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Positive Solutions for a System of Fractional q-Difference Equations with Multi-Point Boundary Conditions

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Abstract: We explore the existence, uniqueness, and multiplicity of positive solutions to a system of fractional q-difference equations that include fractional q-integrals. This investigation is carried out under coupled multi-point boundary conditions featuring q-derivatives and fractional q-derivatives of various orders. The proofs of our principal findings employ a range of fixed-point theorems, including the Guo–Krasnosel'skii fixed-point theorem, the Leggett–Williams fixed-point theorem, the Schauder fixed-point theorem, and the Banach contraction mapping principle.

Keywords: fractional q-difference equations; fractional q-integrals; coupled multi-point boundary conditions; positive solutions; existence; uniqueness; multiplicity

MSC: 39A13; 39A27; 33D05

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1. Introduction

In recent decades, there has been a notable surge in the exploration of nonlocal boundary value problems, including those involving multi-point scenarios, within the realm of ordinary differential or difference equations. This area of research is experiencing rapid growth, spurred not only by theoretical interest but also by the practical applications of modeling various phenomena in engineering, physics, and life sciences. To illustrate, consider systems with feedback controls, such as the steady states of a thermostat. In this context, a second-order ordinary differential equation subject to a three-point boundary condition can capture the dynamics, where a controller at one end adjusts the heat based on the temperature recorded at another point. Another instance is found in the vibrations of a guy wire with a uniform cross-section composed of N parts of varying densities. Such scenarios can be effectively modeled as multi-point boundary value problems (refer to [1]).

In this paper, we will investigate the system of fractional q-difference equations

$$\begin{cases} \left(D_q^{\alpha}\mathfrak{u}\right)(\mathfrak{t}) + \mathfrak{f}\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), I_q^{\delta_1}\mathfrak{u}(\mathfrak{t}), I_q^{\gamma_1}\mathfrak{v}(\mathfrak{t})\right) = 0, \quad \mathfrak{t} \in [0,1], \\ \left(D_q^{\beta}\mathfrak{v}\right)(\mathfrak{t}) + \mathfrak{g}\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), I_q^{\delta_2}\mathfrak{u}(\mathfrak{t}), I_q^{\gamma_2}\mathfrak{v}(\mathfrak{t})\right) = 0, \quad \mathfrak{t} \in [0,1], \end{cases} \tag{1}$$

supplemented with the multi-point boundary conditions

$$\begin{cases} D_q^i\mathfrak{u}(0) = 0, \quad i = 0,\dots,n-2, \quad D_q^{\varsigma}\mathfrak{u}(1) = \sum_{i=1}^{a} a_i D_q^{\varrho_i}\mathfrak{u}(\xi_i) + \sum_{i=1}^{b} b_i D_q^{\sigma_i}\mathfrak{v}(\omega_i), \\ D_q^i\mathfrak{v}(0) = 0, \quad i = 0,\dots,m-2, \quad D_q^{\vartheta}\mathfrak{v}(1) = \sum_{i=1}^{c} c_i D_q^{\eta_i}\mathfrak{u}(\zeta_i) + \sum_{i=1}^{d} d_i D_q^{\rho_i}\mathfrak{v}(\theta_i). \end{cases} \tag{2}$$

Here, $q \in (0,1)$, α, $\beta \in \mathbb{R}$, $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geq 3$; $a,b,c,d \in \mathbb{N}$, $0 \leq \varrho_i \leq \varsigma < \alpha - 1$, $i = 1,\dots,a$, $\varsigma \geq 1$, $0 \leq \eta_i \leq \varsigma$, $i = 1,\dots,c$, $0 \leq \sigma_i \leq \vartheta < \beta - 1$, $i = 1,\dots,b$, $\vartheta \geq 1$, $0 \leq \rho_i \leq \vartheta$, $i = 1,\dots,d$; $\delta_i, \gamma_i > 0$, $i = 1,2$; $\xi_i, \omega_j, \zeta_k, \theta_\iota \in (0,1)$, and $a_i, b_j, c_k, d_\iota \geq 0$ for $i = 1,\dots,a$, $j = 1,\dots,b$, $k = 1,\dots,c$, $\iota = 1,\dots,d$; D_q^{κ} is the fractional

q-derivative of order κ for $\kappa = \alpha, \beta, \varsigma, \vartheta, \varrho_i, \sigma_j, \eta_k, \rho_\iota$, $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, c$, $\iota = 1, \ldots, d$; D_q^i is the q-derivative of order i for $i = 0, \ldots, n-2$ and $i = 0, \ldots, m-2$; and I_q^κ is the fractional q-integral of order κ for $\kappa = \delta_i, \gamma_i$, $i = 1, 2$.

We will establish conditions on the functions f and g under which problem (1),(2) possesses at least one positive solution. Our proofs will leverage several fixed point theorems, including the Guo–Krasnosel'skii fixed point theorem, the Leggett–Williams fixed point theorem, the Schauder fixed point theorem, and the Banach contraction mapping principle. Subsequently, we will provide references to pertinent papers that are closely related to our investigated problem. In [2], the authors studied the solvability for the system of nonlinear fractional q-difference equations

$$\begin{cases} (D_q^\alpha \mathfrak{u})(\mathfrak{t}) + P(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), I_q^{\omega_1}\mathfrak{u}(\mathfrak{t}), I_q^{\delta_1}\mathfrak{v}(\mathfrak{t})) = 0, \ \ \mathfrak{t} \in (0,1), \\ (D_q^\beta \mathfrak{v})(\mathfrak{t}) + Q(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}), I_q^{\omega_2}\mathfrak{u}(\mathfrak{t}), I_q^{\delta_2}\mathfrak{v}(\mathfrak{t})) = 0, \ \ \mathfrak{t} \in (0,1), \end{cases} \tag{3}$$

subject to the coupled nonlocal boundary conditions

$$\begin{cases} D_q^i \mathfrak{u}(0) = 0, \ \ i = 0, \ldots, n-2, \ \ D_q^{\zeta_0}\mathfrak{u}(1) = \int_0^1 D_q^\zeta \mathfrak{v}(\mathfrak{t})\, d_q\mathfrak{H}(\mathfrak{t}), \ \ \mathfrak{t} \in (0,1), \\ D_q^i \mathfrak{v}(0) = 0, \ \ i = 0, \ldots, z-2, \ \ D_q^{\xi_0}\mathfrak{v}(1) = \int_0^1 D_q^\xi \mathfrak{u}(\mathfrak{t})\, d_q\mathfrak{K}(\mathfrak{t}), \ \ \mathfrak{t} \in (0,1), \end{cases} \tag{4}$$

where $q \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, $\alpha \in (n-1, n]$, $\beta \in (z-1, z]$, $n, z \in \mathbb{N}$, $n \geq 2$, $z \geq 2$, $\omega_i > 0$, $\delta_i > 0$, $i = 1, 2$, $\zeta \in [0, \beta - 1)$, $\xi \in [0, \alpha - 1)$, $\zeta_0 \in [0, \alpha - 1)$, $\xi_0 \in [0, \beta - 1)$, and the integrals in (4) are Riemann–Stieltjes integrals with $\mathfrak{H}$- and $\mathfrak{K}$-bounded variation functions. By using varied fixed-point theorems, they obtained the existence and uniqueness results for the solutions of problem (3),(4). In [3], the authors examined the existence of solutions for the fractional q-difference equation with nonlinear integral conditions

$$\begin{cases} ({}^C D_q^\alpha \mathfrak{y})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})), \ \ \text{for a.e. } \mathfrak{t} \in [0, T], \\ \mathfrak{y}(0) - \mathfrak{y}'(0) = \int_0^T \mathfrak{g}(\mathfrak{s}, \mathfrak{y}(\mathfrak{s})\, d\mathfrak{s}, \ \ \mathfrak{y}(T) + \mathfrak{y}'(T) = \int_0^T \mathfrak{h}(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}))\, d\mathfrak{s}, \end{cases} \tag{5}$$

where $T > 0$, $q \in (0,1)$, $\alpha \in (1,2]$, and ${}^C D_q^\alpha$ is the Caputo fractional q-derivative of order α. In the proof of the main result, they utilized measures of noncompactness and the Mönch fixed-point theorem. In [4], the authors investigated the existence, uniqueness, and multiplicity of positive solutions to the fractional q-difference equation with the nonlocal boundary conditions

$$\begin{cases} (D_q^\alpha \mathfrak{u})(\mathfrak{t}) + \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t})) = 0, \ \ \mathfrak{t} \in (0,1), \\ (D_q^i \mathfrak{u})(0) = 0, \ \ i = 0, \ldots, n-2, \ \ (D_q^\beta \mathfrak{u})(1) = a(D_q^\beta \mathfrak{u})(\eta), \end{cases} \tag{6}$$

where $q \in (0,1)$, $\alpha \in (n-1, n]$, $n > 2$, $\beta \in [1, n-2]$, $\eta \in (0,1)$, $a \in [0,1]$, and $\mathfrak{f} : [0,1] \times [0, \infty) \to [0, \infty)$ satisfies Caratheodory-type conditions. In the proof of the main theorems, they applied several fixed-point theorems. In [5], by using the Guo–Krasnosel'skii fixed-point theorem, the author analyzed the existence of positive solutions to the fractional q-difference equation with the boundary conditions

$$\begin{cases} (D_q^\alpha \mathfrak{y})(\mathfrak{t}) = -\mathfrak{f}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})), \ \ \mathfrak{t} \in (0,1), \\ \mathfrak{y}(0) = (D_q \mathfrak{y})(0) = 0, \ \ (D_q \mathfrak{y})(1) = \beta, \end{cases} \tag{7}$$

where $q \in (0,1)$, $\alpha \in (2,3]$, $\beta \geq 0$, and $\mathfrak{f} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. In [6], the author explored the existence of nontrivial solutions to the nonlinear q-fractional boundary value problem

$$\begin{cases} (D_q^{\alpha}\mathfrak{y})(\mathfrak{t}) = -\mathfrak{f}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t})), \ \ \mathfrak{t} \in (0,1), \\ \mathfrak{y}(0) = \mathfrak{y}(1) = 0, \end{cases} \tag{8}$$

where $q \in (0,1)$, $\alpha \in (1,2]$, and $\mathfrak{f} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. To prove the main finding, he also used the Guo–Krasnosel'skii fixed-point theorem. In [7], the authors proved the existence of solutions to the second-order q-difference equation with the boundary conditions

$$\begin{cases} D_q^2\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), D_q\mathfrak{u}(\mathfrak{t})), \ \ \mathfrak{t} \in I, \\ D_q\mathfrak{u}(0) = 0, \ \ D_q\mathfrak{u}(1) = \alpha\mathfrak{u}(1), \end{cases} \tag{9}$$

where $q \in (0,1)$, $I = \{q^n, \ n \in \mathbb{N}\} \cup \{0,1\}$, $\mathfrak{f} \in C(I \times \mathbb{R}^2, \mathbb{R})$, and $\alpha \neq 0$ is a fixed real number. In [8], the authors studied the existence of solutions to the second-order q-difference equation subject to the boundary conditions

$$\begin{cases} D_q^2\mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t})), \ \ \mathfrak{t} \in I, \\ \mathfrak{u}(0) = \eta\mathfrak{u}(T), \ \ D_q\mathfrak{u}(0) = \eta D_q\mathfrak{u}(T), \end{cases} \tag{10}$$

where $I = [0,T] \cap q^{\mathbb{N}}$, $q^{\mathbb{N}} = \{q^n, \ n \in \mathbb{N}\} \cup \{0\}$, $T \in q^{\mathbb{N}}$ is a fixed constant, $\eta \neq 1$ is a fixed real number, and $\mathfrak{f} \in C(I \times \mathbb{R}, \mathbb{R})$. Regarding additional papers that investigate systems of fractional q-difference equations with either coupled or uncoupled boundary conditions or that focus on fractional q-difference equations specifically, we refer to the following papers: [9–16].

The field of q-difference calculus, also known as quantum calculus, traces its origins back to the pioneering work of Jackson ([17,18]). To explore various applications of this discipline, readers are directed to the research of Ernst ([19]). The fractional q-difference calculus originated in the works of Al-Salam ([20]) and Agarwal ([21]). For advancements in this branch, encompassing q-analogs of integral and differential fractional operators, including properties like the fractional Leibniz q-formula, q-analogs of Cauchy's formula, q-Laplace transform, q-Taylor's formula, and q-analogs of the Mittag-Leffler function, refer to the papers [22–26].

The novel aspect of our problem (1),(2), compared to (3),(4) from [2], lies in the inclusion of generalized coupled boundary conditions (2) for the system of q-fractional difference equations in Equation (1). In this formulation, the q-fractional derivative of order ς for the unknown function $\mathfrak{u}$ at the point 1 is contingent upon the q-fractional derivatives of various orders for both functions $\mathfrak{u}$ and $\mathfrak{v}$ at different points within the interval $(0,1)$. Similarly, the q-fractional derivative of order ϑ for the unknown function $\mathfrak{v}$ at the point 1 is linked to the q-fractional derivatives of distinct orders for functions $\mathfrak{u}$ and $\mathfrak{v}$ at diverse points within the interval $(0,1)$. Furthermore, unlike the approach in the paper [2], we have explored the presence of positive solutions to our specific problem.

Our paper is organized as follows: Section 2 introduces key definitions and properties from q-calculus and fractional q-calculus, along with an existence result for the associated linear problem, the relevant Green functions, and their properties. Section 3 will then present the primary existence results for problem (1),(2), while Section 4 will provide illustrative examples to demonstrate the applicability of our theorems. Finally, Section 5 concludes the paper by summarizing the findings and presenting the overall conclusions.

2. Preliminary Results

In this section, we will introduce certain definitions and properties derived from q-calculus and fractional q-calculus. Additionally, we will outline some auxiliary findings that will play a pivotal role in the subsequent section.

Let $q \in (0,1)$. Define the number

$$[\mathfrak{a}]_q = \frac{1-q^{\mathfrak{a}}}{1-q}, \quad \mathfrak{a} \in \mathbb{R}. \tag{11}$$

Next, to introduce the q-gamma function, we present the q-analog of the power function $(\mathfrak{a}-\mathfrak{b})^n$ with $n \in \mathbb{N} \cup \{0\}$:

$$(\mathfrak{a}-\mathfrak{b})^{(0)} = 1, \quad (\mathfrak{a}-\mathfrak{b})^{(n)} = \prod_{k=0}^{n-1}(\mathfrak{a}-\mathfrak{b}q^k), \quad n \in \mathbb{N}, \ \mathfrak{a}, \mathfrak{b} \in \mathbb{R}. \tag{12}$$

For $\alpha \in \mathbb{R}$, define

$$(\mathfrak{a}-\mathfrak{b})^{(\alpha)} = \mathfrak{a}^{\alpha} \frac{\prod_{n=0}^{\infty}\left(1-\frac{\mathfrak{b}}{\mathfrak{a}}q^n\right)}{\prod_{n=0}^{\infty}\left(1-\frac{\mathfrak{b}}{\mathfrak{a}}q^{\alpha+n}\right)}. \tag{13}$$

If $\mathfrak{b} = 0$, then $\mathfrak{a}^{(\alpha)} = \mathfrak{a}^{\alpha}$.
The q-gamma function is defined by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}} = \frac{1}{(1-q)^{\alpha-1}} \frac{\prod_{n=0}^{\infty}(1-q^{n+1})}{\prod_{n=0}^{\infty}(1-q^{n+\alpha})}, \quad \alpha \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}. \tag{14}$$

This function satisfies the relation $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$.

Definition 1. *The q-derivative of a real function $\mathfrak{f}$ is defined by*

$$(D_q\mathfrak{f})(\mathfrak{t}) = \frac{\mathfrak{f}(\mathfrak{t})-\mathfrak{f}(q\mathfrak{t})}{(1-q)\mathfrak{t}}, \quad \mathfrak{t} \neq 0; \quad (D_q\mathfrak{f})(0) = \lim_{\mathfrak{t}\to 0}(D_q\mathfrak{f})(\mathfrak{t}). \tag{15}$$

Definition 2. *The q-derivatives of higher order of a real function $\mathfrak{f}$ are defined by*

$$(D_q^0\mathfrak{f})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}), \quad (D_q^n\mathfrak{f})(\mathfrak{t}) = D_q(D_q^{n-1}\mathfrak{f})(\mathfrak{t}), \quad n \in \mathbb{N}. \tag{16}$$

Definition 3. *The q-integral of a function $\mathfrak{f}$ defined in the interval $[0,b]$ is defined by*

$$(I_q\mathfrak{f})(\mathfrak{t}) = \int_0^{\mathfrak{t}} \mathfrak{f}(\mathfrak{s})\, d_q\mathfrak{s} = \mathfrak{t}(1-q)\sum_{n=0}^{\infty}\mathfrak{f}(\mathfrak{t}q^n)q^n, \quad \mathfrak{t} \in [0,b]. \tag{17}$$

Definition 4. *If $a \in [0,b]$ and $\mathfrak{f}$ is defined in the interval $[0,b]$, then its q-integral from a to b is given by*

$$\int_a^b \mathfrak{f}(\mathfrak{s})\, d_q\mathfrak{s} = \int_0^b \mathfrak{f}(\mathfrak{s})\, d_q\mathfrak{s} - \int_0^a \mathfrak{f}(\mathfrak{s})\, d_q\mathfrak{s}. \tag{18}$$

Definition 5. *The q-integrals of a function $\mathfrak{f}$ of higher order are defined by*

$$(I_q^0\mathfrak{f})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}), \quad (I_q^n\mathfrak{f})(\mathfrak{t}) = I_q(I_q^{n-1}\mathfrak{f})(\mathfrak{t}), \quad n \in \mathbb{N}. \tag{19}$$

The fundamental theorem of q-calculus says that $(D_qI_q\mathfrak{f})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t})$, and if $\mathfrak{f}$ is continuous at $\mathfrak{t} = 0$, then $(I_qD_q\mathfrak{f})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}) - \mathfrak{f}(0)$. The properties of the operators D_q and I_q are presented in [6,27]. Below, we present some properties that will be used later.

Lemma 1. *For* $\mathfrak{a}$, $\mathfrak{t}$, $\mathfrak{s}$, $\alpha \in \mathbb{R}$, *and* $\mathfrak{f} : [0, \mathfrak{b}] \to \mathbb{R}$, *we have*
(i) $[\mathfrak{a}(\mathfrak{t} - \mathfrak{s})]^{(\alpha)} = \mathfrak{a}^{\alpha}(\mathfrak{t} - \mathfrak{s})^{(\alpha)}$;
(ii) ${}_{\mathfrak{t}}D_q(\mathfrak{t} - \mathfrak{s})^{(\alpha)} = [\alpha]_q(\mathfrak{t} - \mathfrak{s})^{(\alpha-1)}$;
(iii) *If* $\alpha > 1$, *then* $D_q(\mathfrak{t}^{\alpha}) = [\alpha]_q \mathfrak{t}^{\alpha-1}$;
(iv) *If* $\alpha > 0$, *then* $I_q(\mathfrak{t}^{\alpha}) = \frac{1}{[\alpha+1]_q}\mathfrak{t}^{\alpha+1}$;
(v) *If* $\alpha > 0$ *and* $\mathfrak{c} \le \mathfrak{d} \le \mathfrak{t}$, *then* $(\mathfrak{t} - \mathfrak{c})^{(\alpha)} \ge (\mathfrak{t} - \mathfrak{d})^{(\alpha)}$;
(vi) ${}_{\mathfrak{t}}D_q\left(\int_0^{\mathfrak{t}} \mathfrak{f}(\mathfrak{t}, \mathfrak{s})\, d_q\mathfrak{s}\right)(\mathfrak{t}) = \int_0^{\mathfrak{t}} {}_{\mathfrak{t}}D_q\mathfrak{f}(\mathfrak{t}, \mathfrak{s})\, d_q\mathfrak{s} + \mathfrak{f}(q\mathfrak{t}, \mathfrak{t}), \ \forall\, \mathfrak{t} \in [0, \mathfrak{b}]$,
where ${}_{\mathfrak{t}}D_q$ *denotes the q-derivative with respect to variable* $\mathfrak{t}$.

Definition 6 ([21])**.** *Let* $\mathfrak{f}$ *be a function defined on* $[0, 1]$. *The fractional q-integral of the Riemann–Liouville type of order* $\alpha \ge 0$ *is defined by* $(I_q^0\mathfrak{f})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t})$ *and*

$$(I_q^{\alpha}\mathfrak{f})(\mathfrak{t}) = \frac{1}{\Gamma_q(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t} - q\mathfrak{s})^{(\alpha-1)} \mathfrak{f}(\mathfrak{s})\, d_q\mathfrak{s}, \ \ \mathfrak{t} \in [0, 1], \ \ \alpha > 0. \tag{20}$$

Definition 7 ([24])**.** *The fractional q-derivative of the Riemann–Liouville type of order* $\alpha \ge 0$ *is defined by* $(D_q^0\mathfrak{f})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t})$ *and*

$$(D_q^{\alpha}\mathfrak{f})(\mathfrak{t}) = (D_q^m I_q^{m-\alpha}\mathfrak{f})(\mathfrak{t}), \ \ \alpha > 0, \tag{21}$$

where m is the smallest integer greater than or equal to α.

Lemma 2 ([21,24])**.** *Let* α, $\beta \ge 0$, $\gamma > 0$, $\zeta \in [0, \gamma)$, *and* $\lambda \ge 0$, *and let* $\mathfrak{f}$ *be a function defined on* $[0, 1]$. *Then, the following relations are satisfied:*
(a) $(I_q^{\beta} I_q^{\alpha}\mathfrak{f})(\mathfrak{t}) = (I_q^{\alpha+\beta}\mathfrak{f})(\mathfrak{t})$;
(b) $(D_q^{\alpha} I_q^{\alpha}\mathfrak{f})(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t})$;
(c) *If* $\alpha \ge \beta > 0$, *then* $(D_q^{\beta} I_q^{\alpha}\mathfrak{f})(\mathfrak{t}) = (D_q^{\beta} I_q^{\beta} I_q^{\alpha-\beta}\mathfrak{f})(\mathfrak{t}) = (I_q^{\alpha-\beta}\mathfrak{f})(\mathfrak{t})$;
(d) $D_q^{\zeta}(\mathfrak{t}^{\gamma}) = \frac{\Gamma_q(\gamma+1)}{\Gamma_q(\gamma-\zeta+1)}\mathfrak{t}^{\gamma-\zeta}$;
(e) $I_q^{\alpha}(\mathfrak{t}^{\lambda}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}\mathfrak{t}^{\alpha+\lambda}$.

Lemma 3 ([6])**.** *Let* $\alpha > 0$ *and* p *be a positive integer. Then, the following relation holds:*

$$(I_q^{\alpha} D_q^{p}\mathfrak{f})(\mathfrak{t}) = (D_q^{p} I_q^{\alpha}\mathfrak{f})(\mathfrak{t}) - \sum_{k=0}^{p-1} \frac{\mathfrak{t}^{\alpha-p+k}}{\Gamma_q(\alpha + k - p + 1)} (D_q^{k}\mathfrak{f})(0). \tag{22}$$

Lemma 4. *If* $w \in C[0, 1]$, *then for* $\kappa > 0$, *we have*

$$|I_q^{\kappa} w(\mathfrak{t})| \le \frac{\|w\|}{\Gamma_q(\kappa + 1)}, \ \ \forall\, t \in [0, 1], \tag{23}$$

where $\|w\| = \sup_{\mathfrak{t}\in[0,1]} |w(\mathfrak{t})|$.

Proof. By Definition 6 and Lemma 2 (e) (with $\alpha = \kappa$ and $\lambda = 0$), we obtain

$$
\begin{aligned}
|I_q^\kappa w(\mathfrak{t})| &= \left|\frac{1}{\Gamma_q(\kappa)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\kappa-1)}w(\mathfrak{s})\,d_q\mathfrak{s}\right| \\
&= \left|\frac{1}{\Gamma_q(\kappa)}\mathfrak{t}(1-q)\sum_{n=0}^{\infty}(\mathfrak{t}-\mathfrak{t}q^{n+1})^{(\kappa-1)}w(\mathfrak{t}q^n)q^n\right| \\
&\leq \|w\|\left|\frac{1}{\Gamma_q(\kappa)}\mathfrak{t}(1-q)\sum_{n=0}^{\infty}(\mathfrak{t}-\mathfrak{t}q^{n+1})^{(\kappa-1)}q^n\right| \\
&= \|w\||(I_q^\kappa 1)(t)| = \|w\|\frac{\mathfrak{t}^\kappa}{\Gamma_q(\kappa+1)} \leq \frac{\|w\|}{\Gamma_q(\kappa+1)},\ \ \forall\, \mathfrak{t}\in[0,1].
\end{aligned}
\tag{24}
$$

□

In what follows, we will study the linear problem associated with our problem (1),(2). We consider the system of fractional q-difference equations

$$
\begin{cases}
(D_q^\alpha \mathfrak{u})(\mathfrak{t}) + \mathfrak{h}(\mathfrak{t}) = 0, & \mathfrak{t}\in[0,1], \\
(D_q^\beta \mathfrak{v})(\mathfrak{t}) + \mathfrak{k}(\mathfrak{t}) = 0, & \mathfrak{t}\in[0,1],
\end{cases}
\tag{25}
$$

subject to boundary conditions (2), where $\mathfrak{h}$, $\mathfrak{k}\in C[0,1]$.

We define

$$
\begin{aligned}
&\Lambda_1 = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varsigma)} - \sum_{i=1}^{a} a_i\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varrho_i)}\xi_i^{\alpha-\varrho_i-1}, \\
&\Lambda_2 = \sum_{i=1}^{b} b_i\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1},\ \ \Lambda_3 = \sum_{i=1}^{c} c_i\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\eta_i)}\zeta_i^{\alpha-\eta_i-1}, \\
&\Lambda_4 = \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)} - \sum_{i=1}^{d} d_i\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}, \\
&\Delta = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3.
\end{aligned}
\tag{26}
$$

Lemma 5. *If $\Delta\neq 0$, then the solution $(\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t})),\ \mathfrak{t}\in[0,1]$, of problem (25),(2) is given by*

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t}) &= -\frac{1}{\Gamma_q(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} + \frac{t^{\alpha-1}}{\Delta}(A\Lambda_4+B\Lambda_2),\ \ \mathfrak{t}\in[0,1], \\
\mathfrak{v}(\mathfrak{t}) &= -\frac{1}{\Gamma_q(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\beta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} + \frac{t^{\beta-1}}{\Delta}(B\Lambda_1+A\Lambda_3),\ \ \mathfrak{t}\in[0,1],
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
A &= \frac{1}{\Gamma_q(\alpha-\varsigma)}\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad - \sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad - \sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}, \\
B &= \frac{1}{\Gamma_q(\beta-\vartheta)}\int_0^1(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad - \sum_{i=1}^{c}\frac{c_i}{\Gamma_q(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad - \sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}.
\end{aligned}
\tag{28}
$$

Proof. We apply I_q^{α} and I_q^{β}, respectively, to the equations of system (25). Then, by Lemma 2 (a),(b) and Lemma 3, we obtain

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) &= -I_q^{\alpha}\mathfrak{h}(\mathfrak{t}) + \widetilde{a}_1\mathfrak{t}^{\alpha-1} + \widetilde{a}_2\mathfrak{t}^{\alpha-2} + \cdots + \widetilde{a}_n\mathfrak{t}^{\alpha-n} \\ &= -\frac{1}{\Gamma_q(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} + \widetilde{a}_1\mathfrak{t}^{\alpha-1} + \widetilde{a}_2\mathfrak{t}^{\alpha-2} + \cdots + \widetilde{a}_n\mathfrak{t}^{\alpha-n}, \quad \mathfrak{t}\in[0,1], \\ \mathfrak{v}(\mathfrak{t}) &= -I_q^{\beta}\mathfrak{k}(\mathfrak{t}) + \widetilde{b}_1\mathfrak{t}^{\beta-1} + \widetilde{b}_2\mathfrak{t}^{\beta-2} + \cdots + \widetilde{b}_m\mathfrak{t}^{\beta-m} \\ &= -\frac{1}{\Gamma_q(\beta)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\beta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} + \widetilde{b}_1\mathfrak{t}^{\beta-1} + \widetilde{b}_2\mathfrak{t}^{\beta-2} + \cdots + \widetilde{b}_m\mathfrak{t}^{\beta-m}, \quad \mathfrak{t}\in[0,1], \end{aligned} \tag{29}$$

for some $\widetilde{a}_i, \widetilde{b}_j \in \mathbb{R}$, $i=1,\ldots,n$, $j=1,\ldots,m$.

From the conditions $D_q^i\mathfrak{u}(0)=0$ for $i=0,\ldots,n-2$, and $D_q^j\mathfrak{v}(0)=0$ for $j=0,\ldots,m-2$, we deduce that $\widetilde{a}_i=0$, $i=2,\ldots,n$, and $\widetilde{b}_j=0$, $j=2,\ldots,m$. So, the solution (29) becomes

$$\begin{cases} \mathfrak{u}(\mathfrak{t}) = \widetilde{a}_1\mathfrak{t}^{\alpha-1} - \dfrac{1}{\Gamma_q(\alpha)}\displaystyle\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}, \quad \mathfrak{t}\in[0,1], \\ \mathfrak{v}(\mathfrak{t}) = \widetilde{b}_1\mathfrak{t}^{\beta-1} - \dfrac{1}{\Gamma_q(\beta)}\displaystyle\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\beta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}, \quad \mathfrak{t}\in[0,1]. \end{cases} \tag{30}$$

On the other hand, we find

$$D_q^{\kappa}\mathfrak{u}(\mathfrak{t}) = \widetilde{a}_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\kappa)}\mathfrak{t}^{\alpha-\kappa-1} - \frac{1}{\Gamma_q(\alpha-\kappa)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-\kappa-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}, \tag{31}$$

for $\kappa=\varsigma,\ \varrho_i,\ \eta_j$, $i=1,\ldots,a$, $j=1,\ldots,c$, and

$$D_q^{\kappa}\mathfrak{v}(\mathfrak{t}) = \widetilde{b}_1\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\kappa)}\mathfrak{t}^{\beta-\kappa-1} - \frac{1}{\Gamma_q(\beta-\kappa)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\beta-\kappa-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}, \tag{32}$$

for $\kappa=\vartheta,\ \sigma_i,\ \rho_j$, $i=1,\ldots,b$, $j=1,\ldots,d$.

By using relations (31) and (32), the boundary conditions $D_q^{\varsigma}\mathfrak{u}(1)=\sum_{i=1}^{a}a_iD_q^{\varrho_i}\mathfrak{u}(\xi_i)$ $+\sum_{i=1}^{b}b_iD_q^{\sigma_i}\mathfrak{v}(\omega_i)$ and $D_q^{\vartheta}\mathfrak{v}(1)=\sum_{i=1}^{c}c_iD_q^{\eta_i}\mathfrak{u}(\zeta_i)+\sum_{i=1}^{d}d_iD_q^{\rho_i}\mathfrak{v}(\theta_i)$ become

$$\begin{cases} \widetilde{a}_1\dfrac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varsigma)} - \dfrac{1}{\Gamma_q(\alpha-\varsigma)}\displaystyle\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\ \quad = \displaystyle\sum_{i=1}^{a}a_i\left[\widetilde{a}_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varrho_i)}\xi_i^{\alpha-\varrho_i-1} - \frac{1}{\Gamma_q(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}\right] \\ \qquad + \displaystyle\sum_{i=1}^{b}b_i\left[\widetilde{b}_1\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1} - \frac{1}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\right], \\ \widetilde{b}_1\dfrac{\Gamma_q(\beta)}{\Gamma(\beta-\vartheta)} - \dfrac{1}{\Gamma_q(\beta-\vartheta)}\displaystyle\int_0^1(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\ \quad = \displaystyle\sum_{i=1}^{c}c_i\left[\widetilde{a}_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\eta_i)}\zeta_i^{\alpha-\eta_i-1} - \frac{1}{\Gamma_q(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}\right] \\ \qquad + \displaystyle\sum_{i=1}^{d}d_i\left[\widetilde{b}_1\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1} - \frac{1}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\right]. \end{cases} \tag{33}$$

or

$$
\begin{cases}
\widetilde{a}_1\left(\dfrac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varsigma)}-\sum\limits_{i=1}^{a}a_i\dfrac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varrho_i)}\xi_i^{\alpha-\varrho_i-1}\right)-\widetilde{b}_1\sum\limits_{i=1}^{b}b_i\dfrac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1} \\
\quad =\dfrac{1}{\Gamma_q(\alpha-\varsigma)}\displaystyle\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
\qquad -\sum\limits_{i=1}^{a}\dfrac{a_i}{\Gamma_q(\alpha-\varrho_i)}\displaystyle\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
\qquad -\sum\limits_{i=1}^{b}\dfrac{b_i}{\Gamma_q(\beta-\sigma_i)}\displaystyle\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}, \\
-\widetilde{a}_1\sum\limits_{i=1}^{c}c_i\dfrac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\eta_i)}\zeta_i^{\alpha-\eta_i-1}+\widetilde{b}_1\left(\dfrac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}-\sum\limits_{i=1}^{d}d_i\dfrac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right) \\
\quad =\dfrac{1}{\Gamma_q(\beta-\vartheta)}\displaystyle\int_0^1(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
\qquad -\sum\limits_{i=1}^{c}\dfrac{c_i}{\Gamma_q(\alpha-\eta_i)}\displaystyle\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
\qquad -\sum\limits_{i=1}^{d}\dfrac{d_i}{\Gamma_q(\beta-\rho_i)}\displaystyle\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}.
\end{cases} \tag{34}
$$

The determinant of system (34) in the unknowns $\widetilde{a}_1$ and $\widetilde{b}_1$ is Δ, which, by the assumption of this lemma, is nonzero. So, system (34) has the unique solution

$$
\widetilde{a}_1=\frac{1}{\Delta}(A\Lambda_4+B\Lambda_2),\ \ \widetilde{b}_1=\frac{1}{\Delta}(B\Lambda_1+A\Lambda_3). \tag{35}
$$

By substituting (35) into (30), we obtain the solution $(\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t})),\ \mathfrak{t}\in[0,1]$, of problem (25),(2) given by (27). □

Lemma 6. *If $\Delta\neq 0$, then the solution $(\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t})),\ \mathfrak{t}\in[0,1]$, of problem (25),(2) can be written as*

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t})&=\int_0^1\mathfrak{G}_1(\mathfrak{t},q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}+\int_0^1\mathfrak{G}_2(\mathfrak{t},q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s},\ \ \mathfrak{t}\in[0,1],\\
\mathfrak{v}(\mathfrak{t})&=\int_0^1\mathfrak{G}_3(\mathfrak{t},q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}+\int_0^1\mathfrak{G}_4(\mathfrak{t},q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s},\ \ \mathfrak{t}\in[0,1],
\end{aligned} \tag{36}
$$

where

$$
\begin{aligned}
\mathfrak{G}_1(\mathfrak{t},\mathfrak{s})&=g_1(\mathfrak{t},\mathfrak{s})+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\left[\Lambda_4\sum_{i=1}^{a}a_ig_{1i}(\xi_i,\mathfrak{s})+\Lambda_2\sum_{i=1}^{c}c_ig_{2i}(\zeta_i,\mathfrak{s})\right],\\
\mathfrak{G}_2(\mathfrak{t},\mathfrak{s})&=\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\left[\Lambda_4\sum_{i=1}^{b}b_ig_{3i}(\omega_i,\mathfrak{s})+\Lambda_2\sum_{i=1}^{d}d_ig_{4i}(\theta_i,\mathfrak{s})\right],\\
\mathfrak{G}_3(\mathfrak{t},\mathfrak{s})&=\frac{\mathfrak{t}^{\beta-1}}{\Delta}\left[\Lambda_3\sum_{i=1}^{a}a_ig_{1i}(\xi_i,\mathfrak{s})+\Lambda_1\sum_{i=1}^{c}c_ig_{2i}(\zeta_i,\mathfrak{s})\right],\\
\mathfrak{G}_4(\mathfrak{t},\mathfrak{s})&=g_2(\mathfrak{t},\mathfrak{s})+\frac{\mathfrak{t}^{\beta-1}}{\Delta}\left[\Lambda_3\sum_{i=1}^{b}b_ig_{3i}(\omega_i,\mathfrak{s})+\Lambda_1\sum_{i=1}^{d}d_ig_{4i}(\theta_i,\mathfrak{s})\right],
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
g_1(\mathfrak{t},\mathfrak{s}) &= \frac{1}{\Gamma_q(\alpha)}\begin{cases} \mathfrak{t}^{\alpha-1}(1-\mathfrak{s})^{(\alpha-\varsigma-1)} - (\mathfrak{t}-\mathfrak{s})^{(\alpha-1)}, & 0\le \mathfrak{s}\le \mathfrak{t}\le 1, \\ \mathfrak{t}^{\alpha-1}(1-\mathfrak{s})^{(\alpha-\varsigma-1)}, & 0\le \mathfrak{t}\le \mathfrak{s}\le 1, \end{cases}\\
g_2(\mathfrak{t},\mathfrak{s}) &= \frac{1}{\Gamma_q(\beta)}\begin{cases} \mathfrak{t}^{\beta-1}(1-\mathfrak{s})^{(\beta-\vartheta-1)} - (\mathfrak{t}-\mathfrak{s})^{(\beta-1)}, & 0\le \mathfrak{s}\le \mathfrak{t}\le 1, \\ \mathfrak{t}^{\beta-1}(1-\mathfrak{s})^{(\beta-\vartheta-1)}, & 0\le \mathfrak{t}\le \mathfrak{s}\le 1, \end{cases}\\
g_{1i}(\mathfrak{t},\mathfrak{s}) &= \frac{1}{\Gamma_q(\alpha-\varrho_i)}\begin{cases} \mathfrak{t}^{\alpha-\varrho_i-1}(1-\mathfrak{s})^{(\alpha-\varsigma-1)} - (\mathfrak{t}-\mathfrak{s})^{(\alpha-\varrho_i-1)}, & 0\le \mathfrak{s}\le \mathfrak{t}\le 1, \\ \mathfrak{t}^{\alpha-\varrho_i-1}(1-\mathfrak{s})^{(\alpha-\varsigma-1)}, & 0\le \mathfrak{t}\le \mathfrak{s}\le 1, \end{cases}\\
g_{2j}(\mathfrak{t},\mathfrak{s}) &= \frac{1}{\Gamma_q(\alpha-\eta_j)}\begin{cases} \mathfrak{t}^{\alpha-\eta_j-1}(1-\mathfrak{s})^{(\alpha-\varsigma-1)} - (\mathfrak{t}-\mathfrak{s})^{(\alpha-\eta_j-1)}, & 0\le \mathfrak{s}\le \mathfrak{t}\le 1, \\ \mathfrak{t}^{\alpha-\eta_j-1}(1-\mathfrak{s})^{(\alpha-\varsigma-1)}, & 0\le \mathfrak{t}\le \mathfrak{s}\le 1, \end{cases}\\
g_{3k}(\mathfrak{t},\mathfrak{s}) &= \frac{1}{\Gamma_q(\beta-\sigma_k)}\begin{cases} \mathfrak{t}^{\beta-\sigma_k-1}(1-\mathfrak{s})^{(\beta-\vartheta-1)} - (\mathfrak{t}-\mathfrak{s})^{(\beta-\sigma_k-1)}, & 0\le \mathfrak{s}\le \mathfrak{t}\le 1, \\ \mathfrak{t}^{\beta-\sigma_k-1}(1-\mathfrak{s})^{(\beta-\vartheta-1)}, & 0\le \mathfrak{t}\le \mathfrak{s}\le 1, \end{cases}\\
g_{4\iota}(\mathfrak{t},\mathfrak{s}) &= \frac{1}{\Gamma_q(\beta-\rho_\iota)}\begin{cases} \mathfrak{t}^{\beta-\rho_\iota-1}(1-\mathfrak{s})^{(\beta-\vartheta-1)} - (\mathfrak{t}-\mathfrak{s})^{(\beta-\rho_\iota-1)}, & 0\le \mathfrak{s}\le \mathfrak{t}\le 1, \\ \mathfrak{t}^{\beta-\rho_\iota-1}(1-\mathfrak{s})^{(\beta-\vartheta-1)}, & 0\le \mathfrak{t}\le \mathfrak{s}\le 1, \end{cases}
\end{aligned}
\tag{38}
$$

for $\mathfrak{t},\mathfrak{s}\in[0,1]$, $i=1,\dots,a$, $j=1,\dots,c$, $k=1,\dots,b$, $\iota=1,\dots,d$.

Proof. For $\mathfrak{u}(\mathfrak{t})$, we deduce

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t}) &= -\frac{1}{\Gamma_q(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\frac{\Lambda_4}{\Gamma_q(\alpha-\varsigma)}\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad-\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Lambda_4\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad-\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Lambda_2\sum_{i=1}^{c}\frac{c_i}{\Gamma_q(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg[-\Lambda_4\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\frac{\Lambda_2}{\Gamma_q(\beta-\vartheta)}\int_0^1(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad-\Lambda_2\sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg] \\
&=\frac{\mathfrak{t}^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^{\mathfrak{t}}(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}-\frac{1}{\Gamma_q(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\frac{1}{\Gamma_q(\alpha)}\int_{\mathfrak{t}}^1\mathfrak{t}^{\alpha-1}(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}-\frac{\mathfrak{t}^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\frac{\Lambda_4}{\Gamma_q(\alpha-\varsigma)}\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad-\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Lambda_4\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad-\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Lambda_2\sum_{i=1}^{c}\frac{c_i}{\Gamma_q(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg[-\Lambda_4\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\frac{\Lambda_2}{\Gamma_q(\beta-\vartheta)}\int_0^1(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad-\Lambda_2\sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg],\quad\forall\,\mathfrak{t}\in[0,1].
\end{aligned}
\tag{39}
$$

Amplifying the fourth term on the right-hand side of the above relation by $\Delta = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3$, we obtain

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t}) &= \frac{1}{\Gamma_q(\alpha)} \int_0^1 \mathfrak{t}^{\alpha-1}(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} - \frac{1}{\Gamma_q(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg\{ -\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varsigma)}\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}\left(\sum_{i=1}^{a} a_i \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varrho_i)}\xi_i^{\alpha-\varrho_i-1}\right)\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varsigma)}\left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \left(\sum_{i=1}^{a} a_i \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\varrho_i)}\xi_i^{\alpha-\varrho_i-1}\right)\left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \left(\sum_{i=1}^{c} c_i \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\eta_i)}\zeta_i^{\alpha-\eta_i-1}\right)\left(\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\eta_i^{\beta-\eta_i-1}\right)\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}\frac{1}{\Gamma_q(\alpha-\varsigma)}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\frac{1}{\Gamma_q(\alpha-\varsigma)}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \left(\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\eta_i^{\beta-\eta_i-1}\right)\sum_{i=1}^{c}\frac{c_i}{\Gamma_q(\alpha-\eta_i)}\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s}\Bigg\} \\
&+ \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg\{ -\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\, d_q(\mathfrak{s}) \\
&+ \left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \left(\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1}\right)\frac{1}{\Gamma_q(\beta-\vartheta)}\int_0^1 (1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \left(\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1}\right)\sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s}\Bigg\} \\
&= \frac{1}{\Gamma_q(\alpha)} \int_0^1 \mathfrak{t}^{\alpha-1}(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} - \frac{1}{\Gamma_q(\alpha)} \int_0^{\mathfrak{t}} (\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&+ \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg\{ \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\Bigg[\xi_i^{\alpha-\varrho_i-1}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s}\Bigg] \\
&- \left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\Bigg[\xi_i^{\alpha-\varrho_i-1}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s}\Bigg] \\
&+ \left(\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\eta_i^{\beta-\eta_i-1}\right)\sum_{i=1}^{c}\frac{c_i}{\Gamma_q(\alpha-\eta_i)}\Bigg[\zeta_i^{\alpha-\eta_i-1}\int_0^1 (1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\
&- \int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s}\Bigg]\Bigg\}
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg\{\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)} \frac{1}{\Gamma_q(\beta-\sigma_i)}\Bigg[\int_0^1 \omega_i^{\beta-\sigma_i-1}(1-q\mathfrak{s})^{(\beta_i-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&-\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg] \\
&-\left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\int_0^1 \omega_i^{\beta-\sigma_i-1}(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&+\left(\sum_{i=1}^{d} d_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&+\left(\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1}\right)\sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\int_0^1 \theta_i^{\beta-\rho_i-1}(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&-\left(\sum_{i=1}^{b} b_i \frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1}\right)\sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg\}, \quad \forall\, \mathfrak{t}\in[0,1].
\end{aligned}
$$

So, we deduce

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t}) &= \frac{1}{\Gamma_q(\alpha)}\int_0^1 \mathfrak{t}^{\alpha-1}(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} - \frac{1}{\Gamma_q(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg\{\Bigg[\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}-\left(\sum_{i=1}^{d} d_i\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right)\Bigg] \\
&\times\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\Bigg[\xi_i^{\alpha-\varrho_i-1}\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg] \\
&+\left(\sum_{i=1}^{b} b_i\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\eta_i^{\beta-\eta_i-1}\right) \\
&\times\sum_{i=1}^{c}\frac{c_i}{\Gamma_q(\alpha-\eta_i)}\Bigg[\zeta_i^{\alpha-\eta_i-1}\int_0^1(1-q\mathfrak{s})^{(\alpha-\zeta-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg]\Bigg\} \\
&+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg\{\sum_{i=1}^{b} b_i\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\vartheta)}\frac{1}{\Gamma_q(\beta-\sigma_i)}\Bigg[\int_0^1\omega_i^{\beta-\sigma_i-1}(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\
&-\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg]-\left(\sum_{i=1}^{d} d_i\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\rho_i)}\theta_i^{\beta-\rho_i-1}\right) \\
&\times\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\Bigg[\int_0^1\omega_i^{\beta-\sigma_i-1}(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg] \\
&+\left(\sum_{i=1}^{b} b_i\frac{\Gamma_q(\beta)}{\Gamma_q(\beta-\sigma_i)}\omega_i^{\beta-\sigma_i-1}\right) \\
&\times\sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\Bigg[\int_0^1\theta_i^{\beta-\rho_i-1}(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg]\Bigg\} \\
&=\frac{1}{\Gamma_q(\alpha)}\int_0^1\mathfrak{t}^{\alpha-1}(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}-\frac{1}{\Gamma_q(\alpha)}\int_0^{\mathfrak{t}}(\mathfrak{t}-q\mathfrak{s})^{(\alpha-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} \\
&+\frac{t^{\alpha-1}}{\Delta}\Bigg\{\Lambda_4\sum_{i=1}^{a}\frac{a_i}{\Gamma_q(\alpha-\varrho_i)}\Bigg[\xi_i^{\alpha-\varrho_i-1}\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\xi_i}(\xi_i-q\mathfrak{s})^{(\alpha-\varrho_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg] \\
&+\Lambda_2\sum_{i=1}^{c}\frac{c_i}{\Gamma_q(\alpha-\eta_i)}\Bigg[\zeta_i^{\alpha-\eta_i-1}\int_0^1(1-q\mathfrak{s})^{(\alpha-\varsigma-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\zeta_i}(\zeta_i-q\mathfrak{s})^{(\alpha-\eta_i-1)}\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg]\Bigg\} \\
&+\frac{\mathfrak{t}^{\alpha-1}}{\Delta}\Bigg\{\Lambda_4\sum_{i=1}^{b}\frac{b_i}{\Gamma_q(\beta-\sigma_i)}\Bigg[\int_0^1\omega_i^{\beta-\sigma_i-1}(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\omega_i}(\omega_i-q\mathfrak{s})^{(\beta-\sigma_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg] \\
&+\Lambda_2\sum_{i=1}^{d}\frac{d_i}{\Gamma_q(\beta-\rho_i)}\Bigg[\int_0^1\theta_i^{\beta-\rho_i-1}(1-q\mathfrak{s})^{(\beta-\vartheta-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}-\int_0^{\theta_i}(\theta_i-q\mathfrak{s})^{(\beta-\rho_i-1)}\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\Bigg]\Bigg\}.
\end{aligned} \tag{41}
$$

Therefore, we obtain

$$\begin{aligned}\mathfrak{u}(\mathfrak{t}) &= \int_0^1 \left\{ g_1(\mathfrak{t}, q\mathfrak{s}) + \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\left[\Lambda_4 \sum_{i=1}^{a} a_i g_{1i}(\xi_i, q\mathfrak{s}) + \Lambda_2 \sum_{i=1}^{c} c_i g_{2i}(\zeta_i, q\mathfrak{s})\right]\right\}\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\ &+ \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\int_0^1 \left[\Lambda_4 \sum_{i=1}^{b} b_i g_{3i}(\omega_i, q\mathfrak{s}) + \Lambda_2 \sum_{i=1}^{d} d_i g_{4i}(\theta_i, q\mathfrak{s})\right]\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s} \\ &= \int_0^1 \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s}, \ \ \forall\, \mathfrak{t} \in [0,1],\end{aligned} \tag{42}$$

where $\mathfrak{G}_1$, $\mathfrak{G}_2$, g_1, g_{1i}, $i = 1,\ldots,a$, g_{2j}, $j = 1,\ldots,c$, g_{3k}, $k = 1,\ldots,b$, and $g_{4\iota}$, $\iota = 1,\ldots,d$, are given by (37) and (38).

In a similar manner, we find

$$\begin{aligned}\mathfrak{v}(\mathfrak{t}) &= \frac{\mathfrak{t}^{\beta-1}}{\Delta}\int_0^1 \left[\Lambda_3 \sum_{i=1}^{a} a_i g_{1i}(\xi_i, q\mathfrak{s}) + \Lambda_1 \sum_{i=1}^{c} c_i g_{2i}(\zeta_i, q\mathfrak{s})\right]\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} \\ &+ \int_0^1 \left\{ g_2(\mathfrak{t}, q\mathfrak{s}) + \frac{\mathfrak{t}^{\beta-1}}{\Delta}\left[\Lambda_3 \sum_{i=1}^{b} b_i g_{3i}(\omega_i, q\mathfrak{s}) + \Lambda_1 \sum_{i=1}^{d} d_i g_{4i}(\theta_i, q\mathfrak{s})\right]\right\}\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s} \\ &= \int_0^1 \mathfrak{G}_3(\mathfrak{t}, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_4(\mathfrak{t}, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s}, \ \ \forall\, \mathfrak{t} \in [0,1],\end{aligned} \tag{43}$$

where $\mathfrak{G}_3$, $\mathfrak{G}_4$, and g_2 are given by (37) and (38). So, we deduce the formulas in (36) for the solution $(\mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$, of problem (25),(2). □

With a similar proof to that of Lemma 12 from [4], we obtain the next result.

Lemma 7. *The functions g_1, g_2, g_{1i}, $i = 1,\ldots,a$, g_{2j}, $j = 1,\ldots,c$, g_{3k}, $k = 1,\ldots,b$, and $g_{4\iota}$, $\iota = 1,\ldots,d$, have the following properties:*

(a) $g_1(\mathfrak{t}, q\mathfrak{s}) \geq 0$, $g_2(\mathfrak{t}, q\mathfrak{s}) \geq 0$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$;

(b) $g_1(\mathfrak{t}, q\mathfrak{s}) \leq g_1(1, q\mathfrak{s})$, $g_2(\mathfrak{t}, q\mathfrak{s}) \leq g_2(1, q\mathfrak{s})$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$;

(c) $g_1(\mathfrak{t}, q\mathfrak{s}) \geq \mathfrak{t}^{\alpha-1} g_1(1, q\mathfrak{s})$, $g_2(\mathfrak{t}, q\mathfrak{s}) \geq \mathfrak{t}^{\beta-1} g_2(1, q\mathfrak{s})$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$;

(d) $g_{1i}(\mathfrak{t}, q\mathfrak{s}) \geq 0$, $g_{2j}(\mathfrak{t}, q\mathfrak{s}) \geq 0$, $g_{3k}(\mathfrak{t}, q\mathfrak{s}) \geq 0$, $g_{4\iota}(\mathfrak{t}, q\mathfrak{s}) \geq 0$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$, $i = 1,\ldots,a$, $j = 1,\ldots,c$, $k = 1,\ldots,b$, $\iota = 1,\ldots,d$.

Remark 1. *If $\varpi = q^n$ with $n \in \mathbb{N}$, then we obtain*

$$\min_{\mathfrak{t}\in[\varpi,1]} g_1(\mathfrak{t}, q\mathfrak{s}) \geq \varpi^{\alpha-1} g_1(1, q\mathfrak{s}), \ \ \min_{\mathfrak{t}\in[\varpi,1]} g_2(\mathfrak{t}, q\mathfrak{s}) \geq \varpi^{\beta-1} g_2(1, q\mathfrak{s}), \ \ \forall\, \mathfrak{s} \in [0,1]. \tag{44}$$

Lemma 8. *If $\Lambda_1 > 0$, $\Lambda_4 > 0$, and $\Delta > 0$, then the functions $\mathfrak{G}_i$, $i = 1,\ldots,4$ satisfy the following inequalities:*

(a) $\mathfrak{t}^{\alpha-1}\mathfrak{G}_1(1, q\mathfrak{s}) \leq \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) \leq \mathfrak{G}_1(1, q\mathfrak{s})$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$;

(b) $\mathfrak{t}^{\alpha-1}\mathfrak{G}_2(1, q\mathfrak{s}) = \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) \leq \mathfrak{G}_2(1, q\mathfrak{s})$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$;

(c) $\mathfrak{t}^{\beta-1}\mathfrak{G}_3(1, q\mathfrak{s}) = \mathfrak{G}_3(\mathfrak{t}, q\mathfrak{s}) \leq \mathfrak{G}_3(1, q\mathfrak{s})$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$;

(d) $\mathfrak{t}^{\beta-1}\mathfrak{G}_4(1, q\mathfrak{s}) \leq \mathfrak{G}_4(\mathfrak{t}, q\mathfrak{s}) \leq \mathfrak{G}_4(1, q\mathfrak{s})$, $\forall\, \mathfrak{t}, \mathfrak{s} \in [0,1]$.

Proof. (a)–(d) By using Lemma 7, we deduce the following for all $\mathfrak{t}, \mathfrak{s} \in [0,1]$:

$$\begin{aligned}
&\mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) = g_1(\mathfrak{t}, q\mathfrak{s}) + \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\left[\Lambda_4 \sum_{i=1}^{a} a_i g_{1i}(\xi_i, q\mathfrak{s}) + \Lambda_2 \sum_{i=1}^{c} c_i g_{2i}(\zeta_i, q\mathfrak{s})\right] \\
&\leq g_1(1, q\mathfrak{s}) + \frac{1}{\Delta}\left[\Lambda_4 \sum_{i=1}^{a} a_i g_{1i}(\xi_i, q\mathfrak{s}) + \Lambda_2 \sum_{i=1}^{c} c_i g_{2i}(\zeta_i, q\mathfrak{s})\right] = \mathfrak{G}_1(1, q\mathfrak{s}); \\
&\mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) \geq \mathfrak{t}^{\alpha-1} g_1(1, q\mathfrak{s}) + \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\left[\Lambda_4 \sum_{i=1}^{a} a_i g_{1i}(\xi_i, q\mathfrak{s}) + \Lambda_2 \sum_{i=1}^{c} c_i g_{2i}(\zeta_i, q\mathfrak{s})\right] \\
&= \mathfrak{t}^{\alpha-1}\mathfrak{G}_1(1, q\mathfrak{s}); \\
&\mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) = \frac{\mathfrak{t}^{\alpha-1}}{\Delta}\left[\Lambda_4 \sum_{i=1}^{b} b_i g_{3i}(\omega_i, q\mathfrak{s}) + \Lambda_2 \sum_{i=1}^{d} d_i g_{4i}(\theta_i, q\mathfrak{s})\right] \\
&\leq \frac{1}{\Delta}\left[\Lambda_4 \sum_{i=1}^{b} b_i g_{3i}(\omega_i, q\mathfrak{s}) + \Lambda_2 \sum_{i=1}^{d} d_i g_{4i}(\theta_i, q\mathfrak{s})\right] = \mathfrak{G}_2(1, q\mathfrak{s}); \\
&\mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) = \mathfrak{t}^{\alpha-1}\mathfrak{G}_2(1, q\mathfrak{s}); \\
&\mathfrak{G}_3(\mathfrak{t}, q\mathfrak{s}) = \frac{\mathfrak{t}^{\beta-1}}{\Delta}\left[\Lambda_3 \sum_{i=1}^{a} a_i g_{1i}(\xi_i, q\mathfrak{s}) + \Lambda_1 \sum_{i=1}^{c} c_i g_{2i}(\zeta_i, q\mathfrak{s})\right] \\
&\leq \frac{1}{\Delta}\left[\Lambda_3 \sum_{i=1}^{a} a_i g_{1i}(\xi_i, q\mathfrak{s}) + \Lambda_1 \sum_{i=1}^{c} c_i g_{2i}(\zeta_i, q\mathfrak{s})\right] = \mathfrak{G}_3(1, q\mathfrak{s}); \\
&\mathfrak{G}_3(\mathfrak{t}, q\mathfrak{s}) = \mathfrak{t}^{\beta-1}\mathfrak{G}_3(1, q\mathfrak{s}); \\
&\mathfrak{G}_4(\mathfrak{t}, q\mathfrak{s}) = g_2(\mathfrak{t}, q\mathfrak{s}) + \frac{\mathfrak{t}^{\beta-1}}{\Delta}\left[\Lambda_3 \sum_{i=1}^{b} b_i g_{3i}(\omega_i, q\mathfrak{s}) + \Lambda_1 \sum_{i=1}^{d} d_i g_{4i}(\theta_i, q\mathfrak{s})\right] \\
&\leq g_2(1, q\mathfrak{s}) + \frac{1}{\Delta}\left[\Lambda_3 \sum_{i=1}^{b} b_i g_{3i}(\omega_i, q\mathfrak{s}) + \Lambda_1 \sum_{i=1}^{d} d_i g_{4i}(\theta_i, q\mathfrak{s})\right] = \mathfrak{G}_4(1, q\mathfrak{s}); \\
&\mathfrak{G}_4(\mathfrak{t}, q\mathfrak{s}) \geq \mathfrak{t}^{\beta-1} g_2(1, q\mathfrak{s}) + \frac{\mathfrak{t}^{\beta-1}}{\Delta}\left[\Lambda_3 \sum_{i=1}^{b} b_i g_{3i}(\omega_i, q\mathfrak{s}) + \Lambda_1 \sum_{i=1}^{d} d_i g_{4i}(\theta_i, q\mathfrak{s})\right] \\
&= \mathfrak{t}^{\beta-1}\mathfrak{G}_4(1, q\mathfrak{s}).
\end{aligned} \tag{45}$$

□

Lemma 9. *If $\Lambda_1 > 0$, $\Lambda_4 > 0$, $\Delta > 0$, $\mathfrak{h}(\mathfrak{t}) \geq 0$, and $\mathfrak{k}(\mathfrak{t}) \geq 0$ for all $\mathfrak{t} \in [0,1]$, then the solution $(\mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$, of problem (25),(2) satisfies the inequalities $\mathfrak{u}(\mathfrak{t}) \geq 0$ and $\mathfrak{v}(\mathfrak{t}) \geq 0$ for all $\mathfrak{t} \in [0,1]$ and $\mathfrak{u}(\mathfrak{t}) \geq \mathfrak{t}^{\alpha-1}\mathfrak{u}(\tau)$ and $\mathfrak{v}(\mathfrak{t}) \geq \mathfrak{t}^{\beta-1}\mathfrak{v}(\tau)$ for all $\mathfrak{t}$, $\tau \in [0,1]$.*

Proof. We can easily verify that $\mathfrak{u}(\mathfrak{t}) \geq 0$ and $\mathfrak{v}(\mathfrak{t}) \geq 0$ for all $\mathfrak{t} \in [0,1]$. In addition, by using Lemma 8, we obtain

$$\begin{aligned}
&\mathfrak{u}(\mathfrak{t}) = \int_0^1 \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\leq \int_0^1 \mathfrak{G}_1(1, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(1, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s}, \quad \forall\, \mathfrak{t} \in [0,1], \\
&\mathfrak{u}(\mathfrak{t}) \geq \int_0^1 \mathfrak{t}^{\alpha-1}\mathfrak{G}_1(1, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{t}^{\alpha-1}\mathfrak{G}_2(1, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s} \\
&= \mathfrak{t}^{\alpha-1}\left(\int_0^1 \mathfrak{G}_1(1, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(1, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s}\right) \geq \mathfrak{t}^{\alpha-1}\mathfrak{u}(\tau), \quad \forall\, \mathfrak{t}, \tau \in [0,1], \\
&\mathfrak{v}(\mathfrak{t}) = \int_0^1 \mathfrak{G}_3(\mathfrak{t}, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_4(\mathfrak{t}, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\leq \int_0^1 \mathfrak{G}_3(1, q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_4(1, q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\, d_q\mathfrak{s}, \quad \forall\, \mathfrak{t} \in [0,1],
\end{aligned} \tag{46}$$

$$\begin{aligned}\mathfrak{v}(\mathfrak{t}) &\geq \int_0^1 \mathfrak{t}^{\beta-1}\mathfrak{G}_3(1,q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} + \int_0^1 \mathfrak{t}^{\beta-1}\mathfrak{G}_4(1,q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s} \\ &= \mathfrak{t}^{\beta-1}\left(\int_0^1 \mathfrak{G}_3(1,q\mathfrak{s})\mathfrak{h}(\mathfrak{s})\,d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_4(1,q\mathfrak{s})\mathfrak{k}(\mathfrak{s})\,d_q\mathfrak{s}\right) \geq \mathfrak{t}^{\beta-1}\mathfrak{v}(\tau), \quad \forall\,\mathfrak{t},\,\tau \in [0,1].\end{aligned}$$

□

Remark 2. *By Lemma 9, we find* $\mathfrak{u}(\mathfrak{t}) \geq \mathfrak{t}^{\alpha-1}\|u\|$ *and* $\mathfrak{v}(\mathfrak{t}) \geq \mathfrak{t}^{\beta-1}\|\mathfrak{v}\|$ *for all* $\mathfrak{t} \in [0,1]$. *In addition, by Lemma 8, for* $\varpi = q^n$ *with* $n \in \mathbb{N}$, *we have*

$$\begin{aligned}&\min_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_1(\mathfrak{t},q\mathfrak{s}) \geq \varpi^{\alpha-1}\mathfrak{G}_1(1,q\mathfrak{s}), \ \ \forall\,\mathfrak{s} \in [0,1]; \\ &\min_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_2(\mathfrak{t},q\mathfrak{s}) = \varpi^{\alpha-1}\mathfrak{G}_2(1,q\mathfrak{s}), \ \ \forall\,\mathfrak{s} \in [0,1]; \\ &\min_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_3(\mathfrak{t},q\mathfrak{s}) = \varpi^{\beta-1}\mathfrak{G}_3(1,q\mathfrak{s}), \ \ \forall\,\mathfrak{s} \in [0,1]; \\ &\min_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_4(\mathfrak{t},q\mathfrak{s}) \geq \varpi^{\beta-1}\mathfrak{G}_4(1,q\mathfrak{s}), \ \ \forall\,\mathfrak{s} \in [0,1],\end{aligned} \tag{47}$$

and so

$$\min_{\mathfrak{t}\in[\varpi,1]} \mathfrak{u}(\mathfrak{t}) \geq \varpi^{\alpha-1}\|\mathfrak{u}\|, \quad \min_{\mathfrak{t}\in[\varpi,1]} \mathfrak{v}(\mathfrak{t}) \geq \varpi^{\beta-1}\|\mathfrak{v}\|. \tag{48}$$

3. Main Results

In this section, we will outline the results concerning the existence of positive solutions for our given problem (1),(2).

Initially, we state our primary assumptions:

$(H1)$ $q \in (0,1)$, α, $\beta \in \mathbb{R}$, $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geq 3$; $a,b,c,d \in \mathbb{N}$, $0 \leq \varrho_i \leq \varsigma < \alpha - 1$, $i = 1,\ldots,a$, $\varsigma \geq 1$, $0 \leq \eta_i \leq \varsigma$, $i = 1,\ldots,c$, $0 \leq \sigma_i \leq \vartheta < \beta - 1$, $i = 1,\ldots,b$, $\vartheta \geq 1$, $0 \leq \rho_i \leq \vartheta$, $i = 1,\ldots,d$; $\delta_i,\gamma_i > 0$, $i = 1,2$; $\xi_i,\omega_j,\zeta_k,\theta_\iota \in (0,1)$, $a_i,b_j,c_k,d_\iota \geq 0$ for $i = 1,\ldots,a$, $j = 1,\ldots,b$, $k = 1,\ldots,c$, $\iota = 1,\ldots,d$; $\Lambda_1 > 0$, $\Lambda_4 > 0$, $\Delta > 0$ (given by (26)).

$(H2)$ $\mathfrak{f}$, $\mathfrak{g} : [0,1] \times \mathbb{R}_+^4 \to \mathbb{R}_+$ are continuous functions ($\mathbb{R}_+ = [0,\infty)$).

Problem (1),(2) can be expressed equivalently to the following system of fractional q-integral equations:

$$\begin{cases} \mathfrak{u}(\mathfrak{t}) = \displaystyle\int_0^1 \mathfrak{G}_1(\mathfrak{t},q\mathfrak{s})\mathfrak{f}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_1}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_1}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s} \\ \qquad + \displaystyle\int_0^1 \mathfrak{G}_2(\mathfrak{t},q\mathfrak{s})\mathfrak{g}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_2}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_2}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s}, \ \ \mathfrak{t} \in [0,1], \\ \mathfrak{v}(\mathfrak{t}) = \displaystyle\int_0^1 \mathfrak{G}_3(\mathfrak{t},q\mathfrak{s})\mathfrak{f}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_1}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_1}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s} \\ \qquad + \displaystyle\int_0^1 \mathfrak{G}_4(\mathfrak{t},q\mathfrak{s})\mathfrak{g}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_2}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_2}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s}, \ \ \mathfrak{t} \in [0,1]. \end{cases} \tag{49}$$

Let $\mathcal{X} = C[0,1]$ be the Banach space endowed with the norm $\|\mathfrak{u}\| = \sup_{\mathfrak{t}\in[0,1]} |\mathfrak{u}(\mathfrak{t})|$, and let $\mathcal{Y} = \mathcal{X} \times \mathcal{X}$ be the Banach space with the norm $\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}} = \|\mathfrak{u}\| + \|\mathfrak{v}\|$. We also introduce the cone $\mathcal{P} \subset \mathcal{Y}$ by

$$\mathcal{P} = \{(\mathfrak{u},\mathfrak{v}) \in \mathcal{Y}, \ \ \mathfrak{u}(\mathfrak{t}) \geq \mathfrak{t}^{\alpha-1}\|\mathfrak{u}\|, \ \ \mathfrak{v}(\mathfrak{t}) \geq \mathfrak{t}^{\beta-1}\|\mathfrak{v}\|, \ \ \forall\,\mathfrak{t} \in [0,1]\}. \tag{50}$$

We now define the operator $\mathcal{A}:\mathcal{P}\to\mathcal{Y}$, $\mathcal{A}(\mathfrak{u},\mathfrak{v})=(\mathcal{A}_1(\mathfrak{u},\mathfrak{v}),\mathcal{A}_2(\mathfrak{u},\mathfrak{v}))$, for $(\mathfrak{u},\mathfrak{v})\in\mathcal{P}$, where $\mathcal{A}_1$, $\mathcal{A}_2:\mathcal{P}\to\mathcal{X}$ are given by

$$\begin{aligned}\mathcal{A}_1(\mathfrak{u},\mathfrak{v})(\mathfrak{t})&=\int_0^1\mathfrak{G}_1(\mathfrak{t},q\mathfrak{s})\mathfrak{f}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_1}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_1}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s}\\&+\int_0^1\mathfrak{G}_2(\mathfrak{t},q\mathfrak{s})\mathfrak{g}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_2}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_2}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s},\\\mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t})&=\int_0^1\mathfrak{G}_3(\mathfrak{t},q\mathfrak{s})\mathfrak{f}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_1}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_1}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s}\\&+\int_0^1\mathfrak{G}_4(\mathfrak{t},q\mathfrak{s})\mathfrak{g}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_2}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_2}\mathfrak{v}(\mathfrak{s}))\,d_q\mathfrak{s},\end{aligned}\tag{51}$$

for all $\mathfrak{t}\in[0,1]$ and $(\mathfrak{u},\mathfrak{v})\in\mathcal{P}$.

We observe that $(\mathfrak{u},\mathfrak{v})$ constitutes a positive solution to problem (49) (or equivalently, (1),(2)) if and only if it serves as a fixed point for the operator $\mathcal{A}$. Consequently, our subsequent analysis will focus on examining the existence of fixed points for $\mathcal{A}$.

Under assumptions $(H1)$ and $(H2)$, using standard arguments, we deduce that operator $\mathcal{A}$ is completely continuous. In addition, by Lemma 9, we obtain

$$\mathcal{A}_1(\mathfrak{u},\mathfrak{v})(\mathfrak{t})\geq\mathfrak{t}^{\alpha-1}\|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\|,\ \ \mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t})\geq\mathfrak{t}^{\beta-1}\|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\|,\ \ \forall\,\mathfrak{t}\in[0,1],\tag{52}$$

that is, $\mathcal{A}(\mathcal{P})\subset\mathcal{P}$.

For $\varpi=q^n$, with $n\in\mathbb{N}$, we introduce the constants

$$\begin{aligned}&L_1=\int_0^1\mathfrak{G}_1(1,q\mathfrak{s})\,d_q\mathfrak{s},\ \ L_2=\int_0^1\mathfrak{G}_2(1,q\mathfrak{s})\,d_q\mathfrak{s},\\&L_3=\int_0^1\mathfrak{G}_3(1,q\mathfrak{s})\,d_q\mathfrak{s},\ \ L_4=\int_0^1\mathfrak{G}_2(1,q\mathfrak{s})\,d_q\mathfrak{s},\\&M_1=L_1+L_3,\ \ M_2=L_2+L_4,\\&\widetilde{L}_1=\int_\varpi^1\mathfrak{G}_1(1,q\mathfrak{s})\,d_q\mathfrak{s},\ \ \widetilde{L}_2=\int_\varpi^1\mathfrak{G}_2(1,q\mathfrak{s})\,d_q\mathfrak{s},\\&\widetilde{L}_3=\int_\varpi^1\mathfrak{G}_3(1,q\mathfrak{s})\,d_q\mathfrak{s},\ \ \widetilde{L}_4=\int_\varpi^1\mathfrak{G}_4(1,q\mathfrak{s})\,d_q\mathfrak{s},\\&\widetilde{M}_1=\varpi^{\alpha-1}\widetilde{L}_1+\varpi^{\beta-1}\widetilde{L}_3,\ \ \widetilde{M}_2=\varpi^{\alpha-1}\widetilde{L}_2+\varpi^{\beta-1}\widetilde{L}_4.\end{aligned}\tag{53}$$

We remark that L_2, L_3, $\widetilde{L}_2$, and $\widetilde{L}_3\geq0$ and $L_1,L_4,\widetilde{L}_1,\widetilde{L}_4,M_1,M_2,\widetilde{M}_1$, and $\widetilde{M}_2>0$.

Our initial existence result for positive solutions to problem (1),(2) relies on the Guo–Krasnosel'skii fixed-point theorem (refer to [28]).

Theorem 1. *Let $\varpi=q^n$ with $n\in\mathbb{N}$. Assume that $(H1)$ and $(H2)$ are satisfied. In addition, we suppose that there exist two positive constants $r_2>r_1>0$ and the constants $\widetilde{\sigma}_1\in(0,M_1^{-1}]$, $\widetilde{\sigma}_2\in(0,M_2^{-1}]$, $\widetilde{\sigma}_3\in[\widetilde{M}_1^{-1},\infty)$, and $\widetilde{\sigma}_4\in[\widetilde{M}_2^{-1},\infty)$ such that*

$$(H3)\begin{cases}\mathfrak{f}(t,u,v,x,y)\geq\frac{\widetilde{\sigma}_3r_1}{2},\ \ \forall\,t\in[\varpi,1],\ u,v\geq0,\ u+v\leq r_1,\\ \qquad x\in\left[0,\frac{r_1}{\Gamma_q(\delta_1+1)}\right],\ y\in\left[0,\frac{r_1}{\Gamma_q(\gamma_1+1)}\right],\\ \mathfrak{g}(t,u,v,x,y)\geq\frac{\widetilde{\sigma}_4r_1}{2},\ \ \forall\,t\in[\varpi,1],\ u,v\geq0,\ u+v\leq r_1,\\ \qquad x\in\left[0,\frac{r_1}{\Gamma_q(\delta_2+1)}\right],\ y\in\left[0,\frac{r_1}{\Gamma_q(\gamma_2+1)}\right];\end{cases}$$

$$(H4)\begin{cases}\mathfrak{f}(t,u,v,x,y)\leq\frac{\widetilde{\sigma}_1r_2}{2},\ \ \forall\,t\in[0,1],\ u,v\geq0,\ u+v\leq r_2,\\ \qquad x\in\left[0,\frac{r_2}{\Gamma_q(\delta_1+1)}\right],\ y\in\left[0,\frac{r_2}{\Gamma_q(\gamma_1+1)}\right],\\ \mathfrak{g}(t,u,v,x,y)\leq\frac{\widetilde{\sigma}_2r_2}{2},\ \ \forall\,t\in[0,1],\ u,v\geq0,\ u+v\leq r_2,\\ \qquad x\in\left[0,\frac{r_2}{\Gamma_q(\delta_2+1)}\right],\ y\in\left[0,\frac{r_2}{\Gamma_q(\gamma_2+1)}\right].\end{cases}$$

Then, problem (1),(2) has at least one positive solution $(\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t}))$, $\mathfrak{t}\in[0,1]$, *such that* $(\mathfrak{u},\mathfrak{v})\in\mathcal{P}$ *and* $r_1\le\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}}\le r_2$.

Proof. We introduce the set $\Omega_2=\{(\mathfrak{u},\mathfrak{v})\in\mathcal{Y},\ \|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}}<r_2\}$. Then, for $(\mathfrak{u},\mathfrak{v})\in\mathcal{P}\cap\partial\Omega_2$, we have $\|\mathfrak{u}\|+\|\mathfrak{v}\|=r_2$, so $\mathfrak{u}(\mathfrak{t})+\mathfrak{v}(\mathfrak{t})\le r_2$ for all $\mathfrak{t}\in[0,1]$. Because $\|\mathfrak{u}\|\le r_2$ and $\|\mathfrak{v}\|\le r_2$, by Lemma 4, we find $|I_q^{\delta_i}\mathfrak{u}(\mathfrak{s})|\le\frac{r_2}{\Gamma_q(\delta_i+1)}$, and $|I_q^{\gamma_j}\mathfrak{v}(\mathfrak{s})|\le\frac{r_2}{\Gamma_q(\gamma_j+1)}$ for all $\mathfrak{s}\in[0,1]$, $i,j=1,2$. We define $F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})=\mathfrak{f}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_1}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_1}\mathfrak{v}(\mathfrak{s}))$ and $G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})=\mathfrak{g}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}),\mathfrak{v}(\mathfrak{s}),I_q^{\delta_2}\mathfrak{u}(\mathfrak{s}),I_q^{\gamma_2}\mathfrak{v}(\mathfrak{s}))$ for $\mathfrak{s}\in[0,1]$.

Then, by Lemma 8 and $(H4)$, we obtain

$$\begin{aligned}
\|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\| &= \sup_{\mathfrak{t}\in[0,1]}|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})(\mathfrak{t})| \\
&\le \sup_{\mathfrak{t}\in[0,1]}\int_0^1\mathfrak{G}_1(\mathfrak{t},q\mathfrak{s})F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\,d_q\mathfrak{s}+\sup_{\mathfrak{t}\in[0,1]}\int_0^1\mathfrak{G}_2(\mathfrak{t},q\mathfrak{s})G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\le \int_0^1\sup_{\mathfrak{t}\in[0,1]}\mathfrak{G}_1(\mathfrak{t},q\mathfrak{s})F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\,d_q\mathfrak{s}+\int_0^1\sup_{\mathfrak{t}\in[0,1]}\mathfrak{G}_2(\mathfrak{t},q\mathfrak{s})G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\le \int_0^1\mathfrak{G}_1(1,q\mathfrak{s})F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\,d_q\mathfrak{s}+\int_0^1\mathfrak{G}_2(1,q\mathfrak{s})G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\le \frac{\widetilde{\sigma}_1 r_2}{2}\int_0^1\mathfrak{G}_1(1,q\mathfrak{s})\,d_q\mathfrak{s}+\frac{\widetilde{\sigma}_2 r_2}{2}\int_0^1\mathfrak{G}_2(1,q\mathfrak{s})\,d_q\mathfrak{s} \\
&= \frac{\widetilde{\sigma}_1 r_2}{2}L_1+\frac{\widetilde{\sigma}_2 r_2}{2}L_2=\left(\frac{\widetilde{\sigma}_1 L_1}{2}+\frac{\widetilde{\sigma}_2 L_2}{2}\right)r_2,
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
\|\mathcal{A}_2(\mathfrak{u},\mathfrak{v})\| &= \sup_{\mathfrak{t}\in[0,1]}|\mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t})| \\
&\le \sup_{\mathfrak{t}\in[0,1]}\int_0^1\mathfrak{G}_3(\mathfrak{t},q\mathfrak{s})F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\,d_q\mathfrak{s}+\sup_{\mathfrak{t}\in[0,1]}\int_0^1\mathfrak{G}_4(\mathfrak{t},q\mathfrak{s})G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\le \int_0^1\sup_{\mathfrak{t}\in[0,1]}\mathfrak{G}_3(\mathfrak{t},q\mathfrak{s})F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\,d_q\mathfrak{s}+\int_0^1\sup_{\mathfrak{t}\in[0,1]}\mathfrak{G}_4(\mathfrak{t},q\mathfrak{s})G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\le \int_0^1\mathfrak{G}_3(1,q\mathfrak{s})F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\,d_q\mathfrak{s}+\int_0^1\mathfrak{G}_4(1,q\mathfrak{s})G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\,d_q\mathfrak{s} \\
&\le \frac{\widetilde{\sigma}_1 r_2}{2}\int_0^1\mathfrak{G}_3(1,q\mathfrak{s})\,d_q\mathfrak{s}+\frac{\widetilde{\sigma}_2 r_2}{2}\int_0^1\mathfrak{G}_4(1,q\mathfrak{s})\,d_q\mathfrak{s} \\
&= \frac{\widetilde{\sigma}_1 r_2}{2}L_3+\frac{\widetilde{\sigma}_2 r_2}{2}L_4=\left(\frac{\widetilde{\sigma}_1 L_3}{2}+\frac{\widetilde{\sigma}_2 L_4}{2}\right)r_2.
\end{aligned} \tag{55}$$

Then, we deduce

$$\begin{aligned}
\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}} &= \|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\|+\|\mathcal{A}_2(\mathfrak{u},\mathfrak{v})\|\le\left[\frac{\widetilde{\sigma}_1(L_1+L_3)}{2}+\frac{\widetilde{\sigma}_2(L_2+L_4)}{2}\right]r_2 \\
&= \left(\frac{\widetilde{\sigma}_1 M_1}{2}+\frac{\widetilde{\sigma}_2 M_2}{2}\right)r_2\le r_2,
\end{aligned} \tag{56}$$

that is,

$$\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}}\le\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}},\ \ \forall\,(\mathfrak{u},\mathfrak{v})\in\mathcal{P}\cap\partial\Omega_2. \tag{57}$$

Now, we define the set $\Omega_1=\{(\mathfrak{u},\mathfrak{v})\in\mathcal{Y},\ \|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}}<r_1\}$. Then, for $(\mathfrak{u},\mathfrak{v})\in\mathcal{P}\cap\partial\Omega_1$, we have $\|\mathfrak{u}\|+\|\mathfrak{v}\|=r_1$, so $\mathfrak{u}(\mathfrak{t})+\mathfrak{v}(\mathfrak{t})\le r_1$ for all $\mathfrak{t}\in[0,1]$.

Therefore, by Lemma 8 and $(H3)$, we obtain

$$\begin{aligned}
\|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\| &= \sup_{\mathfrak{t}\in[0,1]} |\mathcal{A}_1(\mathfrak{u},\mathfrak{v})(\mathfrak{t})| \\
&= \sup_{\mathfrak{t}\in[0,1]} \left(\int_0^1 \mathfrak{G}_1(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \right) \\
&\geq \inf_{\mathfrak{t}\in[\varpi,1]} \left(\int_0^1 \mathfrak{G}_1(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \right) \\
&\geq \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_1(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_2(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\geq \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_1(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_2(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\geq \int_\varpi^1 \varpi^{\alpha-1} \mathfrak{G}_1(1,q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_\varpi^1 \varpi^{\alpha-1} \mathfrak{G}_2(1,q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\geq \frac{\widetilde{\sigma}_3 r_1 \varpi^{\alpha-1}}{2} \int_\varpi^1 \mathfrak{G}_1(1,q\mathfrak{s})\, d_q\mathfrak{s} + \frac{\widetilde{\sigma}_4 r_1 \varpi^{\alpha-1}}{2} \int_\varpi^1 \mathfrak{G}_2(1,q\mathfrak{s})\, d_q\mathfrak{s} \\
&= \frac{\widetilde{\sigma}_3 r_1 \varpi^{\alpha-1}\widetilde{L}_1}{2} + \frac{\widetilde{\sigma}_4 r_1 \varpi^{\alpha-1}\widetilde{L}_2}{2} = \left(\frac{\widetilde{\sigma}_3 \varpi^{\alpha-1}\widetilde{L}_1}{2} + \frac{\widetilde{\sigma}_4 \varpi^{\alpha-1}\widetilde{L}_2}{2} \right) r_1,
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
\|\mathcal{A}_2(\mathfrak{u},\mathfrak{v})\| &= \sup_{\mathfrak{t}\in[0,1]} |\mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t})| \\
&= \sup_{\mathfrak{t}\in[0,1]} \left(\int_0^1 \mathfrak{G}_3(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_4(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \right) \\
&\geq \inf_{\mathfrak{t}\in[\varpi,1]} \left(\int_0^1 \mathfrak{G}_3(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_4(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \right) \\
&\geq \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_3(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_4(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\geq \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_3(\mathfrak{t},q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_4(\mathfrak{t},q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\geq \int_\varpi^1 \varpi^{\beta-1} \mathfrak{G}_3(1,q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_\varpi^1 \varpi^{\beta-1} \mathfrak{G}_4(1,q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\geq \frac{\widetilde{\sigma}_3 r_1 \varpi^{\beta-1}}{2} \int_\varpi^1 \mathfrak{G}_3(1,q\mathfrak{s})\, d_q\mathfrak{s} + \frac{\widetilde{\sigma}_4 r_1 \varpi^{\beta-1}}{2} \int_\varpi^1 \mathfrak{G}_4(1,q\mathfrak{s})\, d_q\mathfrak{s} \\
&= \frac{\widetilde{\sigma}_3 r_1 \varpi^{\beta-1}\widetilde{L}_3}{2} + \frac{\widetilde{\sigma}_4 r_1 \varpi^{\beta-1}\widetilde{L}_4}{2} = \left(\frac{\widetilde{\sigma}_3 \varpi^{\beta-1}\widetilde{L}_3}{2} + \frac{\widetilde{\sigma}_4 \varpi^{\beta-1}\widetilde{L}_4}{2} \right) r_1.
\end{aligned} \tag{59}$$

Then, we conclude

$$\begin{aligned}
\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}} &= \|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\| + \|\mathcal{A}_2(\mathfrak{u},\mathfrak{v})\| \\
&\geq \left[\frac{\widetilde{\sigma}_3(\varpi^{\alpha-1}\widetilde{L}_1 + \varpi^{\beta-1}\widetilde{L}_3)}{2} + \frac{\widetilde{\sigma}_4(\varpi^{\alpha-1}\widetilde{L}_2 + \varpi^{\beta-1}\widetilde{L}_4)}{2} \right] r_1 \\
&= \left(\frac{\widetilde{\sigma}_3 \widetilde{M}_1}{2} + \frac{\widetilde{\sigma}_4 \widetilde{M}_2}{2} \right) r_1 \geq r_1,
\end{aligned} \tag{60}$$

and so

$$\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}} \geq \|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}}, \ \forall\, (\mathfrak{u},\mathfrak{v}) \in \mathcal{P} \cap \partial\Omega_1. \tag{61}$$

As we mentioned before, the operator $\mathcal{A}$ is completely continuous. Then, by (57), (61), and the Guo–Krasnosel'skii fixed-point theorem, we deduce that the operator $\mathcal{A}$ has a fixed point $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, which is a solution of problem (1),(2). This solution satisfies $r_1 \leq \|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}} \leq r_2$, $\mathfrak{u}(\mathfrak{t}) \geq 0$, $\mathfrak{v}(\mathfrak{t}) \geq 0$ for all $\mathfrak{t} \in [0,1]$, $\mathfrak{u}(\mathfrak{t}) \geq \mathfrak{t}^{\alpha-1}\|u\|$, $\mathfrak{v}(\mathfrak{t}) \geq \mathfrak{t}^{\beta-1}\|v\|$ for all $\mathfrak{t} \in [0,1]$; because $\|\mathfrak{u}\| + \|\mathfrak{v}\| \geq r_1$, we obtain $\|\mathfrak{u}\| > 0$ or $\|\mathfrak{v}\| > 0$, that is, $\mathfrak{u}(\mathfrak{t}) > 0$ for all $\mathfrak{t} \in (0,1]$ or $\mathfrak{v}(\mathfrak{t}) > 0$ for all $\mathfrak{t} \in (0,1]$. □

Subsequently, we will establish the existence of at least three positive solutions to problem (1),(2) using the Leggett–Williams fixed-point theorem (refer to Theorem 3.3 in [29] or Theorem 2.3 in [30]).

Theorem 2. *Let $\varpi = q^n$ with $n \in \mathbb{N}$. Assume that $(H1)$ and $(H2)$ are satisfied. In addition, we suppose that there exist positive constants $0 < r_1 < r_2 < r_3$ such that*

$$(H5)\begin{cases} \mathfrak{f}(t,u,v,x,y) < \frac{r_1}{2M_1}, \ \forall\, t \in [0,1],\ u,v \geq 0,\ u+v \leq r_1, \\ \qquad x \in \left[0, \frac{r_1}{\Gamma_q(\delta_1+1)}\right],\ y \in \left[0, \frac{r_1}{\Gamma_q(\gamma_1+1)}\right], \\ \mathfrak{g}(t,u,v,x,y) < \frac{r_1}{2M_2}, \ \forall\, t \in [0,1],\ u,v \geq 0,\ u+v \leq r_1, \\ \qquad x \in \left[0, \frac{r_1}{\Gamma_q(\delta_2+1)}\right],\ y \in \left[0, \frac{r_1}{\Gamma_q(\gamma_2+1)}\right]; \end{cases}$$

$$(H6)\begin{cases} \mathfrak{f}(t,u,v,x,y) > \frac{r_2}{2M_1}, \ \forall\, t \in [\varpi,1],\ u,v \geq 0,\ r_2 \leq u+v \leq r_3, \\ \qquad x \in \left[0, \frac{r_3}{\Gamma_q(\delta_1+1)}\right],\ y \in \left[0, \frac{r_3}{\Gamma_q(\gamma_1+1)}\right], \\ \mathfrak{g}(t,u,v,x,y) > \frac{r_2}{2M_2}, \ \forall\, t \in [\varpi,1],\ u,v \geq 0,\ r_2 \leq u+v \leq r_3, \\ \qquad x \in \left[0, \frac{r_3}{\Gamma_q(\delta_2+1)}\right],\ y \in \left[0, \frac{r_3}{\Gamma_q(\gamma_2+1)}\right]; \end{cases}$$

$$(H7)\begin{cases} \mathfrak{f}(t,u,v,x,y) \leq \frac{r_3}{2M_1}, \ \forall\, t \in [0,1],\ u,v \geq 0,\ u+v \leq r_3, \\ \qquad x \in \left[0, \frac{r_3}{\Gamma_q(\delta_1+1)}\right],\ y \in \left[0, \frac{r_3}{\Gamma_q(\gamma_1+1)}\right], \\ \mathfrak{g}(t,u,v,x,y) \leq \frac{r_3}{2M_2}, \ \forall\, t \in [0,1],\ u,v \geq 0,\ u+v \leq r_3, \\ \qquad x \in \left[0, \frac{r_3}{\Gamma_q(\delta_2+1)}\right],\ y \in \left[0, \frac{r_3}{\Gamma_q(\gamma_2+1)}\right]. \end{cases}$$

Then, problem (1),(2) has at least three positive solutions $(\mathfrak{u}_i(\mathfrak{t}), \mathfrak{v}_i(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$, $i = 1, \ldots, 3$, such that $(\mathfrak{u}_i, \mathfrak{v}_i) \in \mathcal{P}$, $i = 1, \ldots, 3$, $\|(\mathfrak{u}_1, \mathfrak{v}_1)\| < r_1$, $\|(\mathfrak{u}_3, \mathfrak{v}_3)\| > r_1$, $\inf_{\mathfrak{t}\in[\varpi,1]}(\mathfrak{u}_2(\mathfrak{t}) + \mathfrak{v}_2(\mathfrak{t})) > r_2$, and $\inf_{\mathfrak{t}\in[\varpi,1]}(\mathfrak{u}_3(\mathfrak{t}) + \mathfrak{v}_3(\mathfrak{t})) < r_2$.

Proof. Firstly, we prove that $\mathcal{A} : \overline{\mathcal{P}}_{r_3} \to \overline{\mathcal{P}}_{r_3}$, where $\mathcal{P}_{r_3} = \{(\mathfrak{u}, \mathfrak{v}) \in \mathcal{P},\ \|(\mathfrak{u}, \mathfrak{v})\| < r_3\}$. For any $(\mathfrak{u}, \mathfrak{v}) \in \overline{\mathcal{P}}_{r_3}$, we have $\|(\mathfrak{u}, \mathfrak{v})\| \leq r_3$, and so $\|\mathfrak{u}\| + \|\mathfrak{v}\| \leq r_3$ and $0 \leq \mathfrak{u}(\mathfrak{t}) + \mathfrak{v}(\mathfrak{t}) \leq r_3$ for all $\mathfrak{t} \in [0,1]$. By using $(H7)$ and Lemma 8, we find

$$\begin{aligned} \|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\| &= \sup_{\mathfrak{t}\in[0,1]} |\mathcal{A}_1(\mathfrak{u},\mathfrak{v})(\mathfrak{t})| \\ &\leq \int_0^1 \mathfrak{G}_1(1,q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(1,q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\ &\leq \frac{r_3}{2M_1} \int_0^1 \mathfrak{G}_1(1,q\mathfrak{s})\, d_q\mathfrak{s} + \frac{r_3}{2M_2} \int_0^1 \mathfrak{G}_2(1,q\mathfrak{s})\, d_q\mathfrak{s} \\ &= \frac{r_3}{2M_1} L_1 + \frac{r_3}{2M_2} L_2 = \left(\frac{L_1}{2M_1} + \frac{L_2}{2M_2}\right) r_3, \end{aligned} \tag{62}$$

and

$$\begin{aligned} \|\mathcal{A}_2(\mathfrak{u},\mathfrak{v})\| &= \sup_{\mathfrak{t}\in[0,1]} |\mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t})| \\ &\leq \int_0^1 \mathfrak{G}_3(1,q\mathfrak{s}) F_{\mathfrak{u}\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_4(1,q\mathfrak{s}) G_{\mathfrak{u}\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\ &\leq \frac{r_3}{2M_1} \int_0^1 \mathfrak{G}_3(1,q\mathfrak{s})\, d_q\mathfrak{s} + \frac{r_3}{2M_2} \int_0^1 \mathfrak{G}_4(1,q\mathfrak{s})\, d_q\mathfrak{s} \\ &= \frac{r_3}{2M_1} L_3 + \frac{r_3}{2M_2} L_4 = \left(\frac{L_3}{2M_1} + \frac{L_4}{2M_2}\right) r_3. \end{aligned} \tag{63}$$

Then, we obtain

$$\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\| \leq \left(\frac{L_1+L_3}{2M_1} + \frac{L_2+L_4)}{2M_2}\right) r_3 = \left(\frac{M_1}{2M_1} + \frac{M_2}{2M_2}\right) r_3 = r_3,\ \ \forall\,(\mathfrak{u},\mathfrak{v}) \in \overline{\mathcal{P}}_{r_3}. \tag{64}$$

So, $\mathcal{A}(\overline{\mathcal{P}}_{r_3}) \subset \overline{\mathcal{P}}_{r_3}$.

We consider $\widetilde{r}_3 \in (r_2, r_3)$, and we define the concave nonnegative continuous functional Θ on $\mathcal{P}$ by $\Theta(\mathfrak{u},\mathfrak{v}) = \inf_{\mathfrak{t}\in[\varpi,1]}(\mathfrak{u}(\mathfrak{t}) + \mathfrak{v}(\mathfrak{t})),\ (\mathfrak{u},\mathfrak{v}) \in \mathcal{P}$. We see that $\Theta(\mathfrak{u},\mathfrak{v}) \le \|(\mathfrak{u},\mathfrak{v})\|$ for all $(\mathfrak{u},\mathfrak{v}) \in \overline{\mathcal{P}}_{r_3}$.

Next, we will verify conditions (i)–(iii) of Theorem 2.3 from [30], with $E = \mathcal{Y}$, $K = \mathcal{P}$, $A = \mathcal{A}$, $m = r_1$, $c = r_2$, $l = r_3$, and $d = \widetilde{r}_3$.

Firstly, we verify condition (ii). For $(\mathfrak{u},\mathfrak{v}) \in \overline{\mathcal{P}}_{r_1}$, we will show that $\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\| < r_1$. For this, let $(\mathfrak{u},\mathfrak{v}) \in \overline{\mathcal{P}}_{r_1}$. In a similar manner to that in (62) and (63), by using $(H5)$, we obtain

$$\|\mathcal{A}_1(\mathfrak{u},\mathfrak{v})\| < \left(\frac{L_1}{2M_1} + \frac{L_2}{2M_2}\right) r_1,\ \ \|\mathcal{A}_2(\mathfrak{u},\mathfrak{v})\| < \left(\frac{L_3}{2M_1} + \frac{L_4}{2M_2}\right) r_1, \tag{65}$$

and so $\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\| < r_1$. So, we have assumption (ii) of Theorem 2.3 from [30].

We now verify condition (i). We set the element $(\mathfrak{u}_0(\mathfrak{t}), \mathfrak{v}_0(\mathfrak{t})) = (\frac{r_2+\widetilde{r}_3}{4}, \frac{r_2+\widetilde{r}_3}{4})$, $\mathfrak{t} \in [0,1]$. Because $\mathfrak{u}_0(\mathfrak{t}) \ge 0$, $\mathfrak{v}_0(\mathfrak{t}) \ge 0$ for all $\mathfrak{t} \in [0,1]$, $\|(\mathfrak{u}_0,\mathfrak{v}_0)\| = \frac{r_2+\widetilde{r}_3}{2} < \widetilde{r}_3$, and $\Theta(\mathfrak{u}_0,\mathfrak{v}_0) = \frac{r_2+\widetilde{r}_3}{2} > r_2$, we deduce that $(\mathfrak{u}_0,\mathfrak{v}_0) \in \{(\mathfrak{u},\mathfrak{v}),\ (\mathfrak{u},\mathfrak{v}) \in \mathcal{P}(\Theta, r_2, \widetilde{r}_3),\ \Theta(\mathfrak{u},\mathfrak{v}) > r_2\}$. Now, let $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}(\Theta, r_2, \widetilde{r}_3)$, that is, $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}$, $\Theta(\mathfrak{u},\mathfrak{v}) \ge r_2$ and $\|(\mathfrak{u},\mathfrak{v})\| \le \widetilde{r}_3$. So, we have $\mathfrak{u}(\mathfrak{t}) + \mathfrak{v}(\mathfrak{t}) \le \widetilde{r}_3$ for all $\mathfrak{t} \in [0,1]$ and $\inf_{\mathfrak{t}\in[\varpi,1]}(\mathfrak{u}(\mathfrak{t}) + \mathfrak{v}(\mathfrak{t})) \ge r_2$. Then, by $(H6)$ and Lemma 8, we find

$$\begin{aligned}
\Theta(\mathcal{A}(\mathfrak{u},\mathfrak{v})) &= \inf_{\mathfrak{t}\in[\varpi,1]}(\mathcal{A}_1(\mathfrak{u},\mathfrak{v})(\mathfrak{t}) + \mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t})) \\
&\ge \inf_{\mathfrak{t}\in[\varpi,1]} \mathcal{A}_1(\mathfrak{u},\mathfrak{v})(\mathfrak{t}) + \inf_{\mathfrak{t}\in[\varpi,1]} \mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t}) \\
&\ge \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) F_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) G_{\mathfrak{u},\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\quad + \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_3(\mathfrak{t}, q\mathfrak{s}) F_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \inf_{\mathfrak{t}\in[\varpi,1]} \int_0^1 \mathfrak{G}_4(\mathfrak{t}, q\mathfrak{s}) G_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\ge \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) F_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) G_{\mathfrak{u},\mathfrak{v}}^{\delta_2\gamma_2}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\quad + \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_3(\mathfrak{t}, q\mathfrak{s}) F_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \inf_{\mathfrak{t}\in[\varpi,1]} \mathfrak{G}_4(\mathfrak{t}, q\mathfrak{s}) G_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\ge \int_\varpi^1 \varpi^{\alpha-1} \mathfrak{G}_1(1, q\mathfrak{s}) F_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}\, d_q\mathfrak{s} + \int_\varpi^1 \varpi^{\alpha-1} \mathfrak{G}_2(1, q\mathfrak{s}) G_{\mathfrak{u},\mathfrak{v}}^{\delta_2\gamma_2}\, d_q\mathfrak{s} \\
&\quad + \int_\varpi^1 \varpi^{\beta-1} \mathfrak{G}_3(1, q\mathfrak{s}) F_{\mathfrak{u},\mathfrak{v}}^{\delta_1\gamma_1}\, d_q\mathfrak{s} + \int_\varpi^1 \varpi^{\beta-1} \mathfrak{G}_4(1, q\mathfrak{s}) G_{\mathfrak{u},\mathfrak{v}}^{\delta_2\gamma_2}\, d_q\mathfrak{s} \\
&> \frac{r_2\varpi^{\alpha-1}}{2\widetilde{M}_1} \int_\varpi^1 \mathfrak{G}_1(1, q\mathfrak{s})\, d_q\mathfrak{s} + \frac{r_2\varpi^{\alpha-1}}{2\widetilde{M}_2} \int_\varpi^1 \mathfrak{G}_2(1, q\mathfrak{s})\, d_q\mathfrak{s} \\
&\quad + \frac{r_2\varpi^{\beta-1}}{2\widetilde{M}_1} \int_\varpi^1 \mathfrak{G}_3(1, q\mathfrak{s})\, d_q\mathfrak{s} + \frac{r_2\varpi^{\beta-1}}{2\widetilde{M}_2} \int_\varpi^1 \mathfrak{G}_4(1, q\mathfrak{s})\, d_q\mathfrak{s} \\
&= \frac{r_2\varpi^{\alpha-1}}{2\widetilde{M}_1}\widetilde{L}_1 + \frac{r_2\varpi^{\alpha-1}}{2\widetilde{M}_2}\widetilde{L}_2 + \frac{r_2\varpi^{\beta-1}}{2\widetilde{M}_1}\widetilde{L}_3 + \frac{r_2\varpi^{\beta-1}}{2\widetilde{M}_2}\widetilde{L}_4 \\
&= \left[\frac{\varpi^{\alpha-1}\widetilde{L}_1 + \varpi^{\beta-1}\widetilde{L}_3}{2\widetilde{M}_1} + \frac{\varpi^{\alpha-1}\widetilde{L}_2 + \varpi^{\beta-1}\widetilde{L}_4}{2\widetilde{M}_2}\right] r_2 \\
&= \left(\frac{\widetilde{M}_1}{2\widetilde{M}_1} + \frac{\widetilde{M}_2}{2\widetilde{M}_2}\right) r_2 = r_2.
\end{aligned} \tag{66}$$

Therefore, $\Theta(\mathcal{A}(\mathfrak{u},\mathfrak{v})) > r_2$, and we have assumption (i) of Theorem 2.3 from [30].

We now verify condition (iii), namely, $\Theta(\mathcal{A}(\mathfrak{u},\mathfrak{v})) > r_2$ for all $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}(\Theta, r_2, r_3)$ and $\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\| > \widetilde{r}_3$. So, let $(\mathfrak{u},\mathfrak{v}) \in \mathcal{P}(\Theta, r_2, r_3)$ and $\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\| > \widetilde{r}_3$. By using similar arguments to those used before, we deduce that $\Theta(\mathcal{A}(\mathfrak{u},\mathfrak{v})) > r_2$; that is, assumption (iii) of Theorem 2.3 from [30] is satisfied.

We know that $\mathcal{A}$ is a completely continuous operator. Then, by the Leggett–Williams fixed-point theorem (see Theorem 2.3 from [30]), we conclude that problem (1),(2) has at least three positive solutions $(\mathfrak{u}_i(\mathfrak{t}), \mathfrak{v}_i(\mathfrak{t}))$ $\mathfrak{t} \in [0,1]$, $i = 1, \ldots, 3$, with $(\mathfrak{u}_i, \mathfrak{v}_i) \in \mathcal{P}$,

$i = 1, \ldots, 3$ and $\|(\mathfrak{u}_1, \mathfrak{v}_1)\| < r_1$, $\|(\mathfrak{u}_3, \mathfrak{v}_3)\| > r_1$, $\Theta(\mathfrak{u}_2, \mathfrak{v}_2) = \inf_{\mathfrak{t}\in[\varpi,1]}(\mathfrak{u}_2(\mathfrak{t}) + \mathfrak{v}_2(\mathfrak{t})) > r_2$ and $\Theta(\mathfrak{u}_3, \mathfrak{v}_3) = \inf_{\mathfrak{t}\in[\varpi,1]}(\mathfrak{u}_3(\mathfrak{t}) + \mathfrak{v}_3(\mathfrak{t})) < r_2$. □

The following result is derived from the Schauder fixed-point theorem (refer to [31]).

Theorem 3. *Assume that* $(H1)$ *and* $(H2)$ *are satisfied. In addition, we suppose that there exist continuous functions* P_i, $Q_i : [0,1] \to \mathbb{R}_+$, $i = 1, \ldots, 5$, *such that*

$$\begin{aligned} \mathfrak{f}(\mathfrak{t}, u_1, u_2, u_3, u_4) &\le \sum_{i=1}^{4} P_i(\mathfrak{t})u_i + P_5(\mathfrak{t}), \\ \mathfrak{g}(\mathfrak{t}, u_1, u_2, u_3, u_4) &\le \sum_{i=1}^{4} Q_i(\mathfrak{t})u_i + Q_5(\mathfrak{t}), \end{aligned} \tag{67}$$

for all $\mathfrak{t} \in [0,1]$ *and* $u_i \in \mathbb{R}_+$, $i = 1, \ldots, 4$. *If*

$$\Psi_0 = \max\{\Psi_1, \Psi_2\} < 1, \tag{68}$$

where

$$\begin{aligned} \Psi_1 &= (L_1 + L_3)\left(p_1^* + \frac{p_3^*}{\Gamma_q(\delta_1 + 1)}\right) + (L_2 + L_4)\left(q_1^* + \frac{q_3^*}{\Gamma_q(\delta_2 + 1)}\right), \\ \Psi_2 &= (L_1 + L_3)\left(p_2^* + \frac{p_4^*}{\Gamma_q(\gamma_1 + 1)}\right) + (L_2 + L_4)\left(q_2^* + \frac{q_4^*}{\Gamma_q(\gamma_2 + 1)}\right), \\ p_i^* &= \sup_{\mathfrak{t}\in[0,1]} P_i(\mathfrak{t}), \ \ q_i^* = \sup_{\mathfrak{t}\in[0,1]} Q_i(\mathfrak{t}), \ \ i = 1, \ldots, 4, \end{aligned} \tag{69}$$

then the boundary value problem (1),(2) has at least one positive solution $(\mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$.

Proof. We consider a positive number R_0 satisfying the condition

$$R_0 \ge \frac{(L_1 + L_3)p_5^* + (L_2 + L_4)q_5^*}{1 - \Psi_0}, \tag{70}$$

where $p_5^* = \sup_{\mathfrak{t}\in[0,1]} P_5(\mathfrak{t})$, $q_5^* = \sup_{\mathfrak{t}\in[0,1]} Q_5(\mathfrak{t})$. We define the set $\Omega_1 = \{(\mathfrak{u}, \mathfrak{v}) \in \mathcal{P}, \|(\mathfrak{u}, \mathfrak{v})\| \le R_0\}$.

Firstly, we show that $\mathcal{A}(\Omega_1) \subset \Omega_1$. For this, let $(\mathfrak{u}, \mathfrak{v}) \in \Omega_1$, that is, $\|(\mathfrak{u}, \mathfrak{v})\|_{\mathcal{Y}} \le R_0$, so $\|\mathfrak{u}\| + \|\mathfrak{v}\| \le R_0$, which implies $\|\mathfrak{u}\| \le R_0$ and $\|\mathfrak{v}\| \le R_0$. Then, by using (67), Lemma 4 and Lemma 8, we obtain

$$\begin{aligned} &\mathcal{A}_1(\mathfrak{u}, \mathfrak{v})(\mathfrak{t}) \\ &\le \int_0^1 \mathfrak{G}_1(1, q\mathfrak{s})(P_1(\mathfrak{s})\mathfrak{u}(\mathfrak{s}) + P_2(\mathfrak{s})\mathfrak{v}(\mathfrak{s}) + P_3(\mathfrak{s})I_q^{\delta_1}\mathfrak{u}(\mathfrak{s}) + P_4(\mathfrak{s})I_q^{\gamma_1}\mathfrak{v}(\mathfrak{s}) + P_5(\mathfrak{s}))\, d_q\mathfrak{s} \\ &\quad + \int_0^1 \mathfrak{G}_2(1, q\mathfrak{s})(Q_1(\mathfrak{s})\mathfrak{u}(\mathfrak{s}) + Q_2(\mathfrak{s})\mathfrak{v}(\mathfrak{s}) + Q_3(\mathfrak{s})I_q^{\delta_2}\mathfrak{u}(\mathfrak{s}) + Q_4(\mathfrak{s})I_q^{\gamma_2}\mathfrak{v}(\mathfrak{s}) + Q_5(\mathfrak{s}))\, d_q\mathfrak{s} \\ &\le \left(p_1^*\|\mathfrak{u}\| + p_2^*\|\mathfrak{v}\| + p_3^*\frac{\|\mathfrak{u}\|}{\Gamma_q(\delta_1 + 1)} + p_4^*\frac{\|\mathfrak{v}\|}{\Gamma_q(\gamma_1 + 1)} + p_5^*\right)\int_0^1 \mathfrak{G}_1(1, q\mathfrak{s})\, d_q\mathfrak{s} \\ &\quad + \left(q_1^*\|\mathfrak{u}\| + q_2^*\|\mathfrak{v}\| + q_3^*\frac{\|\mathfrak{u}\|}{\Gamma_q(\delta_2 + 1)} + q_4^*\frac{\|\mathfrak{v}\|}{\Gamma_q(\gamma_2 + 1)} + q_5^*\right)\int_0^1 \mathfrak{G}_2(1, q\mathfrak{s})\, d_q\mathfrak{s} \\ &= L_1\left[\left(p_1^* + \frac{p_3^*}{\Gamma_q(\delta_1 + 1)}\right)\|\mathfrak{u}\| + \left(p_2^* + \frac{p_4^*}{\Gamma_q(\gamma_1 + 1)}\right)\|\mathfrak{v}\| + p_5^*\right] \\ &\quad + L_2\left[\left(q_1^* + \frac{q_3^*}{\Gamma_q(\delta_2 + 1)}\right)\|\mathfrak{u}\| + \left(q_2^* + \frac{q_4^*}{\Gamma_q(\gamma_2 + 1)}\right)\|\mathfrak{v}\| + q_5^*\right], \ \ \forall\, \mathfrak{t} \in [0,1], \end{aligned} \tag{71}$$

and

$$\begin{aligned}
&\mathcal{A}_2(\mathfrak{u},\mathfrak{v})(\mathfrak{t}) \\
&\le \int_0^1 \mathfrak{G}_3(1,q\mathfrak{s})(P_1(\mathfrak{s})\mathfrak{u}(\mathfrak{s})+P_2(\mathfrak{s})\mathfrak{v}(\mathfrak{s})+P_3(\mathfrak{s})I_q^{\delta_1}\mathfrak{u}(\mathfrak{s})+P_4(\mathfrak{s})I_q^{\gamma_1}\mathfrak{v}(\mathfrak{s})+P_5(\mathfrak{s}))\,d_q\mathfrak{s} \\
&\quad+\int_0^1 \mathfrak{G}_4(1,q\mathfrak{s})(Q_1(\mathfrak{s})\mathfrak{u}(\mathfrak{s})+Q_2(\mathfrak{s})\mathfrak{v}(\mathfrak{s})+Q_3(\mathfrak{s})I_q^{\delta_2}\mathfrak{u}(\mathfrak{s})+Q_4(\mathfrak{s})I_q^{\gamma_2}\mathfrak{v}(\mathfrak{s})+Q_5(\mathfrak{s}))\,d_q\mathfrak{s} \\
&\le \left(p_1^*\|\mathfrak{u}\|+p_2^*\|\mathfrak{v}\|+p_3^*\frac{\|\mathfrak{u}\|}{\Gamma_q(\delta_1+1)}+p_4^*\frac{\|\mathfrak{v}\|}{\Gamma_q(\gamma_1+1)}+p_5^*\right)\int_0^1\mathfrak{G}_3(1,q\mathfrak{s})\,d_q\mathfrak{s} \\
&\quad+\left(q_1^*\|\mathfrak{u}\|+q_2^*\|\mathfrak{v}\|+q_3^*\frac{\|\mathfrak{u}\|}{\Gamma_q(\delta_2+1)}+q_4^*\frac{\|\mathfrak{v}\|}{\Gamma_q(\gamma_2+1)}+q_5^*\right)\int_0^1\mathfrak{G}_4(1,q\mathfrak{s})\,d_q\mathfrak{s} \\
&= L_3\left[\left(p_1^*+\frac{p_3^*}{\Gamma_q(\delta_1+1)}\right)\|\mathfrak{u}\|+\left(p_2^*+\frac{p_4^*}{\Gamma_q(\gamma_1+1)}\right)\|\mathfrak{v}\|+p_5^*\right] \\
&\quad+L_4\left[\left(q_1^*+\frac{q_3^*}{\Gamma_q(\delta_2+1)}\right)\|\mathfrak{u}\|+\left(q_2^*+\frac{q_4^*}{\Gamma_q(\gamma_2+1)}\right)\|\mathfrak{v}\|+q_5^*\right],\ \ \forall\,\mathfrak{t}\in[0,1].
\end{aligned} \tag{72}$$

Then, by (70)–(72), we deduce

$$\begin{aligned}
\|\mathcal{A}(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}} &\le (L_1+L_3)\left[\left(p_1^*+\frac{p_3^*}{\Gamma_q(\delta_1+1)}\right)\|\mathfrak{u}\|+\left(p_2^*+\frac{p_4^*}{\Gamma_q(\gamma_1+1)}\right)\|\mathfrak{v}\|+p_5^*\right] \\
&\quad+(L_2+L_4)\left[\left(q_1^*+\frac{q_3^*}{\Gamma_q(\delta_2+1)}\right)\|\mathfrak{u}\|+\left(q_2^*+\frac{q_4^*}{\Gamma_q(\gamma_2+1)}\right)\|\mathfrak{v}\|+q_5^*\right] \\
&= \left[(L_1+L_3)\left(p_1^*+\frac{p_3^*}{\Gamma_q(\delta_1+1)}\right)+(L_2+L_4)\left(q_1^*+\frac{q_3^*}{\Gamma_q(\delta_2+1)}\right)\right]\|\mathfrak{u}\| \\
&\quad+\left[(L_1+L_3)\left(p_2^*+\frac{p_4^*}{\Gamma_q(\gamma_1+1)}\right)+(L_2+L_4)\left(q_2^*+\frac{q_4^*}{\Gamma_q(\gamma_2+1)}\right)\right]\|\mathfrak{v}\| \\
&\quad+(L_1+L_3)p_5^*+(L_2+L_4)q_5^* \\
&= \Psi_1\|\mathfrak{u}\|+\Psi_2\|\mathfrak{v}\|+(L_1+L_3)p_5^*+(L_2+L_4)q_5^* \\
&\le \Psi_0\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}}+(L_1+L_3)p_5^*+(L_2+L_4)q_5^* \\
&\le \Psi_0R_0+(L_1+L_3)p_5^*+(L_2+L_4)q_5^*\le R_0.
\end{aligned} \tag{73}$$

Therefore, $\mathcal{A}(\Omega_1)\subset\Omega_1$. Because the operator $\mathcal{A}$ is completely continuous, then by the Schauder fixed-point theorem, we conclude that $\mathcal{A}$ has a fixed point $(\mathfrak{u},\mathfrak{v})\in\mathcal{P}$ with $\|(\mathfrak{u},\mathfrak{v})\|_{\mathcal{Y}}\le R_0$, which is a positive solution of problem (1),(2). □

In the final theorem, we will employ the Banach contraction mapping principle.

Theorem 4. *Assume that* $(H1)$ *and* $(H2)$ *are satisfied. In addition, we suppose that there exist continuous functions* H_i, $K_i:[0,1]\to\mathbb{R}_+$, $i=1,\dots,4$, *such that*

$$\begin{aligned}
|\mathfrak{f}(\mathfrak{t},u_1,u_2,u_3,u_4)-\mathfrak{f}(\mathfrak{t},v_1,v_2,v_3,v_4)| &\le \sum_{i=1}^4 H_i(\mathfrak{t})|u_i-v_i|, \\
|\mathfrak{g}(\mathfrak{t},u_1,u_2,u_3,u_4)-\mathfrak{g}(\mathfrak{t},v_1,v_2,v_3,v_4)| &\le \sum_{i=1}^4 K_i(\mathfrak{t})|u_i-v_i|,
\end{aligned} \tag{74}$$

for all $\mathfrak{t}\in[0,1]$, u_i, $v_i\in\mathbb{R}_+$, $i=1,\dots,4$. *If*

$$\Phi_0=\max\{\Phi_1,\Phi_2\}<1, \tag{75}$$

where

$$\begin{aligned}
\Phi_1 &= (L_1+L_3)\left(h_1^*+\frac{h_3^*}{\Gamma_q(\delta_1+1)}\right)+(L_2+L_4)\left(k_1^*+\frac{k_3^*}{\Gamma_q(\delta_2+1)}\right), \\
\Phi_2 &= (L_1+L_3)\left(h_2^*+\frac{h_4^*}{\Gamma_q(\gamma_1+1)}\right)+(L_2+L_4)\left(k_2^*+\frac{k_4^*}{\Gamma_q(\gamma_2+1)}\right), \\
h_i^* &= \sup_{\mathfrak{t}\in[0,1]}H_i(\mathfrak{t}),\ \ k_i^*=\sup_{\mathfrak{t}\in[0,1]}K_i(\mathfrak{t}),\ \ i=1,\dots,4,
\end{aligned} \tag{76}$$

then the boundary value problem given by (1),(2) possesses a unique positive solution $(\mathfrak{u}^*(\mathfrak{t}), \mathfrak{v}^*(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$. *Moreover, for any initial point* $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathcal{P}$, *the sequence* $((\mathfrak{u}_n, \mathfrak{v}_n))_{n\geq 0}$ *defined as* $(\mathfrak{u}_n, \mathfrak{v}_n) = \mathcal{A}(\mathfrak{u}_{n-1}, \mathfrak{v}_{n-1})$, $n \geq 1$, *converges to* $(\mathfrak{u}^*, \mathfrak{v}^*)$ *as* $n \to \infty$. *Additionally, an error estimate is provided by the following inequality:*

$$\|(\mathfrak{u}_n, \mathfrak{v}_n) - (\mathfrak{u}^*, \mathfrak{v}^*)\|_{\mathcal{Y}} \leq \frac{\Phi_0^n}{1-\Phi_0} \|(\mathfrak{u}_1, \mathfrak{v}_1) - (\mathfrak{u}_0, \mathfrak{v}_0)\|_{\mathcal{Y}}. \tag{77}$$

Proof. By using (74), Lemma 4, and Lemma 8, for any $(\mathfrak{u}_1, \mathfrak{v}_1)$, $(\mathfrak{u}_2, \mathfrak{v}_2) \in \mathcal{P}$, we obtain

$$\begin{aligned}
&|\mathcal{A}_1(\mathfrak{u}_1, \mathfrak{v}_1)(\mathfrak{t}) - \mathcal{A}_1(\mathfrak{u}_2, \mathfrak{v}_2)(\mathfrak{t})| \\
&= \Big| \int_0^1 \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) F^{\delta_1\gamma_1}_{\mathfrak{u}_1\mathfrak{v}_1}(\mathfrak{s})\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) G^{\delta_2\gamma_2}_{\mathfrak{u}_1\mathfrak{v}_1}(\mathfrak{s})\, d_q\mathfrak{s} \\
&\quad - \int_0^1 \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) F^{\delta_1\gamma_1}_{\mathfrak{u}_2\mathfrak{v}_2}(\mathfrak{s})\, d_q\mathfrak{s} - \int_0^1 \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) G^{\delta_2\gamma_2}_{\mathfrak{u}_2\mathfrak{v}_2}(\mathfrak{s})\, d_q\mathfrak{s} \Big| \\
&\leq \int_0^1 \mathfrak{G}_1(\mathfrak{t}, q\mathfrak{s}) \Big| F^{\delta_1\gamma_1}_{\mathfrak{u}_1\mathfrak{v}_1}(\mathfrak{s}) - F^{\delta_1\gamma_1}_{\mathfrak{u}_2\mathfrak{v}_2}(\mathfrak{s}) \Big|\, d_q\mathfrak{s} + \int_0^1 \mathfrak{G}_2(\mathfrak{t}, q\mathfrak{s}) \Big| G^{\delta_2\gamma_2}_{\mathfrak{u}_1\mathfrak{v}_1}(\mathfrak{s}) - G^{\delta_2\gamma_2}_{\mathfrak{u}_2\mathfrak{v}_2}(\mathfrak{s}) \Big|\, d_q\mathfrak{s} \\
&\leq \int_0^1 \mathfrak{G}_1(1, q\mathfrak{s}) [H_1(\mathfrak{s})|\mathfrak{u}_1(\mathfrak{s}) - \mathfrak{u}_2(\mathfrak{s})| + H_2(\mathfrak{s})|\mathfrak{v}_1(\mathfrak{s}) - \mathfrak{v}_2(\mathfrak{s})| \\
&\quad + H_3(\mathfrak{s})|I_q^{\delta_1}\mathfrak{u}_1(\mathfrak{s}) - I_q^{\delta_1}\mathfrak{u}_2(\mathfrak{s})| + H_4(\mathfrak{s})|I_q^{\gamma_1}\mathfrak{v}_1(\mathfrak{s}) - I_q^{\gamma_1}\mathfrak{v}_2(\mathfrak{s})| \Big] d_q\mathfrak{s} \\
&\quad + \int_0^1 \mathfrak{G}_2(1, q\mathfrak{s}) [K_1(\mathfrak{s})|\mathfrak{u}_1(\mathfrak{s}) - \mathfrak{u}_2(\mathfrak{s})| + K_2(\mathfrak{s})|\mathfrak{v}_1(\mathfrak{s}) - \mathfrak{v}_2(\mathfrak{s})| \\
&\quad + K_3(\mathfrak{s})|I_q^{\delta_2}\mathfrak{u}_1(\mathfrak{s}) - I_q^{\delta_2}\mathfrak{u}_2(\mathfrak{s})| + K_4(\mathfrak{s})|I_q^{\gamma_2}\mathfrak{v}_1(\mathfrak{s}) - I_q^{\gamma_2}\mathfrak{v}_2(\mathfrak{s})| \Big] d_q\mathfrak{s} \\
&\leq \int_0^1 \mathfrak{G}_1(1, q\mathfrak{s}) [H_1(\mathfrak{s})\|\mathfrak{u}_1 - \mathfrak{u}_2\| + H_2(\mathfrak{s})\|\mathfrak{v}_1 - \mathfrak{v}_2\| \\
&\quad + H_3(\mathfrak{s}) \frac{1}{\Gamma_q(\delta_1+1)} \|\mathfrak{u}_1 - \mathfrak{u}_2\| + H_4(\mathfrak{s}) \frac{1}{\Gamma_q(\gamma_1+1)} \|\mathfrak{v}_1 - \mathfrak{v}_2\| \Big] d_q\mathfrak{s} \\
&\quad + \int_0^1 \mathfrak{G}_2(1, q\mathfrak{s}) [K_1(\mathfrak{s})\|\mathfrak{u}_1 - \mathfrak{u}_2\| + K_2(\mathfrak{s})\|\mathfrak{v}_1 - \mathfrak{v}_2\| \\
&\quad + K_3(\mathfrak{s}) \frac{1}{\Gamma_q(\delta_2+1)} \|\mathfrak{u}_1 - \mathfrak{u}_2\| + K_4(\mathfrak{s}) \frac{1}{\Gamma_q(\gamma_2+1)} \|\mathfrak{v}_1 - \mathfrak{v}_2\| \Big] d_q\mathfrak{s} \\
&\leq \left(h_1^* + \frac{h_3^*}{\Gamma_q(\delta_1+1)} \right) \|\mathfrak{u}_1 - \mathfrak{u}_2\| \int_0^1 \mathfrak{G}_1(1, q\mathfrak{s})\, d_q\mathfrak{s} \\
&\quad + \left(h_2^* + \frac{h_4^*}{\Gamma_q(\gamma_1+1)} \right) \|\mathfrak{v}_1 - \mathfrak{v}_2\| \int_0^1 \mathfrak{G}_1(1, q\mathfrak{s})\, d_q\mathfrak{s} \\
&\quad + \left(k_1^* + \frac{k_3^*}{\Gamma_q(\delta_2+1)} \right) \|\mathfrak{u}_1 - \mathfrak{u}_2\| \int_0^1 \mathfrak{G}_2(1, q\mathfrak{s})\, d_q\mathfrak{s} \\
&\quad + \left(k_2^* + \frac{k_4^*}{\Gamma_q(\gamma_2+1)} \right) \|\mathfrak{v}_1 - \mathfrak{v}_2\| \int_0^1 \mathfrak{G}_2(1, q\mathfrak{s})\, d_q\mathfrak{s} \\
&= \|\mathfrak{u}_1 - \mathfrak{u}_2\| \left[\left(h_1^* + \frac{h_3^*}{\Gamma_q(\delta_1+1)} \right) L_1 + \left(k_1^* + \frac{k_3^*}{\Gamma_q(\delta_2+1)} \right) L_2 \right] \\
&\quad + \|\mathfrak{v}_1 - \mathfrak{v}_2\| \left[\left(h_2^* + \frac{h_4^*}{\Gamma_q(\gamma_1+1)} \right) L_1 + \left(k_2^* + \frac{k_4^*}{\Gamma_q(\gamma_2+1)} \right) L_2 \right], \ \forall\, \mathfrak{t} \in [0,1].
\end{aligned} \tag{78}$$

In a similar manner, we deduce

$$\begin{aligned}
&|\mathcal{A}_2(\mathfrak{u}_1, \mathfrak{v}_1)(\mathfrak{t}) - \mathcal{A}_2(\mathfrak{u}_2, \mathfrak{v}_2)(\mathfrak{t})| \\
&\leq \|\mathfrak{u}_1 - \mathfrak{u}_2\| \left[\left(h_1^* + \frac{h_3^*}{\Gamma_q(\delta_1+1)} \right) L_3 + \left(k_1^* + \frac{k_3^*}{\Gamma_q(\delta_2+1)} \right) L_4 \right] \\
&\quad + \|\mathfrak{v}_1 - \mathfrak{v}_2\| \left[\left(h_2^* + \frac{h_4^*}{\Gamma_q(\gamma_1+1)} \right) L_3 + \left(k_2^* + \frac{k_4^*}{\Gamma_q(\gamma_2+1)} \right) L_4 \right], \ \forall\, \mathfrak{t} \in [0,1].
\end{aligned} \tag{79}$$

Therefore, by (78) and (79), we conclude

$$\begin{aligned}
&\|\mathcal{A}(\mathfrak{u}_1,\mathfrak{v}_1)-\mathcal{A}(\mathfrak{u}_2,\mathfrak{v}_2)\|_{\mathcal{Y}}\\
&\le \|\mathfrak{u}_1-\mathfrak{u}_2\|\left[\left(h_1^*+\frac{h_3^*}{\Gamma_q(\delta_1+1)}\right)(L_1+L_3)+\left(k_1^*+\frac{k_3^*}{\Gamma_q(\delta_2+1)}\right)(L_2+L_4)\right]\\
&\quad+\|\mathfrak{v}_1-\mathfrak{v}_2\|\left[\left(h_2^*+\frac{h_4^*}{\Gamma_q(\gamma_1+1)}\right)(L_1+L_3)+\left(k_2^*+\frac{k_4^*}{\Gamma_q(\gamma_2+1)}\right)(L_2+L_4)\right]\\
&\le \max\{\Phi_1,\Phi_2\}\|(\mathfrak{u}_1,\mathfrak{v}_1)-(\mathfrak{u}_2,\mathfrak{v}_2)\|_{\mathcal{Y}}=\Phi_0\|(\mathfrak{u}_1,\mathfrak{v}_1)-(\mathfrak{u}_2,\mathfrak{v}_2)\|_{\mathcal{Y}}.
\end{aligned} \tag{80}$$

By satisfying condition (75), we establish that the operator $\mathcal{A}$ is a contraction mapping. Consequently, according to the Banach fixed-point theorem, it follows that $\mathcal{A}$ possesses a unique fixed point $(\mathfrak{u}^*,\mathfrak{v}^*)\in\mathcal{P}$. This fixed point corresponds to the unique positive solution of problem (1),(2). Furthermore, for any $(\mathfrak{u}_0,\mathfrak{v}_0)\in\mathcal{P}$, the sequence $((\mathfrak{u}_n,\mathfrak{v}_n))_{n\ge 0}$ defined by $(\mathfrak{u}_n,\mathfrak{v}_n)=\mathcal{A}(\mathfrak{u}_{n-1},\mathfrak{v}_{n-1})$ for $n\ge 1$ converges to $(\mathfrak{u}^*,\mathfrak{v}^*)$ as $n\to\infty$. The proof of the Banach theorem yields the error estimate (77). □

4. Examples

In this section, we will provide various examples to demonstrate and illustrate our key findings.

Let $q=\frac{1}{2}$, $\alpha=\frac{7}{2}$, $n=4$, $\beta=\frac{19}{4}$, $m=5$, $\varsigma=\frac{5}{3}$, $\vartheta=\frac{12}{5}$, $\delta_1=\frac{46}{5}$, $\gamma_1=\frac{31}{6}$, $\delta_2=\frac{59}{8}$, $\gamma_2=\frac{21}{13}$, $a=1$, $b=1$, $c=1$, $d=1$, $\varrho_1=\frac{14}{9}$, $\sigma_1=\frac{2}{11}$, $\eta_1=1$, $\rho_1=\frac{13}{7}$, $\xi_1=\frac{1}{2}$, $\omega_1=\frac{1}{16}$, $\zeta_1=\frac{1}{8}$, $\theta_1=\frac{1}{4}$, $a_1=\frac{1}{6}$, $b_1=31$, $c_1=25$, $d_1=\frac{1}{2}$, and $\varpi=\frac{1}{4}$.

We consider the system of fractional q-difference equations

$$\begin{cases}\left(D_{1/2}^{7/2}\mathfrak{u}\right)(\mathfrak{t})+\mathfrak{f}\left(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t}),I_{1/2}^{46/5}\mathfrak{u}(\mathfrak{t}),I_{1/2}^{31/6}\mathfrak{v}(\mathfrak{t})\right)=0,\ \ \mathfrak{t}\in[0,1],\\ \left(D_{1/2}^{19/4}\mathfrak{v}\right)(\mathfrak{t})+\mathfrak{g}\left(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t}),I_{1/2}^{59/8}\mathfrak{u}(\mathfrak{t}),I_{1/2}^{21/13}\mathfrak{v}(\mathfrak{t})\right)=0,\ \ \mathfrak{t}\in[0,1],\end{cases} \tag{81}$$

with the boundary conditions

$$\begin{cases}\mathfrak{u}(0)=D_{1/2}\mathfrak{u}(0)=D_{1/2}^2\mathfrak{u}(0)=0,\ \ D_{1/2}^{5/3}\mathfrak{u}(1)=\frac{1}{6}D_{1/2}^{14/9}\mathfrak{u}\left(\frac{1}{2}\right)+31D_{1/2}^{2/11}\mathfrak{v}\left(\frac{1}{16}\right),\\ \mathfrak{v}(0)=D_{1/2}\mathfrak{v}(0)=D_{1/2}^2\mathfrak{v}(0)=D_{1/2}^3\mathfrak{v}(0)=0,\ \ D_{1/2}^{12/5}\mathfrak{v}(1)=25D_{1/2}\mathfrak{u}\left(\frac{1}{8}\right)+\frac{1}{2}D_{1/2}^{13/7}\mathfrak{v}\left(\frac{1}{4}\right).\end{cases} \tag{82}$$

We obtain $\Lambda_1\approx 1.86833829$, $\Lambda_2\approx 0.00175841$, $\Lambda_3\approx 1.8190837$, $\Lambda_4\approx 3.62744165$, and $\Delta=\Lambda_1\Lambda_4-\Lambda_2\Lambda_3\approx 6.77408943>0$. So, assumption $(H1)$ is satisfied. We also have

$$\begin{aligned}
g_1(\mathfrak{t},\mathfrak{s})&=\frac{1}{\Gamma_{1/2}(7/2)}\begin{cases}\mathfrak{t}^{5/2}(1-\mathfrak{s})^{(5/6)}-(\mathfrak{t}-\mathfrak{s})^{(5/2)},\ \ 0\le\mathfrak{s}\le\mathfrak{t}\le 1,\\ \mathfrak{t}^{5/2}(1-\mathfrak{s})^{(5/6)},\ \ 0\le\mathfrak{t}\le\mathfrak{s}\le 1,\end{cases}\\
g_2(\mathfrak{t},\mathfrak{s})&=\frac{1}{\Gamma_{1/2}(19/4)}\begin{cases}\mathfrak{t}^{15/4}(1-\mathfrak{s})^{(27/20)}-(\mathfrak{t}-\mathfrak{s})^{(15/4)},\ \ 0\le\mathfrak{s}\le\mathfrak{t}\le 1,\\ \mathfrak{t}^{15/4}(1-\mathfrak{s})^{(27/20)},\ \ 0\le\mathfrak{t}\le\mathfrak{s}\le 1,\end{cases}\\
g_{11}(\mathfrak{t},\mathfrak{s})&=\frac{1}{\Gamma_{1/2}(35/18)}\begin{cases}\mathfrak{t}^{17/18}(1-\mathfrak{s})^{(5/6)}-(\mathfrak{t}-\mathfrak{s})^{(17/18)},\ \ 0\le\mathfrak{s}\le\mathfrak{t}\le 1,\\ \mathfrak{t}^{17/18}(1-\mathfrak{s})^{(5/6)},\ \ 0\le\mathfrak{t}\le\mathfrak{s}\le 1,\end{cases}\\
g_{21}(\mathfrak{t},\mathfrak{s})&=\frac{1}{\Gamma_{1/2}(5/2)}\begin{cases}\mathfrak{t}^{3/2}(1-\mathfrak{s})^{(5/6)}-(\mathfrak{t}-\mathfrak{s})^{(3/2)},\ \ 0\le\mathfrak{s}\le\mathfrak{t}\le 1,\\ \mathfrak{t}^{3/2}(1-\mathfrak{s})^{(5/6)},\ \ 0\le\mathfrak{t}\le\mathfrak{s}\le 1,\end{cases}\\
g_{31}(\mathfrak{t},\mathfrak{s})&=\frac{1}{\Gamma_{1/2}(201/44)}\begin{cases}\mathfrak{t}^{157/44}(1-\mathfrak{s})^{(27/20)}-(\mathfrak{t}-\mathfrak{s})^{(157/44)},\ \ 0\le\mathfrak{s}\le\mathfrak{t}\le 1,\\ \mathfrak{t}^{157/44}(1-\mathfrak{s})^{(27/20)},\ \ 0\le\mathfrak{t}\le\mathfrak{s}\le 1,\end{cases}\\
g_{41}(\mathfrak{t},\mathfrak{s})&=\frac{1}{\Gamma_{1/2}(81/28)}\begin{cases}\mathfrak{t}^{53/28}(1-\mathfrak{s})^{(27/20)}-(\mathfrak{t}-\mathfrak{s})^{(53/28)},\ \ 0\le\mathfrak{s}\le\mathfrak{t}\le 1,\\ \mathfrak{t}^{53/28}(1-\mathfrak{s})^{(27/20)},\ \ 0\le\mathfrak{t}\le\mathfrak{s}\le 1.\end{cases}
\end{aligned} \tag{83}$$

In addition, we find

$$\begin{aligned}
\mathfrak{G}_1(\mathfrak{t},\mathfrak{s}) &= g_1(\mathfrak{t},\mathfrak{s}) + \frac{\mathfrak{t}^{5/2}}{\Delta}\left[\frac{1}{6}\Lambda_4 g_{11}\left(\frac{1}{2},\mathfrak{s}\right) + 25\Lambda_2 g_{21}\left(\frac{1}{8},\mathfrak{s}\right)\right],\\
\mathfrak{G}_2(\mathfrak{t},\mathfrak{s}) &= \frac{\mathfrak{t}^{5/2}}{\Delta}\left[31\Lambda_4 g_{31}\left(\frac{1}{16},\mathfrak{s}\right) + \frac{1}{2}\Lambda_2 g_{41}\left(\frac{1}{4},\mathfrak{s}\right)\right],\\
\mathfrak{G}_3(\mathfrak{t},\mathfrak{s}) &= \frac{\mathfrak{t}^{15/4}}{\Delta}\left[\frac{1}{6}\Lambda_3 g_{11}\left(\frac{1}{2},\mathfrak{s}\right) + 25\Lambda_1 g_{21}\left(\frac{1}{8},\mathfrak{s}\right)\right],\\
\mathfrak{G}_4(\mathfrak{t},\mathfrak{s}) &= g_2(\mathfrak{t},\mathfrak{s}) + \frac{\mathfrak{t}^{15/4}}{\Delta}\left[31\Lambda_3 g_{31}\left(\frac{1}{16},\mathfrak{s}\right) + \frac{1}{2}\Lambda_1 g_{41}\left(\frac{1}{4},\mathfrak{s}\right)\right],
\end{aligned} \tag{84}$$

for all $\mathfrak{t}$, $\mathfrak{s} \in [0,1]$.

After complex computations using the Mathematica program, we obtain

$$\begin{aligned}
&L_1 = \textstyle\int_0^1 \mathfrak{G}_1(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.09173103,\ \ L_2 = \int_0^1 \mathfrak{G}_2(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.00012801,\\
&L_3 = \textstyle\int_0^1 \mathfrak{G}_3(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.17043259,\ \ L_4 = \int_0^1 \mathfrak{G}_4(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.02776846,\\
&\widetilde{L}_1 = \textstyle\int_{1/4}^1 \mathfrak{G}_1(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.08337987,\ \ \widetilde{L}_2 = \int_{1/4}^1 \mathfrak{G}_2(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.00008466,\\
&\widetilde{L}_3 = \textstyle\int_{1/4}^1 \mathfrak{G}_3(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.13008493,\ \ \widetilde{L}_4 = \int_{1/4}^1 \mathfrak{G}_4(1,q\mathfrak{s})\,d_q\mathfrak{s} \approx 0.02446965,\\
&M_1 \approx 0.26216363,\ \ M_2 \approx 0.02789647,\ \ \widetilde{M}_1 \approx 0.00332425,\ \ \widetilde{M}_2 \approx 0.00013782.
\end{aligned} \tag{85}$$

Example 1. *We consider the functions*

$$\begin{aligned}
\mathfrak{f}(\mathfrak{t},u,v,x,y) &= \frac{(u+v)^2}{25} + \frac{1}{8}\arctan x + \frac{1}{7}\cos^2 y + \frac{3(\mathfrak{t}^2+1)}{\mathfrak{t}+4},\\
\mathfrak{g}(\mathfrak{t},u,v,x,y) &= e^{-(u+v)} + \frac{x^2}{3} + y + \frac{9(\mathfrak{t}+1)}{\mathfrak{t}^3+2},
\end{aligned} \tag{86}$$

for all $\mathfrak{t} \in [0,1]$, $u,v,x,y \in \mathbb{R}_+$. *Assumption* $(H2)$ *is satisfied.*

We choose $r_1 = \frac{1}{1000}$, $r_2 = 10$, $\widetilde{\sigma}_1 = 3 < \frac{1}{M_1} \approx 3.8144$, $\widetilde{\sigma}_2 = 35 < \frac{1}{M_2} \approx 35.8468$, $\widetilde{\sigma}_3 = 301 > \frac{1}{\widetilde{M}_1} \approx 300.8201$, and $\widetilde{\sigma}_4 = 7256 > \frac{1}{\widetilde{M}_2} \approx 7255.6974$.

Then, we obtain

$$\begin{aligned}
&\mathfrak{f}(\mathfrak{t},u,v,x,y) \ge \min_{\mathfrak{t}\in[1/4,1]} \tfrac{3(\mathfrak{t}^2+1)}{\mathfrak{t}+4} = 0.75 \ge \tfrac{\widetilde{\sigma}_3 r_1}{2} = 0.1505,\\
&\quad \forall\, \mathfrak{t} \in \left[\tfrac{1}{4},1\right],\ \ u,v \ge 0,\ \ u+v \le \tfrac{1}{1000},\ \ x \in \left[0, \tfrac{1}{1000\Gamma_{1/2}(51/5)}\right],\ y \in \left[0, \tfrac{1}{1000\Gamma_{1/2}(37/6)}\right],\\
&\mathfrak{g}(\mathfrak{t},u,v,x,y) \ge e^{-1/1000} + \min_{\mathfrak{t}\in[1/4,1]} \tfrac{9(\mathfrak{t}+1)}{\mathfrak{t}^3+2} \approx 6.5804 \ge \tfrac{\widetilde{\sigma}_4 r_1}{2} \approx 3.628,\\
&\quad \forall\, \mathfrak{t} \in \left[\tfrac{1}{4},1\right],\ \ u,v \ge 0,\ \ u+v \le \tfrac{1}{1000},\ \ x \in \left[0, \tfrac{1}{1000\Gamma_{1/2}(67/8)}\right],\ y \in \left[0, \tfrac{1}{1000\Gamma_{1/2}(34/13)}\right],
\end{aligned} \tag{87}$$

that is, assumption $(H3)$ is satisfied. In addition, we have

$$\begin{aligned}
&\mathfrak{f}(\mathfrak{t},u,v,x,y) \le 4 + \tfrac{1}{8}\arctan \tfrac{10}{\Gamma_{1/2}(51/5)} + \tfrac{1}{7} + \max_{\mathfrak{t}\in[0,1]} \tfrac{3(\mathfrak{t}^2+1)}{\mathfrak{t}+4} \approx 5.3502 \le \tfrac{\widetilde{\sigma}_1 r_2}{2} = 15,\\
&\quad \forall\, \mathfrak{t} \in [0,1],\ \ u,v \ge 0,\ \ u+v \le 10,\ \ x \in \left[0, \tfrac{10}{\Gamma_{1/2}(51/5)}\right],\ y \in \left[0, \tfrac{10}{\Gamma_{1/2}(37/6)}\right],\\
&\mathfrak{g}(\mathfrak{t},u,v,x,y) \le 1 + \left(\tfrac{10}{\Gamma_{1/2}(67/8)}\right)^2 + \tfrac{10}{\Gamma_{1/2}(34/13)} + \max_{\mathfrak{t}\in[0,1]} \tfrac{9(\mathfrak{t}+1)}{\mathfrak{t}^3+2} \approx 15.5719 \le \tfrac{\widetilde{\sigma}_2 r_2}{2} = 175,\\
&\quad \forall\, \mathfrak{t} \in [0,1],\ \ u,v \ge 0,\ \ u+v \le 10,\ \ x \in \left[0, \tfrac{10}{\Gamma_{1/2}(67/8)}\right],\ y \in \left[0, \tfrac{10}{\Gamma_{1/2}(34/13)}\right],
\end{aligned} \tag{88}$$

so assumption $(H4)$ is verified.

By Theorem 1, we deduce that problem (81),(82) with the functions in (86) has at least one positive solution $(\mathfrak{u}(\mathfrak{t}),\mathfrak{v}(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$, such that $\frac{1}{1000} \le \|\mathfrak{u}\| + \|\mathfrak{v}\| \le 10$, and $\min_{\mathfrak{t}\in[1/4,1]} \mathfrak{u}(\mathfrak{t}) \ge \left(\frac{1}{4}\right)^{5/2}\|\mathfrak{u}\|$, $\min_{\mathfrak{t}\in[1/4,1]} \mathfrak{v}(\mathfrak{t}) \ge \left(\frac{1}{4}\right)^{15/4}\|\mathfrak{v}\|$.

Example 2. *Let the functions*

$$\mathfrak{f}(\mathfrak{t},u,v,x,y)=\begin{cases}\frac{\mathfrak{t}^2+2}{4}+\frac{u}{3(1+u)}+\frac{v}{5(1+v)}+x^{1/3}+\frac{1}{6}e^{-y},\\ \quad\forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ u+v\leq 1,\\ \frac{\mathfrak{t}^2+2}{4}+\frac{1}{3}\Big(1+4175\Big(\arctan\frac{\sqrt{3}(u+v)}{3}-\frac{\pi}{6}\Big)\Big)\frac{u}{1+u}\\ \quad+\frac{1}{5}\big(1+3985\big(1-\sin\frac{\pi}{2}(u+v)\big)\big)\frac{v}{1+v}+x^{1/3}+\frac{1}{6}e^{-y},\\ \quad\forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ 1<u+v\leq 4,\\ \frac{\mathfrak{t}^2+2}{4}+\frac{1}{3}\Big(1+4175\Big(\arctan\frac{4\sqrt{3}}{3}-\frac{\pi}{6}\Big)\Big)\frac{u}{1+u}\\ \quad+\frac{3986}{5}\frac{v}{1+v}+x^{1/3}+\frac{1}{6}e^{-y},\\ \quad\forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ u+v>4,\end{cases}$$
$$\mathfrak{g}(\mathfrak{t},u,v,x,y)=\begin{cases}\frac{14\mathfrak{t}^5+3}{7}+\frac{4u}{1+u}+\frac{v}{1+v}+\frac{x}{x^2+1}+y^{1/4},\\ \quad\forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ u+v\leq 1,\\ \frac{14\mathfrak{t}^5+3}{7}+4\Big(1+38547\Big(1-\frac{1}{\sqrt{u+v}}\Big)\Big)\frac{u}{1+u}\\ \quad+\big(1+18452\cos^2\frac{3\pi}{2}(u+v)\big)\frac{v}{1+v}+\frac{x}{x^2+1}+y^{1/4},\\ \quad\forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ 1<u+v\leq 4,\\ \frac{14\mathfrak{t}^5+3}{7}+77098\frac{u}{1+u}+18453\frac{v}{1+v}+\frac{x}{x^2+1}+y^{1/4},\\ \quad\forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ u+v>4.\end{cases}\tag{89}$$

Assumption $(H2)$ *is satisfied.*

We choose $r_1=1$, $r_2=4$, and $r_3=6000$. Then, we obtain

$$\begin{aligned}&\mathfrak{f}(\mathfrak{t},u,v,x,y)\leq\frac{3}{4}+\frac{1}{3}+\frac{1}{5}+\left(\frac{1}{\Gamma_{1/2}(51/5)}\right)^{1/3}+\frac{1}{6}\approx 1.63046876<\frac{r_1}{2M_1}\approx 1.90720585,\\ &\quad\forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ u+v\leq 1,\ x\in\left[0,\frac{1}{\Gamma_{1/2}(51/5)}\right],\ y\in\left[0,\frac{1}{\Gamma_{1/2}(37/6)}\right],\\ &\mathfrak{g}(\mathfrak{t},u,v,x,y)\leq\frac{17}{7}+5+\frac{1/\Gamma_{1/2}(67/8)}{(1/\Gamma_{1/2}(67/8))^2+1}+\left(\frac{1}{\Gamma_{1/2}(34/13)}\right)^{1/4}\approx 8.39492155\\ &\quad<\frac{r_1}{2M_2}\approx 17.92341375,\ \ \forall\,\mathfrak{t}\in[0,1],\ u,v\geq 0,\ u+v\leq 1,\\ &\quad x\in\left[0,\frac{1}{\Gamma_{1/2}(67/8)}\right],\ y\in\left[0,\frac{1}{\Gamma_{1/2}(34/13)}\right],\end{aligned}\tag{90}$$

that is, assumption $(H5)$ is satisfied.

Next, we find

$$\begin{aligned}&\mathfrak{f}(\mathfrak{t},u,v,x,y)\geq\frac{1}{64}+\frac{1}{2}+\frac{1}{3}\Big(1+4175\Big(\arctan\frac{4\sqrt{3}}{3}-\frac{\pi}{6}\Big)\Big)\frac{u}{1+u}+\frac{3986}{5}\frac{v}{1+v}+\frac{1}{6}e^{-6000/\Gamma_{1/2}(37/6)}\\ &\geq\frac{33}{64}+\min\Big\{\frac{1}{3}\Big(1+4175\Big(\arctan\frac{4\sqrt{3}}{3}-\frac{\pi}{6}\Big)\Big),\frac{3986}{5}\Big\}\big(\frac{u}{1+u}+\frac{v}{1+v}\big)+\frac{1}{6}e^{-6000/\Gamma_{1/2}(37/6)}\\ &\geq\frac{33}{64}+\min\Big\{\frac{1}{3}\Big(1+4175\Big(\arctan\frac{4\sqrt{3}}{3}-\frac{\pi}{6}\Big)\Big),\frac{3986}{5}\Big\}\frac{u+v}{1+u+v}+\frac{1}{6}e^{-6000/\Gamma_{1/2}(37/6)}\\ &\geq\frac{33}{64}+\min\Big\{\frac{1}{3}\Big(1+4175\Big(\arctan\frac{4\sqrt{3}}{3}-\frac{\pi}{6}\Big)\Big),\frac{3986}{5}\Big\}\frac{4}{5}+\frac{1}{6}e^{-6000/\Gamma_{1/2}(37/6)}\\ &\approx 638.275625>\frac{r_2}{2\widetilde{M}_1}\approx 601.64029336,\\ &\quad\forall\,\mathfrak{t}\in\left[\frac{1}{4},1\right],\ u,v\geq 0,\ 4\leq u+v\leq 6000,\ x\in\left[0,\frac{6000}{\Gamma_{1/2}(51/5)}\right],\ y\in\left[0,\frac{6000}{\Gamma_{1/2}(37/6)}\right],\\ &\mathfrak{g}(\mathfrak{t},u,v,x,y)\geq\frac{14(1/4)^5+3}{7}+\min\{77098,18453\}\big(\frac{u}{1+u}+\frac{v}{1+v}\big)\\ &\geq\frac{14(1/4)^5+3}{7}+18453\frac{u+v}{1+u+v}\geq\frac{14(1/4)^5+3}{7}+18453\cdot\frac{4}{5}\\ &\approx 14762.83052455>\frac{r_2}{2\widetilde{M}_2}\approx 14511.39486712,\\ &\quad\forall\,\mathfrak{t}\in\left[\frac{1}{4},1\right],\ u,v\geq 0,\ 4\leq u+v\leq 6000,\ x\in\left[0,\frac{6000}{\Gamma_{1/2}(67/8)}\right],\ y\in\left[0,\frac{6000}{\Gamma_{1/2}(34/13)}\right],\end{aligned}\tag{91}$$

and so assumption $(H6)$ is verified.

Lastly, we deduce

$$\begin{array}{l} \mathfrak{f}(\mathfrak{t},u,v,x,y) \leq \frac{3}{4} + \frac{1}{3}\left(1 + 4175\left(\arctan\frac{4\sqrt{3}}{3} - \frac{\pi}{6}\right)\right)\frac{6000}{6001} + \frac{3986}{5}\cdot\frac{6000}{6001} \\ \quad + \left(\frac{6000}{\Gamma_{1/2}(51/5)}\right)^{1/3} + \frac{1}{6} \approx 1690.110593 \leq \frac{r_3}{2M_1} \approx 11443.23507, \\ \qquad \forall\, \mathfrak{t} \in [0,1],\ u,v \geq 0,\ u+v \leq 6000,\ x \in \left[0, \frac{6000}{\Gamma_{1/2}(51/5)}\right],\ y \in \left[0, \frac{6000}{\Gamma_{1/2}(37/6)}\right], \\ \mathfrak{g}(\mathfrak{t},u,v,x,y) \leq \frac{17}{7} + 77098\cdot\frac{6000}{6001} + 18453\cdot\frac{6000}{6001} + \frac{1}{2} + \left(\frac{6000}{\Gamma_{1/2}(34/13)}\right)^{1/4} \\ \approx 95546.32860601 \leq \frac{r_3}{2M_2} \approx 107540.48247, \\ \qquad \forall\, \mathfrak{t} \in [0,1],\ u,v \geq 0,\ u+v \leq 6000,\ x \in \left[0, \frac{6000}{\Gamma_{1/2}(67/8)}\right],\ y \in \left[0, \frac{6000}{\Gamma_{1/2}(34/13)}\right], \end{array} \tag{92}$$

that is, assumption $(H7)$ is also satisfied.

Therefore, by Theorem 2, we conclude that problem (81),(82) with the nonlinearities in (89) has at least three positive solutions $(\mathfrak{u}_i, \mathfrak{v}_i) \in \mathcal{P},\ i = 1,\ldots,3$, such that $\|(\mathfrak{u}_1, \mathfrak{v}_1)\|_{\mathcal{Y}} < 1$, $\|(\mathfrak{u}_3, \mathfrak{v}_3)\|_{\mathcal{Y}} > 1$, $\inf_{\mathfrak{t}\in[1/4,1]}(\mathfrak{u}_2(\mathfrak{t}) + \mathfrak{v}_2(\mathfrak{t})) > 4$, and $\inf_{\mathfrak{t}\in[1/4,1]}(\mathfrak{u}_3(\mathfrak{t}) + \mathfrak{v}_3(\mathfrak{t})) < 4$.

Example 3. *We consider the functions*

$$\begin{array}{l} \mathfrak{f}(\mathfrak{t},u,v,x,y) = \dfrac{(\mathfrak{t}+1)^{1/3}e^{-\mathfrak{t}+2}u}{4(1+u^2)} + \dfrac{(\mathfrak{t}-1)^2 v}{2(\mathfrak{t}+5)} + \dfrac{e^{\mathfrak{t}}x}{3} + \dfrac{\mathfrak{t}y}{1+3y} + \dfrac{\mathfrak{t}^{11/3}}{\mathfrak{t}^2+6}, \\ \mathfrak{g}(\mathfrak{t},u,v,x,y) = \dfrac{\mathfrak{t}^{2/7}u}{3(\mathfrak{t}+4)} + \dfrac{(\mathfrak{t}+3)e^{\mathfrak{t}+1}v}{2(\mathfrak{t}^2+3)(1+2v^3)} + \dfrac{x}{(\mathfrak{t}+1)^2(1+2x)} + \dfrac{\mathfrak{t}y}{\mathfrak{t}^3+1} + \dfrac{\mathfrak{t}^{6/5}}{\mathfrak{t}+2}, \end{array} \tag{93}$$

for all $\mathfrak{t} \in [0,1],\ u,v,x,y \geq 0$. *We obtain the following inequalities:*

$$\begin{array}{l} \mathfrak{f}(\mathfrak{t},u,v,x,y) \leq P_1(\mathfrak{t})u + P_2(\mathfrak{t})v + P_3(\mathfrak{t})x + P_4(\mathfrak{t})y + P_5(\mathfrak{t}), \\ \mathfrak{g}(\mathfrak{t},u,v,x,y) \leq Q_1(\mathfrak{t})u + Q_2(\mathfrak{t})v + Q_3(\mathfrak{t})x + Q_4(\mathfrak{t})y + Q_5(\mathfrak{t}), \end{array} \tag{94}$$

for all $\mathfrak{t} \in [0,1]$, $u,v,x,y \geq 0$, *where*

$$\begin{array}{l} P_1(\mathfrak{t}) = \frac{(\mathfrak{t}+1)^{1/3}e^{-\mathfrak{t}+2}}{4},\ P_2(\mathfrak{t}) = \frac{(\mathfrak{t}-1)^2}{2(\mathfrak{t}+5)},\ P_3(\mathfrak{t}) = \frac{e^t}{3},\ P_4(\mathfrak{t}) = t,\ P_5(\mathfrak{t}) = \frac{\mathfrak{t}^{11/3}}{\mathfrak{t}^2+6}, \\ Q_1(\mathfrak{t}) = \frac{\mathfrak{t}^{2/7}}{3(\mathfrak{t}+4)},\ Q_2(\mathfrak{t}) = \frac{(\mathfrak{t}+3)e^{\mathfrak{t}+1}}{2(\mathfrak{t}^2+3)},\ Q_3(\mathfrak{t}) = \frac{1}{(\mathfrak{t}+1)^2},\ Q_4(\mathfrak{t}) = \frac{\mathfrak{t}}{\mathfrak{t}^3+1},\ Q_5(\mathfrak{t}) = \frac{\mathfrak{t}^{6/5}}{\mathfrak{t}+2}, \end{array} \tag{95}$$

for all $\mathfrak{t} \in [0,1]$.

The functions P_i and $Q_i,\ i = 1,\ldots,5$, are continuous and satisfy condition (67). In addition, we find $p_1^* \approx 1.84726402$, $p_2^* = 0.1$, $p_3^* \approx 0.90609394$, $p_4^* = 1$, $q_1^* \approx 0.06666667$, $q_2^* \approx 3.69452805$, $q_3^* = 1$, $q_4^* \approx 0.52913368$, $\Psi_1 \approx 0.48811984$, $\Psi_2 \approx 0.16566027$, and $\Psi_0 \approx 0.4881 < 1$; that is, assumption (68) is satisfied. Therefore, by Theorem 3, we conclude that problem (81),(82) with the nonlinearities in (93) has at least one positive solution $(\mathfrak{u}(\mathfrak{t}), \mathfrak{v}(\mathfrak{t})),\ \mathfrak{t} \in [0,1]$.

Example 4. *Let the functions*

$$\begin{array}{l} \mathfrak{f}(\mathfrak{t},u,v,x,y) = \dfrac{2(\mathfrak{t}+1)^3e^{-5\mathfrak{t}+1}}{3}\sqrt{u^2+4} + \dfrac{(\mathfrak{t}-2)^2}{2}e^{3\mathfrak{t}-2}\cos^2(v+1) \\ \quad + \dfrac{\mathfrak{t}^{15/4}}{\mathfrak{t}^2+2} + \dfrac{\mathfrak{t}x}{4(\mathfrak{t}+1)} + \dfrac{\mathfrak{t}+1}{\mathfrak{t}^2+3}\arctan y, \\ \mathfrak{g}(\mathfrak{t},u,v,x,y) = \dfrac{\mathfrak{t}^3e^{-4\mathfrak{t}+1}}{2}\arctan u + \dfrac{e^{2\mathfrak{t}}}{3(\mathfrak{t}+2)^4}\sqrt{v^2+1} + \dfrac{\mathfrak{t}^{9/8}}{\mathfrak{t}+3} \\ \quad + \dfrac{\mathfrak{t}x}{10(x^2+1)} + \dfrac{e^t}{9}\sin^2 y, \end{array} \tag{96}$$

for all $\mathfrak{t} \in [0,1]$, $u, v, x, y \geq 0$. *We obtain the following inequalities:*

$$\begin{aligned} &|\mathfrak{f}(\mathfrak{t}, u_1, v_1, x_1, y_1) - \mathfrak{f}(\mathfrak{t}, u_2, v_2, x_2, y_2)| \\ &\leq H_1(\mathfrak{t})|u_1 - u_2| + H_2(\mathfrak{t})|v_1 - v_2| + H_3(\mathfrak{t})|x_1 - x_2| + H_4(\mathfrak{t})|y_1 - y_2|, \\ &|\mathfrak{g}(\mathfrak{t}, u_1, v_1, x_1, y_1) - \mathfrak{g}(\mathfrak{t}, u_2, v_2, x_2, y_2)| \\ &\leq K_1(\mathfrak{t})|u_1 - u_2| + K_2(\mathfrak{t})|v_1 - v_2| + K_3(\mathfrak{t})|x_1 - x_2| + K_4(\mathfrak{t})|y_1 - y_2|, \end{aligned} \tag{97}$$

for all $\mathfrak{t} \in [0,1]$, $u_i, v_i, x_i, y_i \geq 0$, $i = 1, 2$, *where*

$$\begin{aligned} &H_1(\mathfrak{t}) = \tfrac{2(\mathfrak{t}+1)^3 e^{-5\mathfrak{t}+1}}{3},\ H_2(\mathfrak{t}) = (\mathfrak{t}-2)^2 e^{3\mathfrak{t}-2},\ H_3(\mathfrak{t}) = \tfrac{\mathfrak{t}}{4(\mathfrak{t}+1)},\ H_4(\mathfrak{t}) = \tfrac{\mathfrak{t}+1}{\mathfrak{t}^2+3}, \\ &K_1(\mathfrak{t}) = \tfrac{\mathfrak{t}^3 e^{-4\mathfrak{t}+1}}{2},\ K_2(\mathfrak{t}) = \tfrac{e^{2\mathfrak{t}}}{3(\mathfrak{t}+2)^4},\ K_3(\mathfrak{t}) = \tfrac{\mathfrak{t}}{10},\ K_4(\mathfrak{t}) = \tfrac{2e^{\mathfrak{t}}}{9}, \end{aligned} \tag{98}$$

for all $\mathfrak{t} \in [0,1]$.

The functions H_i and K_i, $i = 1, \ldots, 4$ are continuous and satisfy condition (74). In addition, we find $h_1^* \approx 1.81218788$, $h_2^* \approx 2.71828183$, $h_3^* = 0.125$, $h_4^* = 0.5$, $k_1^* \approx 0.02854728$, $k_2^* \approx 0.03040764$, $k_3^* = 0.1$, $k_4^* \approx 0.60406263$, $\Phi_1 \approx 0.47613656$, $\Phi_2 \approx 0.73924549$, and $\Phi_0 \approx 0.7392 < 1$; that is, assumption (75) is verified. Then, by Theorem 4, we deduce that problem (81),(82) with the functions in (96) has a unique positive solution $(\mathfrak{u}^*(\mathfrak{t}), \mathfrak{v}^*(\mathfrak{t}))$, $\mathfrak{t} \in [0,1]$.

5. Conclusions

In this paper, we explore the existence, uniqueness, and multiplicity of positive solutions for a system of fractional q-difference equations (Equation (1)). These equations involve fractional q-integrals and are subject to coupled multi-point boundary conditions (2). These boundary conditions encompass q-derivatives and fractional q-derivatives of unknown functions with varying orders. Initially, our focus was on investigating the linear boundary value problem related to (1),(2), along with studying the associated Green functions and their properties. Subsequently, we reformulated our problem equivalently to a system of fractional q-integral equations (Equation (49)). We associated these equations with an operator, and the fixed points of this operator correspond to the positive solutions of (49). Our primary results hinge on the application of the Guo–Krasnosel'skii fixed-point theorem (in Theorem 1), the Leggett–Williams fixed-point theorem (for Theorem 2), the Schauder fixed-point theorem (in the case of Theorem 3), and the Banach contraction mapping principle (in the context of Theorem 4). In the second-to-last section of the paper, we provide several examples to elucidate and illustrate the implications of our results.

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