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Maclaurin-Type Integral Inequalities for GA-Convex Functions Involving Confluent Hypergeometric Function via Hadamard Fractional Integrals

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Abstract: In this manuscript, by using a new identity, we establish some new Maclaurin-type inequalities for functions whose modulus of the first derivatives are GA-convex functions via Hadamard fractional integrals.

Keywords: Hadamard fractional integrals; confluent hypergeometric function; incomplete confluent hypergeometric function; Maclaurin-type integral inequalities; geometrically arithmetically convex functions

1. Introduction

It is well known that convexity plays an important and central role in many fields, such as economics, finance, optimization, and game theory. Due to its various applications, this concept has been extended and generalized in several directions.

This concept is closely related to integral inequalities. The literature in this context is rich. One can easily find papers that deal with different types of inequalities via different kinds of convexity.

Over the past few years, numerous scholars have investigated the error estimates associated with specific quadrature formulas. Their aim has been to develop new refinements, generalizations, and variants. For additional details, readers are encouraged to consult references [1–10] for classical inequalities, and [11–14] for fractional inequalities.

In [15], ˙I¸scan gave the analogue fractional of Hermite–Hadamard inequality for GA-convex functions as follows:

\[ f\left(\sqrt{ab}\right) \leq \frac{1}{2(\ln b - \ln a)^q} \left\{ H_{a}^q f(b) + H_{b}^q f(a) \right\} \leq \frac{f(a) + f(b)}{2}, \]

where \( a > 0 \) and \( 0 < a < b \) and \( f \) is an integrable and GA-convex function on \([a, b]\).

Qi and Xi [16] have derived specific Simpson-type inequalities for GA-ε-convex functions. Within the outcomes obtained for differentiable function \( f: [a, b] \rightarrow \mathbb{R}, \) with \( 0 < a < b \) and \( f' \in L[a, b] \) and \( |f'|^q \) is GA-ε-convex, we have
whose modulus of the first derivatives are \(GA\) where \(q\) with \(\frac{\ln b - \ln a}{2} \leq \left\{ M_1^{1-\frac{q}{2}}(a, b) \left( (M_1(a, b) - M_2(a, b)) \right| f'(a) \right| \right. \\
+ M_2(a, b) \| f'(b) \| + \varepsilon M_1(a, b) \right)^\frac{1}{q} \\
+ M_1^{1-\frac{q}{2}}(b, a) \left( M_2(b, a) \| f'(b) \| + M_1(b, a) \right)^\frac{1}{q} \right\},
\]
where \(q \geq 1\),
\[M_1(x, y) = \frac{2 \left( x^{\frac{1}{2}} L \left( x^{\frac{1}{2}}, y^{\frac{1}{2}} \right) + x^{\frac{1}{2}} (2y^{\frac{1}{2}} - x^{\frac{1}{2}}) - 2x^{\frac{1}{2}} y^{\frac{1}{2}} L \left( x^{\frac{1}{2}}, y^{\frac{1}{2}} \right) \right)}{3(\ln y - \ln x)} \]
and
\[M_1(x, y) = \frac{2x^{\frac{1}{2}} (4y^{\frac{1}{2}} L \left( x^{\frac{1}{2}}, y^{\frac{1}{2}} \right) + y^{\frac{1}{2}} (\ln y - \ln x) - 2x^{\frac{1}{2}} L \left( x^{\frac{1}{2}}, y^{\frac{1}{2}} \right) - 5y^{\frac{1}{2}} + 2x^{\frac{1}{2}} y^{\frac{1}{2}} + x^{\frac{1}{2}} \right)}{3(\ln y - \ln x)^r},\]
with
\[L(x, y) = \left\{ \begin{array}{ll}
\frac{x - y}{\ln x - \ln y} & \text{if } x \neq y \\
x & \text{if } x = y.
\end{array} \right.\]

Motivated by the above results, we propose in this work to study one of the open
three-point Newton–Cotes formulas called Maclaurin inequality, which can be declared as follows:
\[
\left\| \frac{1}{8} \left( 3 f \left( \frac{5a+b}{6} \right) + 2 f \left( \frac{a+b}{2} \right) + 3 f \left( \frac{a+5b}{6} \right) \right) - \frac{1}{8} \int_a^b f(u) du \right\| \leq \frac{7(\tau - 1)^4}{5154} \| f^{(4)} \|_\infty (b - a)^4,
\]
where \(f\) is four times continuously differentiable function on \((a, b)\), and \(\| f^{(4)} \|_\infty = \sup_{x \in (a, b)} | f^{(4)}(x) |\), (see [17]).

For this, we first prove a new identity involving Hadamard fractional integrals. On the
basis of this identity, we establish some new Maclaurin-type inequalities for functions
whose modulus of the first derivatives are \(GA\)-convex.

2. Preliminaries

This section recalls some known definitions. We denote by \(\mathbb{R}\) the set of real numbers,
and by \(\mathbb{R}^+\) the set of non-negative real numbers.

**Definition 1** ([18]). Let \(I\) be a subintervals of \((0, +\infty)\). A function \(f : I \rightarrow \mathbb{R}^+\) is said to be
\(GA\)-convex on \(I\), if
\[f \left( x^{\frac{1}{t}} y^{\frac{1}{1-t}} \right) \leq t f(x) + (1 - t) f(y)\]
holds for all \(x, y \in I\) and \(t \in [0, 1]\).
\textbf{Definition 2 ([19]).} The integral representation of the confluent hypergeometric function is given by

\[
1F_1(a; b; z) = \frac{1}{B(a, b-a)} \int_0^1 t^{a-1} (1-t)^{b-a-1} e^{zt} \, dt,
\]

where $\Re b > \Re a > 0$ and $B$ is the beta function.

\textbf{Definition 3 ([20]).} The integral representation of the incomplete confluent hypergeometric function is given by

\[
1F_1([a, b]; y; z) = \frac{1}{B(a, b-a)} \int_0^y u^{a-1} (1-u)^{b-a-1} e^{zu} \, du
\]

\[
= \frac{y^a}{B(a, b-a)} \int_0^1 u^{a-1} (1-uy)^{b-a-1} e^{zu} \, du,
\]

where $\Re b > \Re a > 0$ and $B$ is the beta function.

\textbf{Definition 4 ([21]).} The left-sided and right-sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ of function $f(x)$ are defined by

\[
H^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln u)^{\alpha-1} f(u) \frac{du}{u}, \quad (0 < a < x \leq b),
\]

and

\[
H^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln u - \ln x)^{\alpha-1} f(u) \frac{du}{u}, \quad (0 < a \leq x < b).
\]

\textbf{Lemma 1 ([22]).} For any $0 \leq a < b$ in $\mathbb{R}$, and a fixed $p \geq 1$, we have

\[
(b-a)^p \leq b^p - a^p.
\]

3. Auxiliary Results

We provide certain lemmas in this section that help with the computations and are utilized in the following section. The following lemma is crucial to establish our main results

\textbf{Lemma 2.} Let $f \colon [a, b] \to \mathbb{R}$ be a differentiable mapping on $[a, b]$ with $0 < a < b$. Assume that $f' \in L[a, b]$. Then, the following equality for fractional integrals holds:

\[
\frac{1}{8} \bigg(3f(a^2 b^3) + 2f(a^2 b^2) + 3f(a b^2)\bigg) - \frac{6^\alpha \Gamma(a+1)}{(\ln b - \ln a)^\alpha} \\
\int_0^\alpha a^{(\alpha-\frac{2}{3})} b^{\frac{1-2\alpha}{3}} f' \bigg(a^{\frac{2-2\alpha}{3}} b^{\frac{1-2\alpha}{3}}\bigg) \, d\xi
\]

\[
- \int_0^\alpha ((1-\xi)^{\alpha} - \frac{3}{8}) a^{\frac{5-2\alpha}{6}} b^{\frac{11-2\alpha}{6}} f' \bigg(a^{\frac{5-2\alpha}{6}} b^{\frac{11-2\alpha}{6}}\bigg) \, d\xi
\]

\[
+ \int_0^\alpha (\xi^{\alpha} - \frac{3}{8}) a^{\frac{3-3\alpha}{6}} b^{\frac{3+2\alpha}{6}} f' \bigg(a^{\frac{3-3\alpha}{6}} b^{\frac{3+2\alpha}{6}}\bigg) \, d\xi
\]

\[
- \int_0^\alpha (1-\xi)^{\alpha} a^{\frac{1-\alpha}{6}} b^{\frac{5+2\alpha}{6}} f' \bigg(a^{\frac{1-\alpha}{6}} b^{\frac{5+2\alpha}{6}}\bigg) \, d\xi
\]
where $\alpha > 0$ and

$$Y = Hf^\alpha_{\left(\frac{1}{a} \frac{\partial}{\partial b} \right)} f(a) + Hf^\alpha_{\left(\frac{1}{b} \frac{\partial}{\partial b} \right)} f(b)$$

$$+ \frac{1}{2^n} \left( Hf^\alpha_{\left(\frac{1}{a} \frac{\partial}{\partial b} \right)} f(a) \right)^2 + Hf^\alpha_{\left(\frac{1}{b} \frac{\partial}{\partial b} \right)} f(b) \right). \quad (1)$$

**Proof.** Let

$$I = I_1 - I_2 + I_3 - I_4,$$ \quad (2)

where

$$I_1 = \int_0^1 \frac{\alpha}{a} a^{\frac{1}{b}} b^{\frac{1}{b}} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha,$$

$$I_2 = \int_0^1 \left( (1 - \alpha)^a - \frac{3}{2} \right) a^{\frac{1}{b}} b^{\frac{1}{b}} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha,$$

$$I_3 = \int_0^1 (\alpha - \frac{3}{2}) a^{\frac{1}{b}} b^{\frac{1}{b}} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha$$

and

$$I_4 = \int_0^1 \frac{1}{4} (1 - \alpha)^a a^{\frac{1}{b}} b^{\frac{1}{b}} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha,$$

Integrating by parts $I_1$, we have

$$I_1 = \frac{1}{4} \int_0^1 (1 - \alpha)^a a^{\frac{1}{b}} b^{\frac{1}{b}} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha$$

$$= \frac{6a^\alpha}{4(\ln b - \ln a)} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) \bigg|_{\alpha = 1} - \frac{6a^\alpha}{4(\ln b - \ln a)} \int_0^1 (1 - \alpha)^a f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha$$

where

$$f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) = \frac{1}{4} \left( \frac{\alpha}{a} a^{\frac{1}{b}} b^{\frac{1}{b}} \right) f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right)$$

Similarly, we have

$$I_2 = \int_0^1 \left( (1 - \alpha)^a - \frac{3}{2} \right) a^{\frac{1}{b}} b^{\frac{1}{b}} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha$$

$$= \frac{3}{4(\ln b - \ln a)} \left( (1 - \alpha)^a \right) \bigg|_{\alpha = 0} - \frac{6a^\alpha}{4(\ln b - \ln a)} \int_0^1 (1 - \alpha)^a f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha$$

$$= \frac{3}{4(\ln b - \ln a)} f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) - \frac{6a^\alpha}{4(\ln b - \ln a)} \int_0^1 (1 - \alpha)^a f^\prime \left( a^{\frac{1}{b}} b^{\frac{1}{b}} \right) d\alpha$$
Lemma 3. Let \( \lambda \) and \( \eta \) be two positive numbers. Then, the following equality holds:

\[
I_1(\lambda, \eta) = \int_0^\eta e^{\lambda u} \, du = \frac{\lambda}{\eta} (e^{\eta \lambda} - 1) \tag{7}
\]
and

$$I_2(\lambda, \eta) = \int_0^\lambda e^{\eta \xi} \, d\xi = \frac{1}{\eta} \lambda e^{\eta \lambda} - \frac{1}{\eta} \left( e^{\eta \lambda} - 1 \right). \tag{8}$$

**Proof.** By computing directly, we have

$$\int_0^\lambda e^{\eta \xi} \, d\xi = \frac{1}{\eta} e^{\eta \lambda} \bigg|_{\xi=0}^{\xi=\lambda} = \frac{1}{\eta} \left( e^{\eta \lambda} - 1 \right).$$

By using the integration by parts, we have

$$\int_0^\lambda e^{\eta \xi} \, d\xi = \frac{1}{\eta} e^{\eta \lambda} \bigg|_{\xi=0}^{\xi=\lambda} - \frac{1}{\eta} \int_0^\lambda e^{\eta \xi} \, d\xi
= \frac{1}{\eta} \lambda e^{\eta \lambda} - \frac{1}{\eta^2} e^{\eta \lambda} \bigg|_{\xi=0}^{\xi=\lambda} = \frac{1}{\eta} \lambda e^{\eta \lambda} - \frac{1}{\eta} \left( e^{\eta \lambda} - 1 \right).$$

The proof is completed. □

**Lemma 4.** Let $\alpha$ and $\theta$ be two positive numbers. Then, the following equality holds:

$$\theta_1(\alpha, \theta) = \int_0^1 x^\alpha \left( 1 - \frac{1}{6} x \right) \theta \bar{\gamma} \, d\xi = \frac{6^{\alpha+1} \Gamma\left(\frac{\alpha+1}{6}\right)}{(\alpha+1)(\alpha+2)} \ln\theta. \tag{9}$$

**Proof.** By computing directly, we have

$$\int_0^1 x^\alpha \left( 1 - \frac{1}{6} x \right) \theta \bar{\gamma} \, d\xi = \int_0^1 x^\alpha \left( 1 - \frac{1}{6} x \right) e^{\frac{1}{6} \ln \theta} \, d\xi
= \frac{6^{\alpha+1} \Gamma\left(\frac{\alpha+1}{6}\right)}{(\alpha+1)(\alpha+2)} \ln\theta.$$

The proof is completed. □

**Lemma 5.** Let $\alpha$ and $\theta$ be two positive numbers. Then, the following equality holds:

$$\int_0^1 \left| x^\alpha - \frac{3}{8} (3 + 2x) \theta \bar{\gamma} \right| \, d\xi = \mu_1(\alpha, \theta) + \mu_2(\alpha, \theta). \tag{10}$$

**Proof.** Clearly, we have

$$\int_0^1 \left| x^\alpha - \frac{3}{8} (3 + 2x) \theta \bar{\gamma} \right| \, d\xi = \int_0^{\frac{3}{8}} (3 + 2x) \theta \bar{\gamma} \, d\xi + \int_0^1 (x^\alpha - \frac{3}{8}) (3 + 2x) \theta \bar{\gamma} \, d\xi. \tag{11}$$
Lemma 6. Let $\alpha$ and $\theta$ be two positive numbers. Then, the following equality holds:

$$
\int_{0}^{1} \left( \alpha^a - \frac{3}{8} \right)(3 + 2\alpha)\theta^\frac{7}{4} d\alpha = \mu_3(\alpha, \theta) + \mu_4(\alpha, \theta).
$$

By computing directly, we obtain

$$
\mu_1(\alpha, \theta) = \left( \frac{1}{3} \right)^{\frac{1}{4}} \int_{0}^{a} \left( \frac{3}{8} - \frac{3}{4} \right)(3 + 2\alpha)\theta^\frac{7}{4} d\alpha
$$

$$
= \left( \frac{1}{3} \right)^{\frac{1}{4}} \frac{1}{4} \int_{0}^{a} \left( 9 + 6\alpha^a - 24\alpha^a - 16\alpha^a \theta^\frac{7}{4} d\alpha
$$

$$
= \left( \frac{1}{3} \right)^{\frac{1}{4}} \frac{1}{4} \int_{0}^{a} e^{\frac{7}{4}\theta} e^{\frac{7}{4}\theta} d\alpha
$$

$$
+ \left( \frac{1}{3} \right)^{\frac{1}{4}} \frac{1}{4} \int_{0}^{a} e^{\frac{7}{4}\theta} e^{\frac{7}{4}\theta} d\alpha
$$

$$
= \left( \frac{1}{3} \right)^{\frac{1}{4}} \frac{1}{4} \int_{0}^{a} e^{\frac{7}{4}\theta} e^{\frac{7}{4}\theta} d\alpha
$$

$$
= \frac{3}{8} \int_{0}^{a} \left( \frac{1}{3} \right)^{\frac{1}{4}} \left( \frac{1}{3} \right)^{\frac{1}{4}} \ln \theta
$$

On the other hand, we have

$$
\mu_2(\alpha, \theta) = \frac{1}{8} \left( \frac{3}{8} - \frac{3}{4} \right)(3 + 2\alpha)\theta^\frac{7}{4} d\alpha
$$

$$
= \frac{1}{8} \left( \frac{1}{3} \right)^{\frac{1}{4}} \frac{1}{4} \int_{0}^{a} \left( 40(1 - \frac{3}{8} \alpha^a) - 16(1 - \frac{3}{8} \alpha^a) - 15 + 6\alpha \right) e^{-\frac{7}{4}\theta} d\alpha
$$

$$
= \frac{1}{8} \left( \frac{1}{3} \right)^{\frac{1}{4}} \frac{1}{4} \int_{0}^{a} \left( 1 - \frac{3}{8} \alpha^a \right) e^{-\frac{7}{4}\theta} d\alpha
$$

$$
- \frac{1}{8} \left( \frac{1}{3} \right)^{\frac{1}{4}} \frac{1}{4} \int_{0}^{a} \left( 1 - \frac{3}{8} \alpha^a \right) e^{-\frac{7}{4}\theta} d\alpha
$$

$$
= \frac{1}{8} \left( \frac{1}{3} \right)^{\frac{1}{4}} \left( \frac{1}{4} \right)^{\frac{1}{4}} \ln \theta
$$

Using (12) and (13) in (11), we obtain the desired result. The proof is completed. \(\square\)
Proof. Clearly, we have

\[
\int_0^1 \left| x^\alpha - \frac{3}{8} \right| (3 - 2x) \theta \bar{\theta} \, d\theta = \left( \frac{3}{8} \right)^\frac{1}{\beta} \int_0^1 \left( \frac{3}{8} - x^\alpha \right) (3 - 2x) \theta \bar{\theta} \, d\theta + \int_0^1 \left( x^\alpha - \frac{3}{8} \right) (3 - 2x) \theta \bar{\theta} \, d\theta. \tag{15}
\]

By computing directly, we obtain

\[
\mu_5(\alpha, \theta) = \int_0^1 \left( \frac{3}{8} - x^\alpha \right) (3 - 2x) \theta \bar{\theta} \, d\theta \tag{16}
\]

And on the other hand, we have

\[
\mu_4(\alpha, \theta) = \int_0^1 \left( x^\alpha - \frac{3}{8} \right) (3 - 2x) \theta \bar{\theta} \, d\theta \tag{17}
\]
Using (15) and (16) in (14), we obtain the desired result. The proof is completed. ☐

**Lemma 7.** Let $\alpha$ and $\theta$ be two positive numbers. Then, the following equality holds:

$$\theta_2(\alpha, \theta) = \int_{0}^{1} x^{\alpha+1} \theta^\pi d\pi = \frac{\Gamma_1(\alpha + 2, \alpha + 3, \frac{1}{\theta} \ln \theta)}{\alpha + 2}. \quad (18)$$

**Proof.** By computing directly, we obtain

$$\int_{0}^{1} x^{\alpha+1} \theta^\pi d\pi = \int_{0}^{1} x^{\alpha+1} e^{\theta \ln \theta} d\pi = \frac{\Gamma_1(\alpha + 2, \alpha + 3, \frac{1}{\theta} \ln \theta)}{\alpha + 2}. \quad (18)$$

The proof is completed. ☐

**Lemma 8.** Let $\lambda$, $\beta$ and $\eta$ be a positive numbers. Then, the following equality holds

$$J_1(\lambda, \eta) = \lambda \int_{0}^{1} x^\beta \theta^{\eta} d\eta = \frac{\Gamma_1(\beta + 1, \beta + 2, \lambda \eta \ln \theta)}{\beta + 1}. \quad (19)$$

and

$$J_2(\lambda, \eta) = \lambda \int_{0}^{1} (1 - x)^\beta \theta^{\eta} d\eta = \frac{\Gamma_1(\lambda, \beta + 2, \lambda \eta \ln \theta)}{\beta + 1}. \quad (20)$$

**Proof.** By computing directly, we have

$$J_1(\lambda, \eta) = \lambda \int_{0}^{1} x^\beta \theta^{\eta} d\eta = J_1(\lambda, \eta) = \frac{\Gamma_1(\beta + 1, \beta + 2, \lambda \eta \ln \theta)}{\beta + 1}. \quad (19)$$

For $J_2(\lambda, \eta)$, we have

$$J_2(\lambda, \eta) = \lambda \int_{0}^{1} (1 - x)^\beta \theta^{\eta} d\eta = \frac{\Gamma_1(1, \beta + 2, \lambda \eta \ln \theta)}{\beta + 1}. \quad (20)$$

The proof is completed. ☐

**Lemma 9.** Let $\alpha$ and $\theta$ be two positive numbers. Then, the following equality holds:

$$\Phi(\alpha, \theta) = \int_{0}^{1} x^\alpha - \frac{1}{2} |\theta^\pi| d\pi$$

where

$$\frac{\theta^\pi}{\ln \pi} \left( e^{\pi \left( \frac{1}{2} \right) \ln \theta} + \theta^\frac{1}{2} \left( 1 - \left( \frac{1}{2} \right) \right) \ln \theta^{-1} - \theta^\frac{1}{2} - 1 \right)$$

$$+ \theta^\frac{1}{2} \Gamma_1(\alpha + 2, \lambda \eta \ln \theta). \quad (21)$$
Proof. By computing directly, we have

\[
\int_0^1 |x^\alpha - \frac{3}{8} \theta | \frac{d\theta}{d\alpha} = \int_0^1 \left( \frac{3}{8} - x^\alpha \right) \frac{d\theta}{d\alpha} + \int_0^1 \left( x^\alpha - \frac{3}{8} \right) \frac{d\theta}{d\alpha} = \left( \frac{3}{8} \right) \frac{d\theta}{d\alpha} + \int_0^1 \left( 1 - x^\alpha \right) \frac{d\theta}{d\alpha} \theta - \frac{3}{8} \theta \frac{d\theta}{d\alpha} d\alpha
\]

which yields

\[
\frac{1}{8} \left( 3 f \left( a^\theta b^\frac{1}{\theta} \right) + 2 f \left( a^\frac{1}{2} b^6 \right) + 4 f \left( a^\frac{3}{2} b^\frac{1}{3} \right) \right) \leq \ln b - \ln a + \frac{\partial f}{\partial Y} \left( \frac{a \times 6^{a+1} \Gamma \left( a + 1 + \frac{3}{2} b^\frac{1}{3} \right) \ln \frac{1}{\theta}}{4(a+1)(a+2) \ln b} + b \times F_1 \left( a + 2, a, \frac{1}{2} b^\frac{1}{3} \right) \right)
\]

4. Main Results

Our first result concerns functions whose absolute values of the first derivatives are GA-convex functions.

Theorem 1. Let \( f: [a, b] \to \mathbb{R} \) be a differentiable mapping on \([a, b]\) with \( 0 < a < b \), and \( f' \in L^1([a, b]) \). If \(|f'|\) is GA-convex function, then the following inequality for fractional integrals holds:

\[
\left| \frac{1}{8} \left( 3 f \left( a^\theta b^\frac{1}{\theta} \right) + 2 f \left( a^\frac{1}{2} b^6 \right) + 4 f \left( a^\frac{3}{2} b^\frac{1}{3} \right) \right) \leq \ln b - \ln a + \frac{\partial f}{\partial Y} \left( \frac{a \times 6^{a+1} \Gamma \left( a + 1 + \frac{3}{2} b^\frac{1}{3} \right) \ln \frac{1}{\theta}}{4(a+1)(a+2) \ln b} + b \times F_1 \left( a + 2, a, \frac{1}{2} b^\frac{1}{3} \right) \right)
\]

where \( a > 0 \), \( Y, \mu_1, \mu_2, \mu_3, \mu_4 \) are defined by (1), (12), (13), (16), and (17), respectively, where \( F_1(\cdot; \cdot; \cdot; \cdot) \) and \( F_1(\cdot; \cdot; \cdot; \cdot) \) are the confluent and the incomplete confluent hypergeometric functions, respectively.

Proof. From Lemma 2, and the properties of modulus and GA-convexity of \(|f'|\), we obtain

\[
\left| \frac{1}{8} \left( 3 f \left( a^\theta b^\frac{1}{\theta} \right) + 2 f \left( a^\frac{1}{2} b^6 \right) + 4 f \left( a^\frac{3}{2} b^\frac{1}{3} \right) \right) \leq \ln b - \ln a + \frac{\partial f}{\partial Y} \left( \frac{a \times 6^{a+1} \Gamma \left( a + 1 + \frac{3}{2} b^\frac{1}{3} \right) \ln \frac{1}{\theta}}{4(a+1)(a+2) \ln b} + b \times F_1 \left( a + 2, a, \frac{1}{2} b^\frac{1}{3} \right) \right)
\]
\[ + \int_0^1 \left| \left( 1 - x^a \right) - \frac{3}{8} a b \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \left| \left( x^a - \frac{3}{8} \right) a b \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \frac{1}{4} \left( 1 - x^a \right) a b \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \]

\[ \leq \frac{\ln b - \ln a}{y} \left( \int_0^1 \frac{1}{4} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right) d\alpha \\
+ \int_0^1 \left| \left( 1 - x^a \right) - \frac{3}{8} a b \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \left| \left( x^a - \frac{3}{8} \right) a b \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \frac{1}{4} \left( 1 - x^a \right) a b \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \]

\[ = \frac{\ln b - \ln a}{y} \left( \int_0^1 \frac{1}{4} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right) d\alpha \\
+ \int_0^1 \left| \left( 1 - x^a \right) - \frac{3}{8} a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \left| \left( x^a - \frac{3}{8} \right) a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \frac{1}{4} \left( 1 - x^a \right) a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \]

\[ = \frac{\ln b - \ln a}{y} \left( \int_0^1 \frac{1}{4} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right) d\alpha \\
+ \int_0^1 \left| \left( 1 - x^a \right) - \frac{3}{8} a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \left| \left( x^a - \frac{3}{8} \right) a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \frac{1}{4} \left( 1 - x^a \right) a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \]

\[ = \frac{\ln b - \ln a}{y} \left( \int_0^1 \frac{1}{4} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right) d\alpha \\
+ \int_0^1 \left| \left( 1 - x^a \right) - \frac{3}{8} a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \left| \left( x^a - \frac{3}{8} \right) a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \\
+ \int_0^1 \frac{1}{4} \left( 1 - x^a \right) a b \frac{1}{b} \frac{1}{b} \ln \left( \frac{a b}{b} \right) f' \left( \frac{a}{b} \right) \right| d\alpha \]
which we have used. The proof is completed.

**Theorem 2.** Let \( f: [a, b] \to \mathbb{R} \) be a differentiable mapping on \([a, b]\) with \( 0 < a < b \), and \( f' \in L^1[a, b] \). If \( |f'|^q \) is GA-convex function and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality for fractional integrals holds:

\[
\int_0^b \left| f'(b) \right| \left( \frac{d}{dx} \int_0^x \left| f'(a) \right|^q d\alpha \right)^{\frac{1}{q}} d\alpha + \frac{a^\frac{1}{p} b^{\frac{1}{q}}}{\frac{1}{q}} \int_0^b \left| f'(a) \right|^q d\alpha + \frac{a^\frac{1}{p} b^{\frac{1}{q}}}{\frac{1}{q}} \int_0^b \left| f'(b) \right|^q d\alpha
\]

which we have used. The proof is completed. \( \square \)

The following result deals with the case where the absolute values of the first derivatives at a certain power holds:

**Theorem 2.** Let \( f: [a, b] \to \mathbb{R} \) be a differentiable mapping on \([a, b]\) with \( 0 < a < b \), and \( f' \in L^1[a, b] \). If \( |f'|^q \) is GA-convex function and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then the following inequality for fractional integrals holds:

\[
\int_0^b \left| f'(b) \right| \left( \frac{d}{dx} \int_0^x \left| f'(a) \right|^q d\alpha \right)^{\frac{1}{q}} d\alpha + \frac{a^\frac{1}{p} b^{\frac{1}{q}}}{\frac{1}{q}} \int_0^b \left| f'(a) \right|^q d\alpha + \frac{a^\frac{1}{p} b^{\frac{1}{q}}}{\frac{1}{q}} \int_0^b \left| f'(b) \right|^q d\alpha
\]

where \( a > 0 \), \( \gamma \) is given by (1), and \( \text{I}_1(\cdot;\cdot;\cdot) \) and \( \text{I}_2(\cdot;\cdot;\cdot) \) are the confluent and the incomplete confluent hypergeometric functions, respectively.
Proof. From Lemma 2, the modulus, Hölder’s inequality, GA-convexity of $|f'|^q$, and Lemma 1, we have

$$\frac{1}{b - a} \left( \frac{1}{\gamma} \left( \int_0^1 \left( \frac{1}{\gamma} \int_0^1 f'(a \frac{x - 1}{b - 1})^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}x \right) \right)^{\frac{1}{\gamma}} \leq \frac{\ln b - \ln a}{b - a} \left( \frac{1}{\gamma} \left( \int_0^1 \left( \frac{1}{\gamma} \int_0^1 f'(a \frac{x - 1}{b - 1})^p \mathrm{d}x \right)^{\frac{1}{p}} \mathrm{d}x \right) \right)^{\frac{1}{\gamma}}$$

$$+ \left( \int_0^1 \left( (1 - x)^\alpha - \frac{3}{8} \left( a^{\frac{5-2\alpha}{3}} b^{\frac{1+2\alpha}{3}} \right)^p \right) \mathrm{d}x \right)^{\frac{1}{p}} \left( \int_0^1 \left( f'(a \frac{x - 1}{b - 1})^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^1 \left( x^\alpha - \frac{3}{8} \left( a^{\frac{3-2\alpha}{3}} b^{\frac{1+2\alpha}{3}} \right)^p \right) \mathrm{d}x \right)^{\frac{1}{p}} \left( \int_0^1 \left( f'(a \frac{x - 1}{b - 1})^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$+ \left( \int_0^1 \left( \frac{1}{\gamma} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$\leq \frac{\ln b - \ln a}{b - a} \left( \frac{1}{\gamma} \left( \int_0^1 \left( \frac{5-2\alpha}{2} f'(a)^{\frac{p}{\gamma}} \mathrm{d}x \right) \right)^{\frac{1}{p}} \left( \int_0^1 \left( \frac{5-2\alpha}{2} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$+ \frac{1}{a b} \left( \int_0^1 \left( \frac{5-2\alpha}{2} f'(a)^{\frac{p}{\gamma}} \mathrm{d}x \right) \right)^{\frac{1}{p}} \left( \int_0^1 \left( \frac{5-2\alpha}{2} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$+ \frac{1}{2} \left( \int_0^1 \left( \frac{5-2\alpha}{2} f'(a)^{\frac{p}{\gamma}} \mathrm{d}x \right) \right)^{\frac{1}{p}} \left( \int_0^1 \left( \frac{5-2\alpha}{2} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$\leq \frac{\ln b - \ln a}{b - a} \left( \frac{1}{\gamma} \left( \int_0^1 \left( \frac{5-2\alpha}{2} f'(a)^{\frac{p}{\gamma}} \mathrm{d}x \right) \right)^{\frac{1}{p}} \left( \int_0^1 \left( \frac{5-2\alpha}{2} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$+ \frac{1}{2} \left( \int_0^1 \left( \frac{5-2\alpha}{2} f'(a)^{\frac{p}{\gamma}} \mathrm{d}x \right) \right)^{\frac{1}{p}} \left( \int_0^1 \left( \frac{5-2\alpha}{2} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$

$$+ \frac{1}{2} \left( \int_0^1 \left( \frac{5-2\alpha}{2} f'(a)^{\frac{p}{\gamma}} \mathrm{d}x \right) \right)^{\frac{1}{p}} \left( \int_0^1 \left( \frac{5-2\alpha}{2} |f'(a)|^q + \frac{3}{8} |f'(b)|^q \right) \mathrm{d}x \right)^{\frac{1}{q}}$$
The proof is completed. 

The following theorem represents a variation of Theorem 2.
Theorem 3. Let \( f: [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \([a, b]\) with \(0 < a < b\), and \( f' \in L^1[a, b] \). If \(|f'|^\alpha\) is \( GA \)-convex function and \( q \geq 1 \), then the following inequality for fractional integrals holds:

\[
\left| \frac{1}{\Gamma(a+1)} \int_0^\infty \left( \frac{\Gamma(a+\frac{\alpha}{\beta})}{\Gamma(a+\frac{\alpha}{\beta}+\frac{\alpha}{\beta^-})} + \frac{\Gamma(a+\frac{\alpha}{\beta})}{\Gamma(a+\frac{\alpha}{\beta}+\frac{\alpha}{\beta^-})} \right) \left( \frac{f(a^\frac{\alpha}{\beta} b^\frac{\alpha}{\beta^-})}{f(a^\frac{\alpha}{\beta} b^\frac{\alpha}{\beta^-})} \right)^{\frac{\alpha}{\beta}} \right|^q dx
\]

where \( \alpha > 0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \Phi \) are defined by (1), (12), (13), (16), (17) and (20), respectively, where \( \gamma_1; \gamma_2; \gamma_3 \) and \( \gamma_1 (\ldots, \gamma_3, \ldots) \) are the confluent and the incomplete confluent hypergeometric functions, respectively.

Proof. From Lemma 2, the power mean inequality, and the \( GA \)-convexity of \(|f'|^\alpha\), we have

\[
\left| \frac{1}{\Gamma(a+1)} \int_0^\infty \left( \frac{\Gamma(a+\frac{\alpha}{\beta})}{\Gamma(a+\frac{\alpha}{\beta}+\frac{\alpha}{\beta^-})} + \frac{\Gamma(a+\frac{\alpha}{\beta})}{\Gamma(a+\frac{\alpha}{\beta}+\frac{\alpha}{\beta^-})} \right) \left( \frac{f(a^\frac{\alpha}{\beta} b^\frac{\alpha}{\beta^-})}{f(a^\frac{\alpha}{\beta} b^\frac{\alpha}{\beta^-})} \right)^{\frac{\alpha}{\beta}} \right|^q dx
\]

where \( \alpha > 0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \Phi \) are defined by (1), (12), (13), (16), (17) and (20), respectively, where \( \gamma_1; \gamma_2; \gamma_3 \) and \( \gamma_1 (\ldots, \gamma_3, \ldots) \) are the confluent and the incomplete confluent hypergeometric functions, respectively.
5. Conclusions

This study deals with the fractional Newton–Cotes-type inequalities involving three points by applying one of a novel generalizations of convexity, called geometrically arithmetically convexity. To study this, we have firstly proved a new integral identity. Based on this identity, we
have establish some new Maclaurin-type inequalities for functions whose modulus of the first derivatives are geometrically arithmetically convex via Hadamard fractional integral operators, which are very useful and important fractional integral operators in fractional calculus. We hope that the obtained results could be motivation researchers working in the of fractional calculus, and serve as inspiration for academics to prove novel results using more generalized forms of convexity together with other fractional integral operators.

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**References**


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