Well-Posedness of Backward Stochastic Differential Equations with Jumps and Irregular Coefficients

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Abstract: In this paper, we focus on investigating the well-posedness of backward stochastic differential equations with jumps (BSDEJs) driven by irregular coefficients. We establish new results regarding the existence and uniqueness of solutions for a specific class of singular BSDEJs. Unlike previous studies, our approach considers terminal data that are square-integrable, eliminating the need for them to be necessarily bounded. The generators in our study encompass a standard drift, a signed measure across the entire real line, and the local time of the unknown process. This broadens the scope to include BSDEJs with quadratic growth in the Brownian component and exponential growth concerning the jump noise. The key methodology involves establishing Krylov-type estimates for a subset of solutions to irregular BSDEJs and subsequently proving the Tanaka-Krylov formula. Additionally, we employ a space transformation technique to simplify the initial BSDEJs, leading to a standard form without singular terms. We also provide various examples and special cases, shedding light on BSDEJs with irregular drift coefficients and contributing to new findings in the field.

Keywords: backward stochastic differential equations with jumps; Poisson random measure; Markov processes; Tanaka-Krylov formula; local time

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1. Introduction and Notations

Backward Stochastic Differential Equations (BSDEs), both with and without jumps, have undergone extensive investigation due to their broad applications in mathematical finance, insurance reserving, optimal control theory, stochastic differential games, and dynamic risk measures [1–7]. Additionally, they establish significant connections with partial differential equations and play a crucial role in utility optimization and dynamic risk measure applications [8–13].

Efforts have been made to relax assumptions on the coefficients, allowing for the consideration of BSDEs with irregular generators. Notably, measurable generators involving the local time of the unknown process have been investigated in continuous cases in references such as [14–17], and in the context of jump processes in [18]. These prior studies encompass a specific class of BSDEs with quadratic growth, extensively explored using various approaches as seen in works like [19–25]. Further investigations for BSDEs with quadratic growth and jumps are found in [26–30]. Singular forms of BSDEs in the Brownian setting have also been explored in [31,32], and for stochastic differential equations (SDEs) with measurable drifts, readers can refer to [33].

This paper aims to extend the findings from previous works, including [14–17], into the setting of jump processes. Additionally, it builds upon results established in [18] when dealing with generators represented as signed measures on $\mathbb{R}$ with finite total variation. More precisely we analyze BSDEs driven by Wiener process and an independent Poisson random measure. The drift term contains a singular expression related to the local time of the unknown process, a signed measure that may charge singletons and a functional of the
integral process with respect to the compensated Poisson random measure. This represent a
generalization of the results established in the former references to the framework of
jumps processes. The difficulties appeared on the jumps parts once the original BSDEJ is
transformed my means of an Itô type formula. A particular focus will be on a new term
generated after this transformation.

Furthermore, this study provides a fresh proof of the converse of Proposition 1 in [14]
without introducing additional assumptions. In particular, we succeeded by deeper analysis
to avoid some assumptions imposed in the former reference.

The key methodology involves establishing Tanaka-Krylov’s formula for a specific
class of solutions of BSDEs with jumps, where irregular drifts are present. This is achieved
through the utilization of the phase space transformation technique introduced by [34],
eliminating the drift term containing the signed measure and the local time of the un-
known process Y. It is noteworthy that this technique has also been applied to the nu-
merical solutions of a class of stochastic differential equations in continuous settings, as
detailed in [33]. Collectively, these studies enhance our understanding of BSDEs with
irregular coefficients, establishing them as valuable tools in various mathematical and
financial domains.

Consider a bounded time interval [0, T], and let \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \) be equipped with the
Borel sigma-algebra \( \mathcal{B}(\mathbb{R}^*) \), where a positive measure \( \sigma \) is defined on \( \mathbb{R}^* \) with \( \sigma(\mathbb{R}^*) \) being
finite. We work within a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]} , \mathbb{P}) \) that supports two
independent stochastic processes:

- \( B = \{B_t\}_{t \in [0,T]} \), a one-dimensional standard Brownian motion.
- \( \mathcal{N}(dr,de) \), a Poisson random measure that is time-homogeneous with compensator
  \( \sigma(d\mathcal{N})dr \) on \([0,T] \times \mathbb{R}^*, \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^*) \).

Let \( \mathcal{N}(dr,de) = \mathcal{N}(dr,de) - \sigma(d\mathcal{N})dr \) represent the compensated jump measure. The
filtration \( \{\mathcal{F}_t\}_{t \in [0,T]} \) is generated by the processes \( W \) and \( \mathcal{N} \), with the completion involving
\( \mathbb{P} \)-null sets and ensuring right continuity.

Now, define the following spaces:

- \( L^2(\Omega) \) denotes the Banach space of real-valued random variables on the probability
  space \( (\Omega, \mathcal{F}, \mathbb{P}) \) that are square integrable. \( S^2 \): The set of càdlàg processes \( Y \) that are
  \( \mathcal{F}_t \)-adapted for which \( \mathbb{E}[\sup_{0 \leq t \leq T} |Y_t|^2] < \infty \).
- \( L^p(\mathbb{R}^*) \): The set of measurable functions \( u \) defined on \( \mathbb{R}^* \) such that the norm \( \|u(\cdot)\|_{p,p} :=
  (\int_{\mathbb{R}^*} |u(\epsilon)|^p \sigma(d\epsilon))^\frac{1}{p} \) is finite.
- \( M^1_{\mathcal{F}_t} \): The space of processes \( Z \) that are \( \mathcal{F}_t \)-adapted and satisfying \( \mathbb{E}[\int_0^T |Z_r|^2 dr] < \infty \).
- \( L^2_{\mathcal{F}_t} \): The space of processes \( Z \) on \([0,T] \) that are \( \mathcal{F}_t \)-adapted and satisfying \( \int_0^T |Z_r|^2 dr < \infty \) \( \mathbb{P} \)-a.s.
- \( M^2_{\mathcal{F}_t} \): The space of \( \mathcal{F}_t \)-predictable processes \( U \) satisfying \( \mathbb{E}[\int_0^T \|U_r(\cdot)\|^2_{p,p} dr] < \infty \).
- \( L^2_{\mathcal{F}_t} \): The space of \( \mathcal{F}_t \)-predictable processes \( U \) on \([0,T] \times \mathbb{R}^+ \) satisfying \( \int_0^T \|U_r(\cdot)\|^2_{p,p} dr < \infty \) \( \mathbb{P} \)-a.s.

Also, denote by \( BV(\mathbb{R}) \) the space of functions \( f : \mathbb{R} \to \mathbb{R} \) with bounded variation on
\( \mathbb{R} \), satisfying the following conditions:

1. \( f \) is right-continuous.
2. There exists \( \varepsilon > 0 \) such that \( f(x) \geq \varepsilon \) for all \( x \in \mathbb{R} \).

Given a function \( f \) in \( BV(\mathbb{R}) \), \( f(x^-) \) denotes the left-limit of \( f \) at a point \( x \), and \( f'(dx) \)
is the bounded measure associated with \( f \) (i.e., \( |f'(x)|(\mathbb{R}) < +\infty \)).

For a continuous function \( g \), let \( g^+ \) and \( g^- \) be the left-hand and right-hand derivatives
of \( g \) when they exist, and \( g'(x) = \frac{g^+(x) + g^-(x)}{2} \) be the associated symmetric derivative
of \( g \).

\( M(\mathbb{R}) \) denotes the space of all signed measures \( \kappa \) on \( \mathbb{R} \) such that the total variation
\( |\kappa| \) \( |\kappa| = \kappa^+ + \kappa^- \) of \( \kappa \) is finite \( |\kappa|(\mathbb{R}) < +\infty \), and \( |\kappa(\{x\})| < 1 \) for all \( x \in \mathbb{R} \). If
\( \kappa \) is in \( M(\mathbb{R}) \), \( \kappa^c \) denotes the continuous part of \( \kappa \), and \( \kappa^+ \) and \( \kappa^- \) are respectively the positive and negative parts of \( \kappa \).

- \( BV^2_{\mathbb{R}}(\mathbb{R}) \) is the space of continuous functions \( g : \mathbb{R} \to \mathbb{R} \) for which the symmetric derivative \( s_g' \) of \( g \) belongs to \( BV(\mathbb{R}) \), and the signed measure \( g''(dx) \) associated with \( s_g' \) satisfies \( |g''|(\mathbb{R}) < +\infty \).
- \( BV^2_{2,loc}(\mathbb{R}) \) is the space of continuous functions \( g \) defined on \( \mathbb{R} \) such that both \( g \) and its generalized derivatives \( s_g' \) are locally integrable on \( \mathbb{R} \), and \( g''(dx) \), the signed measure, is locally bounded. Clearly, \( BV^2_{2,loc}(\mathbb{R}) \subset BV^2_{1,loc}(\mathbb{R}) \).

1.1. Brief Overview of Local Time

In this subsection, we will use specific notation. The function sign denoted \( \text{sgn} \) is defined as follows:

\[
\text{sgn}(x) = \begin{cases} 
1 & \text{if } x \in [0, +\infty[ \\
-1 & \text{if } x \in ]-\infty, 0]. 
\end{cases}
\]

It is crucial to observe that our definition of \( \text{sgn} \) is asymmetric. Throughout the discussion, \( \text{sgn}(x) \) represents the left derivative of \( |x| \), and \( \text{sgn}(x - a) \) signifies the left derivative of \( |x - a| \). Due to the convexity of \( |x - a| \), Tanaka’s formula implies, for a semi-martingale \( Y \):

\[
|Y_{t} - a| = |Y_{0} - a| + \int_{0+}^{t} \text{sgn}(Y_{r} - a) dY_{r} + A_{t}^{a},
\]

(1)

here, \( (A_{t}^{a})_{t \geq 0} \) represents the increasing process associated with the semi-martingale \( |Y - a| \).

**Definition 1.** The local time at \( a \) for \( Y \), denoted as \( L_{t}^{a}(Y) \), is defined by:

\[
L_{t}^{a}(Y) = A_{t}^{a} - \sum_{0 < r < t} \{|Y_{r} - a| - |Y_{r} - a| - \text{sgn}(Y_{r} - a) \Delta Y_{r} \}.
\]

Protter [35] demonstrated that the stochastic integral

\[
\int_{0+}^{t} \text{sgn}(Y_{r} - a) dY_{r}
\]

in (1) possesses a version that is jointly measurable in \( (\omega, t, a) \) and càdlàg in \( t \). Consequently, the process \( (A_{t}^{a})_{t \geq 0} \) also has this property, and consequently, the local time \( L_{t}^{a}(Y) \) does too. We consistently adopt this jointly measurable, càdlàg version of the local time without specific mention.

Moreover, it is worth noting that the jumps of the process \( A_{t}^{a} \) defined in (1) precisely correspond to:

\[
\sum_{0 < r < t} \{|Y_{r} - a| - |Y_{r} - a| - \text{sgn}(Y_{r} - a) \Delta Y_{r} \},
\]

hence, the local time \( (L_{t}^{a}(Y))_{t \geq 0} \) exhibits continuity with respect to \( t \). Specifically, the local time \( L_{t}^{a}(Y) \) corresponds to the continuous part of the increasing process \( A_{t}^{a} \). A well-established result asserts the existence of a version of \( \{L_{t}^{a}(Y) : t \geq 0, a \in \mathbb{R}\} \) that is continuous in \( t \), with both right and left limits in \( a \). This version is given by:

\[
\lim_{x \to a^+} L_{t}^{a}(Y) = L_{t}^{a^+}(Y) = L_{t}^{a}(Y) \quad \text{and} \quad \lim_{x \to a^-} L_{t}^{a}(Y) = L_{t}^{a^-}(Y) \exists.
\]

The following proposition, with a proof available in Protter [35], is both straightforward and essential for establishing the properties of \( L_{t}^{a}(Y) \) that in fact validate its nomenclature. For any real number \( x \), we reintroduce the standard notations \( x^+ = x \vee 0 = \max(x, 0) \) and \( x^- = -(x \wedge 0) = \min(-x, 0) \), hence \( |x| = x^- + x^+ \).
Proposition 1. Let $Y$ be a semi-martingale and let $L^a_t$ be its local time at the level $a$. Then

\[
(Y_t - a)^+ - (Y_0 - a)^+ = \int_{0^+}^t \mathbb{1}_{\{Y_r > a\}} dY_r + \sum_{0 < s < t^+} \mathbb{1}_{\{Y_r < a\}} (Y_r - a)^- + \frac{1}{2} L^a_t(Y)
\]

Furthermore, for almost every $\omega$, the measure in $t$, $dL^a_t(Y(\omega))$, is supported by the set

\[
\{ r : Y_r(\omega) = a \} = \{ r : Y_t(\omega) = a \}.
\]

For details we refer to [35], Theorem 68. and its proof p. 213.

Let $Y$ be càdlàg semi-martingale, let $\tilde{L}^a_t(Y)$ denotes the local time of $Y$ at the level $a$, defined by Tanaka’s formula as follows:

\[
\tilde{L}^a_t(Y) = |Y_t - a| - |Y_0 - a| - \int_0^t \sigma(Y_r, a) dY_r
\]

and

\[
\tilde{\sigma}(x) = \begin{cases} 
1 & \text{for } x \in ]0, \infty[ \\
0 & \text{for } x = 0 \\
-1 & \text{for } x \in ]-\infty, 0[.
\end{cases}
\]

One can write alternatively:

\[
\tilde{L}^a_t(Y) = \frac{L^+_t(Y) + L^-_t(Y)}{2}.
\]

1.2. Problem Formulation

We consider the following BSDEJs that will be referred along the paper as Eq($\zeta, \Theta, \kappa$)

\[
Y_t = \zeta + \int_0^T \Theta(Y_r, Z_r, U_r(\cdot)) dr + \int_{\mathbb{R}} \left( \tilde{L}^a_t(Y) - \tilde{L}^a_t(Y) \right) \kappa(da) - \int_0^T Z_r dB_r - \int_0^T \int_{\mathbb{R}} U_r(\epsilon) \tilde{N}(dr, d\epsilon),
\]

where for any given level parameter $a$, $\tilde{L}^a_t(Y)$ represents the symmetric local time at time $t$ for the unknown semi-martingale $(Y_t)_{t \geq 0}$.

We aim to solve the equation Eq($\zeta, \Theta, \kappa$) under the following conditions:

Condition (A1) The random variable $\zeta$ is $\mathbb{R}$-valued and belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

Condition (A2): The function $\Theta : \mathbb{R} \times \mathbb{R} \times L^2_T \rightarrow \mathbb{R}$ satisfies:

(i) The map $(y, z, u(\cdot)) \mapsto \Theta(y, z, u(\cdot))$ is continuous.

(ii) There exists a constant $K > 0$ such that for any $(y, z) \in \mathbb{R} \times \mathbb{R}$:

\[
|\Theta(y, z, u(\cdot))| \leq K(1 + |y| + |z| + \|u(\cdot)\|_{L^2_T}).
\]

Condition (A3): $\kappa$ belongs to $\mathbb{M}(\mathbb{R})$.

Given $\kappa$ in $\mathbb{M}(\mathbb{R})$, we denote $\kappa^\ast$ as the continuous part of the measure $\kappa$. We also define:

\[
f_\kappa(x) = \exp(2\kappa((\cdot, x))) \prod_{y \leq x} \frac{1 + \kappa((y))}{1 - \kappa((y))}.
\]

(2)

For any $x \in \mathbb{R}$, we denote:
\[ F_\kappa(x) = \int_0^x f_\kappa(y) dy \text{ and } g_\kappa(x) = \frac{f_\kappa(F_\kappa^{-1}(x)) + f_\kappa(F_\kappa^{-1}(x) - )}{2}. \] (3)

Let \( sF'_\kappa \) denote the symmetric derivative of \( F_\kappa \), expressed as:

\[ sF'_\kappa(x) = \frac{f_\kappa(x) + f_\kappa(x-) - }{2} = g_\kappa(F_\kappa(x)). \] (4)

Our focus is on investigating the well-posedness of the \( \mathbb{R} \)-valued BSDEJs \( \text{Eq}(\zeta, \Theta, \kappa) \) for the given generators \( \Theta \) outlined below.

1. \( \Theta(y, z, u(\cdot)) = [u(\cdot)]_{F_\kappa}\sigma(y) = \int_{\mathbb{R}} \frac{F_\kappa(y + u(e)) - F_\kappa(y) - sF'_\kappa(y)u(e)}{sF'_\kappa(y)} \sigma(de) =: \Theta_{\kappa, \sigma}(y, u(\cdot)) \),
2. \( \Theta(y, z, u(\cdot)) = cz - \int_{\mathbb{R}} u(e) \sigma(de) \)
3. \( \Theta(y, z, u(\cdot)) = cz + \Theta_{\kappa, \sigma}(y, u(\cdot)) \)
4. \( \Theta(y, z, u(\cdot)) = \theta(y, u(\cdot)) + cz + \Theta_{\kappa, \sigma}(y, u(\cdot)) \)
5. \( \Theta(y, z, u(\cdot)) = \theta(y, u(\cdot)) + cz \)
6. \( \Theta(x, y, z, u(\cdot)) = f_0(\cdot, x) + cz + \Theta_{\kappa, \sigma}(y, u(\cdot)) \)

Explicit conditions for \( h \) and \( f_0 \) will be detailed in the relevant section.

It is worth noting that the equation \( \text{Eq}(\zeta, \Theta, \kappa) \) encompasses BSDE instances with quadratic growth, particularly when the measure \( \kappa \) is absolutely continuous concerning the Lebesgue measure on \( \mathbb{R} \). This observation becomes apparent through the utilization of the occupation density formula.

1.3. Technical Results

**Definition 2.** Consider an \( \mathcal{F}_t \)-measurable and square integrable random variable \( \zeta \). A triple of processes \( (Y, Z, U(\cdot)) \), where \( Y \) is adapted, and \( Z \) and \( U(\cdot) \) are predictable, is deemed a solution to \( \text{Eq}(\zeta, \Theta, \kappa) \) if it satisfies \( \text{Eq}(\zeta, \Theta, \kappa) \) \( \mathbb{P} \)-almost surely, provided that \( Z \in \mathcal{M}_T^2, U(\cdot) \in \mathcal{M}_T^2, Y \in \mathcal{S}_T^2 \).

The following lemma, pivotal in the proof of Proposition 3 and Theorem 2, is particularly useful. The transformation \( F_\kappa \) eliminates the generator \( \Theta_{\kappa, \sigma} \) and the part involving the local time in the Eq(\( \zeta, \Theta, \kappa \)).

**Lemma 1.** Let \( f \) be a function of bounded variation, where \( f(x-) \) denotes the left limit of \( f \) at a point \( x \), and \( f'(dx) \) represents the bounded measure associated with \( f \). If \( x \) belongs to \( \mathbb{M}(\mathbb{R}) \), uniquely determined up to a multiplicative constant, such that:

\[ f'(dx) - (f(x) + f(x-))\kappa(dx) = 0, \]

if we specify that \( \lim_{x \to -\infty} f(x) = 1 \), then \( f \) is unique and given by

\[ f_\kappa(x) = \exp(2\kappa((-\infty, x])) \prod_{y \leq x} \left( \frac{1 + \kappa(\{y\})}{1 - \kappa(\{y\})} \right). \]

A more concise form of the lemma below, suitable when the measure \( \kappa \) is absolutely continuous with respect to the Lebesgue measure, can be found in [33] for SDEs and [18] for BSDEs in the Brownian motion framework.

**Lemma 2.** The function \( f_\kappa \) defined in (2) is increasing, right-continuous, and satisfies:

\[ 0 < m^{-1}_\kappa \leq f_\kappa(x) \leq m_\kappa \text{ for all } x \in \mathbb{R} \] (5)

for some constant \( m_\kappa \). Moreover, \( f_\kappa \) satisfies:
\begin{align}
\begin{cases}
    f'_κ(dx) - (f_κ(x) + f_κ(x^-))κ(dx) = 0, \\
    or 
    \text{equivalently}
    
    F'_κ(dx) - 2sF'_κ(x)κ(dx) = 0.
\end{cases}
\end{align} (6)

The function $F_κ$, as defined in (3), possesses the following properties:

(i) Both $F_κ$ and $F_κ^{-1}$ are quasi-isometries: that is for any $x, y \in \mathbb{R}$:

$$m_κ^{-1}|x - y| \leq |F_κ(x) - F_κ(y)| \leq m_κ|x - y|,$$

$$m_κ^{-1}|x - y| \leq |F_κ^{-1}(x) - F_κ^{-1}(y)| \leq m_κ|x - y|.$$ (7)

(ii) The function $F_κ$ is injective. Moreover, both $F_κ$ and its inverse, $F_κ^{-1}$, are members of $BV^1_κ(\mathbb{R})$.

\textbf{Proof of (i).} By definition, the functions $F_κ$ and its inverse $F_κ^{-1}$ are continuous, injective, strictly increasing functions. Moreover, $f_κ$ satisfies ordinary differential Equation (6). Additionally, $s(F_κ^{-1})'$ is the symmetric derivative of $F_κ^{-1}$, so for every $x \in \mathbb{R}$:

$$m_κ^{-1} \leq sF_κ(x) \leq m_κ \quad \text{and} \quad m_κ^{-1} \leq s(F_κ^{-1})'(x) \leq m_κ.$$ (8)

\textbf{Proof of (ii).} It can be easily verified that $F_κ^{-1}$ belongs to the class $BV^1_κ(\mathbb{R})$. \qed

\textbf{Lemma 3.} Let $u(\cdot)$ be a measurable function in $L^1_σ$. For a given real number $x$

(i) The operator

$$[u]_{F_κσ}(x) := \int_{\mathbb{R}} F_κ(x + u(\epsilon)) - F_κ(x) - sF'_κ(x)u(\epsilon) \, σ(dε)$$ (9)

is well-defined. Moreover,

$$|[u]_{F_κσ}(x)| \leq (1 + m_κ^2)\|u(\cdot)\|_{σ,1}.$$ (10)

(ii) If $κ$ is a non-negative measure, then $[u]_{F_κσ}(x) \geq 0$ for all $x \in \mathbb{R}$.

\textbf{Proof of (i).} By virtue of the quasi-isometry properties of the function $F_κ$ as defined in (3), for all $x \in \mathbb{R}$, we obtain:

\begin{align}
|[u]_{F_κσ}(x)| & \leq \int_{\mathbb{R}} \left| \frac{F_κ(x + u(\epsilon)) - F_κ(x)}{sF'_κ(x)} - u(\epsilon) \right| \sigma(dε) \\
& \leq \int_{\mathbb{R}} \left| \frac{F_κ(x + u(\epsilon)) - F_κ(x)}{sF'_κ(x)} \right| \sigma(dε) + \|u(\cdot)\|_{σ,1} \\
& \leq (m_κ^2 + 1)\|u(\cdot)\|_{σ,1},
\end{align}

consequently,

$$|[u]_{F_κσ}(x)| \leq (1 + m_κ^2)\|u(\cdot)\|_{σ,1},$$

which means that the operator $[u]_{F_κσ}(\cdot)$ is well-defined. \qed

\textbf{Proof of (ii).} Also, note that for every $x \in \mathbb{R}$, we can express $sF'_κ(x)[u]_{F_κσ}(x)$ as:

\begin{align}
sF'_κ(x)[u]_{F_κσ}(x) &= \int_{\mathbb{R}} (F_κ(x + u(\epsilon)) - F_κ(x) - sF'_κ(x)u(\epsilon)) \sigma(dε) \\
&= \int_{\mathbb{R}} \int_{x}^{x+u(\epsilon)} \mathbf{1}_{\{u(\epsilon) > 0\}} (sF'_κ(y) - sF'_κ(x)) \, dy \, \sigma(dε) \\
&\quad + \int_{\mathbb{R}} \int_{x}^{x+u(\epsilon)} \mathbf{1}_{\{u(\epsilon) < 0\}} (sF'_κ(x) - sF'_κ(x)) \, dy \, \sigma(dε).
\end{align}
The last two terms in the inequality above are non-negative, given that \( F' \) is positive and increasing whenever \( \kappa \) is a non-negative measure. \( \square \)

**Corollary 1.** For a given real number \( x \) and a predictable process \( \tilde{U}(\cdot) \) on \([0, T] \times \mathbb{R}^*\), such that:

\[
\int_0^T \int_{\mathbb{R}^*} |\tilde{U}_t(e)| \sigma(de) dr < +\infty \ \mathbb{P}\text{-a.s.}
\]

Then, from (10), we have:

\[
\int_0^T \mathbb{E}[\|\tilde{U}_t \|_{\mathbb{F}_t,\sigma}(x)] dr \leq \left( 1 + m_x^2 \right) \int_0^T \mathbb{E}[\|\tilde{U}_t \|_{\sigma,1}] dr \ \mathbb{P}\text{-a.s.}
\]

Moreover, if \( \tilde{U}(\cdot) \) in \( \mathcal{M}_{\tilde{N}_t} \), then there exists a constant \( C_{x,\sigma} \) (depending only on \( \kappa \) and \( \sigma \)) such that:

\[
\int_0^T \mathbb{E}[\|\tilde{U}_t \|_{\mathbb{F}_t,\sigma}(x)] dr \leq C_{x,\sigma} \int_0^T \mathbb{E}[\|\tilde{U}_t \|_{\sigma,1}]^2 dr.
\]

### 1.4. Krylov’s Estimates and Tanaka-Krylov’s Formula for BSDEJs

If \( (X_t)_{t \geq 0} \) is a real-valued semi-martingale such that \( \sum_{0 \leq r \leq t} |\Delta X_r| \) is almost surely finite for each \( t > 0 \), and \( g \) represents the difference between two convex functions, according to [35] Tanaka’s formula affirms that for every \( X \in \mathbb{R} \), there exists an adapted process \( (\tilde{L}_t^X)(t)_{t \geq 0} \) such that, for each \( t \geq 0 \), it holds with probability 1:

\[
g(X_t) = g(X_0) + \int_0^t g'(X_r- \Delta X_r) \ dX_r + \frac{1}{2} \int_0^t \tilde{L}_r^X \ g''(dx) + \sum_{0 \leq r \leq t} \{ u(X_r) - u(X_r-) - g'_-(X_r-\Delta X_r) \},
\]

where \( g' \) represents the left first derivative of \( g \), and \( g'' \) is a signed measure, serving as the second derivative of \( g \) in the generalized function sense. Additionally,

\[
\int_0^t \phi(X_r) d[X]_r^c = \int_{-\infty}^\infty \tilde{L}_r^X \phi(x) dx
\]

for any measurable function \( \phi \).

**Proposition 2 (Krylov’s type estimates).** Let \( (Y, Z, U(\cdot)) \) be a solution to Eq(\( \zeta, \Theta, \kappa \)) in the sense of Definition 2 where \( \Theta \) satisfies (A2). Put

\[
\eta = 2 \sup_{0 \leq r \leq T} |Y_r| + 2 \int_0^T \|U_r(\cdot)\|_{\sigma,1} dr + 2 \int_0^T \|\Theta(Y_r, Z_r, U_r(\cdot))\| dr,
\]

then, for any measure \( \kappa \) in \( \mathcal{M}(\mathbb{R}) \), we have

\[
\mathbb{E} \int_{\mathbb{R}} \tilde{L}_r^X(Y) |\kappa|| (da) \leq 2Ce^{2|\eta|} \mathbb{E} [\eta]
\]

**Proof.** For a fixed real number \( x \), we define, for simplification of expressions, \( \varphi_x(y) = (y - x)^- \). Tanaka’s formula implies:
Applying Gronwall’s Lemma to the function \( \varphi_t(Y_t) = \varphi_t(Y_0) + \int_0^t \mathbb{1}_{\{Y_r < x\}} dY_r + \frac{1}{2} \tilde{L}_t^x (Y) \)

\[ + \int_0^t \int_{\mathbb{R}} \left( \varphi_t(Y_r - + U_r(e) - \varphi_t(Y_r -) - \mathbb{1}_{\{Y_r < x\}} U_r(e) \right) \sigma(\text{d}e) \text{d}r \]

\[ = \varphi_t(Y_0) + M_t - \int_0^t \mathbb{1}_{\{Y_r < x\}} d_r \left( \int_{\mathbb{R}} \tilde{L}_r^x (Y) \lambda(\text{d}a) \right) \]

\[ + \frac{1}{2} \tilde{L}_t^x (Y) - \int_0^t \mathbb{1}_{\{Y_r < x\}} \Theta(Y_r, Z_r, U_r(\cdot)) \text{d}r \]

\[ + \int_0^t \int_{\mathbb{R}} \left( \varphi_t(Y_r - + U_r(e) - \varphi_t(Y_r -) - \mathbb{1}_{\{Y_r < x\}} U_r(e) \right) \sigma(\text{d}e) \text{d}r \]

where

\[ M_t = \int_0^t \mathbb{1}_{\{Y_r < x\}} Z_r \text{d}B_r + \int_0^t \int_{\mathbb{R}} \mathbb{1}_{\{Y_r \leq x\}} U_r(e) \tilde{N}(\text{d}r, \text{d}e) \]

is a martingale.

Observe also that thanks to the property of the local time

\[ \int_0^t \mathbb{1}_{\{Y_r < x\}} d_r \left( \int_{\mathbb{R}} \tilde{L}_r^x (Y) \lambda(\text{d}a) \right) = \int_{-\infty}^x \tilde{L}_r^x (Y) \lambda(\text{d}a). \]

Utilizing the one-Lipschitz property of the mapping \( y \mapsto \varphi_t(y) = (y-x)^-, \) we deduce that:

\[ \frac{1}{2} \tilde{L}_t^x (Y) \leq |Y_t - Y_0| + \int_{-\infty}^x \tilde{L}_r^x (Y) |\lambda(\text{d}a) - M_t| \]

\[ + \int_0^t |\Theta(Y_r, Z_r, U_r(\cdot))| \text{d}r + 2 \int_0^t \|U_r(\cdot)\|_{C^1} \text{d}r, \]

hence

\[ 0 \leq \tilde{L}_t^x (Y) \leq 2\eta - 2M_t + 2 \int_{-\infty}^x \tilde{L}_r^x (Y) |\lambda(\text{d}a). \]

(13)

By computing the expectation in both sides of (13), we derive

\[ \mathbb{E}\left[ \tilde{L}_t^x (Y) \right] \leq 2\mathbb{E}[\eta] + 2 \int_{-\infty}^x \mathbb{E}\left[ \tilde{L}_r^x (Y) \right] |\lambda(\text{d}a). \]

Applying Gronwall’s Lemma to the function \( \mathbb{E}[\tilde{L}_t^x (Y)] \) yields:

\[ \sup_x \mathbb{E}\left[ \tilde{L}_t^x (Y) \right] \leq 2\mathbb{E}[\eta] e^{2|\lambda|_{C^1}}. \]

(14)

Now, let \( \kappa \) be in \( M(\mathbb{R}) \), then

\[ \int_{\mathbb{R}} \mathbb{E}\left[ \tilde{L}_t^x (Y) \right] |\lambda(\text{d}a) \leq \sup_x \mathbb{E}\left[ \tilde{L}_t^x (Y) \right] \int_{-\infty}^x |\lambda(\text{d}a) \leq 2|\lambda|_{C^1(\mathbb{R})} e^{2|\lambda|_{C^1(\mathbb{R})}} \mathbb{E}[\eta]. \]

Proposition 2 is proved since \( \mathbb{E}[\eta] \) is finite thanks to the linear growth of \( \Theta \) and Definition 2. \( \square \)

In what follows, we will establish a change of variable formula in the spirit of Tanaka-Krylov for solutions to one-dimensional BSDEs with jumps, incorporating the local time of the unknown process.

**Theorem 1** (Tanaka-Krylov’s formula). Assume that \( \Theta \) satisfies \( (A_2) \). Let \( (Y, Z, U(\cdot)) \) be a solution to Eq(\( \zeta, \Theta, \kappa \)). Then, for any function \( g \) in the space \( \mathcal{B}V_{1,\text{loc}}^2(\mathbb{R}) \), the following holds:
\[ g(Y_t) = g(Y_0) + \int_0^t s'g(Y_{r-})dY_r + \frac{1}{2} \int_\mathbb{R} \tilde{L}^q(Y)g''(da) + \sum_{0 \leq r \leq t} \left( g(Y_r) - g(\bar{Y}_r) - s'g(Y_{r-})\Delta Y_r \right), \] (15)

which can be written as
\[ g(Y_t) = g(Y_0) + \int_0^t s'g(Y_{r-})dY_r + \frac{1}{2} \int_\mathbb{R} \tilde{L}^q(Y)g''(da) + \int_0^t \int_{\mathbb{R}^2} \left( g(Y_r + U_r(e)) - g(Y_r) - s'g(Y_{r-})U_r(e) \right) \mathbb{1}_{\{d \leq 0\}}(de, dr). \]

**Proof.** For \( R > |Y_0|, \) let \( \tau_R := \inf \{ t > 0 : \max(\{|Y_{r-}|, \sup_{e \in \mathbb{R}^+}|Y_{r-} + U_r(e)| \} \geq R \}. \) Given that \( \tau_R \) tends to infinity as \( R \) approaches infinity, we can establish the formula (15) by substituting \( t \) with \( t \land \tau_R. \) The stochastic integral \( \int_0^{t \land \tau_R} s'g(Y_r)dY_r \) is well-defined since \( s'g \) is of bounded variation, and \((Y_r)_{0 \leq r \leq T} \) is a càdlàg semi-martingale. Additionally, the jump term
\[ \int_0^{t \land \tau_R} \int_{\mathbb{R}^2} \left( g(Y_r + U_r(e)) - g(Y_r) - s'g(Y_{r-})U_r(e) \right) \mathbb{1}_{\{d \leq 0\}}(de, dr) \]
is also well defined since
\[ \int_0^{t \land \tau_R} \int_{\mathbb{R}} \left| (g(Y_r + U_r(e)) - g(Y_r) - g'(Y_{r-})U_r(e)) \right| \sigma(de)dr \leq \left( M_R + \max_{x \in [-R, R]} |s'(x)| \right) \int_0^{t \land \tau_R} \|U_r(\cdot)\|_{\mathbb{C}^1}dr. \]

Leveraging the local Lipschitz continuity of \( g \) and \( g' \), along with Proposition 2, the expression \( \int_\mathbb{R} \tilde{L}^q_{t \land \tau_R}(Y)g''(da) \) is properly defined, given that
\[ \mathbb{E} \left| \int_\mathbb{R} \tilde{L}^q_{t \land \tau_R}(Y)g''(da) \right| \leq \sup_{x \in \mathbb{R}} \mathbb{E} \left[ \tilde{L}^q_{t \land \tau_R}(Y) \right] |g''|(\mathbb{R}). \]

Next we consider a sequence of \( C^2 \)-class functions, denoted as \( g_n, \) obtained through classical regularization by convolution, satisfying the conditions:

(i) the sequence \( g_n \) converges uniformly to \( g \) in the interval \([-R, R] \).

(ii) the sequence \( g'_n = s'g'_n \) converges uniformly to \( s'g' \) in the interval \([-R, R] \).

(iii) \( g''_n(da) \) converges weakly in \( L^1([-R, R]) \) to \( g''(da) \).

Classical Itô’s formula applied to \( g_n(Y_{t \land \tau_R}) \) yields:
\[ g_n(Y_{t \land \tau_R}) = g_n(Y_0) + \int_0^{t \land \tau_R} s'g'(Y_{r-})dY_r + \frac{1}{2} \int_\mathbb{R} \tilde{L}^q_n(Y_r)Z_r^2 dr (16) \]
\[ + \int_0^{t \land \tau_R} \int_{\mathbb{R}^2} \left( g_n(Y_r + U_r(e)) - g_n(Y_r) - g'_n(Y_{r-})U_r(e) \right) \mathbb{1}_{\{d \leq 0\}}(de, dr). \]

Passing to the limit as \( n \) tends towards infinity in (16) together with above properties (i), (ii), (iii) and Proposition 2, yield
\[ g(Y_{t \land \tau_R}) = g(Y_0) + \int_0^{t \land \tau_R} s'g'(Y_{r-})dY_r + \frac{1}{2} \int_\mathbb{R} \tilde{L}^q(Y)g''(da) \]
\[ + \int_0^{t \land \tau_R} \int_{\mathbb{R}^2} \left( g(Y_r + U_r(e)) - g(Y_r) - s'g'(Y_{r-})U_r(e) \right) \mathbb{1}_{\{d \leq 0\}}(de, dr). \]
Moving forward, the subsequent variations of Tanaka-Krylov’s formula will be frequently employed in the subsequent sections.

Considering a generator Θ subject to appropriate conditions ensuring the existence of a solution for BSDEJs, the following types of Tanaka-Krylov’s formula will be prevalent in the subsequent discussions.

Let \((Y, Z, U(\cdot))\) be a solution to Eq\((\xi, \Theta, \kappa)\). Applying Tanaka-Krylov’s formula (15) to \(F_k(Y_t)\) results in

\[
F_k(Y_t) = F_k(\xi) + \int_t^T sF'_k(Y_r-)dY_r + \frac{1}{2} \int_t^T \left( \frac{\partial^2 F_k}{\partial \kappa^2}(Y) \right)\kappa'(\kappa(\kappa'Y))d\kappa + \int_t^T F_k(\xi)\kappa'(\kappa(Y))\kappa'(\kappa(\kappa'Y))d\kappa \tag{17}
\]

Due to the characteristic that \(\tilde{L}_t^\kappa(Y)\) only increases on the set \(Y_\kappa = x\), the term \(\int_t^T sF'_k(Y_r-)dY_r\) can be expressed as:

\[
- \int_t^T sF'_k(Y_r-)\Theta(Y_r, Z_r, U_r(\cdot))d\kappa - \int_t^T sF'_k(Y_r-)d\kappa \left( \int_t^T \tilde{L}_t^\kappa(Y)\kappa(dx) \right)
\]

Furthermore,

\[
\sum_{t < r \leq T} (F_k(Y_r) - F_k(Y_r-) - sF'_k(Y_r-)\kappa(\kappa(\kappa'Y))d\kappa)
\]

Taking also into account that

\[
sF'_k(y)[u]_{F_k,e}(y) = \int_{\mathbb{R}^+} (F_k(y + u(c)) - F_k(y) - sF'_k(y)u(c))\kappa(dx)
\]

and

\[
\frac{1}{2} sF''_k(dx) - sF'_k(x)\kappa(dx) = 0,
\]

the Equation (17) reads

\[
F_k(Y_t) = F_k(\xi) - \int_t^T sF'_k(Y_r-)d\kappa + \int_t^T sF'_k(Y_r-)(\Theta(Y_r, Z_r, U_r(\cdot)) - [U_r]_{F_k,e}(Y_r-))d\kappa - \int_t^T \int_{\mathbb{R}^+} (F_k(Y_r- + U_r(\cdot)) - F_k(Y_r-))\kappa(dx)\kappa(\kappa'(\kappa'Y))d\kappa + \int_t^T F_k(\xi)\kappa'(\kappa(Y))\kappa'(\kappa(\kappa'Y))d\kappa \tag{18}
\]

In particular, if \(\Theta(y, z, u(\cdot)) = \Theta_{k,e}(y, z, u(\cdot)) = [u]_{F_k,e}(y)\), we get

\[
F_k(Y_t) = F_k(\xi) - \int_t^T sF'_k(Y_r-)d\kappa + \int_t^T \int_{\mathbb{R}^+} (F_k(Y_r- + U_r(\cdot)) - F_k(Y_r-))\kappa(dx)\kappa(\kappa'(\kappa'Y))d\kappa + \int_t^T F_k(\xi)\kappa'(\kappa(Y))\kappa'(\kappa(\kappa'Y))d\kappa \tag{19}
\]

For each \(0 \leq r \leq T\), we define new processes

\[
\tilde{Y}_r = F_k(Y_r), \quad \tilde{Z}_r = sF'_k(Y_r-)d\kappa
\]
and
\[ \tilde{U}_t(e) = F_k(Y_{t-} + U_t(e)) - F_k(Y_{t-}). \]

These notations will be employed consistently throughout the rest of this paper.

1.5. A Priori Estimates

**Proposition 3.** Let \( \zeta \in L^2(\Omega) \) and \( \kappa \in M(\mathbb{R}) \). If \( (Y, Z, U) \) satisfies the Eq(\( \zeta, \Theta_{\kappa, \rho}, \kappa \)), then we have:

(i) \( (\tilde{Z}_r)_{0 \leq r \leq T}, (Z_r)_{0 \leq r \leq T} \in M^2_W \) and \( (\tilde{U}_r(e))_{0 \leq r \leq T}, (U_r(e))_{0 \leq r \leq T} \in M^2_{N_r} \),

(ii) \( (\tilde{Y}_r)_{0 \leq r \leq T}, (Y_r)_{0 \leq r \leq T} \in S^2, \)

(iii) \( \mathbb{E}[|\int_{\mathbb{R}} \tilde{U}^*_T(Y)(d\alpha)|^2] \) is finite.

**Proof of (i).** Let us first recall an important equality that will be used repeatedly in the proofs. For a given stochastic process \( \psi(\cdot) \) in \( M^2_N \), we have
\[
\mathbb{E}\left| \int_0^T \int_{\mathbb{R}^*} \psi(e)\tilde{N}(dr, de) \right|^2 = \mathbb{E}\left( \int_0^T \int_{\mathbb{R}^*} |\psi(e)|^2 \tilde{N}(dr, de) \right)^2 = \mathbb{E}\left( \int_0^T \int_{\mathbb{R}^*} |\psi(e)|^2 \sigma(de)dr \right),
\]
then from Tanaka-Krylov’s formula (19) we have
\[
F_k(Y_t) = F_k(\xi) - \int_t^T sF'_k(Y_{r-})Z_r \, dB_r - \int_t^T \int_{\mathbb{R}^*} (F_k(Y_{r-} + U_r(e)) - F_k(Y_{r-})) \tilde{N}(dr, de),
\]
since \( F_k \) satisfies (6). For \( t = 0 \) we get
\[
F_k(Y_0) - F_k(\xi) = \int_0^T sF'_k(Y_{r-})Z_r \, dB_r + \int_0^T \int_{\mathbb{R}^*} (F_k(Y_{r-} + U_r(e)) - F_k(Y_{r-})) \tilde{N}(dr, de).
\]

Taking the square of the \( L^2(\Omega) \) norm in (21), thanks to the orthogonality of the martingales \( B_r \) and \( \int_{\mathbb{R}^*} \tilde{N}(dr, de) \) together with the inequalities (7) and (8), we get
\[
m_k^{-2}\left( \int_0^T \mathbb{E}\left[ Z_r^2 \right] dr + \int_0^T \mathbb{E}\left[ |U_r(\cdot)|^2 \right] d\tau \right) \leq \mathbb{E}\left( \int_0^T sF'_k(Y_{r-})Z_r \, dB_r \right)^2 + \mathbb{E}\left( \int_0^T \int_{\mathbb{R}^*} |F_k(Y_{r-} + U_r(e)) - F_k(Y_{r-})|^2 \sigma(de)dr \right)
\]
\[
= \mathbb{E}\left( \int_0^T \tilde{Z}_r \, dB_r \right)^2 + \mathbb{E}\left( \int_0^T \tilde{U}_r(e) \tilde{N}(dr, de) \right)^2 \leq F_k^2(Y_0) + \mathbb{E}\left[ F_k^2(\xi) \right] \leq F_k^2(Y_0) + m_k^2 \mathbb{E}\left[ \tilde{\zeta}^2 \right] < \infty,
\]

consequently \( \tilde{Z}, Z \in M^2_W \) and \( \tilde{U}_r, U_r \in M^2_{N_r} \). \( \square \)
**Proof of (ii).** Again thanks to the Tanaka-Krylov’s formula (19), we get

\[
F_{\kappa}(Y_t) = F_{\kappa}(\zeta) - \int_{t}^{T} sF'_{\kappa}(Y_{t-})Z_r dB_r
- \int_{t}^{T} \int_{\mathbb{R}^*} (F_{\kappa}(Y_{t-} + U_r(\epsilon)) - F_{\kappa}(Y_{t-})) \tilde{N}(dr, de).
\]  

(22)

Thanks to (7) and \(F_{\kappa}(0) = 0\) one has the following estimates:

\[
m_{\kappa}^{-1}|Y_t| \leq |F_{\kappa}(Y_t)| \leq |F_{\kappa}(\zeta)| + \left| \int_{t}^{T} sF'_{\kappa}(Y_{t-})Z_r dB_r \right|
+ \left| \int_{t}^{T} \int_{\mathbb{R}^*} (F_{\kappa}(Y_{t-} + U_r(\epsilon)) - F_{\kappa}(Y_{t-})) \tilde{N}(dr, de) \right|
\leq |F_{\kappa}(\zeta)| + \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \tilde{Z}_r dB_r \right| + \sup_{0 \leq t \leq T} \left| \int_{\mathbb{R}^*} \tilde{U}_r(\epsilon) \tilde{N}(dr, de) \right|
\]

applying convex inequalities and taking the supremum over the interval [0, T] yield:

\[
m_{\kappa}^{-2} \sup_{0 \leq t \leq T} |Y_t|^2 \leq \sup_{0 \leq t \leq T} \left| \tilde{Y}_t \right|^2
\leq 2^2 \left( m_{\kappa}^2 |\zeta|^2 + \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \tilde{Z}_r dB_r \right|^2 \right)
+ \left| \sup_{0 \leq t \leq T} \left[ \int_{\mathbb{R}^*} \tilde{U}_r(\epsilon) \tilde{N}(dr, de) \right] \right|^2.
\]

Then by calculating the expectation and applying Burkholder-Davis-Gundy inequality, we obtain

\[
m_{\kappa}^{-2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\tilde{Y}_t|^2 \right]
\leq 4 \left( m_{\kappa}^2 \mathbb{E}[|\zeta|^2] + \int_{0}^{T} \mathbb{E}[|\tilde{Z}_r|^2] dr + \int_{0}^{T} \mathbb{E}[|\tilde{U}_r(\epsilon)|^2] dr \right).
\]

We deduce that the right-hand side of the above inequality is finite due to the statement (i). □

**Proof of (iii).** Since \((Y, Z, U(\cdot))\) satisfies Eq(\(\zeta, \Theta_{\kappa,\sigma}, \kappa\)), thus

\[
\int_{\mathbb{R}} \tilde{L}^\kappa(Y) \kappa(da) = \int_{0}^{T} Z_r dB_r + \int_{0}^{T} \int_{\mathbb{R}^*} U_r(\epsilon) \tilde{N}(dr, de)
+ Y_0 - \zeta - \int_{0}^{T} [U_r]_{F_{\kappa,\sigma}} \tilde{Y}_r dr.
\]

Again, making use of the convex inequality and taking the expectation in the above inequality we get
\[
\mathbb{E}\left| \int_{\mathbb{R}} \tilde{L}^\kappa_T(Y) \kappa(da) \right|^2 
\leq 2^4 \left( \mathbb{E}\left| \int_0^T Z_r \, dB_r \right|^2 + \mathbb{E}\left| \int_{\mathbb{R}^*} U_r(e) \tilde{\mathcal{N}}(dr, de) \right|^2 \right) 
+ 2^4 \left( |Y_0|^2 + \mathbb{E}|\xi|^2 + T \int_0^T \mathbb{E}[|U_r|^2] \right) 
\leq 2^4 \left( \mathbb{E}\int_0^T |Z_r|^2 \, dr + \mathbb{E}\int_0^T \|U_r(\cdot)\|_{\mathcal{S}^2}^2 \, dr \right) 
+ 2^4 \left( |Y_0|^2 + \mathbb{E}|\xi|^2 + TC_{\kappa,\sigma} \mathbb{E}\int_0^T \|U_r(\cdot)\|_{\mathcal{S}^2}^2 \, dr \right)
\]

thus the square integrability of \( \int_{\mathbb{R}} \tilde{L}^\kappa_T(Y) \kappa(da) \) is ensured because all terms on the right-hand side of the above inequality are finite, as guaranteed by condition (i).

2. Main Results

The objective of this section is to investigate the existence and uniqueness of solutions to the BSDE with jumps, denoted as \( \text{Eq}(\zeta, \Theta_{\kappa,\sigma}, \kappa) \).

It is important to note that the findings in this section serve as natural extensions to the results obtained in previous works, namely [14–18] by allowing the BSDE to be also driven by a compensated Poisson random measure as well as the possibility to include irregular generators. Specifically, the following theorem provides a comprehensive response to the converse of Proposition 1 point (2) in [14] without requiring any additional assumption on the discontinuity points of the function \( g_k \). Notably, the assumption (5) on page 106 in [14] has been removed.

Moreover, this result represents a generalization of the outcomes established in [18] for a class of signed measures within the space \( M(\mathbb{R}) \). In other words, the results obtained in [18] correspond to the particular case where the measure \( \kappa \) is absolutely continuous with respect to the Lebesgue measure.

**Theorem 2.** Under \((A_1) - (A_3)\) the triplet \((Y, Z, U(\cdot))\) is a solution to \( \text{Eq}(\zeta, \Theta_{\kappa,\sigma}, \kappa) \) if and only if \((\bar{Y}, \bar{Z}, \bar{U}(\cdot))\) is a solution to \( \text{Eq}(F_k(\zeta), 0, 0) \).

**Proof of Theorem 2.**

**Necessary condition:** Let \((Y, Z, U(\cdot))\) be a solution of equation \( \text{Eq}(\zeta, \Theta_{\kappa,\sigma}, \kappa) \), then (19) shows that \((\bar{Y}, \bar{Z}, \bar{U}(\cdot))\) satisfies \( \text{Eq}(F_k(\zeta), 0, 0) \). Moreover, thanks to Proposition 3, \((\bar{Y}, \bar{Z}, \bar{U}(\cdot))\) is a solution to \( \text{Eq}(F_k(\zeta), 0, 0) \) in the sense of the Definition 2.

**Sufficient condition:** Consider a solution triplet \((\bar{Y}, \bar{Z}, \bar{U}(\cdot))\) to \( \text{Eq}(F_k(\zeta), 0, 0) \). For simplicity, let us denote \( G_k = F_k^{-1} \). Applying Tanaka-Krylov’s formula (15) to \( Y_t = G_k(\bar{Y}_t) \) (given that \( G_k \) belongs to \( BV^{\mathcal{F}_t}_{\mathbb{R}}(\mathbb{R}) \)), we observe that:

\[
G_k(\bar{Y}_t) = \zeta - \int_t^T sG_k'(\bar{Y}_r-\bar{\bar{Z}}_r) \, dB_r - \int_t^T \int_{\mathbb{R}^*} sG_k'(\bar{Y}_r-\bar{\bar{Z}}_r) U_r(e) \tilde{\mathcal{N}}(dr, de) \\
- \int_{\mathbb{R}} \left( \tilde{L}^\kappa_T(\bar{Y}) - \bar{\bar{L}}^\kappa_T(\bar{Y}) \right) G_k''(da) \\
- \sum_{t < r \leq T} \left( G_k(\bar{Y}_r) - G_k(\bar{Y}_{r-}) - sG_k'(\bar{Y}_{r-}) \Delta \bar{\bar{Y}}_r \right),
\]

then
\[ Y_t = \zeta - \int_t^T sG'_k(\tilde{Y}_r-\tilde{Z}_r) \, dB_r - \int_t^T \int_{\mathbb{R}^+} sG'_k(\tilde{Y}_r-\tilde{U}_r(e)) \tilde{N}(dr,de) \]
\[-\frac{1}{2} \int_{\mathbb{R}} \left( \tilde{L}^a_T(\tilde{Y}) - \tilde{L}^a_T(\tilde{Y}) \right) G'_k(da) \]
\[-\int_t^T \int_{\mathbb{R}^+} \left( G_k(\tilde{Y}_r-\tilde{U}_r(e)) - G_k(\tilde{Y}_r-\tilde{U}_r(e)) - sG'_k(\tilde{Y}_r-\tilde{U}_r(e)) \right) \tilde{N}(dr,de) \]

or equivalently
\[ Y_t = \zeta - \int_t^T s(G_k)'(\tilde{Y}_r-\tilde{Z}_r) \, dB_r - \int_t^T \int_{\mathbb{R}^+} sG'_k(\tilde{Y}_r-\tilde{U}_r(e)) \tilde{N}(dr,de) \] (23)
\[-\frac{1}{2} \int_{\mathbb{R}} \left( \tilde{L}^a_T(\tilde{Y}) - \tilde{L}^a_T(\tilde{Y}) \right) G'_k(da) \]
\[-\int_t^T \int_{\mathbb{R}^+} \left( G_k(\tilde{Y}_r-\tilde{U}_r(e)) - G_k(\tilde{Y}_r-\tilde{U}_r(e)) - sG'_k(\tilde{Y}_r-\tilde{U}_r(e)) \right) \sigma(de)dr \]
\[-\int_t^T \int_{\mathbb{R}^+} \left( G_k(\tilde{Y}_r-\tilde{U}_r(e)) - G_k(\tilde{Y}_r-\tilde{U}_r(e)) - sG'_k(\tilde{Y}_r-\tilde{U}_r(e)) \right) \tilde{N}(dr,de). \]

Notice that the difficulty here is identification of the term
\[ \frac{1}{2} \int_{\mathbb{R}} \tilde{L}_T^a(\tilde{Y}) G'_k(da) \]
with
\[ \int_{\mathbb{R}} \tilde{L}_T^a(\tilde{Y}) \kappa(dx). \]

The idea is to apply once again the transformation \( F_k \) to \( Y \) represented by the expression in (23). To this purpose we set \( C_t = \frac{1}{2} \int_{\mathbb{R}} \tilde{L}_T^a(\tilde{Y}) G'_k(da) \) and notice that
\[ sG'_k(x) = \frac{2}{F_k'(G_k(x)) + F_k'(G_k(x)-)} = \frac{1}{g_k(x)}. \]

therefore
\[ \int_{\mathbb{R}^+} \left( G_k(\tilde{Y}_r-\tilde{U}_r(e)) - G_k(\tilde{Y}_r-\tilde{U}_r(e)) - \frac{\tilde{U}_r(e)}{g_k(\tilde{Y}_r-)} \right) \tilde{N}(dr,de) \]
\[ = \int_{\mathbb{R}^+} \left( U_r(e) - \frac{F_k(\tilde{Y}_r) - F_k(\tilde{Y}_r)}{g_k(\tilde{Y}_r-)} \right) \sigma(de) \]
\[ = - \int_{\mathbb{R}^+} \left( \frac{F_k(\tilde{Y}_r) - F_k(\tilde{Y}_r)}{sF_k'(\tilde{Y}_r)} - U_r(e) \right) \sigma(de) \]
\[ = - [U_r]_{F_k(\tilde{Y}_r)}(\tilde{Y}_r-). \] (24)

thus
\[ Y_t = \zeta - \int_t^T s(G_k)'(\tilde{Y}_r-\tilde{Z}_r) \, dB_r \]
\[-\int_t^T \int_{\mathbb{R}^+} \left( G_k(\tilde{Y}_r-\tilde{U}_r(e)) - G_k(\tilde{Y}_r-\tilde{U}_r(e)) \right) \tilde{N}(dr,de) \]
\[ + C_t - C_T + \int_t^T [U_r]_{F_k(\tilde{Y}_r)}(\tilde{Y}_r-)dr. \]

Set \( Z_r = \frac{\tilde{Z}_r}{g_k(\tilde{Y}_r-)} \) and \( U_r(e) = G_k(\tilde{Y}_r-\tilde{U}_r(e)) - G_k(\tilde{Y}_r-) \) this implies
\[ Y_t = \zeta - \int_t^T Z_r \, dB_r - \int_t^T \int_{\mathbb{R}^+} U_r(e) \tilde{N}(dr,de) \]
\[ + C_t - C_T + \int_t^T [U_r]_{F_k(\tilde{Y}_r)}(\tilde{Y}_r-)dr. \] (25)
Applying again the transformation $F_\kappa$, we get thanks to (18)

$$
\tilde{Y}_t = F_\kappa(Y_t) = F_\kappa(\zeta) + \frac{1}{2} \int_{\mathbb{R}} \left( \tilde{L}_\kappa^+(Y) - \tilde{L}_\kappa^-(Y) \right) F'_\kappa(dx)
- \int_t^T \int_{\mathbb{R}^*} (F_\kappa(Y_{r-} + U_r(e)) - F_\kappa(Y_{r-})) \tilde{N}(dr, de)
- \int_t^T sF'_\kappa(Y_{r-}) Z_r \, dB_r - \int_t^T sF'_\kappa(Y_{r-}) \, dC_r
= F_\kappa(\zeta) - \int_t^T \tilde{Z}_r \, dB_r - \int_t^T \int_{\mathbb{R}^*} \tilde{U}_r(e) \tilde{N}(dr, de)
+ \frac{1}{2} \int_{\mathbb{R}} \left( \tilde{L}_\kappa^+(Y) - \tilde{L}_\kappa^-(Y) \right) F''\kappa(dx) - \int_t^T sF'_\kappa(Y_{r-}) \, dC_r.
$$

Remember that $(\tilde{Y}, \tilde{Z}, \tilde{U}(\cdot))$ is a solution to Eq$(F_\kappa(\zeta), 0, 0)$ and thank to the equation $\frac{1}{2} F''\kappa(dx) = \mathbb{J} F'_\kappa(x) \kappa(dx)$, one gets

$$
\int_t^T sF'_\kappa(Y_{r-}) \, dC_r = \frac{1}{2} \int_{\mathbb{R}} \left( \tilde{L}_\kappa^+(Y) - \tilde{L}_\kappa^-(Y) \right) F'_\kappa(dx)
= \int_{\mathbb{R}} \left( \tilde{L}_\kappa^+(Y) - \tilde{L}_\kappa^-(Y) \right) sF'_\kappa(x) \kappa(dx)
= \int_t^T sF'_\kappa(Y_{r-}) \int_{\mathbb{R}} \tilde{L}_\kappa^+(Y) \kappa(dx),
$$

from which we deduce that

$$
dC_r = \int_{\mathbb{R}} \frac{1}{2} \tilde{L}_\kappa^+(Y) \kappa(dx)
$$

or equivalently

$$
C_t = \int_{\mathbb{R}} \tilde{L}_\kappa^+(Y) \kappa(dx) = \frac{1}{2} \int_{\mathbb{R}} \tilde{L}_\kappa^+(\tilde{Y}) G''_\kappa(da),
$$

finally substituting $C_t$ by this expression, the Equation (25) becomes

$$
Y_t = \zeta + \int_t^T [U_r]_{F,\kappa,\sigma} (Y_{r-}) dr + \int_{\mathbb{R}} \left( \tilde{L}_\kappa^+(Y) - \tilde{L}_\kappa^-(Y) \right) \kappa(dx)
- \int_t^T Z_r \, dB_r - \int_t^T \int_{\mathbb{R}^*} \tilde{U}_r(e) \tilde{N}(dr, de).
$$

Similar to the proof of Proposition 3 and leveraging the properties (7) and (8), we readily demonstrate that:

$$
|Y_t| \leq m_\kappa |\tilde{Y}_t|, |Z_t| \leq m_\kappa |\tilde{Z}_t| \text{ and } |U_r(e)| \leq m_\kappa |\tilde{U}_r(e)|,
$$

consequently,

$$
(Y_r, Z_r, U_r(e)) := (G_\kappa(\tilde{Y}_r), s(G_\kappa)'(\tilde{Y}_{r-}) \tilde{Z}_r, G_\kappa(\tilde{Y}_{r-} + \tilde{U}_r(e)) - G_\kappa(\tilde{Y}_{r-})),
$$

for $0 \leq r \leq T$, $e \in \mathbb{R}^*$ is a solution to Eq$(\tilde{\zeta}, \Theta_{\kappa,\sigma,\kappa})$ in the sense of Definition 2. □

**Theorem 3.** Under $(A_1)$–$(A_3)$ the Eq$(\tilde{\zeta}, \Theta_{\kappa,\sigma,\kappa})$ has a unique solution $(Y, Z, U(\cdot))$ in the sense of Definition 2.

**Proof of Theorem 3.** Given that $F_\kappa$ is globally Lipschitz, $F_\kappa(\zeta)$ is in $L^2(\Omega)$ if and only if $\zeta$ is in $L^2(\Omega)$. Consequently, by the martingale representation theorem, the equation Eq$(F_\kappa(\zeta), 0, 0)$ has a triplet $(\tilde{Y}, \tilde{Z}, \tilde{U}(\cdot))$ as its unique solution according to Definition 2, and
the associated predictable processes \((\tilde{Z})\) and \((\tilde{U}(\cdot))\) belong respectively to \(\mathcal{M}_W^2\) and \(\mathcal{M}_N^2\). Now, with the aid of Theorem 2, the processes defined as follows:

\[
Y_t = G_\kappa(\tilde{Y}_t), \quad Z_t = \frac{\tilde{Z}_t}{\mathbb{s}F_\kappa(G_\kappa(Y_{\tau^*}))}
\]

and

\[
U_t(e) = G_\kappa(\tilde{Y}_{\tau^*} + \tilde{U}_t(e)) - G_\kappa(\tilde{Y}_{\tau^*}).
\]

imply that \((Y, Z, U(\cdot))\) is the unique solution to the equation \(\text{Eq}(\zeta, \Theta, \sigma, \kappa)\). Applying the conditional expectation to both sides of \(\text{Eq}(F_\kappa(\zeta), 0, 0)\) results in:

\[
\tilde{Y}_t = \mathbb{E}[F_\kappa(\zeta) \mid \mathcal{F}_t]
\]

and, consequently,

\[
Y_t = \mathbb{E}[F_\kappa(\zeta) \mid \mathcal{F}_t].
\]

This completes the proof of the theorem. \(\square\)

3. Other Examples of Irregular BSDEs with Jumps

In the upcoming section, we will leverage the findings from the preceding section to address specific instances of BSDEJs with irregular generators in the subsequent examples. We would like to point out that all the examples considered in our previous work [18] are particular case of the ones presented below and correspond to the case where the signed measure \(\kappa\) is absolutely continuous with respect to the Lebesgue measure.

\((A_4)\) \(\zeta\) is square integrable  
\((A_5)\) \(\kappa \in \mathcal{M}(\mathbb{R})\).

Example 1. \(\text{Eq}(\zeta, cz - \int_{\mathbb{R}^*} u(e)\sigma(\kappa, \sigma, \kappa))\). Consider the following BSDEJ

\[
Y_t = \zeta + \int_t^T \left( cZ_r - \int_{\mathbb{R}^*} U_r(e)\sigma(\kappa, \sigma, \kappa) \right) dr + \int_t^T \left( \tilde{L}_r^\circ(Y) - \tilde{L}_r^\circ(\zeta) \right) \kappa(dx)
\]

\[- \int_t^T Z_r dB_r - \int_t^T U_r(e)\tilde{N}(dr, de),
\]

Taking \(\Theta(y, z, u(\cdot)) = cz - \int_{\mathbb{R}^*} u(e)\sigma(\kappa, \sigma, \kappa)\) in (18), we get

\[
F_\kappa(Y_t) = F_\kappa(\zeta) - \int_t^T \int_{\mathbb{R}^*} (F_\kappa(Y_{\tau^*} + U_{\tau^*}(e)) - F_\kappa(Y_{\tau^*}))\sigma(\kappa, \sigma, \kappa) dr
\]

\[+ c\int_t^T \int_{\mathbb{R}^*} sF'_\kappa(Y_{\tau^*})Z_r ds - \int_t^T \int_{\mathbb{R}^*} sF'_\kappa(Y_{\tau^*})Z_r dB_r
\]

\[- \int_t^T \int_{\mathbb{R}^*} (F_\kappa(Y_{\tau^*} + U_{\tau^*}(e)) - F_\kappa(Y_{\tau^*}))\tilde{N}(dr, de),
\]

or equivalently as

\[
\tilde{Y}_t = F_\kappa(\zeta) + \int_t^T \left( cz - \int_{\mathbb{R}^*} \tilde{U}_r(e)\sigma(\kappa, \sigma, \kappa) \right) dr
\]

\[- \int_t^T \tilde{Z}_r dB_r - \int_t^T \tilde{U}_r(e)\tilde{N}(dr, de).
\]

Given that the generator of the aforementioned equation is linear in both \(z\) and \(u\), and \(F_\kappa(\zeta)\) is square integrable, a unique solution exists. Consequently, Equation (26) possesses a unique solution.
Example 2. Eq(\(\zeta, cz + \Theta_{\kappa, \nu}(y, u(\cdot)), \kappa\)). This equation corresponds, for any constant \(c \in \mathbb{R}\), to the generator

\[
\Theta(y, z, u(\cdot)) := cz + \Theta_{\kappa, \nu}(y, u(\cdot)).
\]

The application of the characterization derived in Theorem 2 reveals that: Eq(\(\zeta, \Theta, \kappa\)) is equivalent to Eq(\(F_x(\zeta), cz, 0\)). The equation Eq(\(F_x(\zeta), cz, 0\)) has a unique solution if and only if \(F_x(\zeta)\) is square integrable. This condition is satisfied under assumptions (A_4)–(A_5). Therefore, Eq(\(F_x(\zeta), cz, 0\)) has a unique solution, implying that our original Eq(\(\zeta, \Theta\)) also has a unique solution.

Now, we will introduce supplementary conditions to extend the coverage of generators beyond (A_4)–(A_5).

(A_6) Assume that the signed measure \(\kappa\) is absolutely continuous with respect to the Lebesgue measure such that its Radon-Nikodym derivative is bounded and integrable over the whole space \(\mathbb{R}\).

(A_7) Let \(h : \mathbb{R} \times L_2^F \to \mathbb{R}\) be a measurable function satisfying for all \(u(\cdot), \dot{u}(\cdot)\) in \(L_2^F, z, \dot{z}\) and \(y, \dot{y} \in \mathbb{R}\)

\[
|\theta(y, u(\cdot)) - \theta(y, \dot{u}(\cdot))| \leq L\left(|y - \dot{y}| + \|u(\cdot) - \dot{u}(\cdot)\|_{L^2}\right).
\]

and there exists a constant \(C > 0\) such that \(|\theta(y, u(\cdot))| \leq C\).

Example 3. Eq(\(\zeta, \theta(y, u(\cdot)) + cz + \Theta_{\kappa, \nu}(y, u(\cdot)), \kappa\)). This equation corresponds to the generator

\[
\Theta(y, z, u(\cdot)) := \theta(y, u(\cdot)) + cz + \Theta_{\kappa, \nu}(y, u(\cdot)), c \in \mathbb{R}
\]

shows that Eq(\(\zeta, \Theta, \kappa\)) is equivalent to Eq(\(F_x(\zeta), \Theta_0 + cz, 0\)) where

\[
\Theta_0(y, u(\cdot)) := F_x^\prime(G_\kappa(y))(h(G_\kappa(y), G_\kappa(y + u(\cdot)) - G_\kappa(y))).
\]

The equation Eq(\(F_x(\zeta), \Theta_0 + cz, 0\)) possesses a unique solution if and only if \(\Theta_0\) is Lipschitz and \(F_x(\zeta)\) is square integrable. This condition is satisfied under assumptions (A_4)–(A_7). Consequently, Eq(\(F_x(\zeta), \Theta_0 + cz, 0\)) has a unique solution, leading to the uniqueness of the solution for our original Eq(\(\zeta, \Theta\)).

Example 4. Eq(\(\zeta, cz, \kappa\)). Consider the following BSDE

\[
Y_t = \zeta + \int_t^T cZ_r \, dr + \int_t^T \left(\tilde{L}_r^x(Y) - \tilde{L}_r^z(Y)\right) \kappa(dx)
\]

\[
- \int_t^Z Z_r \, dB_r - \int_t^Z u_r(e) \tilde{N}(dr, de).
\]

then by taking \(\Theta(y, z, u(\cdot)) = cz\) in (18) we arrive at

\[
\tilde{Y}_t = F_x(\zeta) + \int_t^T \left(cZ_r + \int_t^Z \Theta_1(\tilde{Y}_r, \tilde{U}_r(e)) \kappa(de)\right) dr
\]

\[
- \int_t^Z \tilde{Z}_r \, dB_r - \int_t^Z \tilde{U}_r(e) \tilde{N}(dr, de).
\]

where

\[
\Theta_1(y, u(e)) := g_x(G_\kappa(y)(G_\kappa(y + u(e)) - G_\kappa(y))) - u(e).
\]

Simple computations show that

\[
|\Theta_1(y, u(e)) - \Theta_1(y, \dot{u}(e))| \leq L(1 + |u(e)|)|y - \dot{y}| + (1 + e^{4m_x})|u(e) - \dot{u}(e)|.
\]
and
\[ |\Theta_1(y, u(e))| \leq \min\left( (1 + e^{km})|u(e)|; |u(e)| + 2e^{km}(|u(e)| + 2|y|) \right). \]

Consequently, the generator
\[ \Theta(y, z, U_r(\cdot)) = cz + \int_{R^r} \Theta_1(\bar{Y}_{s-}, \bar{U}_r(\cdot))v(d\epsilon). \]
is Lipschitz in both \( z \) and \( u(\cdot) \), continuous in \( y \), and exhibits linear growth in all its arguments. This implies that the BSDEJ (29) may not necessarily have a unique solution (refer to [36]). As a result, the Equation (28) has a solution whenever \( F_{Y}(\xi) \) is square integrable. Thus, (28) has at least one solution.

**Example 5.** Eq(\( \zeta, \theta(y, u(\cdot)) + cz, \kappa \)). Let \( h \) and \( \kappa \) satisfy (A_5)-(A_7) of the Example 3 and consider the BSDEJ

\[
Y_t = \zeta + \int_t^T (h(Y_{r-}, U_r(\cdot)) + cZ_r)dr \\
+ \int_{R} \left( \tilde{L}_t^x(Y) - \tilde{L}^z_t(Y) \right) x(dx) \\
- \int_t^T Z_r dB_r - \int_t^T \int_{R^r} U_r(\cdot)\tilde{N}(dr, dc).
\]

Choosing the function \( \Theta \)
\[
\Theta(y, z, u(\cdot)) = \theta(y, u(\cdot)) + cz,
\]
in the Tanaka-Krylov’s formula (18) the new equation reads
\[
F_{Y}(Y_t) = F_{Y}(\zeta) - \int_t^T sF_{Y}(Y_{r-})Z_r dB_r \\
- \int_t^T \int_{R^r} (F_{Y}(Y_{r-} + U_r(e)) - F_{Y}(Y_{r-}))\tilde{N}(dr, dc) \\
+ \int_t^T sF_{Y}(Y_{r-}) \left( h(Y_{r-}, U_r(\cdot)) + cZ_r \\
- \int_{R^r} (F_{Y}(Y_r) - F_{Y}(Y_{r-}) - sF_{Y}(Y_{r-})U_r(\cdot))\sigma(dc) \right) dr.
\]

which can be written as
\[
\bar{Y}_t = F_{Y}(\zeta) + \int_t^T \left( \Theta_2(\bar{Y}_r, \bar{U}_r(\cdot)) + c\bar{Z}_r \right) dr \\
- \int_t^T \bar{Z}_r dB_r - \int_t^T \int_{R^r} \bar{U}_r(\cdot)\tilde{N}(dr, dc)
\]

where
\[
\Theta_2(y, u(\cdot)) := \Theta_0(y, u(\cdot)) + \Theta_1(y, u(\cdot)).
\]

However, \( \Theta_0 \) exhibits Lipschitz continuity property for under assumption \( (A_5) \) on the measure \( \kappa \), as demonstrated in Example 3. On the other hand, \( \Theta_1 \) is only continuous in \( y \), Lipschitz in \( u(\cdot) \), and has linear growth on \( y \) and \( u(\cdot) \). Consequently, the Equation (31) may not necessarily possess a unique solution (refer, e.g., [36]). Ultimately, the Equation (30) does admit a solution.

**Example 6.** Consider the equation Eq(\( \zeta, f_0(r, x) + cz + \Theta_{x,r}(y, u(\cdot), \kappa) \)), where \( \kappa \) is selected to ensure that the function \( g_{\kappa}(\cdot) \) is Lipschitz. Given an adapted stochastic process \( (X_s)_{0 \leq s \leq T} \) and a measurable bounded function \( f_0 : [0, T] \times R \rightarrow R \) such that \( \int_0^T E[|f_0(r, X_r)|^2]dr \) is finite, the equation takes the form:
\[ Y_t = \zeta + \int_t^T (f_0(r, X_r) + c Z_r + \Theta_{K,\sigma}(Y_r, U_r(\cdot))) \, \text{d}r \]
\[ + \int_\mathbb{R} \left( \Lambda^+_t(x) - \Lambda^+_t(x) \right) \kappa(\text{d}x) \]
\[ - \int_t^T Z_r \, \text{d}B_r - \int_t^T \int_{\mathbb{R}^*} U_r(e) \tilde{N}(\text{d}r, \text{d}e), \]

where the terminal value \( \zeta \) is a square integrable random variable. Employing the same transformation as previously, Equation (32) can be reformulated as the following equivalent equation
\[ \tilde{Y}_t = F_e(\zeta) + \int_t^T \Theta(r, \tilde{Y}_r, \tilde{Z}_r) \, \text{d}r - \int_t^T \tilde{Z}_r \, \text{d}B_r - \int_t^T \int_{\mathbb{R}^*} \tilde{U}_r(e) \tilde{N}(\text{d}r, \text{d}e), \]

having a Lipschitz coefficient that is stochastic in nature, taking the form of
\[ \Theta(r, y, z) = sF'_e(G_e(y)) f_0(r, X_r) + cz \]
\[ = \frac{1}{2} f_e(G_e(y)) + \frac{1}{2} f_e(G_e(y)) - f_0(r, X_r) + cz \]
\[ = g_e(y) f_0(r, X_r) + cz \]

The Equation (33) has a unique solution since the generator \( \Theta(r, y, z) \) is Lipschitz in \( y \) owing to the properties of the function \( g_e \) and is evidently linear in \( z \).

**Remark 1.**
- We learned from the specific instances of singular BSDEJs mentioned above that the selection of \( \Theta_{K,\sigma} \) as the drift for our main BSDEJs, Eq(\( \zeta, \Theta_{K,\sigma}, \kappa \)), was crucial. This choice enables us to obtain a BSDEJ without drift.
- If we replace \( cz \) with \( c|z| \) in all the preceding examples, the results remain valid.

**4. Conclusions**

In this article, we studied well-posedness of BSDEJs and irregular coefficients. The class of BSDEJs in concern contains in particular drifts with local time and a signed measure. It covers for instance BSDEJs with quadratic growth in the \( z \)-variable (the component of the Brownian motion) and measurable drifts term in the \( y \)-variable. To this end, we made use of mathematical analysis and probabilistic techniques to establish Krylov’s type estimates for functionals of solutions and Tanaka-Krylov’s formula for a specific class of BSDEJs with singular drifts. Additionally, we presented several examples leading to new findings in the framework of BSDEJs.

We extended then the findings from previous works such as [14,17] into the setting of jump processes. Additionally, this study builds upon the results established in [18] when dealing with generators represented as signed measures on \( \mathbb{R} \) with finite total variation and also involving the local time of the unknown process.

More precisely, it is crucial to emphasize that the findings presented in this paper naturally extend the results from earlier works, specifically [14–18]. These extensions are notable as they allow the BSDEs to be driven not only by a compensated Poisson random measure but also accommodate irregular generators. This expansion is particularly significant as it addresses the converse of Proposition 1 point (2) in [14] without necessitating any additional assumptions on the discontinuity points of the function \( g_e \). Importantly, the previously assumed condition (5) on page 106 in [14] has been eliminated.

Furthermore, this outcome serves as a broader generalization of the results established in [18] for a specific class of signed measures within the space \( M(\mathbb{R}) \). To clarify, the results obtained in [18] can be viewed as a special case where the measure \( \kappa \) is absolutely continuous with respect to the Lebesgue measure. In essence, the current results encompass
a wider range of situations, offering a more comprehensive response beyond the specific conditions considered in prior research.

In particular several examples cover different situations in which the generators possess singularities in various forms.

In future research, these findings can be exploited as a mathematical tool to provide a probabilistic representation of solutions for a class of Partial Integral Differential Equations (PIDEs) incorporating quadratic terms in the gradient.

We plan also to apply approaches utilized in [33,37] for numerically solving BSDEJs with irregular (non-necessary continuous) generators.

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