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Article

New Applications of Fractional q-Calculus Operator for a New Subclass of q-Starlike Functions Related with the Cardioid Domain

Mohammad Faisal Khan [1,*] and Mohammed AbaOud [2]

[1] Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia
[2] Department of Mathematics and Statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11564, Saudi Arabia; maabaoud@imamu.edu.sa
* Correspondence: f.khan@seu.edu.sa

Abstract: Recently, a number of researchers from different fields have taken a keen interest in the domain of fractional q-calculus on the basis of fractional integrals and derivative operators. This has been used in various scientific research and technology fields, including optics, mathematical biology, plasma physics, electromagnetic theory, and many more. This article explores some mathematical applications of the fractional q-differential and integral operator in the field of geometric function theory. By using the linear multiplier fractional q-differintegral operator $D^m_{q,\lambda}(\rho,\sigma)$ and subordination, we define and develop a collection of q-starlike functions that are linked to the cardioid domain. This study also investigates sharp inequality problems like initial coefficient bounds, the Fekete–Szego problems, and the coefficient inequalities for a new class of q-starlike functions in the open unit disc $\mathcal{U}$. Furthermore, we analyze novel findings with respect to the inverse function $\left(\mu^{-1}\right)$ within the class of q-starlike functions in $\mathcal{U}$. The findings in this paper are easy to understand and show a connection between present and past studies.

Keywords: analytic functions; q-calculus; q-starlike functions; fractional q-calculus; cardioid domain; subordination; fractional q-differential and integral operator

MSC: 11B65; 47B38; 05A30; 30C45

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1. Introduction and Definitions

Let $\mathcal{A}$ be the set of all analytic functions $\mu(z)$ in the open unit disc

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\},$$

that are normalized with the following conditions

$$\mu'(0) = 1 \text{ and } \mu(0) = 0.$$

For each μ in $\mathcal{A}$, the series expansion is as follows:

$$\mu(z) = z + \sum_{n=2}^{\infty} d_n z^n. \tag{1}$$

The letter $\mathcal{S}$ denotes the subset of $\mathcal{A}$ that consists of univalent functions in $\mathcal{U}$.

In 1851, the study of function theory was initiated when Bieberbach [1] studied the coefficient conjecture in 1916, and this topic initially became an attractive field for future research. De Branges [2] verified this concept in 1985. Between 1916 and 1985, a large number of prominent scientists worked on verifying or debunking the Bieberbach hypothesis. It is crucial to understand the theory behind analytic and univalent functions and how they estimate the development of functions within their respective domains. An instance of a highly consequential and pragmatic functional inequality is the Fekete–Szegö inequality. In 1933, Fekete and Szegö found the Fekete–Szego inequality [3]. The inequality being discussed specifically relates to the coefficients of univalent functions and is directly linked to the Bieberbach conjecture. The problem of estimating similar properties for different types of functions is often referred to as the Fekete–Szegö problem. Researchers have shown that maximizing the non-linear function $\left|d_3 - \delta d_2^2\right|$ and other types of univalent functions can yield various results. Fekete–Szegö problems are the name given to this type of problem. For $\mu \in \mathcal{A}$, Fekete and Szegö [3] presents the example of a sharp Fekete–Szegö type inequality below:

$$\left|d_3 - \delta d_2^2\right| \leq \begin{cases} 3 - 4\delta & \text{if} \quad \delta \leq 0, \\ 1 + 2\exp\left(\frac{2\delta}{\delta - 1}\right) & \text{if} \quad 0 \leq \delta < 1, \\ 4\delta - 3 & \text{if} \quad \delta \geq 1. \end{cases}$$

Two analytic functions, μ_1 and μ_2, subordinate to each other, can be expressed as:

$$\mu_1(z) \prec \mu_2(z),\ z \in \mathcal{U},$$

if there is a function w_0 that satisfies the conditions

$$|w_0(z)| < 1,\ w_0(0) = 0$$

and

$$\mu_1(z) = \mu_2(w_0(z)),\ z \in \mathcal{U}.$$

In addition, if the function μ_2 is univalent in the region $\mathcal{U}$, then

$$\mu_1(0) = \mu_2(0) \text{ and } \mu_1(\mathcal{U}) \subset \mu_2(\mathcal{U}).$$

An analytic function $h \in \mathcal{P}$ in $\mathcal{U}$ with $h(0) = 1$ is said to be a Carathéodory function if it satisfies

$$Re(h(z)) > 0.$$

The starlike functions, represented by $\mathcal{S}^*$, are the primary and most important subclass of the set $\mathcal{S}$, and there is a function $\mu \in \mathcal{S}^*$ if $\mu \in \mathcal{A}$, which satisfies

$$Re\left(\frac{z\mu'(z)}{\mu(z)}\right) > 0,\ z \in \mathcal{U}.$$

Alternatively, from the perspective of subordination, it can be written as:

$$\mathcal{S}^* = \left\{\mu \in \mathcal{A} : \frac{z\mu'(z)}{\mu(z)} \prec \frac{1+z}{1-z}\right\}.$$

The class $\mathcal{S}^*(\varphi)$ is described in the perspective of subordination [4]:

$$\mu \in \mathcal{S}^*(\varphi) \Longleftrightarrow \frac{z\mu'(z)}{\mu(z)} \prec \varphi(z),$$

where

$$\varphi(0) = 1, Re(\varphi(z)) > 0 \text{ and } \varphi'(0) > 0.$$

The function φ maps $\mathcal{U}$ onto an area that is starlike with respect to the point 1 and symmetric with respect to the real axis. A recent study investigated several subfamilies within the class $\mathcal{A}$, specifically as instances of the set $\mathcal{S}^*(\varphi)$. Furthermore, the authors of the article [5] studied the class $\mathcal{S}_L^*$. Janowski investigated the starlike Janowski function in [6]. Cho et al. investigated the set $\mathcal{S}_{\sin}^*$ in [7]. The class $\mathcal{S}_{\tan}^*$ was studied in [8]. Finally, the class $\mathcal{S}^*(e^z)$ was considered in [9]. For more studies, see the following articles [10–21].

Every univalent function μ has an inverse μ^{-1} provided on a disc with a radius $|w| \le \frac{1}{4} \le r_0(\mu)$ and a series of μ^{-1} is

$$\mu^{-1}(w) = w + B_2 w^2 + B_3 w^3 + \dots, \tag{2}$$

where

$$B_2 = -d_2, \text{ and } B_3 = \left(2d_2^2 - d_3\right). \tag{3}$$

By implementing the definition of subordination, several subclasses of analytic functions have been developed, taking into account the geometric perspective of their image domains, for example, the right half plane [22], the circular disc [23], the leaf-like domain [24], and many others [25–32]. For the shell-like curve form, the function

$$h(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2}, \tau = \frac{1 - \sqrt{5}}{2} \tag{4}$$

is necessary. The series of $h(z)$ is

$$h(z) = 1 + \sum_{n=1}^{\infty} (u_{n+1} + u_{n-1}) z^n,$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}$$

and u_n generates a series of coefficient of Fibonacci numbers.

If we examine the image of a unit circle through the function h, we obtain the conchoid of Maclaurin, which is

$$h\left(e^{i\varphi}\right) = \frac{\sqrt{5}}{2(3 - 2\cos\varphi)} + \frac{i \sin\varphi(4\cos\varphi - 1)}{2(3 - 2\cos\varphi)(1 + \cos\varphi)}, 0 \le \varphi < 2\pi.$$

Malik et al. [33] presented and analyzed a new geometric structure in the image domain, obtaining ideas from a conceptualization of shell-like shapes and circular disks. For more details, refer to the reference [33].

Definition 1 ([33])**.** *For $-1 < R < K \le 1$, and $\tau = \frac{1-\sqrt{5}}{2}$, $z \in \mathcal{U}$. Let $CP(K, R)$ be the class of functions h that are defined by the subordination relation*

$$h(z) \prec \widetilde{h}(K, R, z),$$

where $\widetilde{h}(K, R, z)$ is given by

$$\widetilde{h}(K, R, z) = \frac{2K\tau^2 z^2 + (K-1)\tau z + 2}{2R\tau^2 z^2 + (R-1)\tau z + 2}. \tag{5}$$

In terms of geometry, describing the function $\widetilde{h}(K, R, z)$ might be an easy way to obtain a deeper understanding of the class $CP(K, R)$. If

$$I_{\widetilde{h}}\left(K, R; e^{i\theta}\right) = v$$

and

$$R_{\widetilde{h}}\left(K, R; e^{i\theta}\right) = u$$

then the image $\widetilde{h}(K,R,e^{i\theta})$ of the unit circle is a curve that looks like a cardioid and is given by

$$\begin{aligned} v &= \frac{(K-R)(\tau-\tau^3)\sin\theta+2\tau^2\sin 2\theta}{4(R-1)(\tau+R\tau^3)\cos\theta+8R\tau^2\cos 2\theta+4+(R-1)^2\tau^2+4R^2\tau^4}, \\ u &= \frac{4+(K-1)(R-1)\tau^2+4KR\tau^4+2\lambda\cos\theta+4(K+R)\tau^2\cos 2\theta}{4(R-1)(\tau+R\tau^3)\cos\theta+8R\tau^2\cos 2\theta+4+(R-1)^2\tau^2+4R^2\tau^4}, \end{aligned} \tag{6}$$

where

$$\lambda = (K+R-2)\tau + (2KR-K-R)\tau^3.$$

We note that

$$\widetilde{h}(K,R,1) = \frac{4(R-K)\sqrt{5}+KR+9(K+R)+1}{R^2+18R+1}$$

and

$$\widetilde{h}(K,R,0) = 1$$

The cusp of the cardioid-like curve is determined by $\gamma(K,R)$:

$$\begin{aligned} \gamma(K,R) &= \widetilde{h}\left(K,R;e^{\pm i\arccos\left(\frac{1}{4}\right)}\right) \\ &= \frac{2+(K-R)\sqrt{5}+2KR-3(K+R)}{2R^2-6R+2}. \end{aligned}$$

When examining the open unit disc $\mathcal{U}$ as a collection of concentric circles centered at the origin, each inner circle image resembles a nested cardioid-shaped curve. The function $\widetilde{h}(K,R,z)$ projects $\mathcal{U}$ onto a region in the shape of a cardioid. Therefore, $\mathcal{U}$ represents a cardioid domain.

Engineers and mathematicians have widely studied the q-fractional differential equations based on q-calculus in the past few years. Jackson [34,35] first proposed the concept of q-calculus (also known as quantum calculus) in 1908. He introduced the notion of the q-calculus operator and provided the formulation of the q-difference operator D_q. Furthermore, Ismail et al. [36] first used D_q in the field of GFT and proposed a novel category of q-starlike functions in the set $\mathcal{U}$. Srivastava's work in [37] introduced the fundamental (or q-) hypergeometric functions to GFT, which proved to be important uses of q-calculus in this field. For more information, refer to the book chapter mentioned in [37]. For recent studies, see the following published articles [38–42].

Derivatives and integrals of real or complex order have recently emerged as a potential new path for the analysis of actual challenges in applied sciences and mathematical modeling, with applications in areas as diverse as geometry and physics. Two examples of this sort are the human liver, which is modeled using Caputo–Fabrizio fractional derivatives with the exponential kernel, as explained in [43], and the study of the dynamics of dengue virus transmission [44]. A new fractional derivative operator that goes beyond the single kernel was studied in [45] for its potential use in heat transfer models. Fractional operators are suggested in [46] to address various real-life problems. This article focuses on the analysis of the linear multiplier fractional q-differintegral operator, denoted as $D^m_{q,\lambda}(\rho,\sigma)$.

Now, the basic definitions and notions of q-calculus and fractional q-calculus need to be defined in order to understand the main idea of this article.

Definition 2. *For* $(0<q<1)$ *and* $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. *The q-number* $[n]_q$ *is defined by*

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q} \\ &= \sum_{j=0}^{n-1} q^j. \end{aligned} \tag{7}$$

In particular, for $n = 0$, *we have*

$$[0]_q = 0,$$

when $q \to 1^-$

$$[n]_q = n.$$

Definition 3 ([46])**.** *For* $v \in \mathbb{C}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *the q-shifted factorial* $(v, q)_n$ *is a mathematical function that is defined as follows:*

$$(v,q)_n = \prod_{j=0}^{n-1}\left(1 - vq^j\right), (n \in \mathbb{N})$$

and

$$(v,q)_0 = 1.$$

In terms of the q-Gamma function

$$(q^v, q)_n = \frac{(1-q)^n \Gamma_q(v+n)}{\Gamma_q(v)}, (n \in \mathbb{N}_0).$$

Definition 4. *The q-Gamma function is formally defined as*

$$\Gamma_q(z) = \frac{(q,q)_\infty (1-q)^{1-z}}{(q^z, q)_\infty}, |q| < 1.$$

We note that

$$(v,q)_\infty = \prod_{j=0}^{\infty}\left(1 - vq^j\right), |q| < 1.$$

Definition 5 ([34])**.** *The q-difference operator is a mathematical operator that is defined for analytic functions as:*

$$D_q\mu(z) = \frac{\mu(qz) - \mu(z)}{z(q-1)}, z \in \mathcal{U}, 0 < q < 1$$

and for $n \in \mathbb{N}$

$$D_q \sum_{n=1}^{\infty} d_n z^n = \sum_{n=1}^{\infty} [n]_q d_n z^{n-1} \text{ and } D_q(z^n) = [n]_q z^{n-1}.$$

Definition 6. *Jackson [34,35] introduced the q-integral for* $\mu \in \mathcal{A}$ *as:*

$$\int_0^z \mu(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n \mu(zq^n).$$

Note that $q \to 1^-$, *then*

$$\int_0^z \mu(t) d_q t = \int_0^z \mu(t) dt,$$

where $\int_0^z \mu(t)dt$ *is the ordinary integral.*

The following is the definition of the fractional q-integral operator:

Definition 7 ([47] page 57, Definition 1, see also [48], page 257, Definition 1.1). *The fractional q-integral operator* $J^{\lambda}_{q,z}$ *of a function* $\mu(z)$ *of order* λ *is precisely defined as:*

$$J^{\lambda}_{q,z}\mu(z) = \frac{1}{\Gamma_q(\lambda)} \int_0^z (z - tq)_{\lambda-1}\mu(t)d_q(t), \lambda > 0, \tag{8}$$

where $\mu(z)$ *is an analytic function in an area of the z-plane that is simply linked and includes the origin. The q-binomial function* $(z - tq)_{\lambda-1}$ *is mathematically defined as follows:*

$$\begin{aligned}(z - tq)_{\lambda-1} &= \prod_{n=0}^{\infty}\left(\frac{1-\left(\frac{qt}{z}\right)q^n}{1-\left(\frac{qt}{z}\right)q^{\lambda+n-1}}\right) \\ &= z^{\lambda-1}{}_1\Theta_0\left(q^{-\lambda+1}, -, q; \frac{tq^{\lambda}}{z}\right).\end{aligned}$$

The series of ${}_1\Theta_0$ *is defined as:*

$${}_1\Theta_0(d, -, q; z) = 1 + \sum_{n=1}^{\infty} \frac{(d, q)_n}{(q, q)_n} z^n, (|q| < 1, |z| < 1).$$

The series ${}_1\Phi_0(d, -, q, z)$ *is single-valued when* $|\arg(z)| < \pi$ *and* $|z| < 1$, *(for further details, see [49], pages 104–106) and so the function* $(z - tq)_{\lambda-1}$ *in (8) is single-valued when* $\left|\arg(-tq^{\lambda}/z)\right| < \pi, \left|(tq^{\lambda}/z)\right| < \pi$ *and* $|\arg(z)| < \pi$.

Definition 8 (see [47], page 58, Definition 2, see also [48], page 257, Definition 1.2). *The fractional q-derivative operator* $D^{\lambda}_{q,z}$ *of order* λ *is defined as:*

$$\begin{aligned}D^{\lambda}_{q,z}\mu(z) &= D_{q,z}\left(J^{1-\lambda}_{q,z}\mu(z)\right) \\ &= \frac{1}{\Gamma_q(1-\lambda)} D_q \int_0^z (z - tq)_{-\lambda}\mu(t)d_q(t), (0 \le \lambda < 1),\end{aligned}$$

where $\mu(z)$ *is appropriately constrained and the multiplicity of* $(z - tq)_{-\lambda}$ *is eliminated, as defined in Definition 7.*

Definition 9 ([47]). *The extended fractional q-derivative* $D^{\lambda}_q\mu(z) : \mathcal{A} \to \mathcal{A}$ *of order* λ *is precisely defined as:*

$$D^{\lambda}_q\mu(z) = D^m_q J^{m-\lambda}_{q,z}\mu(z),$$

where $m \in \mathbb{Z}^+, -1 \le \lambda < n$, *and* $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

Purohit and Raina (see [47], page 59) investigated the fractional q-differintegral operator $\Omega^{\lambda}_{q,z}$:

Definition 10 ([47]). *The fractional q-differintegral operator* $\Omega^{\lambda}_{q,z}\mu(z) : \mathcal{A} \to \mathcal{A}$ *is defined as:*

$$\begin{aligned}\Omega^{\lambda}_{q,z}\mu(z) &= \frac{\Gamma_q(2-\lambda)}{\Gamma_q(2)} z^{\lambda-1} D^{\lambda}_{q,z}\mu(z) \\ &= 1 + \sum_{n=2}^{\infty} \frac{\Gamma_q(2-\lambda)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\lambda)} d_n z^{n-1},\end{aligned}$$

where $\lambda < 2; 0 < q < 1$.

Remark 1 ([50]). *The operator* $\Upsilon^{\lambda}_{q,z}\mu(z) : \mathcal{A} \to \mathcal{A}$ *is defined as follows:*

$$\Upsilon^{\lambda}_{q,z}\mu(z) = z\Omega^{\lambda}_{q,z}\mu(z) \tag{9}$$

and for $\lambda = 0$, *then*

$$\Upsilon^{\lambda}_{q,z}\mu(z) = \Upsilon^{0}_{q,z}\mu(z) = \mu(z).$$

Definition 11 ([50]). *The linear multiplier fractional q-differintegral operator* $D^{m}_{q,\lambda}(\rho,\sigma)\mu(z) : \mathcal{A} \to \mathcal{A}$ *is defined as follows:*

$$\begin{aligned} D^{0}_{q,\lambda}(\rho,\sigma)\mu(z) &= \mu(z), \\ D^{1}_{q,\lambda}(\rho,\sigma)\mu(z) &= (1-\tfrac{\rho}{\sigma+1})\Upsilon^{\lambda}_{q,z}\mu(z) + \tfrac{\rho}{\sigma+1} zD_q\left(\Upsilon^{\lambda}_{q,z}\mu(z)\right) \\ &= D_{q,\lambda}(\rho,\sigma)\mu(z) \\ &= z + \sum_{n=2}^{\infty}\left(\tfrac{\Gamma_q(2-\lambda)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\lambda)}\right)\left(\tfrac{(\sigma+1)+\rho\left([n]_q-1\right)}{\sigma+1}\right)d_n z^n, \\ &\ \vdots \\ D^{m}_{q,\lambda}(\rho,\sigma)\mu(z) &= D_{q,\lambda}(\rho,\sigma)\left(D^{m-1}_{q,\lambda}(\rho,\sigma)\mu(z)\right), \end{aligned} \tag{10}$$

where

$$\rho \geq 0, \sigma > -1, m \in \mathbb{N}_0, \lambda < 2 \text{ and } 0 < q < 1.$$

From (1) and (10), we have

$$D^{m}_{q,\lambda}(\rho,\sigma)\mu(z) = z + \sum_{n=2}^{\infty}\Lambda^{m,\rho,\sigma}_{q,\lambda}(n)d_n z^n \tag{11}$$

where

$$\Lambda^{m,\rho,\sigma}_{q,\lambda}(n) = \left\{\left(\frac{\Gamma_q(2-\lambda)\Gamma_q(n+1)}{\Gamma_q(2)\Gamma_q(n+1-\lambda)}\right)\left(\frac{(\sigma+1)+\rho\left([n]_q-1\right)}{\sigma+1}\right)\right\}^m.$$

By utilizing the Equations (9) and (11), we can express $D^{m}_{q,\lambda}(\rho,\sigma)\mu(z)$ *in terms of convolution:*

$$D^{m}_{q,\lambda}(\rho,\sigma)\mu(z) = \left[\left(\Upsilon^{\lambda}_{q,z}\mu(z) * \Psi^{q}_{\rho,\sigma}(z)\right) * \ldots * \left(\Upsilon^{\lambda}_{q,z}\mu(z) * \Psi^{q}_{\rho,\sigma}(z)\right)\right] * \mu(z), \text{m-time}$$

where

$$\Psi^{q}_{\rho,\sigma}(z) = \frac{z-(1-\rho/(1+\sigma))qz^2}{(1-z)(1-qz)}.$$

Remark 2. *For* $\sigma = 0$, *then* $D^{m}_{q,\lambda}(\rho,\sigma) = D^{m}_{q,\lambda}(\rho)$, *as studied by Abelman et al. [48].*

Remark 3. *For* $\sigma = 0$ *and* $\rho = 0$, $D^{m}_{q,\lambda}(\rho,\sigma) = D^{m}_{q,\lambda}$, *as studied by Aouf et al. [51].*

Remark 4. *For* $\sigma = 0$, $\lambda = 0$ *and* $\rho = 1$, $D^{m}_{q,\lambda}(\rho,\sigma) = S^{m}_{q}$, *as studied by Govindaraj and Sivasubramanian [52].*

Taking inspiration from the article [33], we develop a new class of q-starlike functions in the cardioid domain while taking into account the linear multiplier fractional q-differintegral operator $D^{m}_{q,\lambda}(\rho,\sigma)$ and the technique of subordination.

Definition 12. *We say that a function* μ *of the form (1) belongs to the class* $\mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$ *if it satisfies*

$$\frac{zD_q\left(D^{m}_{q,\lambda}(\rho,\sigma)\mu(z)\right)}{D^{m}_{q,\lambda}(\rho,\sigma)\mu(z)} \prec \widetilde{h}(K,R;z),$$

where $\widetilde{h}(K,R;z)$ *is given in* (5) *and*

$$\rho \geq 0, \sigma > -1, -1 < R < K \leq 1$$

and

$$m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda < 2 \text{ and } 0 < q < 1.$$

Remark 5. *For* $q \to 1^-$, $\sigma = 0$, $\lambda = 0$, $\rho = 1$ *and* $m = 0$ *in Definition 12,* $\mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R) = \mathcal{S}^*(K,R)$*, as investigated by Malik et al. in [53], and this is defined as:*

$$\frac{z\mu'(z)}{\mu(z)} \prec \widetilde{h}(K,R;z).$$

Remark 6. *For* $\rho = 1$, $R = -1$, $K = 1$, $\sigma = 0$, $m = 0$, $q \to 1^-$ *and* $\lambda = 0$ *in Definition 12,* $\mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R) = SL$*, as studied in [32].*

2. Set of Lemmas

Our main results are proved by using the following lemmas:

Lemma 1 ([33])**.** *Let* $-1 < R < K \leq 1$, $\tau = \frac{1-\sqrt{5}}{2}$*. If* $h(z) \prec \widetilde{h}(K,R;z)$*, then*

$$Re(h(z)) > \alpha,$$

where

$$\begin{aligned} \alpha &= \frac{2(2KR-K-R)\tau^3+2(K+R-2)\tau+16(K+R)\tau^2\eta}{4(R-1)(\tau+R\tau^3)+32R\tau^2\eta}, \\ \eta &= \frac{4+\tau^2-R^2\tau^2-4R^2\tau^4-(1-R\tau^2)\chi(R)}{4\tau(1+R^2\tau^2)}, \\ \chi(R) &= \sqrt{5(2R\tau^2+2-(R-1)\tau)(2R\tau^2+2+(R-1)\tau)} \end{aligned}$$

and $\widetilde{h}(K,R;z)$ *is defined by* (5)*.*

Remark 7. *For the disc* $|z| < \tau^2$*, the function* $\widetilde{h}(K,R;z)$ *is univalent.*

Lemma 2 ([33])**.** *If* $\widetilde{h}(K,R;z) = 1 + \sum\limits_{n=1}^{\infty} \widetilde{h}_n z^n$ *and* $\widetilde{h}(K,R;z)$ *is defined in* (5)*,*

$$\widetilde{h}_n = \begin{cases} (K-R)\frac{\tau}{2} & \text{if } n = 1, \\ (K-R)(5-R)\frac{\tau^2}{2^2} & \text{if } n = 2, \\ \frac{1-R}{2}\tau h_{n-1} - R\tau^2 h_{n-2} & \text{if } n = 3,4,5,\ldots, \end{cases} \tag{12}$$

where $-1 < R < K \leq 1$.

Lemma 3 ([33])**.** *Let* $h(z) \prec \widetilde{h}(K,R;z)$ *and* $h(z)$ *be expressed in the form* $h(z) = 1 + \sum\limits_{n=1}^{\infty} h_n z^n$*. Then, for* $v \in \mathbb{C}$

$$\left|h_2 - vh_1^2\right| \leq \frac{|\tau|(K-R)}{4} \max\{2, |\tau(R-5+v(K-R))|\}, \tag{13}$$

where $\widetilde{h}(K,R;z)$*, as defined by* (5)*.*

Lemma 4 ([54]). *Let $h \in \mathcal{P}$ and*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Then,

$$\begin{aligned} \left|c_2 - \tfrac{v}{2}c_1^2\right| &\leq \max\{2, 2|v-1|\} \\ &= \begin{Bmatrix} 2 & \text{if } 0 \leq v \leq 2, \\ 2|v-1|, & \textit{elsewhere} \end{Bmatrix} \end{aligned} \tag{14}$$

and

$$|c_n| \leq 2, \mathit{if} n \geq 1. \tag{15}$$

Lemma 5 ([55]). *Let $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be convex in $\mathcal{U}$ and let $\mu(z) = \sum_{n=1}^{\infty} d_n z^n$ be analytic in $\mathcal{U}$. If*

$$\mu(z) \prec g(z),$$

then

$$|d_n| < |b_1|, n \geq 1$$

In this particular part, we investigate coefficient bounds, Fekete–Szego problems, and coefficient inequalities for the functions $f \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$. Moreover, in this investigation, we discover identical problems while working with inverse functions.

3. Main Results

Theorem 1. *For $-1 \leq R < K \leq 1$, if a function μ is of the form (1) and belongs to the class $\mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$, then*

$$\begin{aligned} |d_2| &\leq \frac{(K-R)|\tau|}{2\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}, \\ |d_3| &\leq \frac{(K-R)|\tau|^2}{4\left([3]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left(-\frac{K-R}{[2]_q - 1} + 5 - R\right). \end{aligned}$$

The result is sharp for the function given in (24).

Proof. Let $\mu \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$. Then,

$$\frac{zD_q\left(D^m_{q,\lambda}(\rho,\sigma)\mu(z)\right)}{D^m_{q,\lambda}(\rho,\sigma)\mu(z)} \prec \widetilde{h}(K,R;z), \tag{16}$$

where

$$\widetilde{h}(K,R,z) = \frac{2K\tau^2 z^2 + (K-1)\tau z + 2}{2R\tau^2 z^2 + (R-1)\tau z + 2}.$$

Using the definition of subordination,

$$\frac{zD_q\left(D^m_{q,\lambda}(\rho,\sigma)\mu(z)\right)}{D^m_{q,\lambda}(\rho,\sigma)\mu(z)} = \widetilde{h}(K,R;u(z)), \tag{17}$$

where u is the Schwarz function along with the conditions

$$|u(z)| < 1 \text{ and } u(0) = 0.$$

Also, consider the function

$$\begin{aligned}
&\widetilde{h}(K,R,z)\\
&= \tfrac{2K\tau^2z^2+(K-1)\tau z+2}{2R\tau^2z^2+(R-1)\tau z+2}\\
&= \tfrac{K\tau^2z^2+\frac{(K-1)}{2}\tau z+1}{R\tau^2z^2+\frac{(R-1)}{2}\tau z+1}\\
&= \left(K\tau^2z^2+\frac{(K-1)}{2}\tau z+1\right)\\
&\times\left(1+\tfrac{1}{2}(1-R)\tau z+\left(\tfrac{R^2-6R+1}{4}\right)\tau^2z^2+\ldots\right)\\
&= 1+\tfrac{1}{2}(K-R)\tau z+\tfrac{1}{4}(K-R)(5-R)\tau^2z^2+\ldots.
\end{aligned} \tag{18}$$

Let

$$\begin{aligned}
u(z) &= (h(z)-1)(h(z)+1)^{-1}\\
&= \tfrac{1}{2}c_1z+\tfrac{1}{2}\left(c_2-\tfrac{1}{2}c_1^2\right)z^2+\tfrac{1}{2}\left(c_3-c_1c_2+\tfrac{1}{4}c_1^3\right)z^3+\cdots.
\end{aligned} \tag{19}$$

Now, we have

$$\begin{aligned}
&\widetilde{h}(K,R;u(z))\\
&= 1+\tfrac{1}{4}(K-R)\tau c_1z+\left(\tfrac{1}{4}(K-R)\tau\left(c_2-\tfrac{1}{2}c_1^2\right)\right.\\
&\left.+\tfrac{(K-R)(5-R)\tau^2c_1^2}{16}\right)z^2+\ldots.
\end{aligned} \tag{20}$$

Since $\mu\in\mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$,

$$\begin{aligned}
&\tfrac{zD_q\left(D^m_{q,\lambda}(\rho,\sigma)\mu(z)\right)}{D^m_{q,\lambda}(\rho,\sigma)\mu(z)}\\
&= 1+\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\left([2]_q-1\right)d_2z\\
&+\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)\left([3]_q-1\right)d_3-\left([2]_q-1\right)\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2d_2^2\right)z^2+\ldots.
\end{aligned} \tag{21}$$

By equating the coefficients from (20) and (21), we obtain

$$d_2=\frac{(K-R)\tau c_1}{4\left([2]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}. \tag{22}$$

By applying the modulus and $|c_n|\leq 2$, for $n\geq 1$, we obtain

$$|d_2|\leq\frac{(K-R)|\tau|}{2\left([2]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}.$$

By equating the coefficients of (20) and (21), we have

$$\begin{aligned}
d_3 &= \tfrac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left(c_2-\tfrac{1}{2}\left(1-\tfrac{(K-R)\tau}{2\left([2]_q-1\right)}-\tfrac{(5-R)\tau}{2}\right)c_1^2\right)\\
&= \tfrac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left\{c_2-\tfrac{v}{2}c_1^2\right\},
\end{aligned} \tag{23}$$

where

$$v=1-\frac{\tau}{2}\left((5-R)+\frac{(K-R)}{[2]_q-1}\right).$$

Thus, $v > 2$ for $K > R$. Therefore, considering the Lemma 4, we obtain the required result. Let a function $\mu_* : \mathcal{U} \to \mathbb{C}$ be defined as:

$$\begin{aligned}\mu_*(z) &= z + \frac{(K-R)\tau}{2\left([2]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)} z^2 \\ &\quad + \left(\frac{(K-R)\tau^2}{4\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}\left(5 - R + \frac{K-R}{[2]_q - 1}\right)\right) z^3 + \ldots.\end{aligned} \tag{24}$$

Then, it is clear that $\mu_*(z)$ satisfies the conditions of normalization and

$$\begin{aligned}\frac{zD_q\left(D_{q,\lambda}^m(\rho,\sigma)\mu(z)\right)}{D_{q,\lambda}^m(\rho,\sigma)\mu(z)} &= 1 + \frac{\tau(K-R)}{2} z + \frac{(5-R)(K-R)\tau^2}{4} z^2 + \ldots \\ &= \widetilde{h}(K, R; z).\end{aligned}$$

This shows that $\mu_* \in \mathcal{S}_{\lambda,q,m}^{*,\rho,\sigma}(K, R)$. Therefore, the result is sharp for the function μ_*. □

Example 1. *Let $R = -0.5$, $K = 0.5$, $q = 0.5$, $\lambda = 1$, $m = 1$, $\rho = 0.1$, and $\sigma = 0.2$ and using the definition of the q-number, we have*

$$[2]_{0.5} = 1.5, \text{ and } [3]_{0.5} = 1.75.$$

Therefore, by the above theorem, we obtain

$$\begin{aligned}|d_2| &\leq 0.05667, \\ |d_3| &\leq 0.2080,\end{aligned}$$

where

$$\begin{aligned}\Lambda_{0.5,1}^{1,0.1,0.2}(2) &= 2.083 \\ \Lambda_{0.5,1}^{1,0.1,0.2}(3) &= 4.5833.\end{aligned}$$

Taking $q \to 1^-$, $\sigma = 0$, $\lambda = 0$, $m = 0$ and $\rho = 1$ in Theorem 1, we obtain the well-established result stated below.

Corollary 1 ([56]). *Let $\mu \in \mathcal{S}^*(K, R)$, where μ is defined by (1), $-1 \leq R < K \leq 1$. Then,*

$$\begin{aligned}|d_2| &\leq \frac{(K-R)|\tau|}{2}, \\ |d_3| &\leq \frac{(K-R)|\tau|^2(K-2R+5)}{8}.\end{aligned}$$

Theorem 2. *Let $\mu \in \mathcal{S}_{\lambda,q,m}^{*,\rho,\sigma}(K, R)$, where μ is defined by (1). Then,*

$$\begin{aligned}&\left|d_3 - \delta d_2^2\right| \\ \leq\ &\frac{(K-R)|\tau|}{4\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)} \max\left\{2, \left|\tau\left(-\left(\frac{K-R}{[2]_q - 1} - (R-5)\right)\right.\right.\right. \\ &\left.\left.\left. + \left(\frac{\delta\left([3]_q - 1\right)(K-R)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}{\left(\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\right)^2\left([2]_q - 1\right)^2}\right)\right)\right|\right\}.\end{aligned}$$

This result is sharp for the function given in (26).

Proof. Since $\mu \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$,

$$\frac{zD_q\left(D^m_{q,\lambda}(\rho,\sigma)\mu(z)\right)}{D^m_{q,\lambda}(\rho,\sigma)\mu(z)} = \widetilde{h}(K,R;u(z)), z \in \mathcal{U},$$

Therefore,

$$\begin{aligned} & z + [2]_q\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)d_2z^2 + [3]_q\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)d_3z^3 + \dots \\ = & \left\{z + \Lambda^{m,\rho,\sigma}_{q,\lambda}(2)d_2z^2 + \Lambda^{m,\rho,\sigma}_{q,\lambda}(3)d_3z^3 + \dots\right\} \\ & \times\left\{1 + h_1z + h_2z^2 + \dots\right\}. \end{aligned}$$

We obtain, by equating the coefficients,

$$d_2 = \frac{h_1}{\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}, \text{ and } d_3 = \frac{\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)h_1d_2 + h_2}{\left([3]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}. \tag{25}$$

This implies that

$$\begin{aligned} & \left|d_3 - \delta d_2^2\right| \\ = & \frac{1}{\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left|h_2 - \frac{1}{[2]_q-1}\left(\delta\frac{\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}{\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2\left([2]_q-1\right)} - 1\right)h_1^2\right| \\ = & \frac{1}{\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left|h_2 - vh_1^2\right|, \end{aligned}$$

where

$$v = \frac{1}{[2]_q - 1}\left(\delta\frac{\left([3]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}{\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2\left([2]_q - 1\right)} - 1\right).$$

We obtain the desired outcome by using Lemma 3. The equality

$$\begin{aligned} \left|d_3 - \delta d_2^2\right| \quad = \quad & \frac{(K-R)|\tau|^2}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left|\left(\frac{K-R}{[2]_q-1} - R + 5\right) - \right. \\ & \left.\left(\frac{\delta\left([3]_q-1\right)(K-R)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}{\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2\left([2]_q-1\right)^2}\right)\right| \end{aligned}$$

holds for μ_*, as given in (24). Let $\mu_0 : \mathcal{U} \to \mathbb{C}$, which is defined as:

$$\mu_0(z) = z + \frac{(K-R)\tau}{4\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}z^3 + \dots. \tag{26}$$

Hence, it is clear that $\mu_0(z)$ satisfies the the conditions of normalization and

$$\frac{zD_q\left(D^m_{q,\lambda}(\rho,\sigma)\mu(z)\right)}{D^m_{q,\lambda}(\rho,\sigma)\mu(z)} = \widetilde{h}\left(K,R;z^2\right).$$

This shows that $\mu_0 \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$. Hence, the equality $\left|d_3 - \delta d_2^2\right| = \frac{(K-R)\tau}{4\left([2]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}$ holds for the function μ_0 defined in (26). □

The known result is proven in [56] when the parameters in the Theorem 1 are set to $q \to 1^-$, $\sigma = 0$, $\lambda = 0$, $\rho = 1$, and $m = 0$.

Corollary 2 ([56]). *Let $\mu \in \mathcal{S}^*(K, R)$. Then,*

$$\left|d_3 - \delta d_2^2\right| \le \frac{(K-R)|\tau|}{8} \max\{2, |\tau(-(K-2R+5) + 2(K-R)\delta)|\}.$$

The result is sharp.

Theorem 3. *If $\mu \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K, R)$, then*

$$|d_2| \le \frac{|\tau||K-R|}{2\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}$$

and

$$|d_n| \le \frac{|\tau||K-R|}{2\left([n]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(n)} \prod_{j=2}^{n-1}\left(1 + \frac{|\tau||K-R|}{2\left([j]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(j)}\right), \text{ for } n \ge 3.$$

Proof. Let $\mu \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K, R)$ and let

$$T(z) = \frac{zD_q\left(D^m_{q,\lambda}(\rho,\sigma)\mu(z)\right)}{D^m_{q,\lambda}(\rho,\sigma)\mu(z)}. \tag{27}$$

Then, according to the Definition 12, we may deduce

$$T(z) \prec \widetilde{h}(K, R; z).$$

Hence, by the Lemma 5, we obtain

$$\left|\frac{T^{(i)}(0)}{i!}\right| = |c_i| \le \left|\widetilde{h}_1\right| = \frac{|\tau|}{2}|K-R|, i \in \mathbb{N}, \tag{28}$$

where

$$T(z) = 1 + c_1 z + c_2 z^2 + \ldots.$$

Since $d_1 = 1$, from (27), we obtain

$$\begin{aligned}\left([n]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(n) d_n &= \{c_{n-1} + c_{n-2}d_2 + \ldots + c_1 d_{n-1}\} \\ &= \sum_{i=1}^{n-1} c_i d_{n-i}.\end{aligned} \tag{29}$$

It is obtained by substituting (28) into (29):

$$\left([n]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(n)|d_n| \le \frac{|\tau|}{2}|K-R| \sum_{i=1}^{n-1}|d_{n-i}|, n \in \mathbb{N}.$$

For $n = 2,3,4$, we have

$$\begin{aligned}|d_2| &\le \frac{|\tau||K-R|}{2\left([2]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)},\\ |d_3| &\le \frac{|\tau||K-R|}{2\left([3]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}(1+|d_2|)\\ &\le \frac{|\tau||K-R|}{2\left([3]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}\left(1+\frac{|\tau||K-R|}{2\left([2]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)}\right)\end{aligned}$$

and

$$\begin{aligned}|d_4| &\le \frac{|\tau||K-R|}{2\left([4]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(4)}(1+|d_2|+|d_3|)\\ &\le \frac{|K-R||\tau|}{2\left([4]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(4)}\left(1+\frac{|\tau||K-R|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\left([2]_q-1\right)}+\frac{|\tau||K-R|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)\left([3]_q-1\right)}\right.\\ &\quad\left.\left(1+\frac{|K-R||\tau|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\left([2]_q-1\right)}\right)\right)\\ &= \frac{|\tau||K-R|}{2\left([4]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(4)}\left(1+\frac{|\tau||K-R|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\left([2]_q-1\right)}+\frac{|\tau||K-R|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)\left([3]_q-1\right)}\right.\\ &\quad\left.+\frac{|\tau||K-R|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)\left([3]_q-1\right)}\frac{|\tau||K-R|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\left([2]_q-1\right)}\right)\\ &= \frac{|\tau||K-R|}{2\left([4]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(4)}\left(1+\frac{|\tau||K-R|}{2\left([3]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}\right)\\ &\quad\times\left(1+\frac{|\tau||K-R|}{2\left([2]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)}\right).\end{aligned}$$

Using the mathematical induction, we obtain

$$|d_n| \le \frac{|\tau||K-R|}{2\left([n]_q-1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(n)}\prod_{j=2}^{n-1}\left(1+\frac{|K-R||\tau|}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(j)\left([j]_q-1\right)}\right), \text{ for } n \ge 3.$$

Hence, our Theorem 3 is proved. □

3.1. Inverse Coefficients for the Function $\mu \in \mathcal{S}_{\lambda,q,m}^{*,\rho,\sigma}(K,R)$

Theorem 4. *Let $\mu \in \mathcal{S}_{\lambda,q,m}^{*,\rho,\sigma}(K,R)$, where μ is of the form (1), and μ^{-1} is of the form (2), $-1 \le R < K \le 1$. Then,*

$$|B_2| \le \frac{(K-R)\tau c_1}{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\left([2]_q-1\right)} \tag{30}$$

and

$$\begin{aligned}|B_3| &\le \frac{(K-R)|\tau|}{4\left([3]_q^m-1\right)}\max\left\{2,\left|\tau\left(\frac{(K-R)}{[2]_q-1}+(5-R)\right.\right.\right.\\ &\quad\left.\left.\left.-\frac{2\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)(K-R)\left([3]_q-1\right)}{\left(\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\right)^2\left([2]_q-1\right)^2}\right)\right|\right\}.\end{aligned} \tag{31}$$

This result is sharp for the functions defined in (24) and (26).

Proof. Let $\mu \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$. From (22) and (23), we have

$$d_2 = \frac{(K-R)\tau c_1}{4\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)} \tag{32}$$

and

$$d_3 = \frac{(K-R)\tau}{4\left([3]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left(c_2 - \frac{c_1^2}{2} + \frac{(5-R)\tau}{4}c_1^2 + \frac{(K-R)\tau}{4\left([2]_q-1\right)}c_1^2\right). \tag{33}$$

Since $\mu\left(\mu^{-1}\right)(w) = w$, and from (2), we obtain

$$B_2 = -d_2. \tag{34}$$

From (32) and (34), we obtain

$$|B_2| \leq \frac{(K-R)\tau c_1}{2\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}.$$

From (3), we obtain

$$B_3 = 2d_2^2 - d_3. \tag{35}$$

Using (32) and (33), in (35), we obtain

$$\begin{aligned}
B_3 &= -\frac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left(c_2 - \frac{c_1^2}{2} + \frac{(5-R)\tau}{4}c_1^2 + \frac{(K-R)\tau}{4\left([2]_q-1\right)}c_1^2\right) \\
&\quad +2\left(\frac{(K-R)^2\tau^2c_1^2}{16\left([2]_q-1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2}\right) \\
&= -\frac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{(K-R)\tau c_1^2}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)} \\
&\quad \times\left(\frac{(5-R)\tau}{4} + \frac{(K-R)\tau}{4\left([2]_q-1\right)}\right) + \frac{(K-R)^2\tau^2c_1^2}{8\left([2]_q-1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2} \\
&= -\frac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left(c_2 - \frac{c_1^2}{2}\right) - \frac{(K-R)\tau^2}{16\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)} \\
&\quad \times\left(-\frac{2\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)(K-R)}{\left([2]_q-1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2} + 5 - R + \frac{K-R}{[2]_q-1}\right)c_1^2.
\end{aligned}$$

$$\begin{aligned}
|B_3| &= \left|-\frac{(K-R)\tau c_2}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)} + \frac{(K-R)\tau c_1^2}{8\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)} - \right. \\
&\quad \frac{(K-R)\tau^2}{16\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left((5-R) + \frac{(K-R)}{[2]_q-1}\right. \\
&\quad \left.\left. - \frac{2\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)(K-R)}{\left([2]_q-1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2}\right)c_1^2\right| \\
&= \frac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left|c_2 - \frac{1}{2}\left(1 - \frac{\tau}{2}\left(\frac{K-R}{[2]_q-1} + 5 - R - \right.\right.\right. \\
&\quad \left.\left.\left.\frac{2\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)(K-R)}{\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2\left([2]_q-1\right)^2}\right)\right)c_1^2\right| \\
&= \frac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}\left|c_2 - \frac{1}{2}Vc_1^2\right|,
\end{aligned}$$

where
$$V = 1 - \frac{\tau}{2}\left((5-R) + \frac{(K-R)}{[2]_q - 1} - \frac{2\left([3]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)(K-R)}{\left([2]_q - 1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2}\right).$$

Therefore, according to Lemma 4, we can conclude that

$$|B_3| \leq \frac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)} \max\left\{2, \left|\tau\left((5-R) + \frac{(K-R)}{[2]_q-1} - \frac{2\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)(K-R)}{\left([2]_q-1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2}\right)\right|\right\}.$$

This proves the result. The first result and the inequality

$$|B_3| \leq \frac{(K-R)\tau}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)} \left|\tau\left((5-R) + \frac{(K-R)}{[2]_q-1} - \frac{2\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)(K-R)}{\left([2]_q-1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2}\right)\right|$$

are sharp for μ_* given in (24). The result

$$|B_3| \leq \frac{(K-R)|\tau|}{4\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}$$

is sharp for the function μ_0 given in (26). □

The corollary proven in [56] follows on from the assumptions that $q \to 1^-$, $\sigma = 0$, $\lambda = 0$, $\rho = 1$, and $m = 0$ are true in Theorem 4.

Corollary 3 ([56])**.** *Let $\mu \in \mathcal{S}^*(K,R)$, where μ is of the form (1), and μ^{-1} is of the form (2), $-1 \leq R < K \leq 1$. Then,*
$$|B_2| \leq \frac{|\tau|(K-R)}{2}$$
and
$$|B_3| \leq \frac{|\tau|(K-R)}{8} \max\{\tau|3K - 2R - 5|, 2\}.$$

Theorem 5. *Let $\mu \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$, where μ is of the form (1), and μ^{-1} is of the form (2), $-1 \leq R < K \leq 1$. Then,*

$$\left|B_3 - \delta B_2^2\right| \leq \frac{(K-R)|\tau|}{4\left([3]_q-1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)} \max\left\{2, \left|\tau\left(\frac{\left([3]_q-1\right)(K-R)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)(2-\delta)}{\left([2]_q-1\right)^2\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)\right)^2} - \frac{K-R}{[2]_q-1} + R - 5\right)\right|\right\}.$$

The result is sharp.

Proof. $\mu \in \mathcal{S}^{*,\rho,\sigma}_{\lambda,q,m}(K,R)$ and from (25), we have

$$d_2 = \frac{h_1}{\left([2]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)}, \text{ and } d_3 = \frac{\left(\Lambda^{m,\rho,\sigma}_{q,\lambda}(2)h_1 d_2 + h_2\right)}{\left([3]_q - 1\right)\Lambda^{m,\rho,\sigma}_{q,\lambda}(3)}.$$

Since

$$B_2 = -d_2 \tag{36}$$

and

$$B_3 = 2d_2^2 - d_3. \tag{37}$$

Now, we calculate $\left|B_3 - \delta B_2^2\right|$ as:

$$\begin{aligned}
&\left|B_3 - \delta B_2^2\right| \\
= &\frac{1}{\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}\left|h_2 + \frac{1}{\left([2]_q - 1\right)}\left(1 - \frac{\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)(2-\delta)}{\left([2]_q - 1\right)\left(\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\right)^2}\right)h_1^2\right| \\
= &\frac{1}{\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}\left|h_2 - \frac{1}{[2]_q - 1}\left(\frac{\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)(2-\delta)}{\left([2]_q - 1\right)\left(\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\right)^2} - 1\right)h_1^2\right| \\
= &\frac{1}{\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)}\left|h_2 - v h_1^2\right|,
\end{aligned}$$

where

$$v = \frac{1}{[2]_q - 1}\left(\frac{\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)(2-\delta)}{\left([2]_q - 1\right)\left(\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\right)^2} - 1\right).$$

Thus, by the Lemma 3, we obtain

$$\begin{aligned}
\left|B_3 - \delta B_2^2\right| \leq &\frac{(K-R)|\tau|}{4\left([3]_q - 1\right)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)} \max 2, \left\{\left|\tau\left(\frac{\left([3]_q - 1\right)(K-R)\Lambda_{q,\lambda}^{m,\rho,\sigma}(3)(2-\delta)}{\left([2]_q - 1\right)^2\left(\Lambda_{q,\lambda}^{m,\rho,\sigma}(2)\right)^2}\right.\right.\right. \\
&\left.\left.\left. - \frac{1}{[2]_q - 1}(K-R) + R - 5\right)\right|\right\}.
\end{aligned}$$

This proves the result.

Both (24) and (26) provide functions for which the equality holds. □

The result studied in [56] is obtained by substituting the values $q \to 1^-$, $\sigma = 0$, $\lambda = 0$, $\rho = 1$, and $m = 0$ in Theorem 5.

Corollary 4 ([56])**.** *Let $\mu \in \mathcal{S}^*(K,R)$, where μ is of the form (1), and μ^{-1} is of the form (2), $-1 \leq R < K \leq 1$. Then, $\delta \in \mathbb{C}$ and $|z| < \tau^2$.*

$$\begin{aligned}
&\left|B_3 - \delta B_2^2\right| \\
\leq &\frac{(K-R)|\tau|}{8}\max\{2, |\tau(3K - 2R - 5 - 2\delta(K-R))|\}.
\end{aligned}$$

The result is sharp.

3.2. Application

Engineers and scientists extensively use fractional calculus in the fields of heat and mass transfer, non-linear differential equations, and fuzzy differential equations in engineering science and technology. Fractional differential calculus now plays a significant part in the diagnosis of diseases in the medical field, (see [57–60]). For the area of complex fractional differential operators, see the following articles [61–63].

4. Conclusions

Using the concept of subordination and the linear multiplier fractional q-differential integral operator $D_{q,\lambda}^{m}(\rho,\sigma)$, we studied a new family of q-starlike functions related to the cardioid domain. There are three sections to the article. Section 1 provides the background and definitions, while Section 2 presents certain recognized lemmas. In Section 3, we explored several new problems for the subclass of q-starlike functions, coefficient bounds, and the Fekete–Szegö-type problem. In Section 4, we similarly examined the inverse (μ^{-1}) function of the newly formed class of analytic functions for similar types of consequences. We showed that all of the results discussed in this paper are sharp. Some of the most fundamental consequences are already well-established, and our research brought them to attention. Numerous scholars will be inspired to develop this concept further by studying this article and applying it to the classes of bi-univalent functions, meromorphic functions, etc.

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