)}{\beta \Gamma(\alpha + 1)} \int_a^\varsigma (\Psi(\varsigma) - \Psi(s))^{\alpha-1}(\Psi(\varsigma) - \Psi(s) - a)\mathcal{F}(s, \kappa(s))\Psi'(s)ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^\varsigma (\Psi(\varsigma) - \Psi(s))^{\alpha-1}\mathcal{F}(s, \kappa(s))\Psi'(s)ds,
\]
and this complete the proof.  

### 3. Existence and Uniqueness

Consider the Banach space denoted as \( C = C(J, \mathbb{R}) \), which encompasses all continuous functions mapping from the interval \( J \) to the real numbers \( \mathbb{R} \). The norm for this space is defined as follows:
\[
\|\kappa\| = \sup \{|\kappa(\varsigma)|, \varsigma \in J\}.
\]

We give this assumption so as to prove the main results:

1. **(Ω1)** There exists constant \( \mu > 0 \), such that
   \[
   |\mathcal{F}(\varsigma, \kappa(\varsigma))| \leq \mu |\kappa(\varsigma)|.
   \]

2. **(Ω2)** There exists constant \( k > 0 \), such that
   \[
   |\mathcal{F}(\varsigma, \kappa(\varsigma)) - \mathcal{F}(\varsigma, y(\varsigma))| \leq k|\kappa(\varsigma) - y(\varsigma)|.
   \]

We achieve the first result on the Banach contraction principle. To simplify, we have coded the following:
\[
\lambda = \frac{\gamma}{\beta \Gamma(\alpha + 2)} \left[ (\Psi(\varsigma) - \Psi(a))^{\alpha + 1} + (\alpha + 1) \left( \frac{|\beta|}{\gamma} + 1 \right) (\Psi(\varsigma) - \Psi(a))^\alpha \right].
\]

**Theorem 3.** Assumption (Ω2) holds. If \( \lambda k < 1 \), then the BVP (1) and (2) has a unique solution on \( J \).

**Proof.** Define the operator \( \Xi : C(J, \mathbb{R}) \to C(J, \mathbb{R}) \),
\[
\Xi \kappa(\varsigma) = \frac{\Psi'(a) + \Psi(\varsigma) - \Psi(a)}{\beta \Gamma(\alpha + 1)} \int_a^\varsigma (\Psi(\varsigma) - \Psi(s))^{\alpha-1}(\Psi(\varsigma) - \Psi(s) - a)\mathcal{F}(s, \kappa(s))\Psi'(s)ds \\
+ \frac{1}{\Gamma(\alpha)} \int_a^\varsigma (\Psi(\varsigma) - \Psi(s))^{\alpha-1}\mathcal{F}(s, \kappa(s))\Psi'(s)ds,
\]

Let us set \( r \geq \frac{M}{1-k\lambda} \), and prove that \( \Xi G_r \subset G_r \), when \( G_r = \{ \kappa \in C(J, \mathbb{R}) : \|\kappa\| \leq r \} \),
\[
M = \sup \{|\mathcal{F}(\varsigma, 0)|, \varsigma \in G_r\},
\]
for \( \varsigma \in G_r \), we have
\[
|\Xi(x)(\xi)| \\
\leq \frac{1}{|\beta|\Gamma(a + 1)} \int_a^\alpha |\Psi(\xi) - \Psi(s)|^{a-1}|(\Psi(\xi) - \Psi(s) - a)||F(s, x(s))||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F(s, x(s))||F'(s)|ds, \\
+ \frac{1}{|\beta|\Gamma(a + 1)} \int_c^\alpha |\Psi(\xi) - \Psi(s)|^{a-1}|(\Psi(\xi) - \Psi(s) - a)||F(s, x(s)) - F(s, 0)||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F(s, x(s)) - F(s, 0)||F'(s)|ds \ \\
\leq \frac{1}{|\beta|\Gamma(a + 1)} \int_a^\alpha |\Psi(\xi) - \Psi(s)|^{a-1}|(\Psi(\xi) - \Psi(s)) - F(s, x(s))||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F'(s)|ds \ \\
\leq \left(\frac{1}{|\beta|\Gamma(a + 1)} \int_a^\alpha |\Psi(\xi) - \Psi(s)|^{a-1}|(\Psi(\xi) - \Psi(s)) - F(s, x(s))||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F'(s)|ds \right) ||\Psi(\xi) - \Psi(s)|| \\
\leq \left(\frac{1}{|\beta|\Gamma(a + 1)} \int_a^\alpha |\Psi(\xi) - \Psi(s)|^{a-1}|(\Psi(\xi) - \Psi(s)) - F(s, x(s))||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F'(s)|ds \right) k ||\Psi(\xi) - \Psi(s)|| \\
\leq \lambda k ||\Psi(\xi) - \Psi(s)||, \\
\leq \lambda k ||\Psi(\xi) - \Psi(s)||.
\]

where \(\Psi'(a) + \Psi(\xi) - \Psi(s) = \gamma\); therefore, \(\Xi \subseteq \mathcal{G}_r \subseteq \mathcal{G}_r\). Now, considering that \(\Xi\) exhibits characteristics of a contraction mapping, let \(\lambda_1, \lambda_2 \in \mathcal{G}_r\), for each \(\xi \in J\). We obtain

\[
|\Xi(\lambda_1)(\xi) - \Xi(\lambda_2)(\xi)| \\
\leq \frac{1}{|\beta|\Gamma(a + 1)} \int_a^\alpha |(\Psi(\xi) - \Psi(s) - a)||F(s, x(s))||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F(s, x(s)) - F(s, 0)||F'(s)|ds, \\
+ \frac{1}{|\beta|\Gamma(a + 1)} \int_c^\alpha |(\Psi(\xi) - \Psi(s) - a)||F(s, x(s)) - F(s, 0)||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F'(s)|ds \ \\
\leq \left(\frac{1}{|\beta|\Gamma(a + 1)} \int_a^\alpha |(\Psi(\xi) - \Psi(s) - a)||F(s, x(s)) - F(s, 0)||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F'(s)|ds \right) ||\Psi(\xi) - \Psi(s)|| \\
\leq \left(\frac{1}{|\beta|\Gamma(a + 1)} \int_a^\alpha |(\Psi(\xi) - \Psi(s) - a)||F(s, x(s)) - F(s, 0)||F'(s)|ds \\
+ \frac{1}{\Gamma(a)} \int_a^c |\Psi(s) - \Psi(s)|^{a-1}|F'(s)|ds \right) k ||\Psi(\xi) - \Psi(s)|| \\
\leq \lambda k ||\Psi(\xi) - \Psi(s)||, \\
\leq \lambda k ||\Psi(\xi) - \Psi(s)||.
\]

Consequently, given the condition \(\lambda k \leq 1\), we can deduce that the operator \(\Xi\) exhibits contraction properties. As a result, we can confidently conclude, based on SFPT, that the operator \(\Xi\) possesses a unique fixed point. This unique fixed point corresponds to solution of the problem (1) and (2).

Next, the result is based on SFPT.

**Theorem 4.** Assume that \((\Omega 1) - (\Omega 2)\) holds. Then, the BVP (1) and (2) has at least one solution.

**Proof.** This proof is structured in four distinct steps:
Step 1: Our initial task is to demonstrate the continuity of $\Xi$. To accomplish this, we consider a sequence $\{\zeta_n\}$ that converges to $\zeta$ in the Banach space $C(J, \mathbb{R})$. For each $\zeta \in J$, we can observe the following:

$$\left| \Xi(\zeta_n)(\zeta) - \Xi(\zeta)(\zeta) \right|$$

$$\leq \frac{\gamma}{|\beta|\Gamma(a+1)} \int_a^\gamma \left| (\Psi(\zeta) - \Psi(s) - a)(\Psi(\zeta) - \Psi(s))^{a-1} |F(s, \zeta_n(s) - F(s, \zeta(s)))|\right| |\Psi'(s)ds$$

$$+ \frac{1}{\Gamma(a)} \int_a^\gamma \left( |\Psi(\zeta) - \Psi(s) - a(|\Psi(\zeta) - \Psi(s)|^{a-1} \right) |F(s, \zeta(s)))|\Psi'(s)ds$$

$$\leq \frac{\gamma}{|\beta|\Gamma(a+1)} \int_a^\gamma \left| (\Psi(\zeta) - \Psi(s) - a(|\Psi(\zeta) - \Psi(s)|^{a-1}) \right) |F(s, \zeta(s)))|\Psi'(s)ds$$

$$+ \frac{1}{\Gamma(a)} \int_a^\gamma \left( |\Psi(\zeta) - \Psi(s) - a(|\Psi(\zeta) - \Psi(s)|^{a-1} \right) |F(s, \zeta(s)))|\Psi'(s)ds$$

$$\leq \frac{\gamma}{|\beta|\Gamma(a+2)} \left[ (|\Psi(\zeta) - \Psi(s)|^{a+1} + (a+1) \left( \frac{|\beta|}{\gamma} + 1 \right) (|\Psi(\zeta) - \Psi(s)|^{a+1} \right) |F(s, \zeta(s)))|\Psi'(s)ds$$

Applying the LDC Theorem, if $n$ approaches infinity, this implies that $\|\Xi(\zeta_n)(\zeta) - \Xi(\zeta)(\zeta)\|_\infty \rightarrow 0$.

Step 2: The operator $\Xi$ transforms bounded sets into other bounded sets within the space $C(J, \mathbb{R})$. Consider any positive value $d$, defining the set $H_d = \{ \zeta \in C : \|\zeta\|_\infty \leq d \}$. It is apparent that $H_d \subset C(J, \mathbb{R})$. Now, let us assume $\zeta$ is an element of $C$, and for every $\zeta \in J$, we can observe the following:

$$\left| \Xi(\zeta)(\zeta) \right|$$

$$\leq \frac{\gamma}{|\beta|\Gamma(a+1)} \int_a^\gamma \left| (\Psi(\zeta) - \Psi(s) - a(|\Psi(\zeta) - \Psi(s)|^{a-1}) \right) \Psi'(s)ds$$

$$+ \frac{1}{\Gamma(a)} \int_a^\gamma \left( |\Psi(\zeta) - \Psi(s) - a(|\Psi(\zeta) - \Psi(s)|^{a-1} \right) \Psi'(s)ds$$

$$\leq \frac{\gamma}{|\beta|\Gamma(a+1)} \int_a^\gamma \left| (\Psi(\zeta) - \Psi(s) - a(|\Psi(\zeta) - \Psi(s)|^{a-1}) \right) \Psi'(s)ds$$

$$+ \frac{1}{\Gamma(a)} \int_a^\gamma \left( |\Psi(\zeta) - \Psi(s) - a(|\Psi(\zeta) - \Psi(s)|^{a-1} \right) \Psi'(s)ds$$

$$\leq \frac{\gamma}{|\beta|\Gamma(a+2)} \left[ (|\Psi(\zeta) - \Psi(s)|^{a+1} + (a+1) \left( \frac{|\beta|}{\gamma} + 1 \right) (|\Psi(\zeta) - \Psi(s)|^{a+1} \right) \Psi'(s)ds$$

$$\|\Xi(\zeta)(\zeta)\|_\infty \leq \lambda m_r = L.$$ 

Thus, $\|\Xi(\zeta)(\zeta)\|_\infty \leq L$, for constant $L$.

Step 3: The operator $\Xi$ transforms the space $C(J, \mathbb{R})$ into a collection of functions within $C(J, \mathbb{R})$ that exhibit equicontinuity.

Let $\zeta \in C(J, \mathbb{R})$ and $\xi_1, \xi_2 \in J$ with $\xi_1 < \xi_2$; then,
\[
|\Xi(\varepsilon)(\xi_2) - \Xi(\varepsilon)(\xi_1)|
\leq \frac{1}{\Gamma(\alpha)} \int_{\xi_1}^{\xi_2} (\Psi(\xi_2) - \Psi(s))^{a-1} |F(s, \varepsilon s)| |\Psi'(s)| ds
+ |\Psi(\xi_2) - \Psi(\xi_1)| \frac{1}{|\beta|\Gamma(\alpha + 1)} \int_{a}^{3} |\Psi(\xi) - \Psi(s) - a| |(\Psi(\xi) - \Psi(s))^{a-1} |F(s, \varepsilon s)| |\Psi'(s)| ds
+ \frac{1}{\Gamma(\alpha)} \int_{a}^{5} |\Psi(\xi_2) - \Psi(s) - a| |(\Psi(\xi) - \Psi(s))^{a-1} |F(s, \varepsilon s)| |\Psi'(s)| ds,
\]

\[
\leq \left[ \frac{(\Psi(\xi_2) - \Psi(\xi_1))^a}{\Gamma(\alpha + 1)} + |\Psi(\xi_2) - \Psi(\xi_1)| \frac{1}{|\beta|\Gamma(\alpha + 1)} \int_{a}^{3} |\Psi(\xi) - \Psi(s) - a| |(\Psi(\xi) - \Psi(s))^{a-1} |F(s, \varepsilon s)| |\Psi'(s)| ds \right] \mu,
\]

As \(\xi_1 \to \xi_2\), the expression on the right-hand side of the inequality above converges.

**Step 4:** Next, our objective is to establish that \(\Xi\) is bounded in advance.

Let \(U = \{ \varepsilon \in C([a, 3]) : \varepsilon = \rho \Xi, \text{ for some } 0 < \rho < 1 \}\). Our objective is to demonstrate that the set \(U\) is bounded. Consider an arbitrary element \(\varepsilon \in U\), and for every \(\xi \in J\), we observe the following:

\[
|\Xi(\varepsilon)(\xi)| \leq \rho \left[ \frac{\gamma}{|\beta|\Gamma(\alpha + 1)} \int_{a}^{3} |(\Psi(\xi) - \Psi(s) - a)| |(\Psi(\xi) - \Psi(s))^{a-1} |F(s, \varepsilon s)| |\Psi'(s)| ds \right]
+ \rho \left[ \frac{1}{\Gamma(\alpha)} \int_{a}^{5} |(\Psi(\xi) - \Psi(s))^{a-1} |F(s, \varepsilon s)| |\Psi'(s)| ds \right]
\leq \frac{\gamma}{|\beta|\Gamma(\alpha + 2)} \left[ (\Psi(\xi) - \Psi(s))^{a+1} + (\alpha + 1) \left( \frac{|\beta|}{\gamma + 1} \right) (\Psi(\xi) - \Psi(a))^{a} \right] \mu
\]

\[\leq \lambda \mu,\]

this implies that the set \(U\) is bounded. According to SFPT, the operator \(\Xi\) must possess at least one fixed point, and this fixed point serves as a solution to Equations (1) and (2).

4. **Stability Theorem**

In the upcoming theorems, we will establish the Ulam–Hyers stability for the equation presented in (1) within \(J = [a, 3]\).

**Theorem 5.** If \((\Omega2)\) holds, the BVP (1)–(2) is Ulam–Hyers stable.

**Proof.** For \(\varepsilon > 0\), and each solution \(w \in C([a, 3])\) of the inequality

\[
|D^{a, \Psi} w(\xi) - F(\xi, w(\xi))| \leq \varepsilon, \; \xi \in J.
\]
Let $\phi \in C(J, \mathbb{R})$ be the unique solution of BVP (1)–(2). So, $\phi(\xi)$ is represented by

$$\phi(\xi) = \frac{\Psi'(a) + \Psi(\xi) - \Psi(a)}{\beta \Gamma(a + 1)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1}(\Psi(\xi) - \Psi(s) - a) F(s, \phi(s)) \Psi'(s) ds$$

$$+ \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1} F(s, \phi(s)) \Psi'(s) ds.$$

Then, we have

$$|w(\xi) - \phi(\xi)|$$

$$\leq \left| w(\xi) - \frac{\Psi'(a) + \Psi(\xi) - \Psi(a)}{\beta \Gamma(a + 1)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1}(\Psi(\xi) - \Psi(s) - a) F(s, \phi(s)) \Psi'(s) ds 
- \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1} F(s, \phi(s)) \Psi'(s) ds \right|$$

$$\leq \frac{1}{|\beta| \Gamma(a + 1)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1} |\Psi(\xi) - \Psi(s) - a||F(s, w(s)) - F(s, \phi(s))| \Psi'(s) ds$$

$$+ \frac{e(\Psi(\xi) - \Psi(a))^a}{\Gamma(a + 1)} + \frac{\gamma k}{\beta \Gamma(a + 1)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1}(\Psi(\xi) - \Psi(s) - a) F(s, \phi(s)) \Psi'(s) ds$$

$$+ k \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1}|w(s) - \phi(s)| \Psi'(s) ds,$$

where

$$\left| w(\xi) - \frac{\Psi'(a) + \Psi(\xi) - \Psi(a)}{\beta \Gamma(a + 1)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1}(\Psi(\xi) - \Psi(s) - a) F(s, \phi(s)) \Psi'(s) ds 
- \frac{1}{\Gamma(a)} \int_a^\xi (\Psi(\xi) - \Psi(s))^{a-1} F(s, \phi(s)) \Psi'(s) ds \right| \leq \frac{e(\Psi(\xi) - \Psi(a))^a}{\Gamma(a + 1)}.$$

Then, we obtain

$$|w(\xi) - \phi(\xi)|$$

$$\leq \frac{e(\Psi(\xi) - \Psi(a))^a}{\Gamma(a + 1)} + \frac{\gamma k}{|\beta| \Gamma(a + 2)} \left[(\Psi(\xi) - \Psi(a))^{a+1} + (a + 1)(\Psi(\xi) - \Psi(a))^a \right]|w(\xi) - \phi(\xi)|$$

$$+ \frac{k}{\Gamma(a + 1)} (\Psi(\xi) - \Psi(a))^a |w(\xi) - \phi(\xi)|.$$

It becomes

$$|w(\xi) - \phi(\xi)| \leq \frac{e(\Psi(\xi) - \Psi(a))^a}{(1 - k)\Gamma(a + 1)}.$$

Set $c_F = \frac{(\Psi(\xi) - \Psi(a))^a}{(1 - k)\Gamma(a + 1)}$

The inequality is

$$|w(\xi) - \phi(\xi)| \leq c_F.$$

If we assume that the BVP represented by Equations (1) and (2) holds, then it exhibits Ulam–Hyers stability. □
5. Example

Suggest the BVP

\[
\begin{aligned}
D^\frac{3}{7} \varphi(\zeta) &= e^{-\zeta} + \frac{x}{1 + \cos(\zeta)}, \quad \zeta \in [a, \Im], \\
\varphi(a) &= \varphi'(a), \quad \varphi(\Im) = \int_a^\Im \varphi(s)\Psi'(s)ds,
\end{aligned}
\] (5)

Let \( \Psi(\zeta) = \log \zeta, a = 1, \Im = e, a = \frac{5}{7}, \) and \( F(\zeta, \varphi(\zeta)) = e^{-\zeta} + \frac{x}{1 + \cos(\zeta)}. \) By using (\( \Omega2 \)), the following result is obtained:

\[ |F(\zeta, \varphi_1(\zeta)) - F(\zeta, \varphi_2(\zeta))| \leq \frac{1}{20 + \cos(\zeta)}|\varphi_1 - \varphi_2|. \]

Since \( k = 0.205 \), from Theorem 3, we have

\[ \lambda = \frac{\gamma}{\beta \Gamma(\alpha + 2)} \left[ (\Psi(\Im) - \Psi(a))^{\alpha + 1} + (\alpha + 1) \left( \frac{|\beta|}{\gamma} + 1 \right) (\Psi(\Im) - \Psi(a))^{\alpha} \right]. \]

Then, \( \lambda k = 0.60353384 < 1 \), so the problem (5) has a unique solution on \([a, \Im]\).

6. Conclusions

In our research, we explored the presence and distinctiveness of solutions for \( \Psi \)-Caputo fractional differential equations under specific boundary conditions. Our findings rely on well-established fixed-point theories, including BFPT and SFPT. To conclude, we have suggested a practical application based on the core outcomes of our study.

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