The Effects of Nonlinear Noise on the Fractional Schrödinger Equation

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Abstract: The aim of this work is to investigate the influence of nonlinear multiplicative noise on the Cauchy problem of the nonlinear fractional Schrödinger equation in the non-radial case. Local well-posedness follows from estimates related to the stochastic convolution and deterministic non-radial Strichartz estimates. Furthermore, the blow-up criterion is presented. Then, with the help of Itô’s lemma and stopping time arguments, the global solution is constructed almost surely. The main innovation is that the non-radial global solution is given under fractional-order derivatives and a nonlinear noise term.

Keywords: nonlinear fractional Schrödinger equation; non-radial Strichartz estimates; local well-posedness; Itô’s lemma; global solution

MSC: 35A01; 35Q41; 60H15

1. Introduction

We consider the following Cauchy problem of the stochastic nonlinear fractional Schrödinger equation:

\[
\begin{align*}
    idu - \left( (-\Delta)^s u + \lambda |u|^{2^*_s} u \right) dt &= g(u) \circ dW, \quad x \in \mathbb{R}^n, t > 0, \\
    u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( u_0(x) \in H^s \) is non-radial, \( s \in (0, 1) \setminus \left\{ \frac{1}{2} \right\} \), \( \lambda = \pm 1 \), \( \sigma > 0 \), and \( \circ \) denotes the Stratonovitch product. The variable \( g(u) \) is a nonlinear complex-valued function. And the assumptions related to \( g(u) \) are given later. The fractional Laplacian operator \( (-\Delta)^s \) is defined as

\[
(-\Delta)^s u = \mathcal{F}^{-1} \left( |\xi|^{2s} \mathcal{F}(u) \right).
\]

To state the noise term precisely, we introduce a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), which depends on a filtration \((\mathcal{F}_t)_{t \geq 0}\) and a sequence of independent real-valued Brownian motions \((\mathcal{B}_k)_{k \in \mathbb{N}}\). The random process

\[
W(t, x) = \sum_{k=1}^{\infty} \mathcal{B}_k(t) \varphi_k(x), \quad k \in \mathbb{N}
\]
is a Winner process in the space of square-integrable functions, where linear operator \( \phi \) is bounded on \( L^2 \), and \( (\phi_k)_{k \in \mathbb{N}} \) is an Hilbertian basis on \( L^2 \). The equivalent Itô equation for Problem (1) is presented by

\[
\begin{aligned}
\begin{cases}
    i du - \left( (-\Delta)^s u + \lambda |u|^{2s} u \right) dt &= g(u) dW - \frac{i}{2} g'(u) g(u) \phi dt,
    \\
    u(x, 0) &= u_0(x),
\end{cases}
\end{aligned}
\]

(2)

where

\[
F_\phi = \sum_{k=1}^{\infty} (\phi e_k(x))^2.
\]

The mild solution to Problem (2) can be formulated as the following:

\[
\begin{aligned}
    u(t) &= S(t)u_0 - i\lambda \int_0^t S(t-t') |u|^{2s} u dt' + i \int_0^t S(t-t') g(u) dW(t') \\
    &\quad - \frac{i}{2} \int_0^t S(t-t') g'(u) g(u) \phi dt' + \frac{i}{2} \int_0^t S(t-t') g'(u) g(u) \phi dt'.
\end{aligned}
\]

(3)

The last term of (3),

\[
\Psi(t) := i \int_0^t S(t-t') g(u) dW(t')
\]

is called stochastic convolution, where \( S(t) \) denotes the linear fractional Schrödinger propagator.

In recent years, fractional differential equations have been widely investigated for applications in physics and other fields (see [1–6]). For example, Laskin [5,6] introduced a deterministic fractional Schrödinger equation:

\[
i \partial_t u - \left( (-\Delta)^s u + \lambda |u|^{2s} u \right) = 0.
\]

(4)

For more detailed results of Equation (4), one can see [7–14]. In order to describe the results of Equation (4) more clearly, we first recall the notion of scaling-critical for Equation (4), which is based on the dilation symmetry

\[
u(t, x) \rightarrow u_\mu(t, x) = \mu^{-\frac{s}{2}} u \left( \mu^{-2s} t, \mu^{-1} x \right)
\]

for \( \mu > 0 \). It is clear that if \( u \) is a solution to Equation (4), then \( u_\mu \) is also a solution to Equation (4) with the corresponding rescaled initial value. In addition, the scaling-critical Sobolev regularity

\[
s_{\text{crit}} = \frac{n}{2} - \frac{s}{\sigma}
\]

is obtained by using the dilation symmetry, which makes the homogeneous \( H^{s_{\text{crit}}} (\mathbb{R}^n) \)-norm invariant. And \( s_{\text{crit}} \) works as a threshold law for the well-posedness and ill-posedness of Equation (4). Local well-posedness in \( H^s \) for the Cauchy problem of Equation (4) has been investigated in the sub-critical case (\( \alpha \geq \max(0, s_{\text{crit}}) \)) (see [10,13]). For the critical case \( \alpha = s_{\text{crit}} \), Dinh [10] discussed the local existence and uniqueness of solutions to the Cauchy problem. Furthermore, Hong and Sire [13] showed ill-posedness for the Cauchy problem in scaling the super-critical regime (\( \alpha < s_{\text{crit}} \)). In this paper, we consider the well-posedness for Problem (1) in the sub-critical case.

Although the propagation of waves is often described by deterministic models, spatial and temporal fluctuation of parameters of the medium have to be taken into account in some circumstances. This often occurs through a random potential or by describing the propagation of dispersive waves in non-homogeneous or random media. So it is
natural to introduce stochastic Schrödinger equations. In [15,16], de Bouard and Debussche investigated the Cauchy problem of the following equation:

\[ iut - \left( \Delta u + \lambda |u|^{2^*_\sigma} u \right) = u \circ dW, \quad x \in \mathbb{R}^n, t > 0. \] (5)

From deterministic Strichartz estimates, de Bouard and Debussche established that the stochastic convolution almost surely belongs to a right Strichartz space, which allowed them to show local well-posedness in \( L^2 \) and \( H^1 \), respectively. In [15], the mass conversation for the deterministic equation and Itô’s lemma were used to construct a global solution in \( L^2 \) almost surely. The authors also proved that the global well-posedness in \( H^1 \) follows from an \( a \) priori \( H^1 \)-bound based on the conservation of the energy for the deterministic classical Schrödinger equation and Itô’s lemma in [16]. Furthermore, Barbu et al. [17,18] discussed the effects of a finite dimensional Wiener process on the local and global well-posedness. They introduced the scaling transformation \( u = e^{-iW} y \) such that they could apply the fixed-point argument related to the deterministic equation and Strichartz estimates of the evolution operator \( A(s) := i(\Delta + b(s) \nabla + c(s)) \). However, the drawback is this method only works on the finite dimensional Wiener process. For nonlinear multiplicative noise, Ondreját [19] studied the pathwise uniqueness and norm continuity of the local solution to the stochastic wave equation. Brzniak et al. [20] proved the existence and uniqueness of the global solution on a compact \( n \)-dimensional Riemannian manifold. Furthermore, Fabian [21] considered local well-posedness for the classical Schrödinger equation with a nonlinear perturbation term in \( L^2 \) by utilizing deterministic and stochastic Strichartz estimates; then, the global solution was given based on the uniform bound of \( u \). Yang and Chen [22] showed the existence of martingale solutions for the stochastic nonlinear fractional Schrödinger equation on a bounded interval. We refer readers to references [23–30] for more details.

Inspired by [10,15,16,19–22], we consider the Cauchy problem of the nonlinear fractional Schrödinger equation with a nonlinear noise term in this paper. Compared to the known results about Equation (5), we overcome two difficulties in this work: one is that the noise term is nonlinear, and the another one is the loss of derivatives. Under more suitable assumptions regarding the noise term, we established estimates related to the stochastic convolution, which allows us to discuss the local well-posedness. Furthermore, effects of the loss of derivatives can be compensated for by using the Sobolev embedding. Based on deterministic non-radial Strichartz estimates and the estimates associated with the stochastic convolution, we prove local existence and uniqueness of the mild solution (see Theorem 1). In order to study the existence of the global solution, our main tool is Itô’s lemma. More precisely, we investigate the generalizations of the mass and energy in random settings via Itô’s formula. Then, with the help of stopping time arguments, we show the global well-posedness in \( H^\alpha \) almost surely (see Theorem 2). The new contribution in this work is that we show the existence of a non-radial global solution under fractional-order derivatives and a nonlinear noise term.

To better understand the main results of this paper, we first introduce some notations.

1. \( \lceil a \rceil \) is the smallest positive integer greater than or equal to \( a \).
2. Throughout this paper, we call \((r, p)\) is an admissible pair if it satisfies

\[ (r, p) \in [2, \infty]^2, \quad \frac{2}{r} = \frac{n}{2} - \frac{n}{p}, (r, p, n) \neq (2, \infty, 2). \] (6)

In addition, for \((r, p) \in [1, \infty]^2\), we set

\[ \gamma_{r, p} = \frac{n}{2} - \frac{n}{p} - \frac{2s}{r}. \] (7)

The inequality \( \gamma_{r, p} > 0 \) holds true for all admissible pairs except for the case of \((r, p) = (\infty, 2)\).
Let $\tilde{g} : [0, \infty) \to R$ be a function of class $C^1$ such that

$$g(u) = \tilde{g}(\lvert u \rvert^2) u.$$ 

Furthermore, there exists $g'(u)g(u) = \left(\tilde{g}(\lvert u \rvert^2)\right)^2 u$ and $g''(u)g(u)\pi = \lvert g(u) \rvert^2$. For any admissible pair $(r, p)$, there exists a probability space

$$V := L^p\left(\Omega; C(0, \tau; H^\gamma) \cap L^r\left(0, \tau; W^{\beta, p}\right)\right),$$

where $\rho \geq r$, $\gamma = \max\{s, \alpha\}$ and $\beta = \alpha - \gamma_{r, p}$. We assume that $g(u) : V \to V$ is a continuous mapping. For any $u_1, u_2 \in V$, it satisfies

1. $g(0) = 0$,
2. $\lVert g(u_1) - g(u_2) \rVert_V \leq C_1 \lVert u_1 - u_2 \rVert_V$,
3. $\lVert g'(u_1)g(u) - g'(u_2)g(u_2) \rVert_V \leq C_2 \lVert u_1 - u_2 \rVert_V$,

where $C_1$ and $C_2$ are two positive constants, and $\gamma = \max\{s, \alpha\}$.

The main results of this paper can be stated as the following:

**Theorem 1.** Assume that, $0 < \sigma < \frac{2s}{n - 2s}$ and $\phi \in R(\mathbb{L}^2; W^{s, \infty})$. Let $\sigma \geq \frac{\alpha}{2}$. If $u_0$ is a $\mathbb{F}_0$-measurable random variable in $H^\alpha$, then Problem (1) has a unique solution $u \in C(0, \tau; H^\alpha) \cap L^r\left(0, \tau; W^{\beta, p}\right) \mathbb{P} - a.s.,$

where $0 < \tau < \tau^*(u_0)$, $\beta = \alpha - \gamma_{r, p}$, and

$$r > \max(2\sigma, 6).$$

Moreover, there is an alternative

1. either $\tau^*(u_0) = \infty$ almost surely,
2. or $\tau^*(u_0) < \infty$ and $\lim_{t \to \tau^*(u_0)} \lVert u \rVert_{L^r(\Omega; C(0, \tau; H^\alpha))} = \infty$ if $\tau^*(u_0) < \infty$.

**Theorem 2.** Let $n \leq 4$. Given $s \geq \alpha$ and $\phi \in R(\mathbb{L}^2; W^{s, \infty})$. If $\sigma < \frac{2s}{n}$ or $\lambda = -1$, then the solution $u$ of Problem (1) given by Theorem 1 is almost surely global.

This paper is organized as follows. Section 2 introduces some notations and inequalities. The proof of local well-posedness for Problem (1) is given in Section 3. In Section 4, we present the proof of the existence of a global solution to Problem (1).

2. Preliminaries

In this section, we introduce some notations and estimates.

We first introduce some notations.

1. Given $\beta \in \mathbb{R}$ and $1 \leq p \leq \infty$, the non-homogeneous Sobolev space is defined by

$$W^{\beta, p} := \{u \in W^{\beta, p} : \lVert u \rVert_{W^{\beta, p}} := \lVert \Lambda^\beta u \rVert_L < \infty\}, \Lambda := \sqrt{1 + (\Delta)}.$$

And the definition of the homogeneous Sobolev space is given by

$$W_0^{\beta, p} := \{u \in W^{\beta, p} : \lVert u \rVert_{W_0^{\beta, p}} := \lVert \nabla^\beta u \rVert_L < \infty\}.$$
(2) For \( r, p, \rho \geq 1 \), the space of random function \( u(t, x, \omega) \) in \( L^r(0, T; W^{r, p}) \) is denoted by 
\[
L^r(\Omega; L^r(0, T; W^{r, p})) := (\mathbb{E}(\|u\|_{L^r(0, T; W^{r, p})}^p))^{1/p} < \infty.
\]

(3) We review the definition of space \( R(L^2; W^{a, \infty}) \), which is a space of Hilbert–Schmidt operator \( \phi \) from \( L^2 \) into \( W^{a, \infty} \). The norm of operator \( \phi \) in \( R(L^2; W^{a, \infty}) \) is defined as 
\[
\| \phi \|_{R(L^2; W^{a, \infty})} = \| \sum_{k=1}^{\infty} \phi e_k \|_{W^{a, \infty}},
\]
where \( (e_k)_{k \in \mathbb{N}} \) is the orthonormal basis on \( L^2 \).

We next introduce some necessary estimates.

In the deterministic case, the Cauchy problem of deterministic fractional Schrödinger equation
\[
\begin{cases}
  i\partial_t u - (-\Delta)^{\alpha} u = F(x, t), & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+,
  
  u(x, 0) = u_0(x), & x \in \mathbb{R}^n,
\end{cases}
\tag{9}
\]
exists a solution
\[
u(x, t) = S(t)u_0 - i \int_0^t S(t-s)F(s)ds,
\]
where \( S(t) := e^{-i(-\Delta)^{\alpha} t} \) satisfies the following lemma.

**Lemma 1** (Deterministic non-radial Strichartz estimates [13]). Given \( u_0 \in L^2 \) and \( F \in L^r(\mathbb{R}; L^p) \), then for all \((r, p)\) and \((a, b)\) admissible pairs, the group \( S(t) := e^{-i(-\Delta)^{\alpha} t} (t \in \mathbb{R}) \) satisfies 
\[
\| S(t)u_0 \|_{L^r(\mathbb{R}; L^p)} \leq C \| -\nabla \|^{\alpha p} u_0 \|_{L^2}
\]
and
\[
\left\| \int_0^t S(t-s)F(s)ds \right\|_{L^r(\mathbb{R}; L^p)} \leq C \| -\nabla \|^{\alpha p - \gamma p} \| F \|_{L^2(\mathbb{R}; L^{p'})},
\]
where \( \frac{1}{r} + \frac{1}{p} = \frac{1}{p'} + \frac{1}{p} = 1 \).

We recall the following lemmas that are needed for the proof of local well-posedness.

**Lemma 2** (Nonlinear estimate [10]). Let \( f(u) = |u|^{2\sigma} u \) with \( \sigma > 0 \). Given \( 1 < p, q < \infty \) and \( 1 < m \leq \infty \). Assume that
\[
\frac{1}{p} = \frac{1}{q} + \frac{2\sigma}{m}
\]
holds.

If \( 0 \leq [\alpha] < 2\sigma \), there exists \( C = C(n, \sigma, \alpha, p, q, m) \) such that for \( u, v \in L^m \cap W^{a, p} \), we have
\[
\| f(u) - f(v) \|_{W^{a, p}} \leq C(\| u \|_{W^{a, p}}^{2\sigma} + \| v \|_{L^\infty}^{2\sigma}) \| u - v \|_{W^{a, p}} + C(\| u \|_{W^{a, p}}^{2\sigma - 1} + \| v \|_{L^\infty}^{2\sigma - 1})(\| u \|_{W^{a, p}} + \| v \|_{W^{a, p}}) \| u - v \|_{L^\infty}.
\]

In order to estimate the stochastic integral, we introduce the following lemmas.

**Lemma 3** (Burkholder inequality [31]). Given \( \xi \in L^2(0, \infty; L^p) \). Assume that \( \int_0^T g(s)d\mathcal{B}(s) \) 
\((t \geq 0)\) is an \( L^p \)-valued square-integrable martingale with continuous modification and zero mean.

If \( \rho \in [2, \infty) \), then there exists a constant \( C \) such that
\[
\mathbb{E}(\sup_{0 \leq t \leq T} \| \int_0^t g(s)d\mathcal{B}(s) \|_{L^p}^\rho) \leq C \mathbb{E}[\int_0^T \| g(s) \|_{L^p}^2 ds]^\rho,
\]
where \( C \) is independent of \( T \) and \( g \).
3. Local Well-Posedness

3.1. Estimations

In this subsection, we introduce some estimations that are important for proving Theorem 1. First, we need to estimate the stochastic convolution

\[ \Psi(t) = i \int_0^t S(t - t') g(u) dW(s) \]

in \( L^\rho(\Omega; C(0, T; H^p)) \cap L^{\rho'}(0, T; W^{\beta, p'}) \).

**Lemma 4.** Let the conditions for \( r \) and \( q \) be the same as in Theorem 1. If \( T > 0, \rho \geq r, \) and \( u \in L^q(\Omega; L^\infty(0, T; H^p)), \) it holds that \( \Psi(t) \in L^q(\Omega; L^{\rho}(0, T; W^{\beta, p'})) \) and

\[ \| \Psi(t) \|_{L^q(\Omega; L^{\rho}(0, T; W^{\beta, p'}))} \leq C \| \phi \|_{R(L^2; W^{\alpha, \infty})} T^{1 - \frac{2}{p} - \frac{1}{2}} \| u \|_{L^q(\Omega; L^\infty(0, T; H^p))}. \]

**Proof.** In an application of the Hölder’s inequality and \( \rho \geq r, \) we have

\[
\begin{align*}
&\| \Psi(t) \|^\rho_{L^q(\Omega; L^{\rho}(0, T; W^{\beta, p'}))} \\
&\leq T^{\frac{1}{2} - 1} E \left( \int_0^T \| \sum_{k=0}^\infty \int_0^t S(t - t') \Lambda^\beta (g(u) \phi e_k) d\mathcal{B}_k \|^\rho_{L^p} dt \right) \\
&\leq T^{\frac{1}{2} - 1} E \left( \sup_{0 \leq t \leq T} \| \sum_{k=0}^\infty \int_0^t S(t - t') \Lambda^\beta (g(u) \phi e_k) d\mathcal{B}_k \|^\rho_{L^p} \right).
\end{align*}
\]

Employing the Burkholder inequality yields

\[
E \left( \sup_{0 \leq t \leq T} \| \sum_{k=0}^\infty \int_0^t S(t - t') \Lambda^\beta (g(u) \phi e_k) d\mathcal{B}_k \|^\rho_{L^p} \right) \leq C E \left( \sum_{k=0}^\infty \int_0^T \| S(t - t') \Lambda^\beta (g(u) \phi e_k) \|^2_{L^p} dt' \right)^{\frac{\rho}{2}}.
\]

According to the decay estimate in [13]:

\[ \| S(t)f \|_{L^p} \leq c |t|^{-\frac{n}{2}(1 - \frac{2}{p})} \| \nabla |n| f \|_{L^{p'}} \]

we have

\[ \| S(t - t') \Lambda^\beta (g(u) \phi e_k) \|_{L^p} \leq c |t - t'|^{-\frac{n}{2}(1 - \frac{2}{p})} \| \Lambda^\alpha (g(u)) \phi e_k \|_{L^{p'}} \]

where \( \frac{1}{p} + \frac{1}{p'} = 1. \) Application of the Hölder’s inequality and \( \frac{1}{p} - \frac{\beta}{n} \leq \frac{1}{p} - \frac{n}{p} \) implies that

\[
\begin{align*}
&\| S(t - t') \Lambda^\beta (g(u) \phi e_k) \|_{L^p} \\
&\leq C |t - t'|^{-\frac{n}{2}} \| \Lambda^\alpha (g(u)) \phi e_k \|_{L^\infty} + \| g(u) \|_{L^{p'}} \| \Lambda^\alpha e_k \|_{L^\infty} \\
&\leq C |t - t'|^{-\frac{n}{2}} \| g(u) \|_{W^{\beta, p'}} \| \phi e_k \|_{W^{\alpha, \infty}}.
\end{align*}
\]

Combining (11) and (12), we obtain

\[
E \left( \sum_{k=0}^\infty \int_0^T \| S(t - t') \Lambda^\beta (g(u) \phi e_k) \|^2_{L^p} dt' \right)^{\frac{\rho}{2}} \leq C \| \phi \|_{R(L^2; W^{\alpha, \infty})} T^{\frac{1}{2} - 1} E \left( \| u \|^\rho_{L^q(\Omega; L^{\rho}(0, T; W^{\beta, p'}))}. \right)
\]
Then, substituting (13) into (10), for \( \rho \geq r > \max(2\sigma, 6) \), we have
\[
\| \Psi(t) \|_{L^p(\Omega; L^\infty(0,T;W^{n,p}))}^p \leq C \| \phi \|_{R(L^2; W^{n,\infty})}^p T^{n(\frac{1}{2} - \frac{3}{2} - \frac{1}{r})} \| u \|_{L^p(\Omega; L^\infty(0,T;H^p))}^p
\]
which complements the proof of the lemma. \( \square \)

The estimate about \( \Psi(t) \) on \( L^p(\Omega; L^\infty(0,T;H^a)) \) is presented as follows:

**Lemma 5.** Suppose that the conditions for \( r \) and \( q \) are the same as in Theorem 1. Let \( T > 0 \) and \( \rho \geq r \). If \( u \in L^p(\Omega; L^r(0,T;W^{\beta,p})) \), we have \( \Psi(t) \in L^p(\Omega; L^\infty(0,T;H^a)) \) and
\[
\| \Psi(t) \|_{L^p(\Omega; L^\infty(0,T;H^a))} \leq C \| \phi \|_{R(L^2; W^{n,\infty})} T^{n(\frac{1}{2} - \frac{1}{r})} \| u \|_{L^p(\Omega; L^r(0,T;W^{\beta,p}))}. \tag{14}
\]

**Proof.** Applying the Burkholder inequality, we get
\[
\| \Psi(t) \|_{L^p(\Omega; L^\infty(0,T;H^a))} \leq CE\left( \sum_{k=0}^{\infty} \int_0^T \| S(t-t') \Lambda^a(\mathcal{G}(u)\phi \delta_k) \|_{L^2}^2 dt' \right)^{\frac{p}{2}}.
\tag{15}
\]

Furthermore, employing the fact that linear operator \( S(t) \) is isometrically isomorphic on \( L^2 \) and the Hölder’s inequality leads to
\[
\mathbb{E}\left( \left( \sum_{k=0}^{\infty} \int_0^T \| S(t-t') \Lambda^a(\mathcal{G}(u)\phi \delta_k) \|_{L^2}^2 dt' \right)^{\frac{p}{2}} \right)
\leq C \| \phi \|_{R(L^2; W^{n,\infty})} \mathbb{E}\left( \left( \int_0^T \| \mathcal{G}(u) \|_{L^2}^2 dt' \right)^{\frac{p}{2}} \right).
\tag{16}
\]

Thanks to \( \frac{1}{p} - \frac{\beta}{r} \leq \frac{1}{2} - \frac{\beta}{a} \), according to the Sobolev embedding theorem, we obtain \( W^{\beta,p} \subset H^a \). Then, substituting (16) into (15), we get
\[
\| \Psi(t) \|_{L^p(\Omega; L^\infty(0,T;H^a))} \leq C \| \phi \|_{R(L^2; W^{n,\infty})} T^{n(\frac{1}{2} - \frac{1}{r})} \mathbb{E}\left( \| u \|_{L^p(0,T;W^{\beta,p})}^p \right).
\]

We derive the conclusion (14). \( \square \)

Moreover, the following lemmas are necessary for later processing.

**Lemma 6 ([15]).** Let \( u_1, u_2 \in Y_T \), and \( u_3 \in L^r(\Omega; L^1(0,T;W^{\beta,p})) \). Then, we have
\[
\| (\theta_R(u_1) - \theta_R(u_2))u_3 \|_{L^r(\Omega; L^1(0,T;W^{\beta,p}))} \leq C \| u_1 - u_2 \|_{Y_T} \| u_3 \|_{L^r(\Omega; L^1(0,T;W^{\beta,p}))},
\]
for some positive constant \( c \).

**Lemma 7.** Suppose that the stopping time \( \tau \) satisfies \( \tau \leq \tau^*(u_0) \) and \( \Psi(t) \in L^r(\Omega; L^r(0,T;W^{\beta,p})) \). Given a solution to Problem (1), \( u \in L^\infty(0,\tau; H^a) \) almost surely. Then, there exists
\[
h(\omega) = C_0(1 + \| u \|_{L^{2r+1}(\Omega; L^\infty(0,T;H^p))} + \| \Psi(t) \|_{L^r(\Omega; L^r(0,T;W^{\beta,p}))}),
\]
such that
\[
\| u \|_{L^r(\Omega; L^r(0,T;W^{\beta,p}))} \leq C(\tau^*(u_0))h(\omega)^{\frac{2r}{r+1}}.
\]
Proof. For $0 < T_1 \leq \tau$, according to the Young’s inequality and Lemmas 4–5, we obtain

$$\| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))} \leq C \| u_0 \|_{L'(\Omega; H^p)} + C T_1^{1-\frac{\alpha}{r}} \left( \frac{1}{2^{\frac{\beta}{p-1}}} \| u \|_{L'(\Omega; L^\infty(0,T_1; H^p))} + \frac{2^\alpha}{2^{\frac{\beta}{p-1}}} \| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))} \right) + C \| \phi \|_{R(L^2; W^{\alpha,\infty})} T_1^{1-\frac{1}{r}} \| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))}.$$ 

Let $T_1 \leq T_2$ and $C T_2^{1-\frac{1}{r}} \| \phi \|_{R(L^2; W^{\alpha,\infty})} \leq \frac{1}{2}$. If there exists a large enough

$$C_0 := C(\| u_0 \|_{L'(\Omega; H^p)}, \tau^r(u_0), r, \sigma)$$

then

$$\| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))} \leq h(\omega) + C T_1^{\lambda} \| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))}$$

where $\lambda = 1 - \frac{2\beta}{r}$ and

$$h(\omega) = C_0 (1 + \| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))} + \| \Psi(t) \|_{L'(\Omega; L^r(0,T; W^{\beta,p}))}).$$

If we choose $T_1 = \inf\{\tau, T'\}$ with

$$T' = 2^{\frac{1}{\lambda}} h(\omega) \frac{2\beta}{\lambda},$$

it follows that

$$\| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))} \leq 2h(\omega).$$

We divide the interval $[0, \tau]$ into many sub-intervals: namely,

$$[0, \tau] = \sum_{j=0}^{\frac{\tau}{t_1}} [j T_1, (j+1) T_1].$$

Then, we have

$$\| u \|_{L'(\Omega; L^r(T_j, (j+1) T_1; W^{\beta,p}))} \leq 2h(\omega).$$

Hence, we arrive at

$$\| u \|_{L'(\Omega; L^r(0,T_1; W^{\beta,p}))} \leq \frac{1}{t_1} h(\omega) \leq C(\tau^r(u_0)) h(\omega) \frac{2\beta}{\lambda} + 1.$$

\[ \tag{17} \]

3.2. Proof of Theorem 1

Now, we give the proof of Theorem 1. Denote

$$X_t = C(0,t; H^p) \cap L'(0,t; W^{\beta,p}).$$

We consider $\theta \in C^\infty$ as a cut-off function on $\mathbb{R}^+$, and

$$\theta(u) = \theta \left( \frac{\| u \|_{Y_t}}{R} \right) := \begin{cases} 1, & r \in [0,R], \\ 0, & r \in [2R, \infty), \end{cases} \tag{18}$$

where $R \in \mathbb{N}^+$ and $Y_t := L'(\Omega; X_t)$. 

Since the nonlinear term of Problem (1) is local Lipschitz, we introduce the following Cauchy problem of a truncated equation
\[
\left\{
\begin{array}{l}
    idu_R - \left( -\Delta \right)^{\sigma} u_R + \lambda \theta(u_R) |u_R|^{2^*} u_R \, dt \, = \, \theta(u_R) g(u_R) \circ d\mathbb{W}, \quad x \in \mathbb{R}^n, \, t > 0, \quad (19)
    u_R(x, 0) = u_0(x), \quad x \in \mathbb{R}^n,
\end{array}
\right.
\]

The mild solution to Problem (19) is
\[
u_R(t) = S(t)u_0 - i\lambda \int_0^t S(t - t') \theta(u_R) |u_R|^{2^*} u_R \, dt' - i \int_0^t S(t - t') \theta(u_R) g(u_R) \circ d\mathbb{W}(t') - \frac{1}{2} \int_0^t S(t - t') \theta(u_R) g'(u_R) g(u_R) F_q \circ d\mathbb{W}(t').
\]

Define a mapping as following
\[
\psi_R(u_R) = S(t)u_0 - i\lambda \int_0^t S(t - t') \theta(u_R) |u_R|^{2^*} u_R \, dt' - i \int_0^t S(t - t') \theta(u_R) g(u_R) \circ d\mathbb{W}(t') - \frac{1}{2} \int_0^t S(t - t') \theta(u_R) g'(u_R) g(u_R) F_q \circ d\mathbb{W}(t').
\]

Step 1: We show that for a fixed \( R > 0 \), there exists \( T > 0 \) depending only on \( R \) such that Problem (8) has a unique solution \( u_R \) that satisfies \( u_R \in X_\nu \) almost surely.

We consider the space
\[
E(T_R, R) = \{ u_R \in Y_{T_R} : \| u_R \|_{Y_{T_R}} \leq R \}
\]

with the following metric:
\[
d(u_R, v_R) = \| u_R - v_R \|_{L^1(\Omega; L^\infty(0, T_{R}; H^p))} + \| u_R - v_R \|_{L^1(\Omega; L^\infty(0, T_{R}; \mathbb{W}^p)).}
\]

We need to prove that \( \psi_R \) is a contraction on \( E(T_R, R) \) if \( T_R \) depends on \( R \).

Owing to \( r > \max \{2\sigma, 6\} \) and \( \gamma_{r, p} > 0 \), it holds that \( \alpha > \frac{2}{2} - \frac{3}{r} \). Combining (6) and (7), we can see \( \frac{1}{p} - \frac{\alpha - \gamma_{r, p}}{n} < 0 \). Then, the Sobolev embedding implies
\[
W^{\delta, p} \subset L^\infty.
\]

For any \( u_R \in E(T_R, R) \), using Strichartz estimates, \( W^{\delta, p} \subset L^\infty \) and Lemmas 4–5, we get
\[
\begin{align*}
\| \psi_R(u_R) \|_{L^1(\Omega; L^\infty(0, T_{R}; H^p))} & \leq C \| u_0 \|_{L^1(\Omega; H^p)} + C(\mathbb{E} \| u_R \|^{2^*} u_R \| L^1(0, T_{R}; H^p))^{\frac{1}{2}} + C(\mathbb{E} g'(u_R) g(u_R) F_q \| L^1(0, T_{R}; H^p))^{\frac{1}{2}} \\
& \leq C \| u_0 \|_{L^1(\Omega; H^p)} + C T_{R}^{1 - \frac{2^*}{2}} (\mathbb{E} \| u_R \|^{2^*} \| L^1(0, T_{R}; H^p))^{\frac{1}{2}} \\
& \leq C \| u_0 \|_{L^1(\Omega; H^p)} + C(\| \phi \|_{L^2(\mathbb{W}^p, \infty)}) T_{R}^{1 - \frac{2^*}{2}} (\mathbb{E} \| u_R \|^{2^*} \| L^1(0, T_{R}; \mathbb{W}^p)))^{\frac{1}{2}} \\
& \leq C \| u_0 \|_{L^1(\Omega; H^p)} + C(\| \phi \|_{L^2(\mathbb{W}^p, \infty)}) T_{R}^{1 - \frac{2^*}{2}} (\mathbb{E} \| u_R \|^{2^*} \| L^1(0, T_{R}; \mathbb{W}^p)))^{\frac{1}{2}} \\
& \leq C \| u_0 \|_{L^1(\Omega; H^p)} + C(\| \phi \|_{L^2(\mathbb{W}^p, \infty)}) T_{R}^{1 + \frac{2^*}{2}}
\end{align*}
\]

where \( \lambda = \min \{1 - \frac{2^*}{2}, \frac{1}{2} - \frac{3}{r} \} \). In a similar way, we obtain
\[
\| \psi_R(u_R) \|_{L^1(\Omega; L^\infty(0, T_{R}; H^p))} \leq C \| u_0 \|_{L^1(\Omega; H^p)} + C(\| \phi \|_{L^2(\mathbb{W}^p, \infty)}) T_{R}^{1 + \frac{2^*}{2}} R^{2^* + 1}.
\]

Fix \( C \| u_0 \|_{L^1(\Omega; H^p)} = \frac{\delta}{2} \) such that
\[
2^{2^* + 1} C 2^{2^* + 1} T_{R}^{1 + \frac{2^*}{2}} \| u_0 \|^{2^* + 1} \| L^1(\Omega; H^p)) < 1,
\]
then, we obtain that \( \psi_R \) is well-defined on \( E(T_R, R) \) and \( T_R \lesssim R^{-\frac{2\sigma}{T}} \).

Taking \( u_{R1}, u_{R2} \in E(T_R, R) \), it is clear that

\[
\begin{align*}
\| \psi_R u_{R1} - \psi_R u_{R2} \| \lesssim & \| \psi_R u_{R1} - \psi_R u_{R2} \|_{L_t'(\Omega; L^\infty(0,T_R; H^s))} \\
& + \| \psi_R u_{R1} - \psi_R u_{R2} \|_{L_t'(\Omega; L^\infty(0,T_R; W^\beta_\rho))}.
\end{align*}
\] (22)

By the Strichartz estimates, it follows that

\[
\| \psi_R u_{R1} - \psi_R u_{R2} \|_{L_t'(\Omega; L^\infty(0,T_R; W^\beta_\rho))} \lesssim C \| \theta(u_{R1}) - \theta(u_{R2}) \|_{L_t'(\Omega; L^\infty(0,T_R; H^s))} + \| \theta(u_{R2}) \|_{L_t'(\Omega; L^\infty(0,T_R; W^\beta_\rho))} + \| \theta(u_{R1}) \|_{L_t'(\Omega; L^\infty(0,T_R; W^\beta_\rho))}
\]

\[
= I + II + III.
\]

For the purpose of estimating \( I \) for \( \ell = 1, 2 \), we set

\[
i^R_\ell = \sup \{ t \in [0, T_R], \| u_{Rt} \|_{H^s} \leq 2R \},
\]

where

\[
[0, T_R] = \left[ 0, i^R_1 \right] \cup \left[ i^R_1, i^R_2 \right] \cup \left[ i^R_2, T_R \right].
\]

Without loss of generality, we assume \( i^R_1 \leq i^R_2 \) such that

\[
I \leq \| \theta(u_{R1}) - \theta(u_{R2}) \|_{H^s} + \| \theta(u_{R2}) \|_{H^s} + \| \theta(u_{R1}) \|_{H^s}.
\] (23)

To estimate \( I_1 \), we recall the following lemma from [15]. According to Lemma 6, it is easy to get

\[
I_1 \leq CT^{1 - \frac{2\sigma}{T}} R^{2\sigma + 1} \| u_{R1} - u_{R2} \|_{Y_T}.
\] (24)

For \( t \in \left[ i^R_1, i^R_2 \right] \), there exists \( \theta(u_{R1}) = 0 \). Thus, we can rewrite \( I_3 \) as

\[
I_3 = \| \theta(u_{R2}) - \theta(u_{R1}) \|_{H^s}.
\]

Using Lemma 2, it follows that

\[
I_2 \leq CT^{1 - \frac{2\sigma}{T}} R^{2\sigma} \| u_{R1} - u_{R2} \|_{Y_{T_R}}.
\] (25)

Similarly, we obtain

\[
I_3 \leq CT^{1 - \frac{2\sigma}{T}} R^{2\sigma + 1} \| u_{R1} - u_{R2} \|_{Y_{T_R}}
\] (26)

Then, it follows from (23)–(26) that

\[
I \leq CT^{1 - \frac{2\sigma}{T}} R^{2\sigma + 1} \| u_{R1} - u_{R2} \|_{Y_{T_R}}.
\]

Thanks to Hölder’s inequality, one can check that

\[
III \leq C \| \phi \|_{R(\Omega; W^{\beta_\rho}_\rho)} T^{1 - \frac{2\sigma}{T}} R^{2\sigma + 1} \| u_{R1} - u_{R2} \|_{Y_{T_R}}.
\]
Combining all the above estimates, we conclude
\[
\| \psi_R u_{R1} - \psi_R u_{R2} \|_{L^p((0,T_R];W^{\beta,p})} \leq C(\| \phi \|_{R(L^2;W^{\alpha,\infty})}) R^{2\alpha + 1} T_R^3 \| u_{R1} - u_{R2} \|_{Y_{\psi}}.
\]

Similar to the estimate of the first term on the right-hand side of (22), we arrive at
\[
\| \psi_R u_{R1} - \psi_R u_{R2} \|_{L^p((0,T_R);W^{\beta,p})} \leq C(\| \phi \|_{R(L^2;W^{\alpha,\infty})}) R^{2\alpha + 1} T_R^3 \| u_{R1} - u_{R2} \|_{Y_{\psi}}
\]
as long as \( \rho \geq r \). Thus, we have
\[
d(\psi_R u_{R1} - \psi_R u_{R2}) \leq C(\| \phi \|_{R(L^2;W^{\alpha,\infty})}) R^{2\alpha + 1} T_R^3 d(u_{R1} - u_{R2}),
\]
which means that \( \psi_R \) is a contraction mapping on \( E(T_R, R) \).

Step 2: We show that problem (19) has a unique solution on \([0, T_0]\).

Let \( m \in \mathbb{N}^+ \) and \( u^m_R \in Y_{mT_R} \). In order to extend \( u^m_R \) to \([mT_R, (m + 1)T_R] \), we introduce
\[
u^{m+1}_R := \begin{cases} u^m_R(t), & t \in [0, mT_R], \\
u^m_R(t - mT_R), & t \in [mT_R, (m + 1)T_R]. \end{cases}
\]

An application of the Duhamel formulation gives rise to
\[
y^m_R(\tilde{t}) = S(\tilde{t}) u(mT_R) - i \lambda \int_0^{\tilde{t}} S(\tilde{t} - s) \varphi^m_R(y^m_R)|y^m_R|^{2\alpha} y^m_R dt' - i \int_0^{\tilde{t}} S(t - t') \varphi^m_R(y^m_R) g(y^m_R) dW(t') - \frac{1}{2} \int_0^{\tilde{t}} S(t - t') \varphi^m_R(y^m_R) \delta(y^m_R) g(y^m_R) F_{\delta} dt',
\]
where \( \tilde{t} = t - mT_R, t \in [mT_R, (m + 1)T_R] \), and \( \varphi^m_R \in C^\infty \) is a cut-off function given by
\[
\varphi^m_R(y^m_R) = \varphi\left( \frac{\| y^m_R \|_{Y_{\psi,R}}} {R} \right) := \begin{cases} 1, & r \in [0, R], \\
0, & r \in [2R, \infty). \end{cases}
\]

We define a map
\[
\psi y^m_R(\tilde{t}) = S(\tilde{t}) u(mT_R) - i \lambda \int_0^{\tilde{t}} S(\tilde{t} - s) \varphi^m_R(y^m_R)|y^m_R|^{2\alpha} y^m_R dt' - i \int_0^{\tilde{t}} S(t - t') \varphi^m_R(y^m_R) g(y^m_R) dW(t') - \frac{1}{2} \int_0^{\tilde{t}} S(t - t') \varphi^m_R(y^m_R) \delta(y^m_R) g(y^m_R) F_{\delta} dt'.
\]

In the same spirit as the proof of Step 1, we obtain
\[
\| \psi y^m_R - \psi y^m_R \|_{Y_{\tilde{t}}} \leq C(\| \phi \|_{R(L^2;W^{\alpha,\infty})}) R^{2\alpha + 1} T_R^3 \| y^m_R - y^m_R \|_{Y_{\psi}}.
\]

The constant on the right-hand side of (30) is the same as the constant in (27), so we can conclude that the mapping \( \psi \) is a contraction map on
\[
E(T_R, R) := \{ u \in Y_{\tilde{t}} \| u \|_{Y_{\psi}} \leq R \}.
\]

From (28), we know that for \( t \in [0, mT_R] \), there exists
\[
u^{m+1}_R = u^m_R.
\]

Then, we turn our attention to \( t \in [mT_R, (m + 1)T_R] \). It follows from
\[
S(t_1)S(t_2) = S(t_2)S(t_1) = S(t_1 + t_2).
\]
Thus, there exists a local solution $u$ and $t$ and for $t \in [0, mT_R]$:

$$\theta_R(u_{m}^{m}) = \theta_R\left(u_{R}^{m+1}\right)$$

and for $t \in [mT_R, (m + 1)T_R]$:

$$\varphi_R(u_{m+1}^{m}) = \theta_R\left(u_{R}^{m+1}\right).$$

From (31)–(33), we obtain

$$S\left(\int_{0}^{t} S(t - t') \theta_R\left(u_{R}^{m+1}\right) \left|u_{R}^{m+1}\right|^{2} \left|u_{R}^{m+1}\right| ds\right)$$

$$= -i \int_{0}^{t} S(t - t') \theta_R\left(u_{R}^{m+1}\right) \left|u_{R}^{m+1}\right| \left|u_{R}^{m+1}\right| ds$$

$$= \frac{1}{2} \int_{0}^{t} S(t - t') \theta_R\left(u_{R}^{m+1}\right) \left|u_{R}^{m} - u_{R}^{m+1}\right|^{2} \left|u_{R}^{m+1}\right| ds,$$

Based on (28), (31) and (34), we can show that for $t \in [0, (m + 1)T_R]$:

$$u_{R}^{m+1}(t) = \Gamma_R u_{R}^{m+1}(t)$$

and

$$\|\psi_Ru_{R}^{m+1} - \psi_R\tilde{u}_{R}^{m+1}\|_{Y_{(m+1)T_R}} \leq C(\|\phi\|_{R(L^2;W^{g,0})}) R^{2g+1} T_R^{\lambda} \|u_{R}^{m+1} - \tilde{u}_{R}^{m+1}\|_{Y_{(m+1)T_R}}.$$

As a result, we conclude that $\psi_R$ is a contraction map in $Y_{(m+1)T_R}$. Taking $m = \left[\frac{T_0}{T_R}\right] + 1$ gives rise to $u_R := u_R^m$ on $[0, T_0]$.

Then, we need to present the uniqueness. Suppose that $\pi_R(t, x)$ with $t \in [0, T]$ is another local solution to (19). It follows that

$$\|u_{R} - \pi_R\|_{Y_{R} \times T} \leq C(\|\phi\|_{R(L^2;W^{g,0})}) R^{2g+1} T_R^{\lambda} \|\tilde{u}_{R}^{m+1} - \tilde{u}_{R}^{m+1}\|_{Y_{R} \times T},$$

which means that $u = \pi$ almost surely for $t \in [0, T_R \wedge T]$. By iterating, it follows that $u = \pi$ almost surely for $t \in [0, T_m)$, where $T_m = mT_R \wedge T$. The assertion follows from $T_m = T$ if $m$ is large enough.

We set

$$T_R = \sup\{t \in [0, T_0], \|u\|_{Y_t} \leq R\}$$

and denote

$$u = u_R$$

on $[0, T_R]$. Moreover, we know from [15] that $T_R$ increases with $R$ and $u_R = u_{R+1}$ on $[0, T_R]$. Thus, there exists a local solution $u$ of Problem (1) on $[0, \tau^*(u_0)]$ for which

$$\tau^*(u_0) = \lim_{R \to \infty} T_R.$$
almost surely. Then, it is obvious that
\[ u \in Y_\tau \]
for all stopping times \( \tau < \tau^*(u_0) \). Furthermore, the uniqueness of this local solution is consistent with the uniqueness of the solution to Equation (20) in \( Y_\tau \).

Step 3: We give
\[
\lim_{t \to \tau^*(u_0)} \sup_{s \leq t} \| u(s) \|_{H^\kappa} = \infty \quad \text{P} - \text{a.s.}
\]
for \( \tau^*(u_0) < \infty \).

For \( \tau^*(u_0) < \infty \), set
\[
\bar{T}_R = \sup \{ t \in [0, \tau^*(u_0)), \| u \|_{L'((\Omega;L^\infty(0,t;H^\kappa)))} \leq R \}.
\]

Using (17), we derive
\[
\begin{align*}
\| u \|_{L'((\Omega;L^\infty(0,\bar{T}_R;W^{\beta,p}))} & \leq C(\tau^*(u_0))h(\omega)\left(\frac{2}{\pi}+1\right) \\
& \leq C(\tau^*(u_0))(1 + R^{2\alpha+1} + C(\| \phi \|_{R(L;W^{\alpha,\infty})}) R^{\frac{2\alpha+1}{2}}.
\end{align*}
\]

Suppose that
\[
P(\| u \|_{L'((\Omega;L^\infty(0,\tau^*(u_0);H^\kappa)))} < \infty \quad \text{and} \quad \tau^*(u_0) < \infty) > 0.
\]

Thus, if \( R \) is large enough, we get
\[
P\left( \bar{T}_R = \tau^*(u_0) \right) > 0,
\]
which implies that for \( \tau^*(u_0) < \infty \):
\[
\lim_{t \to \tau^*(u_0)} \| u \|_{L'((\Omega;L^\infty(0,t;H^\kappa)))} < \infty.
\]

From Equation (38), it follows that
\[
\lim_{t \to \tau^*(u_0)} \| u \|_{L'((\Omega;L^\infty(0,t;W^{\beta,p}))} < \infty.
\]

Thus, we arrive at
\[
\lim_{t \to \tau^*(u_0)} \| u \|_{Y_t} < \infty.
\]

However, (35)–(37) lead to that, for \( \tau^*(u_0) < \infty \),
\[
\lim_{t \to \tau^*(u_0)} \| u \|_{Y_t} = \infty,
\]
which contradicts (39).

Hence, we get
\[
\lim_{t \to \tau^*(u_0)} \| u \|_{L'((\Omega;L^\infty(0,t;H^\kappa)))} = \infty \quad \text{if} \quad \tau^*(u_0) < \infty.
\]

The proof of Theorem 1 is now complete.
4. The Existence of a Global Solution

In this section, we study the existence of the global solution. Firstly, some conclusions about invariant quantities of the deterministic nonlinear fractional Schrödinger equation are recalled. It holds that the mass

\[ M(u) = \int_{\mathbb{R}^n} |u|^2 \, dx \]

and the energy

\[ Q(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\nabla)^{\sigma} u|^2 \, dx - \lambda \int_{\mathbb{R}^n} |u|^{2s+2} \, dx. \tag{40} \]

4.1. The Stochastic Identity of \( M(u) \) and \( H(u) \)

For later use, it is convenient to give Itô’s lemmas for \( M(u) \) and \( H(u) \).

Proposition 1. The assumptions about \( u_0, \sigma, \) and \( \phi \) are the same as in Theorem 1. If \( u \) is the solution to Problem (1), then for all stopping times \( \tau < \tau^*(u_0) \), there exists

\[ M(u(\tau)) = M(u_0). \]

Proof. Since the nonlinear term of Problem (1) is local Lipschitz, it is natural to use a truncation argument. Thus, we consider Problem (19). Applying Itô’s lemma to \( M(u_R) \), we have

\[
M(u_R(t)) = M(u_0) + 2\text{Im} \left( \int_0^t \int_{\mathbb{R}^n} \theta(u_R) |u_R|^{2s+2} \, dx \, ds \right) \\
- \text{Re} \left( \int_0^t \int_{\mathbb{R}^n} \theta(u_R) \pi' \chi(u_R) F \phi \, dx \, ds \right) \\
+ \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^n} \theta(u_R) |g(u_R)|^2 |\phi_k|^2 \, dx \, ds \\
+ 2\text{Im} \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^n} \theta(u_R) \pi g(u_R) (\phi_k) \, dx \, dB_k.
\]

Taking a sufficiently large \( R \) and \( t = \tau \), we conclude

\[ M(u(\tau)) = M(u_0). \]

□

Proposition 2. The assumptions about \( u_0, \sigma, \) and \( \phi \) are the same as in Theorem 1. For all stopping times \( \tau < \tau^*(u_0) \), there exists

\[
Q(u(\tau)) = Q(u_0) + \text{Im} \left( \int_0^\tau \int_{\mathbb{R}^n} (-\nabla)^{\sigma}(\nabla)^{\sigma}(g(u)\phi_k) \, dx \, dB_k \right) \\
- \frac{1}{2} \text{Re} \left( \int_0^\tau \int_{\mathbb{R}^n} \chi(u) F \phi_k (-\Delta)^{\tau} \phi_k \, dx \, ds \right) \\
+ \frac{1}{2} \sum_{k=1}^{\infty} \int_0^\tau \left( (-\nabla)^{\sigma}(g(u)\phi_k) \right)^2 \, dx \, ds, \tag{41}
\]

where \( u \) is given by Theorem 1.
Proof. Applying Itô’s lemma and fractional integration by parts to $Q(u)$, we obtain

$$Q(u(t)) = Q(u_0) + \text{Im} \sum_{k=0}^{\infty} \int_{R^k} \int_0^t (-\nabla)^s \pi_R(-\nabla)^s (g(u_R) \phi e_k) d\mathbb{B}_k dx$$

$$- \text{Im} \int_0^t \int_{R^k} \lambda(1-\theta(u_R)) \times (\mu g^{2\nu} u_R) ((-\Delta)^s \pi_R) dx ds$$

$$- \frac{1}{2} \text{Re} \int_0^t \int_{R^k} (g(u_R) g(u_R) F_{\phi}) (-\Delta)^s \pi_R dx ds$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \int_0^t \int_{R^k} (\nabla)^s (g(u_R) \phi e_k)^2 dx ds$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \int_0^t \int_{R^k} |g(u_R)|^2 F_{\phi} dx ds$$

Choosing a large enough $R$ and taking $t = \tau$, we arrive at

$$Q(u(\tau)) = Q(u_0) + \text{Im} \sum_{k=1}^{\infty} \int_0^\tau \int_{R^k} \nabla^s \pi_R \nabla^s (g(u) \phi e_k) dx d\mathbb{B}_k$$

$$- \frac{1}{2} \text{Re} \int_0^\tau \int_{R^k} (g(u) g(u) F_{\phi}) (-\Delta)^s \pi_R dx ds$$

$$+ \frac{1}{2} \sum_{k=1}^{\infty} \int_0^\tau \int_{R^k} \nabla^s (g(u) \phi e_k)^2 dx ds.$$

\[\square\]

4.2 Proof of Theorem 2

Thanks to the assumption $n \leq 4$, we can choose $\sigma < \frac{2n}{2n-4}$ such that $\frac{2n}{2n-4} \leq \sigma < \frac{2n}{2n}$.

Let $u_0 \in L^{2+\frac{4n}{2n-4}}(\Omega; L^2) \cap L^2(\Omega; H^s)$ and $E(Q(u_0))$ be bounded. It is enough to prove that for $\tau^*(u_0) < \infty$, $\| u \|_{H^s}$ is almost surely bounded. Denote stopping time $\tau^* < \inf \{ T_0, \tau^*(u_0) \}$, and

$$\tilde{\tau}_R = \inf \{ t < \tau^*(u_0), \| u \|_{H^s} \geq R \text{almost surely} \}$$

for $R > 0$. It is apparent for Proposition 2 that

$$\mathbb{E} \left( \sup_{t \leq \tau^* \wedge \tilde{\tau}_R} |Q(u(t))|^2 \right)$$

$$\leq \mathbb{E}(Q(u_0)^2) + \mathbb{E} \left( \sup_{t \leq \tau^* \wedge \tilde{\tau}_R} \left| \int_0^t \int_{R^k} \nabla^s \pi_R \nabla^s (g(u) \phi e_k) dx d\mathbb{B}_k \right|^2 \right)$$

$$+ \frac{1}{4} \mathbb{E} \left( \sup_{t \leq \tau^* \wedge \tilde{\tau}_R} \left| \int_0^t \int_{R^k} (g(u) g(u) F_{\phi}) (-\Delta)^s \pi_R dx ds \right|^2 \right)$$

$$+ \frac{1}{4} \mathbb{E} \left( \sup_{t \leq \tau^* \wedge \tilde{\tau}_R} \left| \int_0^t \int_{R^k} \nabla^s(g(u) \phi e_k)^2 dx ds \right|^2 \right)$$

$$= \mathbb{E}(Q(u_0)^2) + I + II + III.$$

From Bürkholder and Hölder’s inequalities, we have

$$I \leq C \| \phi \|_{L^{2+\frac{4n}{2n-4}}(\Omega; L^2)}^2 \mathbb{E} \left( \int_0^t |(-\nabla)^s \pi|^4_{L^2} ds \right).$$

Applying Hölder’s inequality to the variable $s$, we get

$$II \leq \frac{C}{4} \| F_{\phi} \|_{L^{\infty}} T_0 \mathbb{E} \left( \int_0^t |(-\nabla)^s u|^4_{L^2} ds \right).$$
According to Proposition 1, it follows that

\[ 111 \leq C \| \phi \|_{R(L^2;W^{\infty,\infty})} T_0 \mathbb{E}\left( \int_0^T \| (-\nabla)^s u \|_{L^2}^2 \, ds \right) + C \| \phi \|_{R(L^2;W^{\infty,\infty})} T_0^2 \mathbb{E}\left( M^2(u_0) \right). \]  

(45)

If we plug (43)–(45) back into (42), we obtain

\[
\mathbb{E}\left( \sup_{t \leq \tau' \wedge T_k} |Q(u(t))|^2 \right) 
\leq C \left( \| \phi \|_{R(L^2;W^{\infty,\infty})}, T_0 \right) \mathbb{E}\left( \int_0^T \| (-\nabla)^s u \|_{L^2}^2 \, ds \right) + \mathbb{E}\left( |Q(u_0)|^2 \right) + C \| \phi \|_{R(L^2;W^{\infty,\infty})} T_0^2 \mathbb{E}\left( M^2(u_0) \right). \]  

(46)

In the case of \( \lambda = 1 \), from \( \sigma < \frac{2s}{n} \), the following Gagliardo–Nirenberg inequality holds (see [32]):

\[
\| u \|_{L^{2\sigma+2}}^2 \leq C \| u \|_{L^2}^{2\sigma+2-\frac{2\sigma}{n}} \| (-\nabla)^s u \|_{L^2}^{\frac{2\sigma}{n}} \leq \frac{1}{4} \| (-\nabla)^s u \|_{L^2}^2 + C \| u \|_{L^2}^{2\sigma+2-\frac{2\sigma}{n}}. \]  

(47)

Application of (47) and (40) leads to

\[
Q(u) \gtrsim \| (-\nabla)^s u \|_{L^2}^2. \]  

(48)

Combining (46) and (48), it follows that

\[
\mathbb{E}\left( \| (-\nabla)^s u \|_{L^2}^2 \right) \gtrsim C \left( \| \phi \|_{R(L^2;W^{\infty,\infty})}, T_0 \right) \mathbb{E}\left( \int_0^T \| (-\nabla)^s u \|_{L^2}^2 \, ds \right) + \mathbb{E}\left( |Q(u_0)|^2 \right) + C \| \phi \|_{R(L^2;W^{\infty,\infty})} T_0^2 \mathbb{E}\left( M^2(u_0) \right). \]  

(49)

For \( s \geq \alpha \), utilizing Gronwall’s inequality and Sobolev embedding, we have

\[
\mathbb{E}\left( \sup_{t \leq \tau' \wedge T_k} \| u \|_{H^s}^2 \right) \leq \mathbb{E}\left( \sup_{t \leq \tau' \wedge T_k} \| u \|_{H^s}^2 \right) < \infty,
\]

which means that the boundedness of \( \mathbb{E}\left( \sup_{t \leq \tau' \wedge T_k} \| u \|_{H^s}^2 \right) \) is independent of \( R \). Letting \( R \to \infty \), it follows that

\[
\mathbb{E}\left( \sup_{t \leq \tau'} \| u \|_{H^s}^2 \right) \leq C \left( \| \phi \|_{R(L^2;W^{\infty,\infty})}, \mathbb{E}(M(u_0)), T_0 \right). \]

For \( \lambda = -1 \), from (40), it is clear that

\[
Q(u) \gtrsim \| (-\nabla)^s u \|_{L^2}^2. \]

Then, according to (46), we deduce that (49) still holds. Then, applying Gronwall’s inequality and letting \( R \to \infty \), we get

\[
\mathbb{E}\left( \sup_{t \leq \tau'} \| u \|_{H^s}^2 \right) \leq C \left( \| \phi \|_{R(L^2;W^{\infty,\infty})}, \mathbb{E}(M(u_0)), T_0 \right). \]

In summary, we complete the proof of Theorem 2.

5. Conclusions

This paper is dedicated to non-radial solutions to the Cauchy problem of the nonlinear fractional Schrödinger equation with nonlinear multiplicative noise. Local well-posedness in \( H^s \) almost surely follows from the estimates related to the stochastic convolution and deterministic non-radial Strichartz estimates. Furthermore, the blow-up criterion is presented. Then, the global existence of a solution in \( H^s \) almost surely follows from an a priori estimate based on the conservation of energy and Itô’s formula. Comparing to
known results about Equation (5) (see [10,15,16,19–22]), there are two difficulties we have overcome in this paper: one is that the noise term is nonlinear, and the another one is the loss of derivatives. The main innovation is to show the existence of a non-radial global solution under fractional-order derivatives and a nonlinear noise term.

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