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Boundary Value Problem for a Coupled System of Nonlinear Fractional q-Difference Equations with Caputo Fractional Derivatives

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Abstract: This paper focuses on the analysis of a coupled system governed by a Caputo-fractional derivative with q-integral-coupled boundary conditions. This system is particularly relevant in modeling multi-atomic systems, including scenarios involving adsorbed atoms or clusters on crystalline surfaces, surface–atom scattering, and atomic friction. To investigate this system, we introduce an operator that exhibits fixed points corresponding to the solutions of the problem, effectively transforming the system into an equivalent fixed-point problem. We established the necessary conditions for the existence and uniqueness of solutions using the Leray–Schauder nonlinear alternative and the Banach contraction mapping principle, respectively. Stability results in the Ulam sense for the coupled system are also discussed, along with a sensitivity analysis of the range parameters. To support the validity of their findings, we provide illustrative examples. Overall, the paper offers a thorough examination and analysis of the considered coupled system, making important contributions to the understanding of multi-atomic systems and their mathematical modeling.

Keywords: fractional q-integral; boundary conditions; Riemann–Liouville fractional q-derivative; fixed point theorems

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1. Introduction

The fundamental concept of fractional calculus involves replacing natural numbers with rational numbers in the order of derivation operators. Although this concept may seem simple, it has far-reaching consequences and results that pertain to phenomena in various fields, such as bioengineering, dynamics, modeling, control theory, and medicine [1–4]. Additionally, Lopez et al. presented a new definition of fractional curvature of plane curves, specifically when the fractional derivative is in the Caputo sense [5]. Salati et al. [6] studied the numerical solutions of Bagley–Torvik and fractional oscillation equations in the Coputo sense. Asaduzzaman et al. [7] studied the existence criteria of at least one or at least three positive solutions to the Caputo-type nonlinear fractional differential equation by using Guo–Krasnoselskii's fixed point theorem.

In the 20th century, significant research activity focused on q-difference equations, which emerged as an intriguing subject in mathematics and its applications. These equations found applications in areas like orthogonal polynomials and mathematical control theories [8–10]. The book [11] provides comprehensive definitions and properties of q-difference calculus. The extension of fractional differential equations to fractional q-difference equations has attracted the attention of many researchers. For detailed discussions and examples of nonlinear fractional q-difference equations subject to various boundary conditions involving q-derivatives and q-integrals, the book by Annaby and Mansour [12] is a valuable resource. Furthermore, extensive research has been conducted on q-difference and fractional q-difference equations, as evidenced by works such as [13–15].

Recently, Laledj et al. [16] conducted a study focusing on the existence and Ulam stability of implicit fractional q-difference equations in both Banach spaces and Banach algebras. They employed fixed point theory, specifically the nonlinear alternative of Schaefer's type proven by Dhage, as well as Dhage's random fixed point theorem in Banach algebras. Another study conducted by Allouch et al. [17] focused on the existence of solutions for a class of boundary value problems involving fractional q-difference equations in a Banach space. They utilized Mönch's fixed point theorem and the technique of measures of non-compactness. Boutiara et al. [18] examined a system of fractional boundary value problems, specifically addressing the existence of unbounded solutions for a class of nonlinear fractional q-difference equations on an infinite interval. The study was conducted within the context of the Riemann–Liouville fractional q-derivative. Rajkovic et al. [19] present the properties of fractional integrals and derivatives in q-calculus. El-Shahed et al. [20] studied the properties of positive solutions of the q-difference equation. Ahmad et al. [21–24] studied the existence of solutions for nonlinear fractional q-difference equations and inclusions with nonlocal conditions.

The nonlinear Langevin equation (NLE), formulated by the brilliant French physicist Paul Langevin [25] in the early 20th century, played a crucial role in accurately describing Brownian motion. The Langevin equation has found diverse applications, ranging from analyzing stock market behavior and modeling evacuation processes to studying fluid suspensions, self-organization in complex systems, photo-electron counting, and protein dynamics.

The Langevin equation serves as a valuable tool for investigating the temporal evolution of physical phenomena. However, when it comes to dynamics in complex media, the standard Langevin equation falls short of providing an accurate description. To address this limitation, several generalizations of the Langevin equation have been proposed. One such generalization is the generalized Langevin equation, which incorporates fractal and memory features through a dissipative memory kernel. Recent research indicates that introducing fractional derivatives of non-integer orders into the Langevin equation offers a more adaptable model for fractal processes. Notably, the investigation of the Langevin equation involving q-fractional derivatives of various orders remains an unexplored area of research.

Almalahi et al. [26] considered the nonlinear fractional integro-differential Langevin equation with the φ-ABC fractional derivative of the type:

$$\begin{cases} {}^{ABC}\mathbb{D}^{\alpha;\phi}\left({}^{ABC}\mathbb{D}^{\varrho;\phi}+\lambda\right)z(\omega)=g(\omega,z(\omega),{}^{AB}\,\mathbb{I}_{0^+}^{\alpha,\phi}z(\omega)),\omega\in(0,b), \\ z(0)=a_1,z'_{\phi}(0)=a_2, \end{cases} \tag{1}$$

where ${}^{ABC}\mathbb{D}^{\alpha;\phi}$ and ${}^{ABC}\mathbb{D}^{\varrho;\phi}$ are the ϕ-ABC fractional derivatives of order α and ϱ, respectively such that $\alpha,\varrho\in(0,1]$, ${}^{AB}\mathbb{I}_{0^+}^{\alpha,\phi}$ is a ϕ-Atangana–Baleanu-fractional integral of order α, ϕ is an increasing function, having a continuous derivative ϕ' on $(0,b)$, such that $\phi'(\omega)\neq 0$, for all $\omega\in(0,b)$ and $g:\mho\times\mathbb{R}^2\to\mathbb{R}$ is continuous and differentiable function such that $g(0,\omega(0),{}^{AB}\,\mathbb{I}_{0^+}^{\alpha,\phi}\omega(0))=0$ and $g'_{\phi}(0,\omega(0),{}^{AB}\,\mathbb{I}_{0^+}^{\alpha,\phi}\omega(0))=0$.

In [27], Ahmad et al investigated the existence of solutions for the Caputo fractional q-difference integral equation with two different fractional orders and nonlocal boundary conditions

$$\begin{cases} {}^{C}\mathbb{D}_q^{\Psi}\left({}^{C}\mathbb{D}_q^{\Upsilon}+\lambda\right)z(\omega)=\alpha f(\omega,z(\omega))+\gamma\mathbb{I}_q^{\zeta}g(\omega,z(\omega)),0\le\omega<1, \\ \mu_1 z(0)-\sigma\left(\omega^{(1-\Upsilon)}\mathbb{D}_q z(0)\right)_{\omega=0}=\eta_1 z(\beta_1), \\ \mu_2 z(1)+\sigma_2\mathbb{D}_q z(1)=\eta_2 z(\beta_2), \end{cases}$$

where $\Psi,\Upsilon,\zeta\in(0,1)$, f,g are given continuous functions, λ,α,γ are real constants and $\mu_i,\sigma_i,\eta_i\in\mathbb{R}$, $\beta_i\in(0,1)$ $(i=1,2)$.

Boutiara et al. [28] utilized the eigenvalue of an operator to establish the existence and uniqueness of solutions by employing techniques based on condensing operators and

Sadovskii's measure to investigate the following specific Caputo fractional q-difference boundary value problem

$$\begin{cases} {}^{C}\mathbb{D}_q^{\Psi}\Big({}^{C}\mathbb{D}_q^{\Upsilon}z(\omega) - g(\omega, z(\omega))\Big)z(\omega) = f(\omega, z(\omega)), & \omega \in [0, T], \\ \mu_1 z(0) + \sigma_1 {}^{C}\mathbb{D}_q^{\zeta_1}z(0) = \eta_1 \int_0^{\lambda_1} \frac{(\lambda_1 - qx)^{(\varrho_1 - 1)}}{\Gamma_q(\varrho_1)} z(x) d_q x, \lambda_1 \in (0, T), \varrho_1 > 0, \\ \mu_2 z(0) + \sigma_2 {}^{C}\mathbb{D}_q^{\zeta_2}z(0) = \eta_2 \int_0^{\lambda_2} \frac{(\lambda_2 - qx)^{(\varrho_2 - 1)}}{\Gamma_q(\varrho_2)} z(x) d_q x, \lambda_2 \in (0, T), \varrho_2 > 0, \end{cases}$$

where ${}^{C}\mathbb{D}_q^{\Psi}$, ${}^{C}\mathbb{D}_q^{\Upsilon}$ and ${}^{C}\mathbb{D}_q^{\zeta_i} (i = 1, 2)$ are the fractional q-derivatives of the Caputo type of orders $0 < \Psi, \Upsilon, \zeta_i \leq 1$, $i = 1, 2$. $g, f : [0, T] \times \mathbb{R} \to \mathbb{R}$ continuous functions and $\mu_i, \sigma_i, \eta_i \in \mathbb{R}^+$, $(i = 1, 2)$.

Based on the justification provided, we are motivated to thoroughly evaluate and investigate the necessary conditions for the existence and uniqueness of solutions for a coupled system through the application of Caputo-fractional q-difference equations. Our aim is to carefully examine and determine the specific requirements that must be satisfied to ensure the existence and uniqueness of solutions for the following problem

$$\begin{cases} {}^{C}\mathbb{D}_q^{\Psi_1}\Big({}^{C}\mathbb{D}_q^{\Upsilon_1} + \lambda_1\Big)z_1(\omega) = \alpha_1 f_1(\omega, z_1(\omega), z_2(\omega)) + \gamma_1 \mathbb{I}_q^{\zeta_1} g_1(\omega, z_1(\omega), z_2(\omega)), \\ {}^{C}\mathbb{D}_q^{\Psi_2}\Big({}^{C}\mathbb{D}_q^{\Upsilon_2} + \lambda_2\Big)z_2(\omega) = \alpha_2 f_2(\omega, z_1(\omega), z_2(\omega)) + \gamma_2 \mathbb{I}_q^{\zeta_2} g_2(\omega, z_1(\omega), z_2(\omega)), \end{cases} \tag{2}$$

equipped with q-integral-coupled boundary conditions

$$\begin{cases} \mu_1 z_1(0) - \sigma_1 \Big(\omega^{(1-\Upsilon_1)} \mathbb{D}_q z_1(0)\Big)_{\omega=0} = \eta_1 z_1(\beta_1), \\ \mu_2 z_1(1) + \sigma_2 \mathbb{D}_q z_1(1) = \eta_2 z_1(\beta_2), \\ \mu_3 z_2(0) - \sigma_3 \Big(\omega^{(1-\Upsilon_2)} \mathbb{D}_q z_2(0)\Big)_{\omega=0} = \eta_3 z_2(\beta_3), \\ \mu_4 z_2(1) + \sigma_4 \mathbb{D}_q z_2(1) = \eta_4 z_2(\beta_4), \end{cases} \tag{3}$$

where

1. $0 \leq \omega \leq 1, 0 < q < 1$.
2. ${}^{C}\mathbb{D}_q^{\Psi_i}$ and ${}^{C}\mathbb{D}_q^{\Upsilon_i}$ denote the fractional q-derivatives of the Caputo type of orders $0 < \Psi_i$, $\Upsilon_i \leq 1, i = 1, 2$.
3. $\mathbb{I}_q^{\zeta_i}$ denotes Riemann–Liouville integral of order $\zeta_i \in (0, 1), i = 1, 2$.
4. $\lambda_i, \alpha_i, \gamma_i$, $i = 1, 2$ are real constants and $\mu_i, \sigma_i, \eta_i \in \mathbb{R}, \beta_i \in (0, 1), i = 1, 2, 3, 4$.
5. $f_i, g_i : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}, (i = 1, 2)$ are given continuous functions satisfied the following hypotheses:

($\mathbf{H}_1$) There exist constants $L_i, K_i > 0, i = 1, 2$, such that, for each $\omega \in [0, 1]$ and $z_1, z_2, z_1^*, z_2^* \in \mathbb{R}$, we have

$$\begin{aligned} |f_i(\omega, z_1, z_2) - f_i(\omega, z_1^*, z_2^*)| &\leq L_i(|z_1 - z_1^*| + |z_2 - z_2^*|), \\ |g_i(\omega, z_1, z_2) - g_i(\omega, z_1^*, z_2^*)| &\leq K_i(|z_1 - z_1^*| + |z_2 - z_2^*|). \end{aligned}$$

($\mathbf{H}_2$) There exist real numbers $m_i, \widetilde{m}_i, n_i, \widetilde{n}_i \geq 0$ $(i = 1, 2)$, and $m_0, \widetilde{m}_0, n_0, \widetilde{n}_0 > 0$ such that, $\forall z_1, z_2 \in \mathbb{R}$, we have

$$|f_1(\omega, z_1, z_2)| \leq m_0 + m_1|z_1| + m_2|z_2|,$$

$$|f_2(\omega, z_1, z_2)| \leq \widetilde{m}_0 + \widetilde{m}_1|z_1| + \widetilde{m}_2|z_2|,$$

$$|g_1(\omega, z_1, z_2)| \leq n_0 + n_1|z_1| + n_2|z_2|,$$

and

$$|g_2(\omega, z_1, z_2)| \leq \widetilde{n}_0 + \widetilde{n}_1|z_1| + \widetilde{n}_2|z_2|.$$

1.1. Contributions of This Paper

In this context, it is important to highlight that system (2) with conditions (3) involves q-fractional type Langevin equations with distinct fractional orders. The nonlinearity present in these equations encompasses both non-integral and Riemann–Liouville-type q-integral terms. However, it is possible to reduce the nonlinearity to either a purely non-integral case or an integral nonlinearity case, corresponding to α_i and γ_i (for $i = 1, 2$) respectively. Additionally, as q approaches 1^-, system (2) can be reduced to a system of Langevin equations with two different fractional orders, or a system of second-order q-difference equations with the values Ψ_i and Υ_i (for $i = 1, 2$). An alternative and flexible approach involving ζ_i (for $i = 1, 2$) is provided by the integral type nonlinearity, which is expressed in terms of the q-integral of the Riemann–Liouville type with the order ζ_i in the range $(0, 1)$. Moreover, in feedback control problems such as determining the steady-states of a thermostat, four-point nonlocal boundary conditions arise. These conditions are associated with a controller positioned at the domain's edge, which either adds or removes heat based on temperature variations caused by two variable (nonlocal) positions within the domain under consideration.

Overall, the combination of applying Langevin equations to multi-atomic systems, analyzing a coupled system with a Caputo-fractional derivative, introducing an operator for the fixed-point formulation, establishing necessary conditions for existence and uniqueness, and validating the results through illustrative examples contributes to the novelty and significance of this work.

1.2. Construction of This Paper

The remainder of this paper is organized as follows: In Section 2, we provide a review of fractional calculus notations, definitions, and relevant lemmas that are essential to our research. Additionally, we present an important lemma that allows us to convert the coupled system of Caputo-fractional q-difference Equation (2) into an equivalent integral equation. Section 3 presents the main findings regarding the existence and uniqueness of solutions for the coupled system of Caputo-fractional q-difference Equation (2). In Section 4, we discuss the stability results with parameters sensitivity analysis. To illustrate these results, we present a numerical example in Section 5. Finally, we conclude this paper with a summary of our findings in the last section.

2. Preliminary Results and Essential Concepts

In this section, we provide a review of fractional calculus notations, definitions, and relevant lemmas that are essential to our research. Additionally, we present an important lemma that allows us to convert the coupled system of Caputo-fractional q-difference Equation (2) into an equivalent integral equation. Let $S = \{z \in C([0,1], \mathbb{R})\}$ be the space equipped with the norm $\|z\| = \sup_{\omega \in [0,1]} |z(\omega)|$, Clearly, $(S, \|\cdot\|)$ is a Banach space. Let $S \times S$ be the product space with the norm $\|(z_1, z_2)\| = \|z_1\| + \|z_2\|$ for $(z_1, z_2) \in S \times S$.

For every $a \in \mathbb{R}$, the q-number $[a]_q$ is defined by $[a]_q = \frac{1-q^a}{1-q}$, where $q \in (0,1)$, is an arbitrary real number. Also, the q-shifted factorial of real number a is defined by $(a,q)_0 = 1$, and $(a,q)_n = \Pi_{j=0}^{n-1}(1 - aq^j)$ for $n \in \mathbb{N} \cup \{\infty\}$. For $a, b \in \mathbb{R}$, the q-analog of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is given by

$$(a-b)^0 = 1, \qquad (a-b)^n = \Pi_{j=0}^{n-1}(a - bq^j).$$

In general, if ϱ is a real number, then $(a-b)^{(\varrho)} = a^{\varrho}\Pi_{j=0}^{\infty}(\frac{a-bq^j}{a-bq^{j+\varrho}})$ and $a^{(\varrho)} = a^{\varrho}$ when $b = 0$. If $\varrho > 0$ and $0 \le a \le b \le \omega$, then $(\omega - b)^{(\varrho)} \le (\omega - a)^{(\varrho)}$.

Definition 1 ([12,29])**.** *Let $\varrho \geq 0, q \in (0,1)$, and $z : [0,1] \to \mathbb{R}$ be a continuous function. Then the Riemann–Liouville fractional q-integral for the function z of order ϱ is defined by*

$$\begin{cases} (\mathbb{I}_q^0 z)(\omega) = z(\omega), \\ (\mathbb{I}_q^\varrho z)(\omega) = \frac{1}{\Gamma_q(\varrho)} \int_0^\omega (\omega - qs)^{(\varrho-1)} z(s) d_q s, \qquad \varrho > 0, \end{cases}$$

provided that the right-hand side is point-wise defined on $[0,1]$ and $\omega \in [0,1]$, also, the q-Gamma function $\Gamma_q(\varrho)$ is defined by

$$\Gamma_q(\varrho) = \frac{(1-q)^{(\varrho-1)}}{(1-q)^{\varrho-1}}, \ \varrho \in \mathbb{R}/\{0,-1,-2,...\},$$

which satisfies the relation $\Gamma_q(\varrho+1) = [\varrho]_q \Gamma_q(\varrho)$.

Also, for any $x, y > 0$, we define the q-Beta function $B_q(x,y)$ as

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{4}$$

Definition 2 ([12,29])**.** *The Riemann–Liouville fractional q-derivative of order $n-1 < \varrho < n$, $n \geq 1$, for a function $z : [0,1] \to \mathbb{R}$ is defined by*

$$(\mathbb{D}_q^\varrho z)(\omega) = \frac{1}{\Gamma_q(n-\varrho)} \int_0^\omega \frac{z(s)}{(\omega - qs)^{\varrho - n + 1}} d_q s.$$

Lemma 1 ([12,29])**.** *For $\Psi, \varrho \in \mathbb{R}^+$, and let z be a function defined on $[0,1]$. Then,*

$$\begin{aligned} (\mathbb{I}_q^\Psi \mathbb{I}_q^\varrho z)(\omega) &= (\mathbb{I}_q^{\Psi+\varrho} z)(\omega), \\ (\mathbb{D}_q^\Psi \mathbb{I}_q^\varrho z)(\omega) &= z(\omega), \\ \mathbb{I}_q^\varrho \omega^\Psi &= \frac{\Gamma_q(\Psi+1)}{\Gamma_q(\varrho+\Psi+1)} \omega^{\varrho+\Psi}, \ \Psi \in (-1,\infty), \ \varrho \geq 0, \ \omega > 0. \end{aligned}$$

If $z = 1$, then $\mathbb{I}_q^\varrho(1)(\omega) = \frac{1}{\Gamma_q(\varrho+1)} \omega^\varrho$, for all $\omega > 0$.

Lemma 2 ([12,29])**.** *Let $\varrho > 0$. Then, we have*

$$(\mathbb{I}_q^\varrho \, {}^c\mathbb{D}_q^\varrho z)(\omega) = z(\omega) - \sum_{k=0}^{[\varrho]-1} \frac{\omega^k}{\Gamma_q(k+1)} \left(\mathbb{D}_q^\varrho z\right)(0).$$

In the case $\varrho \in (0,1)$, we have

$$(\mathbb{I}_q^\varrho \, {}^c\mathbb{D}_q^\varrho z)(\omega) = z(\omega) - z(0).$$

Theorem 1 ([30])**.** *Let $C(\mathcal{J}, \mathbb{R})$ be a Banach space. The operator $\mathcal{T} : S \to S$ is a contraction if there exists a constant $0 < L < 1$, such that, i.e., $\|\mathcal{T}(z) - \mathcal{T}(z^*)\| \leq L\|z - z^*\|$ for all $z, z^* \in S$.*

Theorem 2 ([31])**.** *Let S be a non-empty, closed-convex subset of a Banach space X. If $\mathcal{T} : S \to S$ is a completely continuous operator and $\Phi(\mathcal{T}) = \{(z \in S, z = \xi\mathcal{T}(z), 0 < \xi < 1\}$, then either $\Phi(\mathcal{T})$ is unbounded or $\mathcal{T}$ has a fixed point.*

Notations

To improve readability, we fix the following notations and, subsequently, refer to them in our analysis without any additional explanations

$$\begin{aligned}
A_1 &= \frac{\eta_1}{\Delta_1}\Big[(\eta_2-\mu_2)\omega^{\Upsilon_1}-\Big(\eta_2\beta_2^{\Upsilon_1}-\mu_2+\sigma_2[\Upsilon_1]_q\Big)\Big],\\
A_2 &= \frac{\eta_2}{\Delta_1}\Big[(\eta_1-\mu_1)\omega^{\Upsilon_1}-\Big(\eta_1\beta_1^{\Upsilon_1}+\sigma_1[\Upsilon_1]_q\Big)\Big],\\
A_3 &= \frac{\eta_2}{\Delta_1}\Big[(\eta_1-\mu_1)\omega^{\Upsilon_1}-\Big(\eta_1\beta_1^{\Upsilon_1}-\sigma_1[\Upsilon_1]_q\Big)\Big],\\
A_4 &= \frac{\sigma_2}{\Delta_1}\Big[(\eta_1-\mu_1)\omega^{\Upsilon_1}-\Big(\eta_1\beta_1^{\Upsilon_1}-\sigma_1[\Upsilon_1]_q\Big)\Big],\\
A_5 &= \frac{\eta_3}{\Delta_2}\Big[(\eta_4-\mu_4)\omega^{\Upsilon_2}-\Big(\eta_4\beta_4^{\Upsilon_2}-\mu_4+\sigma_4[\Upsilon_2]_q\Big)\Big],\\
A_6 &= \frac{\eta_4}{\Delta_2}\Big[(\eta_3-\mu_3)\omega^{\Upsilon_2}-\Big(\eta_3\beta_3^{\Upsilon_1}+\sigma_3[\Upsilon_2]_q\Big)\Big],\\
A_7 &= \frac{\eta_4}{\Delta_2}\Big[(\eta_3-\mu_3)\omega^{\Upsilon_2}-\Big(\eta_3\beta_3^{\Upsilon_2}-\sigma_3[\Upsilon_2]_q\Big)\Big],\\
A_8 &= \frac{\sigma_4}{\Delta_2}\Big[(\eta_3-\mu_3)\omega^{\Upsilon_2}-\Big(\eta_3\beta_3^{\Upsilon_2}-\sigma_3[\Upsilon_2]_q\Big)\Big],
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\Delta_1 &= (\eta_1-\mu_1)\Big(\eta_2\beta_2^{\Upsilon_1}-\mu_2+\sigma_2[\Upsilon_1]_q\Big)-(\eta_2-\mu_2)\omega^{\Upsilon_1}\Big(\eta_1\beta_1^{\Upsilon_1}+\sigma_1[\Upsilon_1]_q\Big),\\
\Delta_2 &= (\eta_3-\mu_3)\Big(\eta_4\beta_4^{\Upsilon_2}-\mu_4+\sigma_4[\Upsilon_2]_q\Big)-(\eta_4-\mu_4)\omega^{\Upsilon_2}\Big(\eta_3\beta_3^{\Upsilon_2}+\sigma_3[\Upsilon_2]_q\Big).
\end{aligned}$$

In the sequel, we set

$$\begin{aligned}
\rho_1 &= \frac{\Big(1+|A_1|\beta_1^{\Upsilon_1}+|A_2|\beta_2^{\Upsilon_1}+|A_3|+|A_4|\Big)}{\Gamma_q(\Upsilon_1+1)},\\
\rho_2 &= \frac{\Big(1+|A_1|\beta_1^{\Upsilon_1+\Psi_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1}+|A_3|+|A_4|\Big)}{\Gamma_q(\Upsilon_1+\Psi_1+1)},\\
\rho_3 &= \frac{\Big(1+|A_1|\beta_1^{\Upsilon_1+\Psi_1+\zeta_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1+\zeta_1}+|A_3|+|A_4|\Big)}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)},\\
\rho_4 &= \frac{\Big(1+|A_5|\beta_2^{\Upsilon_2}+|A_6|\beta_1^{\Upsilon_2}+|A_7|+|A_8|\Big)}{\Gamma_q(\Upsilon_2+1)},\\
\rho_5 &= \frac{\Big(1+|A_5|\beta_2^{\Upsilon_2+\Psi_2}+|A_6|\beta_1^{\Upsilon_2+\Psi_2}+|A_7|+|A_8|\Big)}{\Gamma_q(\Upsilon_2+\Psi_2+1)},\\
\rho_6 &= \frac{\Big(1+|A_5|\beta_2^{\Upsilon_2+\Psi_2+\zeta_2}+|A_6|\beta_1^{\Upsilon_2+\Psi_2+\zeta_2}+|A_7|+|A_8|\Big)}{\Gamma_q(\Upsilon_2+\Psi_2+\zeta_2+1)}.
\end{aligned} \tag{6}$$

3. Main Results

In this section, we will discuss the existence and uniqueness of the solution for system (2). The study of existence results for fractional q-difference equations is an active area of research, and researchers continue to develop new methods and techniques to address this problem. By establishing the existence of solutions, researchers can provide a solid foundation for further analysis, numerical simulations, and applications of these equations in various fields of science and engineering.

3.1. Equivalent Integral Equation

In this subsection, we will begin by obtaining the equivalent integral equation of the following linear fractional system:

$$\begin{cases} {}^{C}\mathbb{D}_q^{\Psi_1}\left({}^{C}\mathbb{D}_q^{\Upsilon_1}+\lambda_1\right)z_1(\omega)=h_1(\omega), \\ {}^{C}\mathbb{D}_q^{\Psi_2}\left({}^{C}\mathbb{D}_q^{\Upsilon_2}+\lambda_2\right)z_2(\omega)=h_2(\omega), \end{cases} \tag{7}$$

equipped with q-integral-coupled boundary conditions,

$$\begin{cases} \mu_1 z_1(0)-\sigma_1\left(\omega^{(1-\Upsilon_1)}\mathbb{D}_q z_1(0)\right)_{\omega=0}=\eta_1 z_1(\beta_1), \\ \mu_2 z_1(1)+\sigma_2\mathbb{D}_q z_1(1)=\eta_2 z_1(\beta_2), \\ \mu_3 z_2(0)-\sigma_3\left(\omega^{(1-\Upsilon_2)}\mathbb{D}_q z_2(0)\right)_{\omega=0}=\eta_3 z_2(\beta_3), \\ \mu_4 z_2(1)+\sigma_4\mathbb{D}_q z_2(1)=\eta_4 z_2(\beta_4), \end{cases} \tag{8}$$

where $0\le\omega\le 1, 0<q<1$ and $h_1,h_2\in S$.

Theorem 3. *Let $h_1,h_2\in S$. If $(z_1,z_2)\in S\times S$, then (z_1,z_2) satisfies system (7) with condition (8) if and only if z_1 and z_2 are given by*

$$\begin{aligned} z_1(\omega) \;=\; & \int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\left(\mathbb{I}_q^{\Psi_1}h_1(x)-\lambda_1 z_1(x)\right)d_qx \\ & +A_1\int_0^{\beta_1}\frac{(\beta_1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\left(\mathbb{I}_q^{\Psi_1}h_1(x)-\lambda_1 z_1(x)\right)d_qx \\ & -A_2\int_0^{\beta_2}\frac{(\beta_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\left(\mathbb{I}_q^{\Psi_1}h_1(x)-\lambda_1 z_1(x)\right)d_qx \\ & +A_3\int_0^{1}\frac{(1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\left(\mathbb{I}_q^{\Psi_1}h_1(x)-\lambda_1 z_1(x)\right)d_qx \\ & -A_4\int_0^{1}\frac{(1-qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)}\left(\mathbb{I}_q^{\Psi_1}h_1(x)-\lambda_1 z_1(x)\right)d_qx, \end{aligned} \tag{9}$$

and

$$\begin{aligned} z_2(\omega) \;=\; & \int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\left(\mathbb{I}_q^{\Psi_2}h_2(x)-\lambda_2 z_2(x)\right)d_qx \\ & +A_5\int_0^{\beta_3}\frac{(\beta_3-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\left(\mathbb{I}_q^{\Psi_2}h_2(x)-\lambda_2 z_2(x)\right)d_qx \\ & -A_6\int_0^{\beta_4}\frac{(\beta_4-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\left(\mathbb{I}_q^{\Psi_2}h_2(x)-\lambda_2 z_2(x)\right)d_qx \\ & +A_7\int_0^{1}\frac{(1-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\left(\mathbb{I}_q^{\Psi_2}h_2(x)-\lambda_2 z_2(x)\right)d_qx \\ & +A_8\int_0^{1}\frac{(1-qx)^{(\Upsilon_2-2)}}{\Gamma_q(\Upsilon_2-1)}\left(\mathbb{I}_q^{\Psi_2}h_2(x)-\lambda_2 z_2(x)\right)d_qx, \end{aligned} \tag{10}$$

Proof. Applying the operators $\mathbb{I}_q^{\Psi_1}$ and $\mathbb{I}_q^{\Psi_2}$ to both sides of Equation (7), respectively, and using Lemma 2, we have

$$\begin{cases} \left({}^{C}\mathbb{D}_q^{\Upsilon_1}+\lambda_1\right)z_1(\omega)=\mathbb{I}_q^{\Psi_1}h_1(\omega)-\lambda_1 z_1(\omega)-c_0, \\ \left({}^{C}\mathbb{D}_q^{\Upsilon_2}+\lambda_2\right)z_2(\omega)=\mathbb{I}_q^{\Psi_2}h_2(\omega)-\lambda_2 z_2(\omega)-c_2. \end{cases} \tag{11}$$

Applying the operators $\mathbb{I}_q^{\Upsilon_1}$ and $\mathbb{I}_q^{\Upsilon_2}$ to both sides of Equation (11), respectively, and using Lemma 2, we have

$$\begin{cases} z_1(\omega) = \int_0^{\omega} \frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x - \frac{\omega^{\Upsilon_1}}{\Gamma_q(\Upsilon_1+1)} c_0 - c_1, \\ z_2(\omega) = \int_0^{\omega} \frac{(\omega-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)} \Big(\mathbb{I}_q^{\Psi_2} h_2(x) - \lambda_2 z_2(x) \Big) d_q x - \frac{\omega^{\Upsilon_2}}{\Gamma_q(\Upsilon_2+1)} c_2 - c_3. \end{cases} \tag{12}$$

The q-derivative of Equation (12) is

$$\begin{cases} \mathbb{D}_q z_1(\omega) = \int_0^{\omega} \frac{(\omega-qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x - \frac{[\Upsilon_1]_q \omega^{\Upsilon_1-1}}{\Gamma_q(\Upsilon_1+1)} c_0, \\ \mathbb{D}_q z_2(\omega) = \int_0^{\omega} \frac{(\omega-qx)^{(\Upsilon_2-2)}}{\Gamma_q(\Upsilon_2-1)} \Big(\mathbb{I}_q^{\Psi_2} h_2(x) - \lambda_2 z_2(x) \Big) d_q x - \frac{[\Upsilon_2]_q \omega^{\Upsilon_2-1}}{\Gamma_q(\Upsilon_2+1)} c_2. \end{cases}$$

Using the boundary conditions (8) in Equation (12) and the definition of the q-beta function together with the property of (4), we find that

$$\begin{aligned} c_0 \quad = \quad & \frac{\Gamma_q(\Upsilon_1+1)}{\Delta} \Bigg[-\eta_1(\eta_2-\mu_2) \int_0^{\beta_1} \frac{(\beta_1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \\ & + \eta_2(\eta_1-\mu_1) \int_0^{\beta_2} \frac{(\beta_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \\ & - \sigma_2(\eta_1-\mu_1) \int_0^{1} \frac{(1-qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \\ & - \eta_2(\eta_1-\mu_1) \int_0^{1} \frac{(1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \Bigg], \end{aligned}$$

$$\begin{aligned} c_1 \quad = \quad & \frac{1}{\Delta} \Bigg[\eta_1 \Big(\eta_2 \beta_2^{\Upsilon_1} - \mu_2 + \sigma_2 [\Upsilon_1]_q \Big) \int_0^{\beta_1} \frac{(\beta_1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \\ & - \eta_2 \Big(\eta_1 \beta_1^{\Upsilon_1} + \sigma_1 [\Upsilon_1]_q \Big) \int_0^{\beta_2} \frac{(\beta_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \\ & + \sigma_2 \Big(\eta_1 \beta_1^{\Upsilon_1} - \sigma_1 [\Upsilon_1]_q \Big) \int_0^{1} \frac{(1-qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \\ & + \eta_2 \Big(\eta_1 \beta_1^{\Upsilon_1} - \sigma_1 [\Upsilon_1]_q \Big) \int_0^{1} \frac{(1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)} \Big(\mathbb{I}_q^{\Psi_1} h_1(x) - \lambda_1 z_1(x) \Big) d_q x \Bigg], \end{aligned}$$

$$\begin{aligned} c_2 \quad = \quad & \frac{\Gamma_q(\Upsilon_2+1)}{\Delta} \Bigg[-\eta_3(\eta_4-\mu_4) \int_0^{\beta_3} \frac{(\beta_3-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)} \Big(\mathbb{I}_q^{\Psi_2} h_2(x) - \lambda_2 z_2(x) \Big) d_q x \Bigg] \\ & + \eta_4(\eta_3-\mu_3) \int_0^{\beta_4} \frac{(\beta_4-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)} \Big(\mathbb{I}_q^{\Psi_2} h_2(x) - \lambda_2 z_2(x) \Big) d_q x \\ & - \sigma_4(\eta_3-\mu_3) \int_0^{1} \frac{(1-qx)^{(\Upsilon_2-2)}}{\Gamma_q(\Upsilon_2-1)} \Big(\mathbb{I}_q^{\Psi_2} h_2(x) - \lambda_2 z_2(x) \Big) d_q x, \end{aligned}$$

and

$$c_3 = \frac{1}{\Delta}\Bigg[\eta_3\Big(\eta_4\beta_4^{\Upsilon_2} - \mu_4 + \sigma_4[\Upsilon_2]_q\Big)\int_0^{\beta_2}\frac{(\beta_2 - qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\mathbb{I}_q^{\Psi_2}h_2(x) - \lambda_2 z_2(x)\Big)d_qx$$
$$-\eta_4\Big(\eta_3\beta_3^{\Upsilon_1} + \sigma_3[\Upsilon_2]_q\Big)\int_0^{\beta_4}\frac{(\beta_4 - qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\mathbb{I}_q^{\Psi_2}h_2(x) - \lambda_2 z_2(x)\Big)d_qx$$
$$+\sigma_4\Big(\eta_3\beta_3^{\Upsilon_2} - \sigma_3[\Upsilon_2]_q\Big)\int_0^{1}\frac{(1 - qx)^{(\Upsilon_2-2)}}{\Gamma_q(\Upsilon_2-1)}\Big(\mathbb{I}_q^{\Psi_2}h_2(x) - \lambda_2 z_2(x)\Big)d_qx$$
$$+\eta_4\Big(\eta_3\beta_3^{\Upsilon_2} - \sigma_3[\Upsilon_2]_q\Big)\int_0^{1}\frac{(1 - qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\mathbb{I}_q^{\Psi_2}h_2(x) - \lambda_2 z_2(x)\Big)d_qx\Bigg].$$

Substituting the values of c_0, c_1, c_2, and c_3 in Equation (12) yields solutions (9) and (10). This completes the proof. □

As a result of Theorem (3), we obtain the following theorem:

Theorem 4. *Let $(z_1, z_2) \in S \times S$. Then, (z_1, z_2) satisfies system (2) with condition (3) if and only if z_1 and z_2 are given by*

$$z_1(\omega) = \int_0^{\omega}\frac{(\omega - qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{z_1,z_2}(x) + \gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{z_1,z_2}(x) - \lambda_1 z_1(x)\Big)d_qx$$
$$+A_1\Bigg[\int_0^{\beta_1}\frac{(\beta_1 - qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{z_1,z_2}(x) + \gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{z_1,z_2}(x) - \lambda_1 z_1(x)\Big)d_qx\Bigg]$$
$$-A_2\Bigg[\int_0^{\beta_2}\frac{(\beta_2 - qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{z_1,z_2}(x) + \gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{z_1,z_2}(x) - \lambda_1 z_1(x)\Big)d_qx\Bigg]$$
$$+A_3\Bigg[\int_0^{1}\frac{(1 - qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{z_1,z_2}(x) + \gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{z_1,z_2}(x) - \lambda_1 z_1(x)\Big)d_qx\Bigg]$$
$$-A_4\Bigg[\int_0^{1}\frac{(1 - qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{z_1,z_2}(x) + \gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{z_1,z_2}(x) - \lambda_1 z_1(x)\Big)d_qx\Bigg],$$

and

$$z_2(\omega) = \int_0^{\omega}\frac{(\omega - qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{z_1,z_2}(x) + \gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{z_1,z_2}(x) - \lambda_2 z_2(x)\Big)d_qx$$
$$+A_5\Bigg[\int_0^{\beta_3}\frac{(\beta_3 - qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{z_1,z_2}(x) + \gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{z_1,z_2}(x) - \lambda_2 z_2(x)\Big)d_qx\Bigg]$$
$$-A_6\Bigg[\int_0^{\beta_4}\frac{(\beta_4 - qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{z_1,z_2}(x) + \gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{z_1,z_2}(x) - \lambda_2 z_2(x)\Big)d_qx\Bigg]$$
$$+A_7\Bigg[\int_0^{1}\frac{(1 - qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{z_1,z_2}(x) + \gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{z_1,z_2}(x) - \lambda_2 z_2(x)\Big)d_qx\Bigg]$$
$$+A_8\Bigg[\int_0^{1}\frac{(1 - qx)^{(\Upsilon_2-2)}}{\Gamma_q(\Upsilon_2-1)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{z_1,z_2}(x) + \gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{z_1,z_2}(x) - \lambda_2 z_2(x)\Big)d_qx\Bigg],$$

where $F^i_{z_1,z_2}(x) = f_i(x, z_1(x), z_2(x))$, $G^i_{z_1,z_2}(x) = g_i(x, z_1(x), z_2(x)), i = 1, 2$

To obtain results using the fixed point technique, we define an operator $\mathcal{T} : S \times S \to S \times S$ by

$$\mathcal{T}(z_1, z_2)(\omega) = \begin{pmatrix} \mathcal{T}_1(z_1, z_2)(\omega) \\ \mathcal{T}_2(z_1, z_2)(\omega) \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned}
\mathcal{T}_1(z_1,z_2)(\omega) \;=\; & \int_0^{\omega} \frac{(\omega - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \Big(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{z_1,z_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1+\zeta_1-1} G^1_{z_1,z_2}(x) - \lambda_1 z_1(x) \Big) d_q x \\
& + A_1 \left[\int_0^{\beta_1} \frac{(\beta_1 - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \Big(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{z_1,z_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1+\zeta_1-1} G^1_{z_1,z_2}(x) - \lambda_1 z_1(x) \Big) d_q x \right] \\
& - A_2 \left[\int_0^{\beta_2} \frac{(\beta_2 - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \Big(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{z_1,z_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1+\zeta_1-1} G^1_{z_1,z_2}(x) - \lambda_1 z_1(x) \Big) d_q x \right] \\
& + A_3 \left[\int_0^{1} \frac{(1 - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \Big(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{z_1,z_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1+\zeta_1-1} G^1_{z_1,z_2}(x) - \lambda_1 z_1(x) \Big) d_q x \right] \\
& - A_4 \left[\int_0^{1} \frac{(1 - qx)^{(\Upsilon_1 - 2)}}{\Gamma_q(\Upsilon_1 - 1)} \Big(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{z_1,z_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1+\zeta_1-1} G^1_{z_1,z_2}(x) - \lambda_1 z_1(x) \Big) d_q x \right],
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\mathcal{T}_2(z_1,z_2)(\omega) \;=\; & \int_0^{\omega} \frac{(\omega - qx)^{(\Upsilon_2 - 1)}}{\Gamma_q(\Upsilon_2)} \Big(\alpha_2 \mathbb{I}_q^{\Psi_2} F^2_{z_1,z_2}(x) + \gamma_2 \mathbb{I}_q^{\Psi_2+\zeta_2-1} G^2_{z_1,z_2}(x) - \lambda_2 z_2(x) \Big) d_q x \\
& + A_5 \left[\int_0^{\beta_3} \frac{(\beta_3 - qx)^{(\Upsilon_2 - 1)}}{\Gamma_q(\Upsilon_2)} \Big(\alpha_2 \mathbb{I}_q^{\Psi_2} F^2_{z_1,z_2}(x) + \gamma_2 \mathbb{I}_q^{\Psi_2+\zeta_2-1} G^2_{z_1,z_2}(x) - \lambda_2 z_2(x) \Big) d_q x \right] \\
& - A_6 \left[\int_0^{\beta_4} \frac{(\beta_4 - qx)^{(\Upsilon_2 - 1)}}{\Gamma_q(\Upsilon_2)} \Big(\alpha_2 \mathbb{I}_q^{\Psi_2} F^2_{z_1,z_2}(x) + \gamma_2 \mathbb{I}_q^{\Psi_2+\zeta_2-1} G^2_{z_1,z_2}(x) - \lambda_2 z_2(x) \Big) d_q x \right] \\
& + A_7 \left[\int_0^{1} \frac{(1 - qx)^{(\Upsilon_2 - 1)}}{\Gamma_q(\Upsilon_2)} \Big(\alpha_2 \mathbb{I}_q^{\Psi_2} F^2_{z_1,z_2}(x) + \gamma_2 \mathbb{I}_q^{\Psi_2+\zeta_2-1} G^2_{z_1,z_2}(x) - \lambda_2 z_2(x) \Big) d_q x \right] \\
& + A_8 \left[\int_0^{1} \frac{(1 - qx)^{(\Upsilon_2 - 2)}}{\Gamma_q(\Upsilon_2 - 1)} \Big(\alpha_2 \mathbb{I}_q^{\Psi_2} F^2_{z_1,z_2}(x) + \gamma_2 \mathbb{I}_q^{\Psi_2+\zeta_2-1} G^2_{z_1,z_2}(x) - \lambda_2 z_2(x) \Big) d_q x \right]
\end{aligned} \tag{15}$$

Observe that system (2) with conditions (3) has solutions only if the operator equation $\mathcal{T}(z_1, z_2)(\omega)$ has fixed points, where $\mathcal{T}$ is given by Equation (13).

3.2. Uniqueness of Solutions

Theorem 5. *Assume that (H_1) holds. Then the system as described in (2) has a unique solution on $[0,1]$, provided that*

$$(L_1\rho_2\alpha_1 + K_1\rho_3\gamma_1 + L_2\rho_5\alpha_2 + K_2\rho_6\gamma_2) - \max\{|\lambda_1|\rho_1, |\lambda_2|\rho_4\} < 1. \tag{16}$$

Proof. Define the closed ball $B_r = \{(z_1, z_2) \in S \times S : \|(z_1, z_2)\| \le r\}$, with

$$r \ge \frac{N_1\rho_2\alpha_1 + M_1\rho_3\gamma_1 + N_2\rho_5\alpha_2 + M_2\rho_6\gamma_2}{1 - (L_1\rho_2\alpha_1 + K_1\rho_3\gamma_1 + L_2\rho_5\alpha_2 + K_2\rho_6\gamma_2) - \max\{|\lambda_1|\rho_1, |\lambda_2|\rho_4\}},$$

where $N_1 = \sup_{\omega\in[0,1]}|f_1(\omega, 0, 0)|$, $N_2 = \sup_{\omega\in[0,1]}|f_2(\omega, 0, 0)|$, $M_1 = \sup_{\omega\in[0,1]}|g_1(\omega, 0, 0)|$ and $M_2 = \sup_{\omega\in[0,1]}|g_2(\omega, 0, 0)|$. Now, we show that $\mathcal{T}(B_r) \subset B_r$, where $\mathcal{T} : B_r \to S \times S$ is defined by (13). For any $z_1, z_2 \in B_r$, $\omega \in [0,1]$ with (H_1), we have

$$\begin{aligned} |f_1(\omega,z_1(\omega),z_2(\omega))| &\le |f_1(\omega,z_1(\omega),z_2(\omega))-f_1(\omega,0,0)|+|f_1(\omega,0,0)| \\ &\le L_1(|z_1(\omega)|+|z_2(\omega)|)+|f_1(\omega,0,0)| \\ &\le L_1(\|z_1\|+\|z_2\|)+N_1 \\ &\le L_1r+N_1. \end{aligned} \tag{17}$$

Similarly, we can find that

$$|f_2(\omega,z_1(\omega),z_2(\omega))| \le L_2r+N_2, \tag{18}$$

$$|g_1(\omega,z_1(\omega),z_2(\omega))| \le K_1r+M_1, \tag{19}$$

$$|g_2(\omega,z_1(\omega),z_2(\omega))| \le K_2r+M_2. \tag{20}$$

Thus, we obtain

$$\begin{aligned} &\|\mathcal{T}_1(z_1,z_2)\| \\ \le\ & \sup_{\omega\in[0,1]}\Bigg\{\int_0^\omega \frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\Big|F^1_{z_1,z_2}(x)\Big|+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\Big|G^1_{z_1,z_2}(x)\Big|+|\lambda_1||z_1(x)|\Big)d_qx \\ &+|A_1|\Bigg[\int_0^{\beta_1} \frac{(\beta_1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\Big|F^1_{z_1,z_2}(x)\Big|+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\Big|G^1_{z_1,z_2}(x)\Big|+|\lambda_1||z_1(x)|\Big)d_qx\Bigg] \\ &+|A_2|\Bigg[\int_0^{\beta_2} \frac{(\beta_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\Big|F^1_{z_1,z_2}(x)\Big|+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\Big|G^1_{z_1,z_2}(x)\Big|+|\lambda_1||z_1(x)|\Big)d_qx\Bigg] \\ &+|A_3|\Bigg[\int_0^{1} \frac{(1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\Big|F^1_{z_1,z_2}(x)\Big|+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\Big|G^1_{z_1,z_2}(x)\Big|+|\lambda_1||z_1(x)|\Big)d_qx\Bigg] \\ &+|A_4|\Bigg[\int_0^{1} \frac{(1-qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\Big|F^1_{z_1,z_2}(x)\Big|+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\Big|G^1_{z_1,z_2}(x)\Big|+|\lambda_1||z_1(x)|\Big)d_qx\Bigg]\Bigg\}. \end{aligned}$$

By (17)–(20), we have

$$\begin{aligned} &\|\mathcal{T}_1(z_1,z_2)\| \\ \le\ & \left(\frac{(L_1r+N_1)\alpha_1}{\Gamma_q(\Upsilon_1+\Psi_1+1)}\right)\Big(\omega^{\Upsilon_1+\Psi_1}+|A_1|\beta_1^{\Upsilon_1+\Psi_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1}+|A_3|+|A_4|\Big) \\ &+\left(\frac{(K_1r+M_1)\gamma_1}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)}\right)\Big(\omega^{\Upsilon_1+\Psi_1+\zeta_1}+|A_1|\beta_1^{\Upsilon_1+\Psi_1+\zeta_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1+\zeta_1}+|A_3|+|A_4|\Big) \\ &+\frac{|\lambda_1|\|z_1\|}{\Gamma_q(\Upsilon_1+1)}\Big(\omega^{\Upsilon_1}+|A_1|\beta_1^{\Upsilon_1}+|A_2|\beta_2^{\Upsilon_1}+|A_3|+|A_4|\Big) \\ \le\ & (L_1\rho_2\alpha_1+K_1\rho_3\gamma_1)r+|\lambda_1|\rho_1\|z_1\|+(N_1\rho_2\alpha_1+M_1\rho_3\gamma_1). \end{aligned}$$

In the same way, we can find that

$$\|\mathcal{T}_2(z_1,z_2)\| \le (L_2\rho_5\alpha_2+K_2\rho_6\gamma_2)r+|\lambda_2|\rho_4\|z_2\|+(N_2\rho_5\alpha_2+M_2\rho_6\gamma_2).$$

From the above inequalities, we have

$$\begin{aligned} \|\mathcal{T}(z_1,z_2)\| &\le \|\mathcal{T}_1(z_1,z_2)\|+\|\mathcal{T}_2(z_1,z_2)\| \\ &\le ((L_1\rho_2\alpha_1+K_1\rho_3\gamma_1+L_2\rho_5\alpha_2+K_2\rho_6\gamma_2)+\max\{|\lambda_1|\rho_1,|\lambda_2|\rho_4\})r \\ &\quad +(N_1\rho_2\alpha_1+M_1\rho_3\gamma_1+N_2\rho_5\alpha_2+M_2\rho_6\gamma_2) \\ &\le r, \end{aligned}$$

which indicates $\mathcal{T}(B_r)\subset B_r$. Now, by applying conditions (H_1) and (H_2), and for any $(z_1,z_2), \left(z_1^*,z_2^*\right)\in B_r, \omega\in[0,1]$, we have

$$
\begin{aligned}
& \|\mathcal{T}_1(z_1,z_2)-\mathcal{T}_1(z_1^*,z_2^*)\| \\
\leq\; & \left(\frac{L_1(\|z_1-z_1^*\|+\|z_2-z_2^*\|)\alpha_1}{\Gamma_q(\Upsilon_1+\Psi_1+1)}\right)\Big(\omega^{\Upsilon_1+\Psi_1}+|A_1|\beta_1^{\Upsilon_1+\Psi_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1}+|A_3|+|A_4|\Big) \\
& +\left(\frac{K_1(\|z_1-z_1^*\|+\|z_2-z_2^*\|)\gamma_1}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)}\right)\Big(\omega^{\Upsilon_1+\Psi_1+\zeta_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1+\zeta_1}+|A_4| \\
& |A_1|\beta_1^{\Upsilon_1+\Psi_1+\zeta_1}+|A_3|\Big)+\frac{|\lambda_1|\|z_1-z_1^*\|}{\Gamma_q(\Upsilon_1+1)}\Big(\omega^{\Upsilon_1}+|A_1|\beta_1^{\Upsilon_1}+|A_2|\beta_2^{\Upsilon_1}+|A_3|+|A_4|\Big) \\
\leq\; & (\|z_1-z_1^*\|+\|z_2-z_2^*\|)(L_1\rho_2\alpha_1+K_1\rho_3\gamma_1)+|\lambda_1|\rho_1\|z_1-z_1^*\|.
\end{aligned}
$$

Similarly, one can obtain

$$
\begin{aligned}
& \|\mathcal{T}_2(z_1,z_2)-\mathcal{T}_2(z_1^*,z_2^*)\| \\
\leq\; & (\|z_1-z_1^*\|+\|z_2-z_2^*\|)(L_2\rho_5\alpha_2+K_2\rho_6\gamma_2)+|\lambda_2|\rho_4\|z_2-z_2^*\|.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
& \|\mathcal{T}(z_1,z_2)-\mathcal{T}(z_1^*,z_2^*)\| \\
\leq\; & \|\mathcal{T}_1(z_1,z_2)-\mathcal{T}_1(z_1^*,z_2^*)\|+\|\mathcal{T}_2(z_1,z_2)-\mathcal{T}_2(z_1^*,z_2^*)\| \\
\leq\; & (L_1\rho_2\alpha_1+K_1\rho_3\gamma_1+L_2\rho_5\alpha_2+K_2\rho_6\gamma_2+\max\{|\lambda_1|\rho_1,|\lambda_2|\rho_4\}) \\
& \times(\|z_1-z_1^*\|+\|z_2-z_2^*\|).
\end{aligned}
$$

Since

$$(L_1\rho_2\alpha_1+K_1\rho_3\gamma_1+L_2\rho_5\alpha_2+K_2\rho_6\gamma_2+\max\{|\lambda_1|\rho_1,|\lambda_2|\rho_4\})<1.$$

We conclude that the operator $\mathcal{T}$ is a contraction. As a result of the conclusion of the Banach contraction principle, we deduce that the operator $\mathcal{T}$ has a unique fixed point and, hence, system (2) has a unique solution. □

3.3. Existence of Solutions

In the following results, we establish the existence of solutions for the system (2) by employing the Leray–Schauder alternative [31].

Theorem 6. *Assume that (H_2) holds. Then, system (2) has at least one solution on $[0,1]$, provided that $0<\wp_1,\wp_2<1$, where*

$$
\begin{aligned}
\wp_1 &= [(\mathrm{m}_1\alpha_1\rho_2+\mathrm{n}_1\gamma_1\rho_3)+(\widetilde{\mathrm{m}}_1\alpha_2\rho_5+\widetilde{\mathrm{n}}_1\gamma_2\rho_6)+|\lambda_1|\rho_1], \\
\wp_2 &= [(\mathrm{m}_2\alpha_1\rho_2+\mathrm{n}_2\gamma_1\rho_3)+(\widetilde{\mathrm{m}}_2\alpha_2\rho_5+\widetilde{\mathrm{n}}_2\gamma_2\rho_6)+|\lambda_2|z_2\rho_4],
\end{aligned}
$$

and $\rho_i,(i=1,\cdots,6)$ are given in (6).

Proof. We demonstrate in the first step that the operator $\mathcal{T}:S\times S\to S\times S$, defined by (13), is completely continuous. By the continuity of functions $f_i,g_i,i=1,2$, we conclude that the operator $\mathcal{T}$ is continuous. Let $Z\subset S\times S$ be bounded. Then, for all $(z_1,z_2)\in Z$, there exist constants $\kappa_i,\tau_i,i=1,2$, such that

$$
\begin{aligned}
|f_i(\omega,z_1(\omega),z_2(\omega))| &\leq \kappa_i, \\
|g_i(\omega,z_1(\omega),z_2(\omega))| &\leq \tau_i.
\end{aligned}
$$

Let $(z_1,z_2)\in Z$. Then there exists Q, such that $\|(z_1,z_2)\|\leq\|z_1\|+\|z_2\|\leq Q$. Then, for any $(z_1,z_2)\in Z$, we have

$$\begin{aligned}
&\|\mathcal{T}_1(z_1,z_2)\| \\
\le\ & \int_0^{\omega} \frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\kappa_1+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\tau_1+|\lambda_1|\|z_1\|\Big)d_qx \\
&+|A_1|\int_0^{\beta_1} \frac{(\beta_1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\kappa_1+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\tau_1+|\lambda_1|\|z_1\|\Big)d_qx \\
&+|A_2|\int_0^{\beta_2} \frac{(\beta_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\kappa_1+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\tau_1+|\lambda_1|\|z_1\|\Big)d_qx \\
&+|A_3|\int_0^{1} \frac{(1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\kappa_1+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\tau_1+|\lambda_1|\|z_1\|\Big)d_qx \\
&+|A_4|\int_0^{1} \frac{(1-qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\kappa_1+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\tau_1+|\lambda_1|\|z_1\|\Big)d_qx \\
\le\ & \left(\frac{\kappa_1\alpha_1}{\Gamma_q(\Upsilon_1+\Psi_1+1)}\right)\Big(\omega^{\Upsilon_1+\Psi_1}+|A_1|\beta_1^{\Upsilon_1+\Psi_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1}+|A_3|+|A_4|\Big) \\
&+\left(\frac{\tau_1\gamma_1}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)}\right)\Big(\omega^{\Upsilon_1+\Psi_1+\zeta_1}+|A_1|\beta_1^{\Upsilon_1+\Psi_1+\zeta_1}+|A_2|\beta_2^{\Upsilon_1+\Psi_1+\zeta_1}+|A_3|+|A_4|\Big) \\
&+\frac{|\lambda_1|\|z_1\|}{\Gamma_q(\Upsilon_1+1)}\Big(\omega^{\Upsilon_1}+|A_1|\beta_1^{\Upsilon_1}+|A_2|\beta_2^{\Upsilon_1}+|A_3|+|A_4|\Big) \\
\le\ & (\kappa_1\rho_2\alpha_1+\tau_1\rho_3\gamma_1)+\|z_1\||\lambda_1|\rho_1.
\end{aligned}$$

Similarly, we can find that

$$\|\mathcal{T}_2(z_1,z_2)\| \le (\kappa_2\rho_5\alpha_2+\tau_2\rho_6\gamma_2)+|\lambda_2|\|z_2\|\rho_4.$$

Consequently, we have

$$\|\mathcal{T}(z_1,z_2)\| \le (\kappa_1\rho_2\alpha_1+\tau_1\rho_3\gamma_1+\kappa_2\rho_5\alpha_2+\tau_2\rho_6\gamma_2)+\max(|\lambda_1|\rho_1+|\lambda_2|\rho_4)Q.$$

Therefore, the operator $\mathcal{T}$ is uniformly bounded.

Next, we show that the operator $\mathcal{T}$ is equicontinuous. Let $\omega_1,\omega_2\in[0,1]$ with $\omega_1<\omega_2$. Then, we have

$$\begin{aligned}
&|\mathcal{T}_1(z_1(\omega_2),z_2(\omega_2))-\mathcal{T}_1(z_1(\omega_1),z_2(\omega_1)| \\
\le\ & \int_0^{\omega_1}\left(\frac{(\omega_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}-\frac{(\omega_1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\right)\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\kappa_1+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\tau_1+|\lambda_1|\|z_1\|\Big)d_qx \\
&+\int_{\omega_1}^{\omega_2}\frac{(\omega_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}\kappa_1+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\tau_1+|\lambda_1|\|z_1\|\Big)d_qx \\
\le\ & \left(\frac{\kappa_1\alpha_1}{\Gamma_q(\Upsilon_1+\Psi_1+1)}\right)\Big(2(\omega_2-\omega_1)^{\Upsilon_1+\Psi_1}-(\omega_2^{\Upsilon_1+\Psi_1}-\omega_1^{\Upsilon_1+\Psi_1})\Big) \\
&+\left(\frac{\tau_1\gamma_1}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)}\right)\Big(2(\omega_2-\omega_1)^{\Upsilon_1+\Psi_1}-(\omega_2^{\Upsilon_1+\Psi_1}-\omega_1^{\Upsilon_1+\Psi_1})\Big) \\
&+\frac{|\lambda_1|\|z_1\|}{\Gamma_q(\Upsilon_1+1)}\Big(2(\omega_2-\omega_1)^{\Upsilon_1+\Psi_1}-(\omega_2^{\Upsilon_1+\Psi_1}-\omega_1^{\Upsilon_1+\Psi_1})\Big). \\
\le\ & (\kappa_1\alpha_1\rho_2+\tau_1\gamma_1\rho_3+|\lambda_1|\|z_1\|\rho_1)\Big(2(\omega_2-\omega_1)^{\Upsilon_1+\Psi_1}-(\omega_2^{\Upsilon_1+\Psi_1}-\omega_1^{\Upsilon_1+\Psi_1})\Big).
\end{aligned}$$

Thus, we have

$$|\mathcal{T}_1(z_1(\omega_2),z_2(\omega_2))-\mathcal{T}_1(z_1(\omega_1),z_2(\omega_1)|\to 0.$$

as $(\omega_2-\omega_1)\to 0$.

Analogously, we can obtain

$$|\mathcal{T}_2(z_1(\omega_2), z_2(\omega_2)) - \mathcal{T}_2(z_1(\omega_1), z_2(\omega_1)| \to 0,$$

as $(\omega_2 - \omega_1) \to 0$, This shows the equicontinuous of $\mathcal{T}(z_1, z_2)$. Based on the preceding arguments, we conclude that the operator $\mathcal{T}(z_1, z_2)$ is completely continuous. Finally, we prove that $\Phi = \{(z_1, z_2) \in S \times S | (z_1, z_2) = \xi \mathcal{T}(z_1, z_2), 0 < \xi < 1\}$ is bounded. Let $(z_1, z_2) \in \Phi$ with $(z_1, z_2)(\omega) = \xi \mathcal{T}(z_1, z_2)(\omega)$. Then, for any $\omega \in [0, 1]$, we have

$$\begin{aligned} z_i(\omega) &= \xi \mathcal{T}_i(z_1, z_2)(\omega) \\ &\leq \mathcal{T}_i(z_1, z_2)(\omega), i = 1, 2. \end{aligned}$$

In view of condition (H_2), with some calculations, we can find that

$$\begin{aligned} & |z_1(\omega)| \\ \leq & \left(\frac{(m_0 + m_1|z_1| + m_2|z_2|)\alpha_1}{\Gamma_q(\Upsilon_1 + \Psi_1 + 1)} \right) \left(\omega^{\Upsilon_1 + \Psi_1} + |A_1|\beta_1^{\Upsilon_1 + \Psi_1} + |A_2|\beta_2^{\Upsilon_1 + \Psi_1} + |A_3| + |A_4| \right) \\ & + \left(\frac{(n_0 + n_1|z_1| + n_2|z_2|)\gamma_1}{\Gamma_q(\Upsilon_1 + \Psi_1 + \zeta_1 + 1)} \right) \left(\omega^{\Upsilon_1 + \Psi_1 + \zeta_1} + |A_1|\beta_1^{\Upsilon_1 + \Psi_1 + \zeta_1} + |A_2|\beta_2^{\Upsilon_1 + \Psi_1 + \zeta_1} + |A_3| + |A_4| \right) \\ & + \frac{|\lambda_1||z_1|}{\Gamma_q(\Upsilon_1 + 1)} \left(\omega^{\Upsilon_1} + |A_1|\beta_1^{\Upsilon_1} + |A_2|\beta_2^{\Upsilon_1} + |A_3| + |A_4| \right) \\ \leq & (m_0 + m_1|z_1| + m_2|z_2|)\alpha_1\rho_2 + (n_0 + n_1|z_1| + n_2|z_2|)\gamma_1\rho_3 + |\lambda_1||z_1|\rho_1, \end{aligned}$$

and

$$|z_2(\omega)| \leq (\widetilde{m}_0 + \widetilde{m}_1|z_1| + \widetilde{m}_2|z_2|)\alpha_2\rho_5 + (\widetilde{n}_0 + \widetilde{n}_1|z_1| + \widetilde{n}_2|z_2|)\gamma_2\rho_6 + |\lambda_2||z_2|\rho_4.$$

Thus, we have

$$\begin{aligned} \|z_1\| \leq & (m_0\alpha_1\rho_2 + n_0\gamma_1\rho_3) + (m_1\alpha_1\rho_2 + n_1\gamma_1\rho_3 + |\lambda_1|\rho_1)\|z_1\| \\ & + (m_2\alpha_1\rho_2 + n_2\gamma_1\rho_3)\|z_2\|, \end{aligned}$$

and

$$\begin{aligned} \|z_2\| \leq & (\widetilde{m}_0\alpha_2\rho_5 + \widetilde{n}_0\gamma_2\rho_6) + (\widetilde{m}_1\alpha_2\rho_5 + \widetilde{n}_1\gamma_2\rho_6)\|z_1\| \\ & + (\widetilde{m}_2\alpha_2\rho_5 + \widetilde{n}_2\gamma_2\rho_6 + |\lambda_2|z_2\rho_4)\|z_2\|. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|(z_1, z_2)\| &\leq \|z_1\| + \|z_2\| \\ &\leq m_0\alpha_1\rho_2 + n_0\gamma_1\rho_3 + \widetilde{m}_0\alpha_2\rho_5 + \widetilde{n}_0\gamma_2\rho_6 \\ &\quad + \max\{\wp_1, \wp_2\}(\|z_1\| + \|z_2\|) \\ &\leq \frac{1}{G_0}[m_0\alpha_1\rho_2 + n_0\gamma_1\rho_3 + \widetilde{m}_0\alpha_2\rho_5 + \widetilde{n}_0\gamma_2\rho_6], \end{aligned}$$

where

$$G_0 = \min\{\wp_1, \wp_2\},$$

which proves the Φ is bounded. Thus, according to the Leray–Schauder alternative [31], the operator $\mathcal{T}$ has at least one solution, which means that there exists a solution of system (2) on $[0, 1]$. □

4. Stability Analysis

Stability analysis plays a crucial role in understanding the behavior and dynamics of a coupled system of nonlinear fractional q-difference equations with Caputo fractional derivatives in a boundary value problem context. By investigating the stability properties of the system, we can determine whether small perturbations in the initial or boundary conditions lead to significant changes in the system's solutions over time. Stability analysis helps identify stable solutions that are robust and converge to a desired equilibrium or periodic behavior, providing valuable insights into the system's long-term behavior and predictability. Moreover, stability analysis aids in determining the critical parameter ranges or conditions under which the system exhibits stability or undergoes bifurcations, which are characteristic of qualitative changes in the system's dynamics. Understanding the stability properties of the coupled system is essential for ensuring the reliability and applicability of the model in real-world scenarios, guiding parameter selection, and assessing the system's response to uncertainties or variations in the governing equations [32,33]. In this section, we shall discuss the Ulam–Hyers stability of system (2).

Remark 1. *A function* $(\widehat{z}_1, \widehat{z}_2) \in S \times S$ *satisfies the following inequalities*

$$\begin{cases} \left| {}^C\mathbb{D}_q^{\Psi_1}\left({}^C\mathbb{D}_q^{\Upsilon_1} + \lambda_1\right)\widehat{z}_1(\omega) - \alpha_1 f_1(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) - \gamma_1 \mathbb{I}_q^{\zeta_1} g_1(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) \right| \le \varepsilon_1, \\ \left| {}^C\mathbb{D}_q^{\Psi_2}\left({}^C\mathbb{D}_q^{\Upsilon_2} + \lambda_2\right)\widehat{z}_2(\omega) - \alpha_2 f_2(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) - \gamma_2 \mathbb{I}_q^{\zeta_2} g_2(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) \right| \le \varepsilon_2, \end{cases} \tag{21}$$

if and only if there exist functions $\hbar_1, \hbar_2 \in D$, *such that*

(i) $\begin{cases} |\hbar_1(\omega)| \le \varepsilon_1, \\ |\hbar_2(\omega)| \le \varepsilon_2 . \end{cases}$

(ii) $\begin{cases} {}^C\mathbb{D}_q^{\Psi_1}\left({}^C\mathbb{D}_q^{\Upsilon_1} + \lambda_1\right)\widehat{z}_1(\omega) = \alpha_1 f_1(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) + \gamma_1 \mathbb{I}_q^{\zeta_1} g_1(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) + \hbar_1(\omega), \\ {}^C\mathbb{D}_q^{\Psi_2}\left({}^C\mathbb{D}_q^{\Upsilon_2} + \lambda_2\right)\widehat{z}_2(\omega) = \alpha_2 f_2(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) + \gamma_2 \mathbb{I}_q^{\zeta_2} g_2(\omega, \widehat{z}_1(\omega), \widehat{z}_2(\omega)) + \hbar_2(\omega). \end{cases}.$

Definition 3. *System (2) is UH-stable if there exists* $\mathcal{W} > 0$, *such that, for each* $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$ *and each solution* $(\widehat{z}_1, \widehat{z}_2) \in S \times S$ *of the inequalities (21), there exists a solution* $(z_1, z_2) \in S \times S$ *of system (2) with*

$$\|(\widehat{z}_1, \widehat{z}_2) - (z_1, z_2)\| \le \mathcal{W}\varepsilon, \ \ \sigma \in \mathcal{J}.$$

Lemma 3. *If a function* $(\widehat{z}_1, \widehat{z}_2) \in S \times S$ *satisfies the inequalities (21), then* $(\widehat{z}_1, \widehat{z}_2)$ *satisfies the following integral inequalities*

$$\begin{cases} \left| \widehat{z}_1(\omega) - \mathfrak{R}_{\widehat{z}_1} - \int_0^\omega \frac{(\omega - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \left(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{\widehat{z}_1, \widehat{z}_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1 + \zeta_1 - 1} G^1_{\widehat{z}_1, \widehat{z}_2}(x) - \lambda \widehat{z}_1(x) \right) d_q x \right| \\ \qquad \le \mathcal{M}_1 \varepsilon_1, \\ \left| \widehat{z}_2(\omega) - \mathfrak{R}_{\widehat{z}_2} - \int_0^\omega \frac{(\omega - qx)^{(\Upsilon_2 - 1)}}{\Gamma_q(\Upsilon_2)} \left(\alpha_2 \mathbb{I}_q^{\Psi_2} F^2_{\widehat{z}_1, \widehat{z}_2}(x) + \gamma_2 \mathbb{I}_q^{\Psi_2 + \zeta_2 - 1} G^2_{\widehat{z}_1, \widehat{z}_2}(x) - \lambda \widehat{z}_2(x) \right) d_q x \right| \\ \qquad \le \mathcal{M}_2 \varepsilon_2, \end{cases}$$

where

$$\begin{aligned} \mathfrak{R}_{\widehat{z}_1} \;=\; & A_1 \left[\int_0^{\beta_1} \frac{(\beta_1 - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \left(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{\widehat{z}_1, \widehat{z}_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1 + \zeta_1 - 1} G^1_{\widehat{z}_1, \widehat{z}_2}(x) - \lambda_1 \widehat{z}_1(x) \right) d_q x \right] \\ & - A_2 \left[\int_0^{\beta_2} \frac{(\beta_2 - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \left(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{\widehat{z}_1, \widehat{z}_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1 + \zeta_1 - 1} G^1_{\widehat{z}_1, \widehat{z}_2}(x) - \lambda_1 \widehat{z}_1(x) \right) d_q x \right] \\ & + A_3 \left[\int_0^{1} \frac{(1 - qx)^{(\Upsilon_1 - 1)}}{\Gamma_q(\Upsilon_1)} \left(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{\widehat{z}_1, \widehat{z}_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1 + \zeta_1 - 1} G^1_{\widehat{z}_1, \widehat{z}_2}(x) - \lambda_1 \widehat{z}_1(x) \right) d_q x \right] \\ & - A_4 \left[\int_0^{1} \frac{(1 - qx)^{(\Upsilon_1 - 2)}}{\Gamma_q(\Upsilon_1 - 1)} \left(\alpha_1 \mathbb{I}_q^{\Psi_1} F^1_{\widehat{z}_1, \widehat{z}_2}(x) + \gamma_1 \mathbb{I}_q^{\Psi_1 + \zeta_1 - 1} G^1_{\widehat{z}_1, \widehat{z}_2}(x) - \lambda_1 \widehat{z}_1(x) \right) d_q x \right], \end{aligned}$$

$$\begin{aligned}\mathfrak{R}_{\widehat{z}_2} &= A_5\left[\int_0^{\beta_3}\frac{(\beta_3-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{\widehat{z}_1,\widehat{z}_2}(x)+\gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{\widehat{z}_1,\widehat{z}_2}(x)-\lambda_2\widehat{z}_2(x)\Big)d_qx\right]\\&+A_6\left[\int_0^{\beta_4}\frac{(\beta_4-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{\widehat{z}_1,\widehat{z}_2}(x)+\gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{\widehat{z}_1,\widehat{z}_2}(x)-\lambda_2\widehat{z}_2(x)\Big)d_qx\right]\\&+A_7\left[\int_0^{1}\frac{(1-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{\widehat{z}_1,\widehat{z}_2}(x)+\gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{\widehat{z}_1,\widehat{z}_2}(x)-\lambda_2\widehat{z}_2(x)\Big)d_qx\right]\\&+A_8\left[\int_0^{1}\frac{(1-qx)^{(\Upsilon_2-2)}}{\Gamma_q(\Upsilon_2-1)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{\widehat{z}_1,\widehat{z}_2}(x)+\gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{\widehat{z}_1,\widehat{z}_2}(x)-\lambda_2\widehat{z}_2(x)\Big)d_qx\right],\end{aligned}$$

$$\mathcal{M}_1=\frac{\Big(|A_1|\beta_1^{\Upsilon_1}+|A_2|\beta_2^{\Upsilon_1}+|A_3|+|A_4|\Big)}{\Gamma_q(\Upsilon_1+1)},$$

and

$$\mathcal{M}_2=\frac{\Big(|A_5|\beta_3^{\Upsilon_2}+|A_6|\beta_4^{\Upsilon_2}+|A_7|+|A_8|\Big)}{\Gamma_q(\Upsilon_2+1)}.$$

Proof. By Remark 1, we have

$${}^C\mathbb{D}_q^{\Psi_1}\Big({}^C\mathbb{D}_q^{\Upsilon_1}+\lambda_1\Big)\widehat{z}_1(\omega)=\alpha_1f_1(\omega,\widehat{z}_1(\omega),\widehat{z}_2(\omega))+\gamma_1\mathbb{I}_q^{\zeta_1}g_1(\omega,\widehat{z}_1(\omega),\widehat{z}_2(\omega))+\hbar_1(\omega).$$

Then, in view of Lemma 3, we have

$$\begin{aligned}&\left|\widehat{z}_1(\omega)-\mathfrak{R}_{\widehat{z}_1}-\int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\Big(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{\widehat{z}_1,\widehat{z}_2}(x)+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{\widehat{z}_1,\widehat{z}_2}(x)-\lambda\widehat{z}_1(x)\Big)d_qx\right|\\\leq\;&|A_1|\left[\int_0^{\beta_1}\frac{(\beta_1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}(\hbar_1(x))d_qx\right]+|A_2|\left[\int_0^{\beta_2}\frac{(\beta_2-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}(\hbar_1(x))d_qx\right]\\&+|A_3|\left[\int_0^{1}\frac{(1-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}(\hbar_1(x))d_qx\right]+|A_4|\left[\int_0^{1}\frac{(1-qx)^{(\Upsilon_1-2)}}{\Gamma_q(\Upsilon_1-1)}(\hbar_1(x))d_qx\right]\\\leq\;&\frac{\Big(|A_1|\beta_1^{\Upsilon_1}+|A_2|\beta_2^{\Upsilon_1}+|A_3|+|A_4|\Big)}{\Gamma_q(\Upsilon_1+1)}\varepsilon_1\\\leq\;&\mathcal{M}_1\varepsilon_1.\end{aligned}$$

In the same way, one can obtain

$$\begin{aligned}&\left|\widehat{z}_2(\omega)-\mathfrak{R}_{\widehat{z}_2}-\int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\Big(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{\widehat{z}_1,\widehat{z}_2}(x)+\gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{\widehat{z}_1,\widehat{z}_2}(x)-\lambda\widehat{z}_2(x)\Big)d_qx\right|\\\leq\;&\mathcal{M}_2\varepsilon_2.\end{aligned}$$

□

Theorem 7. *Assume that (H_2) holds. If $\mathcal{Y}_2+\mathcal{Y}_1<1$, where*

$$\mathcal{Y}_1=\left[\left(\frac{L_1\alpha_1\omega^{\Upsilon_1+\Psi_1}}{\Gamma_q(\Upsilon_1+\Psi_1+1)}\right)+\left(\frac{K_1\gamma_1\omega^{\Upsilon_1+\Psi_1+\zeta_1}}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)}\right)\right],$$

and

$$\mathcal{Y}_2=\left[\left(\frac{L_2\alpha_2\omega^{\Upsilon_2+\Psi_2}}{\Gamma_q(\Upsilon_2+\Psi_2+1)}\right)+\left(\frac{K_2\gamma_2\omega^{\Upsilon_2+\Psi_2+\zeta_2}}{\Gamma_q(\Upsilon_2+\Psi_2+\zeta_2+1)}\right)\right].$$

Then

$$\begin{cases} {}^{C}\mathbb{D}_q^{\Psi_1}\left({}^{C}\mathbb{D}_q^{\Upsilon_1}+\lambda_1\right)\widehat{z}_1(\omega)=\alpha_1 f_1(\omega,\widehat{z}_1(\omega),\widehat{z}_2(\omega))+\gamma_1\mathbb{I}_q^{\zeta_1}g_1(\omega,\widehat{z}_1(\omega),\widehat{z}_2(\omega)),\\ {}^{C}\mathbb{D}_q^{\Psi_2}\left({}^{C}\mathbb{D}_q^{\Upsilon_2}+\lambda_2\right)\widehat{z}_2(\omega)=\alpha_2 f_2(\omega,\widehat{z}_1(\omega),\widehat{z}_2(\omega))+\gamma_2\mathbb{I}_q^{\zeta_2}g_2(\omega,\widehat{z}_1(\omega),\widehat{z}_2(\omega)),\end{cases} \tag{22}$$

are Ulam–Hyers stable.

Proof. Let $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}>0$ and $(\widehat{z}_1,\widehat{z}_2)\in S\times S$ be a function that satisfies the inequalities (21) and let $(z_1,z_2)\in S\times S$ be the unique solution of the following system

$$\begin{cases} {}^{C}\mathbb{D}_q^{\Psi_1}\left({}^{C}\mathbb{D}_q^{\Upsilon_1}+\lambda_1\right)z_1(\omega)=\alpha_1 f_1(\omega,z_1(\omega),z_2(\omega))+\gamma_1\mathbb{I}_q^{\zeta_1}g_1(\omega,z_1(\omega),z_2(\omega)),\\ {}^{C}\mathbb{D}_q^{\Psi_2}\left({}^{C}\mathbb{D}_q^{\Upsilon_2}+\lambda_2\right)z_2(\omega)=\alpha_2 f_2(\omega,z_1(\omega),z_2(\omega))+\gamma_2\mathbb{I}_q^{\zeta_2}g_2(\omega,z_1(\omega),z_2(\omega)),\end{cases} \tag{23}$$

Now, by Lemma 3, we have

$$z_1(\omega)=\mathfrak{R}_{z_1}+\int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\left(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{z_1,z_2}(x)+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{z_1,z_2}(x)-\lambda_1 z_1(x)\right)d_qx,$$

and

$$z_2(\omega)=\mathfrak{R}_{z_2}+\int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_2-1)}}{\Gamma_q(\Upsilon_2)}\left(\alpha_2\mathbb{I}_q^{\Psi_2}F^2_{z_1,z_2}(x)+\gamma_2\mathbb{I}_q^{\Psi_2+\zeta_2-1}G^2_{z_1,z_2}(x)-\lambda_2 z_2(x)\right)d_qx.$$

Hence, from (H_2) with Lemma 3, and for each $\omega\in[0,1]$, we have

$$\begin{aligned} |\widehat{z}_1(\omega)-z_1(\omega)| \leq\ & \left|\widehat{z}_1(\omega)-\mathfrak{R}_{\widehat{z}_1}-\int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\left(\alpha_1\mathbb{I}_q^{\Psi_1}F^1_{\widehat{z}_1,\widehat{z}_2}(x)+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}G^1_{\widehat{z}_1,\widehat{z}_2}(x)-\lambda_1\widehat{z}_1(x)\right)d_qx\right| \\ & +\int_0^{\omega}\frac{(\omega-qx)^{(\Upsilon_1-1)}}{\Gamma_q(\Upsilon_1)}\left(\alpha_1\mathbb{I}_q^{\Psi_1}\left|F^1_{\widehat{z}_1,\widehat{z}_2}(x)-F^1_{z_1,z_2}(x)\right|\right. \\ & \left.+\gamma_1\mathbb{I}_q^{\Psi_1+\zeta_1-1}\left|G^1_{\widehat{z}_1,\widehat{z}_2}(x)-G^1_{z_1,z_2}(x)\right|+\lambda_1|\widehat{z}_1(x)-z_1(x)|d_qx\right) \\ \leq\ & \mathcal{M}_1\varepsilon_1+(\|\widehat{z}_1-z_1\|+\|\widehat{z}_2-z_2\|)\mathcal{Y}_1. \end{aligned} \tag{24}$$

where

$$\mathcal{Y}_1=\left[\left(\frac{L_1\alpha_1\omega^{\Upsilon_1+\Psi_1}}{\Gamma_q(\Upsilon_1+\Psi_1+1)}\right)+\left(\frac{K_1\gamma_1\omega^{\Upsilon_1+\Psi_1+\zeta_1}}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)}\right)\right].$$

Hence,

$$\|\widehat{z}_1-z_1\|\leq\mathcal{M}_1\varepsilon_1+(\|\widehat{z}_1-z_1\|+\|\widehat{z}_2-z_2\|)\mathcal{Y}_1. \tag{25}$$

By the same technique, we have

$$\|\widehat{z}_2-z_2\|\leq\mathcal{M}_2\varepsilon_2+(\|\widehat{z}_1-z_1\|+\|\widehat{z}_2-z_2\|)\mathcal{Y}_2, \tag{26}$$

where

$$\mathcal{Y}_2=\left[\left(\frac{L_2\alpha_2\omega^{\Upsilon_2+\Psi_2}}{\Gamma_q(\Upsilon_2+\Psi_2+1)}\right)+\left(\frac{K_2\gamma_2\omega^{\Upsilon_2+\Psi_2+\zeta_2}}{\Gamma_q(\Upsilon_2+\Psi_2+\zeta_2+1)}\right)\right].$$

Thus,

$$\begin{aligned} & \|(\widehat{z}_1,\widehat{z}_2)-(z_1,z_2)\| \\ \leq\ & \|\widehat{z}_1-z_1\|+\|\widehat{z}_2-z_2\| \\ \leq\ & \mathcal{M}_1\varepsilon_1+\mathcal{M}_2\varepsilon_2+(\|\widehat{z}_1-z_1\|+\|\widehat{z}_2-z_2\|)(\mathcal{Y}_2+\mathcal{Y}_1) \\ \leq\ & \varepsilon(\mathcal{M}_1+\mathcal{M}_2)+\|(\widehat{z}_1,\widehat{z}_2)-(z_1,z_2)\|(\mathcal{Y}_2+\mathcal{Y}_1) \\ \leq\ & \varepsilon\mathcal{W}, \end{aligned} \tag{27}$$

where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$ and

$$\mathcal{W} = \frac{(\mathcal{M}_1 + \mathcal{M}_2)}{1 - (\mathcal{Y}_2 + \mathcal{Y}_1)} > 0.$$

Hence, from (27) and Definition 3, we deduce that the coupled system (22) is Ulam–Hyers (UH)-stable. □

Parameter Sensitivity Analysis

The sensitivity of the behavior of the coupled system (2) comprising nonlinear fractional q-difference equations with Caputo fractional derivatives, to parameter changes, depends on the specific ranges of these parameters and constants. Moreover, their interrelationships may vary depending on the specific problem being modeled, along with any additional constraints or considerations. In this section, we will discuss the parameter sensitivity analysis for the sufficient conditions required for the existence and uniqueness of solutions, as well as for the conditions of stability within the aforementioned ranges.

- The existence of a solution is crucial for validating the model, establishing feasibility and robustness, solving boundary value problems, enabling mathematical analysis, supporting practical applications, and developing a fundamental understanding of the system's behavior. It ensures that the system can be adequately described, analyzed, and utilized in various domains and applications. For the system to be solvable, the parameters must be chosen within specific ranges so that the following condition is met:

$$0 < \max\{\wp_1, \wp_2\} < 1,$$

where

$$\begin{aligned} \wp_1 &= [(m_1\alpha_1\rho_2 + n_1\gamma_1\rho_3) + (\widetilde{m}_1\alpha_2\rho_5 + \widetilde{n}_1\gamma_2\rho_6) + |\lambda_1|\rho_1], \\ \wp_2 &= [(m_2\alpha_1\rho_2 + n_2\gamma_1\rho_3) + (\widetilde{m}_2\alpha_2\rho_5 + \widetilde{n}_2\gamma_2\rho_6) + |\lambda_2|z_2\rho_4]. \end{aligned}$$

- The uniqueness of the solution ensures the predictability, reliability, stability, and validity of the mathematical model. It plays a crucial role in understanding and analyzing the behavior of systems in diverse fields, enabling accurate predictions, decision-making, and parameter estimation. For the solution of system (2) to be unique, the parameters must be chosen so that the following conditions are met:

$$(L_1\rho_2\alpha_1 + K_1\rho_3\gamma_1 + L_2\rho_5\alpha_2 + K_2\rho_6\gamma_2) + \max\{|\lambda_1|\rho_1, |\lambda_2|\rho_4\} < 1.$$

- Stable solutions that are robust and converge to a desired equilibrium or periodic behavior, as described in system (1), provide valuable insights into the system's long-term behavior and predictability. To ensure the reliability and applicability of the model in real-world scenarios, it is crucial to select parameter ranges that satisfy the following condition: $\mathcal{Y}_2 + \mathcal{Y}_1 < 1$, where

$$\mathcal{Y}_1 = \left[\left(\frac{L_1\alpha_1\omega^{\Upsilon_1+\Psi_1}}{\Gamma_q(\Upsilon_1+\Psi_1+1)}\right) + \left(\frac{K_1\gamma_1\omega^{\Upsilon_1+\Psi_1+\zeta_1}}{\Gamma_q(\Upsilon_1+\Psi_1+\zeta_1+1)}\right)\right],$$

and

$$\mathcal{Y}_2 = \left[\left(\frac{L_2\alpha_2\omega^{\Upsilon_2+\Psi_2}}{\Gamma_q(\Upsilon_2+\Psi_2+1)}\right) + \left(\frac{K_2\gamma_2\omega^{\Upsilon_2+\Psi_2+\zeta_2}}{\Gamma_q(\Upsilon_2+\Psi_2+\zeta_2+1)}\right)\right].$$

5. Examples

This section presents an illustrative example that focuses on the coupled system governed by the Caputo-fractional derivative. The purpose of this example is to emphasize and reinforce our main conclusions. The selection of these examples takes into account the conditions stated in the employed theorems, the formulated conditions derived from our proposed results, and the consideration of various parameter values and fractional-order derivatives. Through these carefully chosen examples, we aim to provide strong support for all the arguments presented in the preceding section.

Example 1. *Consider the following coupled system of the Caputo-fractional derivative:*

$$\begin{cases} {}^C\mathbb{D}_q^{\Psi_1}\Big({}^C\mathbb{D}_q^{\Upsilon_1}+\lambda_1\Big)z_1(\omega)=\alpha_1 f_1(\omega,z_1(\omega),z_2(\omega))+\gamma_1\mathbb{I}_q^{\zeta_1}g_1(\omega,z_1(\omega),z_2(\omega)),\\ {}^C\mathbb{D}_q^{\Psi_2}\Big({}^C\mathbb{D}_q^{\Upsilon_2}+\lambda_2\Big)z_2(\omega)=\alpha_2 f_2(\omega,z_1(\omega),z_2(\omega))+\gamma_2\mathbb{I}_q^{\zeta_2}g_2(\omega,z_1(\omega),z_2(\omega)),\end{cases}$$

equipped with q-integral-coupled boundary conditions

$$\begin{cases} \mu_1 z_1(0)-\sigma_1\Big(\omega^{(1-\Upsilon_1)}\mathbb{D}_q z_1(0)\Big)_{\omega=0}=\eta_1 z_1(\beta_1),\\ \mu_2 z_1(1)+\sigma_2\mathbb{D}_q z_1(1)=\eta_2 z_1(\beta_2),\\ \mu_3 z_2(0)-\sigma_3\Big(\omega^{(1-\Upsilon_2)}\mathbb{D}_q z_2(0)\Big)_{\omega=0}=\eta_3 z_2(\beta_3),\\ \mu_4 z_2(1)+\sigma_4\mathbb{D}_q z_2(1)=\eta_4 z_2(\beta_4),\end{cases}$$

with $\alpha_1=\frac{1}{5},\alpha_2=\frac{1}{7},\gamma_1=\frac{1}{9},\gamma_2=\frac{1}{8},\lambda_1=\frac{1}{10},\lambda_2=\frac{1}{12},\Psi_1=\Psi_2=\frac{1}{3},\Upsilon_1=\Upsilon_2=\frac{1}{2},q=\frac{1}{2},$ $\zeta_1=\zeta_2=\frac{1}{2},\mu_1=\mu_2=\sigma_1=\sigma_2=1,\mu_3=\mu_4=\sigma_3=\sigma_4=\frac{1}{2},\beta_1=\beta_2=\beta_3=\beta_4=\frac{1}{3},$ $\eta_1=\eta_2=\eta_3=\eta_4=1$, *and*

$$\begin{aligned} f_1(\omega,z_1(\omega),z_2(\omega)) &= \frac{1}{100}\sin^2 z_1(\omega)+\frac{1}{4(\omega+6)^2}\frac{|z_2(\omega)|}{1+z_2(\omega)}+2,\\ f_2(\omega,z_1(\omega),z_2(\omega)) &= \frac{1}{3(\omega^3+144)^{\frac{1}{2}}}(\cos z_1(\omega)+z_2(\omega))+5e^{\omega},\\ g_1(\omega,z_1(\omega),z_2(\omega)) &= \frac{1}{40+\omega^2}(\sin z_1(\omega)+|z_2(\omega)|)-8\omega,\\ g_2(\omega,z_1(\omega),z_2(\omega)) &= \frac{1}{30}\Big(z_1(\omega)+\tan^{-1}z_2(\omega)+\sin\omega\Big).\end{aligned}$$

For each $\omega\in[0,1]$ *and* $z_1,z_2,z_1^*,z_2^*\in\mathbb{R}$

$$\begin{aligned} |f_1(\omega,z_1,z_2)-f_1(\omega,z_1^*,z_2^*)| &\le \frac{1}{100}(|z_1-z_1^*|+|z_2-z_2^*|),\\ |f_2(\omega,z_1,z_2)-f_2(\omega,z_1^*,z_2^*)| &\le \frac{1}{36}(|z_1-z_1^*|+|z_2-z_2^*|),\\ |g_1(\omega,z_1,z_2)-g_1(\omega,z_1^*,z_2^*)| &\le \frac{1}{40}(|z_1-z_1^*|+|z_2-z_2^*|),\\ |g_2(\omega,z_1,z_2)-g_2(\omega,z_1^*,z_2^*)| &\le \frac{1}{30}(|z_1-z_1^*|+|z_2-z_2^*|).\end{aligned}$$

Hence, (H_1) *holds. From the given data, we have* $L_1=\frac{1}{100},L_2=\frac{1}{36},K_1=\frac{1}{40},K_2=\frac{1}{30}$. *By some calculations, we have* $\rho_1\simeq 0.19,\rho_2\simeq 0.0450,\rho_3\simeq 0.240,\rho_4\simeq 0.183,\rho_5\simeq 0.2,\rho_6\simeq 0.025$. *Also,* $\Theta_1\simeq 0.2578<1$. *Thus, all conditions in Theorem 5 are satisfied and, hence, system (2) has a unique solution.*

Example 2. *Consider the following coupled system of the Caputo-fractional derivative*

$$\begin{cases} {}^C\mathbb{D}_q^{\Psi_1}\Big({}^C\mathbb{D}_q^{\Upsilon_1}+\lambda_1\Big)z_1(\omega)=\alpha_1 f_1(\omega,z_1(\omega),z_2(\omega))+\gamma_1\mathbb{I}_q^{\zeta_1}g_1(\omega,z_1(\omega),z_2(\omega)),\\ {}^C\mathbb{D}_q^{\Psi_2}\Big({}^C\mathbb{D}_q^{\Upsilon_2}+\lambda_2\Big)z_2(\omega)=\alpha_2 f_2(\omega,z_1(\omega),z_2(\omega))+\gamma_2\mathbb{I}_q^{\zeta_2}g_2(\omega,z_1(\omega),z_2(\omega)),\end{cases}$$

equipped with q-integral-coupled boundary conditions

$$\begin{cases} \mu_1 z_1(0) - \sigma_1 \Big(\omega^{(1-\Upsilon_1)} \mathbb{D}_q z_1(0)\Big)_{\omega=0} = \eta_1 z_1(\beta_1), \\ \mu_2 z_1(1) + \sigma_2 \mathbb{D}_q z_1(1) = \eta_2 z_1(\beta_2), \\ \mu_3 z_2(0) - \sigma_3 \Big(\omega^{(1-\Upsilon_2)} \mathbb{D}_q z_2(0)\Big)_{\omega=0} = \eta_3 z_2(\beta_3), \\ \mu_4 z_2(1) + \sigma_4 \mathbb{D}_q z_2(1) = \eta_4 z_2(\beta_4), \end{cases}$$

with $\alpha_1 = \frac{1}{5}, \alpha_2 = \frac{1}{7}, \gamma_1 = \frac{1}{9}, \gamma_2 = \frac{1}{8}, \lambda_1 = \frac{1}{10}, \lambda_2 = \frac{1}{12}, \Psi_1 = \Psi_2 = \frac{1}{3}, \Upsilon_1 = \Upsilon_2 = \frac{1}{2}, q = \frac{1}{2}$, $\zeta_1 = \zeta_2 = \frac{1}{2}, \mu_1 = \mu_2 = \sigma_1 = \sigma_2 = 1, \mu_3 = \mu_4 = \sigma_3 = \sigma_4 = \frac{1}{2}, \beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{1}{3}$, $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 1$, *and*

$$\begin{aligned} f_1(\omega, z_1(\omega), z_2(\omega)) &= \frac{1}{12} + \frac{|z_1(\omega)|}{210(1+|z_1(\omega)|)} + \frac{|z_2(\omega)|}{(\omega+2)^3(1+|z_2(\omega)|)}, \\ f_2(\omega, z_1(\omega), z_2(\omega)) &= \frac{|z_1(\omega)|}{2(1+|z_1(\omega)|)} + \frac{|z_2(\omega)|}{(\omega+3)^3(1+|z_2(\omega)|)} + \frac{1}{6}, \\ g_1(\omega, z_1(\omega), z_2(\omega)) &= \frac{\cos\omega}{13} + \frac{|z_1(\omega)|}{(14+\omega^2)} + \frac{z_2(\omega)\sin\omega}{90}, \\ g_2(\omega, z_1(\omega), z_2(\omega)) &= \frac{1}{6} + \frac{|z_1(\omega)|}{2(1+|z_1(\omega)|)} + \frac{|z_2(\omega)|}{45(1+|z_2(\omega)|)}. \end{aligned}$$

For each $\omega \in [0,1]$ *and* $z_1, z_2 \in \mathbb{R}$

$$\begin{aligned} |f_1(\omega, z_1, z_2)| &\leq \frac{1}{12} + \frac{1}{210}|z_1(\omega)| + \frac{1}{8}|z_2(\omega)|, \\ |f_2(\omega, z_1, z_2)| &\leq \frac{1}{6} + \frac{1}{2}|z_1(\omega)| + \frac{1}{27}|z_2(\omega)|, \\ |g_1(\omega, z_1, z_2)| &\leq \frac{1}{13} + \frac{1}{14}|z_1(\omega)| + \frac{1}{90}|z_2(\omega)|, \\ |g_2(\omega, z_1, z_2)| &\leq \frac{1}{6} + \frac{1}{2}|z_1(\omega)| + \frac{1}{45}|z_2(\omega)|. \end{aligned}$$

Hence, (H_2) holds with $\mathrm{m}_0 = \frac{1}{12}, \mathrm{m}_1 = \frac{1}{210}, \mathrm{m}_2 = \frac{1}{8}, \widetilde{\mathrm{m}}_0 = \frac{1}{6}, \widetilde{\mathrm{m}}_1 = \frac{1}{2}\widetilde{\mathrm{m}}_2 = \frac{1}{27}, \mathrm{n}_0 = \frac{1}{13}$, $\mathrm{n}_1 = \frac{1}{14}, \mathrm{n}_2 = \frac{1}{90}, \widetilde{\mathrm{n}}_0 = \frac{1}{6}, \widetilde{\mathrm{n}}_1 = \frac{1}{2}$, *and* $\widetilde{\mathrm{n}}_2 = \frac{1}{45}$. *By some calculations, we have*

$$\begin{aligned} \wp_1 &\simeq 0.675 < 1, \\ \wp_2 &\simeq 0.456 < 1. \end{aligned}$$

Thus, all conditions in Theorem 6 are satisfied and, hence, the system (2) has at least one solution. Also, condition (H_1) holds and $\Theta_1 \simeq 0.745 < 1$. *Thus, all conditions in Theorem 5 are satisfied and, hence, system (2) has a unique solution.*

6. Conclusions

We discussed the essential requirements for a coupled system of fractional q-integro-difference equations, utilizing Riemann–Liouville fractional q-derivatives and q-integrals of different orders, along with q-integral boundary conditions. Applying tools from fixed-point theory, we have obtained novel findings that contribute to and generalize the existing literature on this topic. Our results encompass various new results as special cases, marking a significant contribution to the field of boundary value problems associated with fractional q-integro-difference equations.

In future work, we will aim to explore the stability properties of the obtained solutions and investigate numerical methods for effectively solving the coupled system. Additionally, we plan to extend our analysis to more complex scenarios or higher dimensions, which may involve considering additional factors or variables. Furthermore, we intend to explore the potential applications of the obtained results in diverse fields such as physics, engineering, and biology. By doing so, we aim to demonstrate the practical significance and relevance of our findings in real-world contexts.

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