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Solutions for Hilfer-Type Linear Fractional Integro-Differential Equations with a Variable Coefficient

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Abstract: In this paper, we derive an explicit formula of solutions to Hilfer linear fractional integro-differential equations with a variable coefficient in a weighted space, and obtain the existence and uniqueness of solutions for fractional kinetic equations and fractional integro-differential equations with a generalized Mittag–Leffler function. An example is given to illustrate the result obtained.

Keywords: Hilfer fractional differential equation; variable coefficient; generalized Mittag–Leffler function; weighted space

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1. Introduction

Fractional calculus is an important tool which is used to describe abundant phenomena, such as viscoelasticity, relaxation vibrations, nonlinear oscillations of earthquakes, mechanics, etc. It has attracted the interest of researchers from various areas [1–7].

The majority of previous studies have been devoted to the Riemann–Liouville and Caputo fractional derivatives. In 2000, Hilfer [8] introduced a fractional differential operator ${}^{H}D^{\alpha,\beta}_{a^+}$. It becomes the Riemann–Liouville differential operator $D^{\alpha}_{a^+}$ when $\beta = 0$, and becomes the Caputo differential operator ${}^{c}D^{\alpha}_{a^+}$ when $\beta = 1$. Recently, Hilfer fractional differential equations (FDEs) have attracted many researchers (see [8,9] and references therein). For instance, the Hilfer derivative can be applied in regular variation in thermodynamics (see, e.g., [8]).

Due to the complexity of FDEs with variable coefficients, it is very difficult to obtain their exact solutions, so there are very few results on this topic. Some results for exact solutions of linear FDEs can be found in [10–14]. In [14], the authors used a modified Hilfer derivative to study the following initial value problem:

$$\left({}^{H}D^{\alpha,\beta}_{a^+}x\right)(t) + \sigma(t)x(t) = f(t),\ t \in (a,T],\ a > 0,$$
$$\left(I^{1-\gamma}_{a^+}x\right)(a^+) = x_0,$$

where $\sigma(t)$ is a continuous function. Under the assumption $\| \sigma \|_C I^{\alpha}_{a^+} e^{vt} \leq Ce^{vt}$ for some $v \in \mathbb{R}$ and $0 < C < 1$, the authors obtained the existence and uniqueness of solutions for the above problem. In many cases, how to obtain exact solutions of FDEs with variable coefficients is an open question.

In this paper, we study the following initial value problem of fractional kinetic differential equations:

$$\begin{cases} \left({}^{H}D^{\alpha,\beta}_{a^+}x\right)(t) - \lambda(t)\left(I^{\eta}_{a^+}x\right)(t) = f(t), \\ \left(I^{1-\gamma}_{a^+}x\right)(a^+) = x_0, \end{cases} \tag{1}$$

and the following initial value problem of fractional integro-differential equations:

$$\begin{cases} \left({}^{H}D_{a^+}^{\alpha,\beta}x\right)(t) - \lambda(t)\left(\mathbf{E}_{\rho,\omega,\theta;a^+}x\right)(t) = f(t), \\ \left(I_{a^+}^{1-\gamma}x\right)(a^+) = x_0, \end{cases} \tag{2}$$

where $\mathbf{E}_{\rho,\omega,\theta;a^+}$ is an integral operator with a generalized Mittag–Leffler function as follows:

$$\left(\mathbf{E}_{\rho,\,\omega,\,\theta;\,a^+}x\right)(t) := \int_a^t (t-s)^{\omega-1}E_{\rho,\,\omega}\left(-\theta(t-s)^{\rho}\right)x(s)ds, t > a.$$

Firstly, we present the characterization of the space $C_{1-\eta}^{\eta}$. Secondly, based on Definition 3, without the contraction assumption on the coefficient function $\lambda(t)$, we obtain the existence and uniqueness of solutions for the above problems in the weighted space $C_{1-\gamma,\nu}^{\alpha,\beta}$. To the best of our knowledge, very few references can be found in the literature discussing the existence and uniqueness of solutions for a class of fractional integro-differential equations with Hilfer fractional derivatives and variable coefficients.

This article is arranged as follows: In Section 2, we recall the main definitions and properties of the Hilfer fractional derivative and generalized Mittag–Leffler functions. In Sections 3 and 4, we investigate the existence and uniqueness of solutions for Problems (1) and (2) using the generalized Banach's fixed point theorem. Finally, we present an example to illustrate our result.

2. Preliminaries

In this section we present some definitions and conclusions that are essential throughout the paper.

2.1. Fractional Integral and Derivatives

Definition 1 ([3])**.** *Let $\alpha \in (0,1]$. The left-sided Riemann–Liouville fractional integral of $f(t)$ is defined by*

$$\left(I_{a^+}^{\alpha}f\right)(t) = \tfrac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, t > a,$$

if the integral exists, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 ([3])**.** *Let $\alpha \in (0,1)$. The left-sided Riemann–Liouville fractional derivative of $f(t)$ is defined by*

$$\left(D_{a^+}^{\alpha}f\right)(t) = \tfrac{d}{dt}\left(I_{a^+}^{1-\alpha}f\right)(t) = \tfrac{1}{\Gamma(1-\alpha)}\tfrac{d}{dt}\int_a^t (t-s)^{-\alpha}f(s)ds,$$

if the integral exists.

In [14], the authors presented the following modified Hilfer derivative:

Definition 3 ([14])**.** *The left-sided Hilfer fractional derivative of $f(t)$ is defined by:*

$${}^{H}D_{a^+}^{\alpha,\beta}f(t) = D_{a^+}^{1-\gamma+\alpha}\left[\left(I_{a^+}^{1-\gamma}f\right)(t) - \left(I_{a^+}^{1-\gamma}f\right)(a^+)\right],$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$ and $\gamma = \alpha + \beta(1-\alpha)$.

We list some function spaces in the following definition.

Definition 4 ([9,14])**.** *Let $0 \le \mu < 1$, $0 \le \nu < 1$ and $0 \le \eta < 1$.*

(i) *We denote by* $C[a,b]$ *the space of continuous functions* x *on* $[a,b]$ *with the norm* $\| x \|_C = \max_{t\in[a,b]} |x(t)|$.

(ii) *We denote the weighted space*

$$C_\mu[a,b] := \{x : (a,b] \to \mathbb{R};\ (t-a)^\mu x(t) \in C[a,b]\} and \| x \|_{C_\mu} = \max_{t\in[a,b]} |(t-a)^\mu x(t)|.$$

(iii) *We denote the weighted space*

$$C^n_\mu[a,b] := \left\{x \in C^{n-1}[a,b];\ x^{(n)} \in C_\mu[a,b],\ n \in \mathbb{N}\right\},$$

with the norm $\| x \|_{C^n_\mu} = \sum_{k=0}^{n-1} \| x^{(k)} \|_C + \| x^{(n)} \|_{C_\mu}$.

(iv) *We denote the weighted space*

$$C^\eta_\mu[a,b] = \left\{x \in C_\mu[a,b];\ D^\eta_{a^+} x \in C_\mu[a,b]\right\},$$

with the norm $\| x \|_{C^\eta_\mu} = \| x \|_{C_\mu} + \| D^\eta_{a^+} x \|_{C_\mu}$.

(v) *We denote the weighted space*

$$C^{\alpha,\beta}_{\mu,\nu}[a,b] = \left\{x \in C_\mu[a,b];\ {}^H D^{\alpha,\beta}_{a^+} x \in C_\nu[a,b]\right\},$$

with the norm $\| x \|_{C^{\alpha,\beta}_{\mu,\nu}} = \| x \|_{C_\mu} + \| {}^H D^{\alpha,\beta}_{a^+} x \|_{C_\nu}$.

Clearly, $C_0[a,b] = C[a,b]$. *We abbreviate* $C[a,T]$, $C_\mu[a,T]$, $C^n_\mu[a,T]$, $C^\eta_\mu[a,T]$, $C^{\alpha,\beta}_{\mu,\nu}[a,T]$ *with* $\mathcal{C}$, C_μ , C^n_μ , C^η_μ , $C^{\alpha,\beta}_{\mu,\nu}$, *respectively.*

Theorem 1 ([3]). *Let* $0 < \eta < 1$, *then,* $D^\eta_{a^+} x(t) = 0$ *if, and only if,* $x(t) = C(t-a)^{\eta-1}$, *where* C *is an arbitrary constant.*

Lemma 1 ([3]). *Let* $\omega > 0$, $\nu \in [0,1)$ *and* $\nu \le \omega$, *if* $y \in C_\nu$, *then,* $\left(I^\omega_{a^+} y\right)(t) \in \mathcal{C}$.

Lemma 2 ([3]). *Let* $\omega, \nu > 0$, *if* $\nu > \omega$ *and* $y \in C_\nu$, *then,* $\left(I^\omega_{a^+} y\right)(t) \in C_{\nu-\omega}$.

Lemma 3 ([9]). *Let* $\omega \in (0,1)$ *and* $\omega > \nu$, *if* $y \in C_\nu$, *then,* $\left(I^\omega_{a^+} y\right)(a^+) = \lim_{t\to a^+} \left(I^\omega_{a^+} y\right)(t) = 0$.

Lemma 4 ([3]). *Let* $\omega_1, \omega_2 > 0$, $0 \le \nu < 1$, *then, for* $\varphi \in C_\nu$, *the following assertions are valid:*

$$\begin{aligned} \left(I^{\omega_1}_{a^+} I^{\omega_2}_{a^+} \varphi\right)(t) &= \left(I^{\omega_1+\omega_2}_{a^+} \varphi\right)(t), \\ \left(D^{\omega_1}_{a^+} I^{\omega_1}_{a^+} \varphi\right)(t) &= \varphi(t), \\ \left(D^{\omega_2}_{a^+} I^{\omega_1}_{a^+} \varphi\right)(t) &= \left(I^{\omega_1-\omega_2}_{a^+} \varphi\right)(t), for \omega_1 > \omega_2. \end{aligned}$$

Lemma 5 ([3,9]). *Let* ω, $\xi > 0$ *and* $\theta > 0$, *then,*

$$\begin{aligned} \left[I^\omega_{a^+} (s-a)^{\xi-1}\right](t) &= \frac{\Gamma(\xi)}{\Gamma(\xi+\omega)} (t-a)^{\xi+\omega-1}, t > a, \\ \left[D^\omega_{a^+} (s-a)^{\omega-j}\right](t) &= 0, j = 1, 2, \cdots, [\omega]+1, t > a, \\ \left[D^\omega_{a^+} (s-a)^{\xi-1}\right](t) &= \frac{\Gamma(\xi)}{\Gamma(\xi-\omega)} (t-a)^{\xi-\omega-1}, t > a,\ \omega < \xi. \end{aligned}$$

Lemma 6 ([3]). *Let* $\omega \in (0,1)$ *and* $\nu \in [0,1)$. *If* $y \in C_\nu$ *and* $I^{1-\omega}_{a^+} y \in C^1_\nu$, *then,*

$$\left(I_{a^+}^{\omega} D_{a^+}^{\omega} y\right)(t) = y(t) - \frac{\left(I_{a^+}^{1-\omega} y\right)(a^+)}{\Gamma(\omega)}(t-a)^{\omega-1}.$$

Theorem 2. *Let $0 < \eta < 1$, then, $f \in C_{1-\eta}^{\eta}$ if, and only if, there exists a function $\varphi \in C_{1-\eta}$ such that*

$$f(t) = \left(I_{a^+}^{\eta} \varphi\right)(t) + c(t-a)^{\eta-1}, t \in (a, T], \tag{3}$$

where c is an arbitrary constant.

Proof. Let $f \in C_{1-\eta}^{\eta}$. According to Definition 4(iv), there exists a function $\varphi(t) \in C_{1-\eta}$ such that $\left(D_{a^+}^{\eta} f\right)(t) = \varphi(t)$,

that is

$$D_{a^+}^{\eta}\left[f(t) - I_{a^+}^{\eta} \varphi(t)\right] = 0.$$

From Theorem 1, for an arbitrary constant c, we obtain $f(t) = \left(I_{a^+}^{\eta} \varphi\right)(t) + c(t-a)^{\eta-1}$; clearly, $\frac{\left(I_{a^+}^{1-\eta} f\right)(a^+)}{\Gamma(\eta)} = c$. Conversely, if f satisfies Equation (3), then, $f(t) \in C_{1-\eta}$, and

$$D_{a^+}^{\eta} f(t) = \varphi(t) \in C_{1-\eta},$$

hence, $f \in C_{1-\eta}^{\eta}$. □

2.2. Generalized Mittag–Leffler Functions

Definition 5 ([3,15]). *For $\rho, \omega, \zeta > 0, l, z \in \mathbb{R}$, the generalized Mittag–Leffler function $E_{\rho,\omega}(z)$ and the generalized Mittag–Leffler function $E_{\rho,\zeta,l}(z)$ are defined by the following series, respectively:*

$$\begin{aligned} E_{\rho,\omega}(z) &= \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\rho k+\omega)}, \\ E_{\rho,\zeta,l}(z) &= \sum_{k=0}^{+\infty} d_k z^k, \rho(i\zeta + l) + 1 \notin \mathbb{Z}^-(i \in \mathbb{N}), \end{aligned}$$

where $d_0 = 1$ and

$$d_k = \prod_{i=0}^{k-1} \frac{\Gamma[\rho(i\zeta + l) + 1]}{\Gamma[\rho(i\zeta + l + 1) + 1]}.$$

Definition 6 ([16]). *For $\rho, \omega, \sigma > 0, z \in \mathbb{R}$, the generalized Mittag–Leffler function $E_{\rho,\omega}^{\sigma}(z)$ is defined by the following series*

$$E_{\rho,\omega}^{\sigma}(z) = \sum_{k=0}^{+\infty} \frac{(\sigma)_k}{\Gamma(\rho k + \omega)} \frac{z^k}{k!},$$

where $(\sigma)_k$ is the Pochhammer symbol

$$(\sigma)_0 = 1, (\sigma)_k = \sigma(\sigma+1)\cdots(\sigma+k-1), k = 1, 2, \cdots .$$

Clearly, $E_{\rho,\omega}^{1}(z) = E_{\rho,\omega}(z)$.

Lemma 7 ([15]). *Let $\rho, \omega, \delta \in (0,1), \theta > 0$. The generalized Mittag–Leffler functions $E_{\rho,\omega}(\cdot)$, $E_{\rho,\rho+\delta}(\cdot)$ are non-negative and have the following properties:*

(i) for $t > a$, $E_{\rho,\omega}\left(-\theta(t-a)^{\rho}\right) \leq \frac{1}{\Gamma(\omega)}$ and $E_{\rho,\rho+\delta}\left(-\theta(t-a)^{\rho}\right) \leq \frac{1}{\Gamma(\rho+\delta)}$;

(ii) *for* $t_1, t_2 > 0$, $\left|E_{\rho,\omega}\left(-\theta t_2^{\rho}\right) - E_{\rho,\omega}\left(-\theta t_1^{\rho}\right)\right| := O(|t_2 - t_1|^{\rho}),\ t_2 \to t_1$;

(iii) *for* $\kappa \in (0,1)$,

$$\int_a^t (t-s)^{\omega-1} E_{\rho,\omega}\left(-\theta(t-s)^{\rho}\right)(s-a)^{\kappa-1} ds = \Gamma(\kappa)(t-a)^{\omega+\kappa-1} E_{\rho,\omega+\kappa}\left(-\theta(t-a)^{\rho}\right).$$

We denote the integral operator $\mathbf{E}^{\sigma}_{\rho,\omega,\theta;a^+}$ (see [17]) as follows:

$$\left(\mathbf{E}^{\sigma}_{\rho,\omega,\theta;a^+}\varphi\right)(t) := \int_a^t (t-s)^{\omega-1} E^{\sigma}_{\rho,\omega}\left(-\theta(t-s)^{\rho}\right)\varphi(s)ds, t > a,\ \varphi \in C_{\nu}.$$

In particular, $E^{0}_{\rho,\omega}(-\theta t^{\rho}) = \frac{1}{\Gamma(\omega)}$ (see [17]); hence,

$$\left(\mathbf{E}^{0}_{\rho,\omega,\theta;a^+}\varphi\right)(t) = \left(I^{\omega}_{a^+}\varphi\right)(t), t > a.$$

Lemma 8 ([17])**.** *For* $\rho,\ \omega_1,\ \omega_2, \sigma_1,\ \sigma_2,\ \alpha,\ \theta > 0$, $\kappa \in (0,1)$ *and* $\varphi \in C_{\nu}$,

$$\begin{aligned}
\mathbf{E}^{\sigma_1}_{\rho,\omega_1,\theta;a^+}\mathbf{E}^{\sigma_2}_{\rho,\omega_2,\theta;a^+}\varphi(t) &= \mathbf{E}^{\sigma_1+\sigma_2}_{\rho,\omega_1+\omega_2,\theta;a^+}\varphi(t);\\
I^{\alpha}_{a^+}\mathbf{E}^{\sigma_1}_{\rho,\omega_1,\theta;a^+}\varphi(t) &= \mathbf{E}^{\sigma_1}_{\rho,\omega_1,\theta;a^+}I^{\alpha}_{a^+}\varphi(t) = \mathbf{E}^{\sigma_1}_{\rho,\omega_1+\alpha,\theta;a^+}\varphi(t);\\
\left(\mathbf{E}^{\sigma_1}_{\rho,\omega_1,\theta;a^+}(s-a)^{\kappa-1}\right)(t) &= \Gamma(\kappa)(t-a)^{\omega_1+\kappa-1}E^{\sigma_1}_{\rho,\omega_1+\kappa}\left(-\theta(t-a)^{\rho}\right).
\end{aligned}$$

Lemma 9. *Let* $\rho, \omega \in (0,1)$, $\theta > 0$ *and* $\nu \in [0,1)$ *, then, the integral operator* $\boldsymbol{E}_{\rho,\omega,\theta;a^+}$ *has the following properties:*

(i) $\boldsymbol{E}_{\rho,\omega,\theta;a^+}$ *is a bounded operator from* C_{ν} *into* C_{ν} *;*

(ii) *if* $\nu > \omega$ *and* $\varphi \in C_{\nu}$ *, then,* $\left(\boldsymbol{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t) \in C_{\nu-\omega}$.

Proof. It is easy to see that

$$\left|\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t)\right| \le \frac{\|\varphi\|_{C_{\nu}}}{\Gamma(\omega)}\int_a^t (t-s)^{\omega-1}(s-a)^{-\nu}ds \le \frac{(t-a)^{\omega-\nu}\Gamma(1-\nu)}{\Gamma(\omega+1-\nu)}\|\varphi\|_{C_{\nu}}.$$

Let $h > 0$, $t,\ t+h \in [a,T]$; we have

$$\begin{aligned}
&\left|(t+h-a)^{\nu}\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t+h) - (t-a)^{\nu}\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t)\right|\\
\le\ & |(t+h-a)^{\nu} - (t-a)^{\nu}|\left|\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t+h)\right|\\
&+(t-a)^{\nu}\left|\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t+h) - \left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t)\right|\\
\le\ & |(t+h-a)^{\nu} - (t-a)^{\nu}|\left|\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t+h)\right|\\
&+(t-a)^{\nu}\int_a^t \left|(t+h-s)^{\omega-1}E_{\rho,\omega}\left(-\theta(t+h-s)^{\rho}\right)\right.\\
&\left.-(t-s)^{\omega-1}E_{\rho,\omega}\left(-\theta(t-s)^{\rho}\right)\right||\varphi(s)|ds\\
&+(t-a)^{\nu}\int_t^{t+h}\left|(t+h-s)^{\omega-1}E_{\rho,\omega}\left(-\theta(t+h-s)^{\rho}\right)\right||\varphi(s)|ds\\
:=\ & J_1 + J_2 + J_3,
\end{aligned}$$

where

$$J_1 = |(t+h-a)^{\nu} - (t-a)^{\nu}|\left|\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t+h)\right|,$$

$$J_2 = (t-a)^{\nu}\int_a^t \left|(t+h-s)^{\omega-1}E_{\rho,\omega}\left(-\theta(t+h-s)^{\rho}\right) - (t-s)^{\omega-1}E_{\rho,\omega}\left(-\theta(t-s)^{\rho}\right)\right||\varphi(s)|ds,$$

$$J_3 = (t-a)^{\nu}\int_t^{t+h}\left|(t+h-s)^{\omega-1}E_{\rho,\omega}\left(-\theta(t+h-s)^{\rho}\right)\right||\varphi(s)|ds.$$

For J_1 , we have

$$J_1 \leq \left|1 - \left(\frac{t-a}{t+h-a}\right)^{\nu}\right| \frac{(t+h-a)^{\omega}\Gamma(1-\nu)}{\Gamma(\omega+1-\nu)} \| \varphi \|_{C_\nu} \to 0, \; h \to 0^+.$$

Using Lemma 7(ii), we find

$$J_2 = \frac{(t-a)\| \varphi \|_{C_\nu}}{1-\nu} O(h^{\rho}) \to 0, \; h \to 0^+.$$

For J_3 , we derive the estimate

$$\begin{array}{rcl} J_3 & \leq & (t-a)^{\nu} \int_a^{t+h} |(t+h-s)^{\omega-1} E_{\rho,\omega}(-\theta(t+h-s)^{\rho})||\varphi(s)|ds \\ & \leq & \left(\frac{t-a}{t+h-a}\right)^{\nu} \frac{\Gamma(1-\nu)}{\Gamma(\omega+1-\nu)} (t+h-a)^{\omega} \| \varphi \|_{C_\nu}; \end{array}$$

hence, $J_3 \to 0$ as $h \to 0^+$. Therefore,

$$\left|(t+h-a)^{\nu}\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t+h) - (t-a)^{\nu}\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t)\right| \to 0, h \to 0^+.$$

Similarly, we can prove that

$$\left|(t+h-a)^{\nu}\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t+h) - (t-a)^{\nu}\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t)\right| \to 0, h \to 0^-.$$

Thus, $\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t) \in C_\nu$. Moreover,

$$\begin{array}{rcl} (t-a)^{\nu}\left|\left(\mathbf{E}_{\rho,\omega,\theta;a^+}\varphi\right)(t)\right| & \leq & \frac{(t-a)^{\nu}}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1}(s-a)^{-\nu} ds \| \varphi \|_{C_\nu} \\ & \leq & \frac{(T-a)^{\omega}\Gamma(1-\nu)}{\Gamma(\omega+1-\nu)} \| \varphi \|_{C_\nu}, \end{array}$$

which completes the proof of (i). The proof of (ii) is similar to that of (i). □

3. Equivalent Integral Equation

We consider the following linear Hilfer fractional differential equation

$$\left({}^{H}D_{a^+}^{\alpha,\beta} x\right)(t) = f(t), \; t \in (a,T] \tag{4}$$

with the initial condition

$$\left(I_{a^+}^{1-\gamma} x\right)(a^+) = x_0. \tag{5}$$

We study the above problem in the weighted space $C_{1-\gamma,\nu}^{\alpha,\beta}$, where

$$\alpha < \nu < 1 - \gamma + \alpha. \tag{6}$$

Theorem 3. *Let $f : (a,T] \to \mathbb{R}$ and $f(t) \in C_\nu$, then, $x \in C_{1-\gamma,\nu}^{\alpha,\beta}$ satisfies Problems (4) and (5) if, and only if, x satisfies the following relation*

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} x_0 + \left(I_{a^+}^{\alpha} f\right)(t). \tag{7}$$

Proof. In view of Equation (6) and Lemma 3, for $x(t) \in C_{1-\gamma,\nu}^{\alpha,\beta}$ and $f \in C_\nu$, we obtain

$$\left(I_{a^+}^{1-\gamma+\alpha} x\right)(a^+) = 0, \left(I_{a^+}^{1-\gamma+\alpha} f\right)(a^+) = 0.$$

Since $x(t) \in C^{\alpha,\beta}_{1-\gamma,\nu}$, from Definition 3 and Definition 4(iv), we can see that

$$\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+) \in C^{1-\gamma+\alpha}_{\nu}.$$

By Theorem 2 and $I^{\gamma-\alpha}_{a^+}\left(\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right)(a^+) = 0$, there exists a function $\varphi(t) \in C_\nu$ such that

$$\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+) = \left(I^{1-\gamma+\alpha}_{a^+}\varphi\right)(t),$$

and hence,

$$I^{\gamma-\alpha}_{a^+}\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right] = \left(I^{1}_{a^+}\varphi\right)(t).$$

This means $I^{\gamma-\alpha}_{a^+}\left(\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right) \in C^1_\nu$, from Lemma 6 and the fact

$$\left\{I^{\gamma-\alpha}_{a^+}\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right]\right\}(a^+) = 0,$$

one obtains

$$\begin{aligned}
& I^{1-\gamma+\alpha}_{a^+}\, D^{1-\gamma+\alpha}_{a^+}\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right] \\
= & \left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+) - \frac{\left\{I^{\gamma-\alpha}_{a^+}\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right]\right\}(a^+)(t-a)^{\alpha-\gamma}}{\Gamma(1-\gamma+\alpha)} \\
= & \left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I^{\alpha}_{a^+}\, {}^{H}D^{\alpha,\beta}_{a^+}x(t) &= I^{\alpha}_{a^+}\, D^{1-\gamma+\alpha}_{a^+}\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right] \\
&= \left[D^{1-\gamma}_{a^+}I^{1-\gamma}_{a^+}\right] I^{\alpha}_{a^+}\, D^{1-\gamma+\alpha}_{a^+}\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right] \\
&= D^{1-\gamma}_{a^+}\left[I^{1-\gamma+\alpha}_{a^+}\, D^{1-\gamma+\alpha}_{a^+}\right]\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right] \\
&= D^{1-\gamma}_{a^+}\left[\left(I^{1-\gamma}_{a^+}x\right)(t) - \left(I^{1-\gamma}_{a^+}x\right)(a^+)\right] \\
&= x(t) - \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0.
\end{aligned}$$

Now, applying $I^{\alpha}_{a^+}$ to both sides of Equation (4), we obtain Equation (7).

If $x(t)$ satisfies Equation (7), since $I^{\alpha}_{a^+}f \in C_{\nu-\alpha} \subset C_{1-\gamma}$ and $\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 \in C_{1-\gamma}$, then, $x \in C_{1-\gamma}$. Moreover, note that

$$\begin{aligned}
I^{1-\gamma}_{a^+}x(t) &= I^{1-\gamma}_{a^+}\left[\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0\right] + \left(I^{1-\gamma+\alpha}_{a^+}f\right)(t) \\
&= x_0 + \left(I^{1-\gamma+\alpha}_{a^+}f\right)(t),
\end{aligned} \tag{8}$$

and $\left(I^{1-\gamma+\alpha}_{a^+}f\right)(a^+) = 0$, then, by Equation (8), one has $\left(I^{1-\gamma}_{a^+}x\right)(a^+) = x_0$ and

$$D^{1-\gamma+\alpha}_{a^+}\left[I^{1-\gamma}_{a^+}x(t) - I^{1-\gamma}_{a^+}x(a^+)\right] = f(t) \in C_\nu.$$

This means $x(t) \in C^{\alpha,\beta}_{1-\gamma,\nu}$ and ${}^{H}D^{\alpha,\beta}_{a^+}x(t) = f(t)$; now, we obtain Equation (4). The results are proved completely. □

Similarly, we can obtain the result for the nonlinear case; we omit the proof here.

Theorem 4. *Let $f:(a,T]\to\mathbb{R}$ be a function such that $f\Big(t,\,\Big(I^{\eta}_{a^+}x\Big)(t)\Big)\in C_\nu$ for any $x(t)\in C_{1-\gamma}$ and $\eta\in(0,1)$. If $x(t)\in C^{\alpha,\beta}_{1-\gamma,\nu}$, then, x satisfies*

$$\begin{cases} \Big({}^{H}D^{\alpha,\beta}_{a^+}x\Big)(t)=f\Big(t,\,\Big(I^{\eta}_{a^+}x\Big)(t)\Big),\\ \Big(I^{1-\gamma}_{a^+}x\Big)(a^+)=x_0,\end{cases}$$

if, and only if, x satisfies

$$x(t)=\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f\Big(s,\,\Big(I^{\eta}_{a^+}x\Big)(s)\Big)ds.$$

Theorem 5. *Let $f:(a,T]\to\mathbb{R}$ be a function such that $f\Big(t,\,\Big(\mathbf{E}_{\rho,\omega,\theta;a^+}x\Big)(t)\Big)\in C_\nu$ for any $x(t)\in C_{1-\gamma}$ and $\rho,\omega\in(0,1)$, $\theta>0$. If $x(t)\in C^{\alpha,\beta}_{1-\gamma,\nu}$, then, x satisfies*

$$\begin{cases} \Big({}^{H}D^{\alpha,\beta}_{a^+}x\Big)(t)=f\Big(t,\,\Big(\mathbf{E}_{\rho,\omega,\theta;a^+}x\Big)(t)\Big),\\ \Big(I^{1-\gamma}_{a^+}x\Big)(a^+)=x_0,\end{cases}$$

if, and only if, x satisfies

$$x(t)=\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f\Big(s,\,\Big(\mathbf{E}_{\rho,\omega,\theta;a^+}x\Big)(s)\Big)ds.$$

4. Exact Solutions of Two Kinds of Fractional Integro-Differential Equations

In this section, we study two kinds of initial value problems for fractional integro-differential equations with a variable coefficient.

4.1. Solutions of a Fractional Kinetic Equation

Firstly, we present the following lemma:

Lemma 10. *For $0\le\zeta<\eta<\nu$, $\lambda(t)\in C_\zeta$ and $\Phi(t)\in C_\nu$, the integral equation*

$$y(t)=\Phi(t)+\lambda(t)\Big(I^{\eta}_{a^+}y\Big)(t) \tag{9}$$

has a unique solution $y\in C_\nu$ given by

$$y(t)=\sum_{k=0}^{+\infty}\Big[\lambda(t)I^{\eta}_{a^+}\Big]^k\Phi(t).$$

Proof. Since $\Big(I^{\eta}_{a^+}y\Big)(t)\in C_{\nu-\eta}$ and $\lambda(t)\Big(I^{\eta}_{a^+}y\Big)(t)\in C_{\nu-\eta+\zeta}\subset C_\nu$, we define an operator $\mathcal{F}:C_\nu\to C_\nu$ as follows:

$$(\mathcal{F}y)(t)=\Phi(t)+\lambda(t)\Big(I^{\eta}_{a^+}y\Big)(t).$$

Clearly, $\mathcal{F}$ is well defined and a fixed point of $\mathcal{F}$ is a solution of Equation (9). Since

$$\int_a^t(t-s)^{\theta_1-1}(s-a)^{\theta_2-1}ds=(t-a)^{\theta_1+\theta_2-1}\cdot\tfrac{\Gamma(\theta_1)\Gamma(\theta_2)}{\Gamma(\theta_1+\theta_2)},0<\theta_1,\,\theta_2<1,$$

for $y,\,\widetilde{y}\in C_\nu$, we have

$$\begin{aligned} & (t-a)^{\nu}|(\mathcal{F}y)(t)-(\mathcal{F}\widetilde{y})(t)| \\ \leq \ & \frac{\|\lambda\|_{C_\zeta}(t-a)^{\nu-\zeta}}{\Gamma(\eta)}\int_a^t (t-s)^{\eta-1}(s-a)^{-\nu}ds\cdot\| y-\widetilde{y} \|_{C_\nu} \\ = \ & \frac{\|\lambda\|_{C_\zeta}\Gamma(1-\nu)}{\Gamma(1+\eta-\nu)}(t-a)^{\eta-\zeta}\cdot\| y-\widetilde{y} \|_{C_\nu}. \end{aligned}$$

Similarly, we find

$$\begin{aligned} & (t-a)^{\nu}\left|(\mathcal{F}^2y)(t)-(\mathcal{F}^2\widetilde{y})(t)\right| \\ \leq \ & \frac{\|\lambda\|_{C_\zeta}(t-a)^{\nu-\zeta}}{\Gamma(\eta)}\int_a^t (t-s)^{\eta-1}(s-a)^{\eta-\zeta-\nu}(s-a)^{\nu}|(\mathcal{F}y)(s)-(\mathcal{F}\widetilde{y})(s)|ds \\ \leq \ & \| \lambda \|_{C_\zeta}^2 \prod_{i=0}^{1}\frac{\Gamma(i(\eta-\zeta)+1-\nu)}{\Gamma(i(\eta-\zeta)+\eta+1-\nu)}(t-a)^{2(\eta-\zeta)}\cdot\| y-\widetilde{y} \|_{C_\nu}. \end{aligned}$$

By induction, we deduce that

$$(t-a)^{\nu}\left|\left(\mathcal{F}^k y\right)(t)-\left(\mathcal{F}^k\widetilde{y}\right)(t)\right| \leq \prod_{i=0}^{k-1}\frac{\Gamma(i(\eta-\zeta)+1-\nu)}{\Gamma(i(\eta-\zeta)+\eta+1-\nu)}\| \lambda \|_{C_\zeta}^k (T-a)^{k(\eta-\zeta)}\cdot\| y-\widetilde{y} \|_{C_\nu}. \quad (10)$$

We write

$$d_k=\prod_{i=0}^{k-1}\frac{\Gamma(i(\eta-\zeta)+1-\nu)}{\Gamma(i(\eta-\zeta)+\eta+1-\nu)}, k=1,2,\cdots;$$

it follows from ([15], Equation (5.2.13)) that

$$\lim_{k\to+\infty}\frac{k\ln k}{\ln\left(\frac{1}{d_k}\right)}=\frac{1}{\eta}.$$

This means that for sufficiently large k, the right side of Equation (10) is smaller than $L\| y-\widetilde{y} \|_{C_\nu} \|(L\in(0,1))$; by the generalized Banach's fixed point theorem [18], $\mathcal{F}$ has a unique fixed point $y\in C_\nu$ satisfying Equation (9). Then, the following sequence $\{y_n\}$ is convergent in C_ν :

$$\begin{cases} y_0(t)=\Phi(t), \\ y_n(t)=y_0(t)+\lambda(t)\left(I_{a^+}^{\eta}y_{n-1}\right)(t), n=1,2,\cdots. \end{cases}$$

Furthermore, we find

$$\begin{aligned} y_1(t) &= y_0(t)+\lambda(t)\left(I_{a^+}^{\eta}y_0\right)(t), \\ y_2(t) &= y_0(t)+\sum_{k=1}^{2}\left[\lambda(t)I_{a^+}^{\eta}\right]^k y_0(t), \\ &\cdots \\ y_n(t) &= y_0(t)+\sum_{k=1}^{n}\left[\lambda(t)I_{a^+}^{\eta}\right]^k y_0(t). \end{aligned}$$

Taking the limit as $n\to+\infty$ in the last identity, we obtain

$$y(t)=\lim_{n\to+\infty}\sum_{k=0}^{n}\left[\lambda(t)I_{a^+}^{\eta}\right]^k\Phi(t)=\sum_{k=0}^{+\infty}\left[\lambda(t)I_{a^+}^{\eta}\right]^k\Phi(t)$$

and $y(t)$ is the unique solution of Equation (9). □

Next, we consider the initial value problem for the inhomogeneous fractional kinetic equation with a variable coefficient:

$$\begin{cases} \left({}^{H}D_{a^+}^{\alpha,\beta}x\right)(t) - \lambda(t)I_{a^+}^{\tau}x(t) = g(t), \\ \left(I_{a^+}^{1-\gamma}x\right)(a^+) = x_0, \end{cases} \tag{11}$$

where $0 < \tau < \nu - \alpha$ and $\lambda(t) \in C_\zeta(0 \le \zeta < \nu + \gamma - 1 + \tau)$.

Theorem 6. *Let $g(t) \in C_\nu$, then, Problem (11) has a unique solution $x(t) \in C_{1-\gamma,\nu}^{\alpha,\beta}$ given by*

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 + I_{a^+}^{\alpha}\sum_{k=0}^{+\infty}\left(\lambda(t)I_{a^+}^{\tau+\alpha}\right)^k\left[\frac{\lambda(t)}{\Gamma(\gamma+\tau)}(t-a)^{\gamma+\tau-1}x_0 + g(t)\right].$$

Proof. From Theorem 4, $x \in C_{1-\gamma,\nu}^{\alpha,\beta}$ satisfies Problem (11) if, and only if, $x(t)$ satisfies

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 + I_{a^+}^{\alpha}\left[\lambda(t)I_{a^+}^{\tau}x(t) + g(t)\right]. \tag{12}$$

Let $y(t) = {}^{H}D_{a^+}^{\alpha,\beta}x(t) \in C_\nu$, then, $x(t) = I_{a^+}^{\alpha}y(t) + \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0$, and, hence, Equation (12) can be transformed into

$$I_{a^+}^{\alpha}y(t) = I_{a^+}^{\alpha}\left[\lambda(t)I_{a^+}^{\tau+\alpha}y(t) + \frac{\lambda(t)}{\Gamma(\gamma+\tau)}(t-a)^{\gamma+\tau-1}x_0 + g(t)\right];$$

thus,

$$y(t) = \lambda(t)I_{a^+}^{\tau+\alpha}y(t) + \frac{\lambda(t)}{\Gamma(\gamma+\tau)}(t-a)^{\gamma+\tau-1}x_0 + g(t). \tag{13}$$

By Lemma 10 ($\eta = \tau + \alpha$), Equation (13) has a unique solution

$$y(t) = \sum_{k=0}^{+\infty}\left(\lambda(t)I_{a^+}^{\tau+\alpha}\right)^k\left[\frac{\lambda(t)}{\Gamma(\gamma+\tau)}(t-a)^{\gamma+\tau-1}x_0 + g(t)\right].$$

This yields

$$\begin{aligned} x(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 + I_{a^+}^{\alpha}y(t) \\ &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 + I_{a^+}^{\alpha}\sum_{k=0}^{+\infty}\left(\lambda(t)I_{a^+}^{\tau+\alpha}\right)^k\left[\frac{\lambda(t)}{\Gamma(\gamma+\tau)}(t-a)^{\gamma+\tau-1}x_0 + g(t)\right]. \end{aligned}$$

□

When $\lambda(t) \equiv \lambda$ is a constant, we have

$$\begin{aligned} x(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 + I_{a^+}^{\alpha}\sum_{k=0}^{+\infty}\lambda^{k+1}\left(I_{a^+}^{\tau+\alpha}\right)^k\left[\frac{(t-a)^{\gamma+\tau-1}}{\Gamma(\gamma+\tau)}\right]x_0 + I_{a^+}^{\alpha}\sum_{k=0}^{+\infty}\lambda^k\left(I_{a^+}^{\tau+\alpha}\right)^k g(t) \\ &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 + (t-a)^{\gamma-1}\sum_{k=0}^{+\infty}\frac{\left(\lambda(t-a)^{\tau+\alpha}\right)^{k+1}}{\Gamma[(\tau+\alpha)(k+1)+\gamma]}x_0 + \sum_{k=0}^{+\infty}\lambda^k I_{a^+}^{k(\alpha+\tau)+\alpha}g(t) \\ &= (t-a)^{\gamma-1}x_0E_{\tau+\alpha,\gamma}\left(\lambda(t-a)^{\tau+\alpha}\right) + \int_a^t(t-s)^{\alpha-1}E_{\tau+\alpha,\alpha}\left(\lambda(t-s)^{\tau+\alpha}\right)g(s)ds. \end{aligned}$$

Then, we obtain the following conclusion:

Theorem 7. *Let $g(t) \in C_\nu$, then, the fractional kinetic initial value problem*

$$\begin{cases} a\,{}^H D_{0^+}^{\alpha,\beta} x(t) - b I_{0^+}^{\tau} x(t) = N_0 g(t), \\ \left(I_{0^+}^{1-\gamma} x\right)(0^+) = d, \end{cases}$$

has a unique solution $x(t)$ given by

$$x(t) = d t^{\gamma-1} E_{\tau+\alpha,\gamma}\left(\frac{b}{a} t^{\tau+\alpha}\right) + \frac{N_0}{a}\int_0^t (t-s)^{\alpha-1} E_{\tau+\alpha,\alpha}\left(\frac{b}{a}(t-s)^{\tau+\alpha}\right) g(s)ds, a \neq 0.$$

This result coincides with ([19], Theorem 10), in which the result is obtained by using the Laplace transform.

4.2. Solutions of a Fractional Integro-Differential Equation with a Generalized Mittag–Leffler Function

Lemma 11. *For $0 \le \xi < \omega < \nu$, $\lambda(t) \in C_\xi$ and $\Psi(t) \in C_\nu$, the integral equation*

$$y(t) = \Psi(t) + \lambda(t)\left(\mathbf{E}_{\rho,\omega,\theta;a^+} y\right)(t) \tag{14}$$

has a unique solution $y \in C_\nu$ given by

$$y(t) = \sum_{k=0}^{+\infty}\left[\lambda(t)\mathbf{E}_{\rho,\omega,\theta;a^+}\right]^k \Psi(t).$$

Proof. Since $\left(\mathbf{E}_{\rho,\omega,\theta;a^+} y\right)(t) \in C_{\nu-\omega}$ and $\lambda(t)\left(\mathbf{E}_{\rho,\omega,\theta;a^+} y\right)(t) \in C_{\nu-\omega+\xi} \subset C_\nu$, we define an operator $\mathcal{T} : C_\nu \to C_\nu$ as follows:

$$(\mathcal{T}y)(t) = \Psi(t) + \lambda(t)\left(\mathbf{E}_{\rho,\omega,\theta;a^+} y\right)(t).$$

It is obvious that a fixed point of $\mathcal{T}$ is a solution of Equation (14). For $y, \widetilde{y} \in C_\nu$; we have

$$\begin{aligned} & (t-a)^\nu |(\mathcal{T}y)(t) - (\mathcal{T}\widetilde{y})(t)| \\ \le\ & (t-a)^\nu |\lambda(t)| \int_a^t (t-s)^{\omega-1} E_{\rho,\omega}\left(-\theta(t-s)^\rho\right) |y(s) - \widetilde{y}(s)| ds \\ \le\ & \frac{\|\lambda\|_{C_\xi}(t-a)^{\nu-\xi}}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1}(s-a)^{-\nu} ds \cdot \| y - \widetilde{y} \|_{C_\nu} \\ =\ & \frac{\|\lambda\|_{C_\xi}\Gamma(1-\nu)}{\Gamma(1+\omega-\nu)}(t-a)^{\omega-\xi} \cdot \| y - \widetilde{y} \|_{C_\nu}. \end{aligned}$$

Furthermore,

$$\begin{aligned} & (t-a)^\nu \left|\left(\mathcal{T}^2 y\right)(t) - \left(\mathcal{T}^2 \widetilde{y}\right)(t)\right| \\ \le\ & \frac{\|\lambda\|_{C_\xi}(t-a)^{\nu-\xi}}{\Gamma(\omega)} \int_a^t (t-s)^{\omega-1}(s-a)^{\omega-\xi-\nu}(s-a)^\nu |(\mathcal{T}y)(s) - (\mathcal{T}\widetilde{y})(s)| ds \\ \le\ & \| \lambda \|_{C_\xi}^2 \prod_{i=0}^{1} \frac{\Gamma(i(\omega-\xi)+1-\nu)}{\Gamma(i(\omega-\xi)+\omega+1-\nu)}(t-a)^{2(\omega-\xi)} \cdot \| y - \widetilde{y} \|_{C_\nu}, \end{aligned}$$

similar to the proof of Lemma 10, we can deduce that the operator $\mathcal{T}$ has a unique fixed point $y \in C_\nu$ satisfying Equation (14) and $y(t)$ is the limit of the following sequence $\{y_n\}$:

$$\begin{cases} y_0(t) = \Psi(t), \\ y_n(t) = y_0(t) + \lambda(t)\left(\mathbf{E}_{\rho,\omega,\theta;a^+} y_{n-1}\right)(t), n = 1, 2, \cdots. \end{cases}$$

Thus, we obtain

$$y(t) = \sum_{k=0}^{+\infty}\Big[\lambda(t)\mathbf{E}_{\rho,\omega,\theta;a^+}\Big]^k\Psi(t).$$

□

Theorem 8. *Let $h:(a,T]\to\mathbb{R}$ and $h(t)\in C_\nu$, $\lambda(t)\in C_\xi$, if $0<\widetilde{\omega}<\nu-\alpha$ and $0\le\xi<\nu+\gamma-1+\widetilde{\omega}$, then, the initial value problem*

$$\begin{cases} \Big({}^HD^{\alpha,\beta}_{a^+}x\Big)(t)-\lambda(t)\int_a^t(t-s)^{\widetilde{\omega}-1}E_{\rho,\widetilde{\omega}}\big(-\theta(t-s)^\rho\big)x(s)ds=h(t), \\ \Big(I^{1-\gamma}_{a^+}x\Big)(a^+)=x_0, \end{cases} \tag{15}$$

has a unique solution $x\in C^{\alpha,\beta}_{1-\gamma,\nu}$ given by

$$\begin{aligned} x(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 \\ &\quad + I^\alpha_{a^+}\sum_{k=0}^{+\infty}\Big(\lambda(t)\mathbf{E}_{\rho,\widetilde{\omega}+\alpha,\theta;a^+}\Big)^k\Big[x_0\lambda(t)(t-a)^{\widetilde{\omega}+\gamma-1}E_{\rho,\widetilde{\omega}+\gamma}\big(-\theta(t-a)^\rho\big)+h(t)\Big]. \end{aligned}$$

Proof. Let $f(t,x(t))=\lambda(t)\Big(\mathbf{E}_{\rho,\widetilde{\omega},\theta;a^+}x\Big)(t)+h(t)$; from Theorem 5, $x\in C^{\alpha,\beta}_{1-\gamma,\nu}$ satisfies Problem (15), and only $x(t)$ satisfies

$$x(t)=\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0+I^\alpha_{a^+}\Big[\lambda(t)\Big(\mathbf{E}_{\rho,\widetilde{\omega},\theta;a^+}x\Big)(t)+h(t)\Big]. \tag{16}$$

Let $z(t)={}^HD^{\alpha,\beta}_{a^+}x(t)$, then, $x(t)=I^\alpha_{a^+}z(t)+\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0$, and, hence, Equation (16) can be transformed into

$$z(t) = \lambda(t)\mathbf{E}_{\rho,\widetilde{\omega}+\alpha,\theta;a^+}z(t)+\lambda(t)\mathbf{E}_{\rho,\widetilde{\omega},\theta;a^+}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0+h(t).$$

From Lemma 2, one has $\lambda(t)\mathbf{E}_{\rho,\widetilde{\omega}+\alpha,\theta;a^+}z(t)$, $\lambda(t)\mathbf{E}_{\rho,\widetilde{\omega},\theta;a^+}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\in C_\nu$. By Lemma 11, there exists a unique function $z(t)\in C_\nu$ such that

$$z(t)=\sum_{k=0}^{+\infty}\Big(\lambda(t)\mathbf{E}_{\rho,\widetilde{\omega}+\alpha,\theta;a^+}\Big)^k\left[\lambda(t)\mathbf{E}_{\rho,\widetilde{\omega},\theta;a^+}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0+h(t)\right].$$

By Lemma 7(iii), we find

$$\begin{aligned} x(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0+I^\alpha_{a^+}z(t) \\ &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 \\ &\quad + I^\alpha_{a^+}\sum_{k=0}^{+\infty}\Big(\lambda(t)\mathbf{E}_{\rho,\widetilde{\omega}+\alpha,\theta;a^+}\Big)^k\Big[x_0\lambda(t)(t-a)^{\widetilde{\omega}+\gamma-1}E_{\rho,\widetilde{\omega}+\gamma}\big(-\theta(t-a)^\rho\big)+h(t)\Big]. \end{aligned}$$

We complete the proof. □

When $\lambda(t)\equiv\lambda$, from Lemma 8, we obtain

$$\sum_{k=0}^{+\infty}\Big(\mathbf{E}_{\rho,\widetilde{\omega}+\alpha,\theta;a^+}\Big)^k\mathbf{E}_{\rho,\widetilde{\omega},\theta;a^+}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0=\sum_{k=0}^{+\infty}\mathbf{E}^{k+1}_{\rho,k(\widetilde{\omega}+\alpha)+\widetilde{\omega},\theta;a^+}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0,$$

and

$$\begin{aligned} & I_{a^+}^{\alpha}\left[\sum_{k=0}^{+\infty}\left(\mathbf{E}_{\rho,\widetilde{\omega}+\alpha,\theta;a^+}\right)^k \mathbf{E}_{\rho,\widetilde{\omega},\theta;a^+}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0\right] \\ = & I_{a^+}^{\alpha}\sum_{k=0}^{+\infty}\mathbf{E}^{k+1}_{\rho,k(\widetilde{\omega}+\alpha)+\widetilde{\omega},\theta;a^+}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0 \\ = & \sum_{k=0}^{+\infty}(t-a)^{(k+1)(\widetilde{\omega}+\alpha)+\gamma-1}E^{k+1}_{\rho,(k+1)(\widetilde{\omega}+\alpha)+\gamma}\left(-\theta(t-a)^{\rho}\right)x_0. \end{aligned}$$

Hence, we have

$$\begin{aligned} x(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}x_0+\sum_{k=0}^{+\infty}\lambda^{k+1}(t-a)^{(k+1)(\widetilde{\omega}+\alpha)+\gamma-1}E^{k+1}_{\rho,(k+1)(\widetilde{\omega}+\alpha)+\gamma}\left(-\theta(t-a)^{\rho}\right)x_0 \\ &\quad +\sum_{k=0}^{+\infty}\lambda^k\mathbf{E}^k_{\rho,k(\widetilde{\omega}+\alpha)+\alpha,\theta;a^+}h(t) \\ &= \sum_{k=0}^{+\infty}\lambda^k(t-a)^{k(\widetilde{\omega}+\alpha)+\gamma-1}E^k_{\rho,k(\widetilde{\omega}+\alpha)+\gamma}\left(-\theta(t-a)^{\rho}\right)x_0+\sum_{k=0}^{+\infty}\lambda^k\mathbf{E}^k_{\rho,k(\widetilde{\omega}+\alpha)+\alpha,\theta;a^+}h(t). \end{aligned}$$

Now, we obtain the following conclusion.

Theorem 9. *Let $h(t)\in C_\nu$ ($2\alpha<\nu<1-\gamma+\alpha$), then, the following initial value problem*

$$\begin{cases} {}^H D^{\alpha,\beta}_{a^+}x(t)-\lambda\int_a^t(t-s)^{\alpha-1}E_{\rho,\alpha}\left(-\theta(t-s)^{\rho}\right)x(s)ds=h(t), t\in(a,T], \\ \left(I^{1-\gamma}_{a^+}x\right)(a^+)=x_0, \end{cases}$$

has a unique solution $x(t)$ given by

$$x(t) = \sum_{k=0}^{+\infty}\lambda^k(t-a)^{2k\alpha+\gamma-1}E^k_{\rho,2k\alpha+\gamma}\left(-\theta(t-a)^{\rho}\right)x_0+\sum_{k=0}^{+\infty}\lambda^k\mathbf{E}^k_{\rho,2k\alpha+\alpha,\theta;a^+}h(t).$$

This result coincides with Theorem 9, $\gamma=1$ in [19], in which the result is obtained by using the Laplace transform.

Example 1. *We consider the following initial value problem for the fractional kinetic differential equation:*

$$\begin{cases} \left({}^H D^{\frac{1}{10},\frac{1}{3}}_{0^+}x\right)(t)-t^{-\frac{1}{20}}I^{\frac{1}{5}}_{0^+}x(t)=t^{-\frac{1}{2}}, \\ \left(I^{\frac{3}{5}}_{0^+}x\right)(0^+)=1. \end{cases} \tag{17}$$

We set $\alpha=\frac{1}{10}$, $\beta=\frac{1}{3}$, $\gamma=\frac{2}{5}$, $\lambda(t)=t^{-\frac{1}{20}}$, $\tau=\frac{1}{5}$ and $g(t)=t^{-\frac{1}{2}}$. From Theorem 6, Problem (17) has a unique solution $x(t)\in C^{\frac{1}{10},\frac{1}{3}}_{\frac{3}{5},\frac{1}{2}}$ and

$$x(t)=\frac{t^{-\frac{3}{5}}}{\Gamma\left(\frac{2}{5}\right)}+I^{\frac{1}{10}}_{0^+}\sum_{k=0}^{+\infty}\left(t^{-\frac{1}{20}}I^{\frac{3}{10}}_{0^+}\right)^k\left[\frac{t^{-\frac{9}{20}}}{\Gamma\left(\frac{3}{5}\right)}+t^{-\frac{1}{2}}\right]. \tag{18}$$

Note that from the fact

$$\begin{aligned} & \left(t^{-a}I^{v}_{0^+}\right)^k t^b=\prod_{i=0}^{k-1}\frac{\Gamma[i(v-a)+b+1]}{\Gamma[i(v-a)+b+v+1]}t^{k(v-a)+b}, k=1,2,\cdots, b>-1, 1-\frac{a}{v}>0, \\ & t^b+\sum_{k=1}^{+\infty}\prod_{i=0}^{k-1}\frac{\Gamma[i(v-a)+b+1]}{\Gamma[i(v-a)+b+v+1]}t^{k(v-a)+b}=t^bE_{v,1-\frac{a}{v},\frac{b}{v}}\left(t^{v-a}\right), \end{aligned}$$

we obtain

$$\sum_{k=0}^{+\infty}\left(t^{-\frac{1}{20}} I_{0^+}^{\frac{3}{10}}\right)^k \frac{t^{-\frac{9}{20}}}{\Gamma(\frac{3}{5})} = \frac{t^{-\frac{9}{20}}}{\Gamma(\frac{3}{5})}\left[1+\sum_{k=1}^{+\infty}\prod_{i=0}^{k-1}\frac{\Gamma(\frac{i}{4}+\frac{11}{20})}{\Gamma(\frac{i}{4}+\frac{17}{20})}t^{\frac{k}{4}}\right] = \frac{t^{-\frac{9}{20}}}{\Gamma(\frac{3}{5})}E_{\frac{3}{10},\frac{5}{6},-\frac{3}{2}}(t^{\frac{1}{4}}),$$

$$\sum_{k=0}^{+\infty}\left(t^{-\frac{1}{20}} I_{0^+}^{\frac{3}{10}}\right)^k t^{-\frac{1}{2}} = t^{-\frac{1}{2}}\left[1+\sum_{k=1}^{+\infty}\prod_{i=0}^{k-1}\frac{\Gamma(\frac{i}{4}+\frac{1}{2})}{\Gamma(\frac{i}{4}+\frac{4}{5})}t^{\frac{k}{4}}\right] = t^{-\frac{1}{2}}E_{\frac{3}{10},\frac{5}{6},-\frac{5}{3}}(t^{\frac{1}{4}}).$$

Combining with Equation (18), the explicit solution for Problem (17) can be represented by

$$x(t) = \frac{t^{-\frac{3}{5}}}{\Gamma(\frac{2}{5})} + I_{0^+}^{\frac{1}{10}}\left[\frac{t^{-\frac{9}{20}}}{\Gamma(\frac{3}{5})}E_{\frac{3}{10},\frac{5}{6},-\frac{3}{2}}(t^{\frac{1}{4}}) + t^{-\frac{1}{2}}E_{\frac{3}{10},\frac{5}{6},-\frac{5}{3}}(t^{\frac{1}{4}})\right].$$

5. Conclusions

In this paper, we obtain the existence and uniqueness of solutions for two kinds of fractional integro-differential equations with a variable coefficient $\lambda(t)$ in the weighted space $C_{1-\gamma,\nu}^{\alpha,\beta}$, where $\lambda(t)$ belongs to a weighted space. For example, $\lambda(t)$ can be taken as a negative power function $t^{-\mu}$, where μ is an appropriate positive number in $(0,1)$. Therefore, the technique used in this paper can be applied to solve the more general equations.

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