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(ω, c)-Periodic Solution to Semilinear Integro-Differential Equations with Hadamard Derivatives

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Abstract: This study aims to prove the existence and uniqueness of the (ω, c)-periodic solution as a specific solution to Hadamard impulsive boundary value integro-differential equations with fixed lower limits. The results are proven using the Banach contraction, Schaefer's fixed point theorem, and the Arzelà–Ascoli theorem. Furthermore, we establish the necessary conditions for a set of solutions to the explored boundary values with impulsive fractional differentials. Finally, we present two examples as applications for our results.

Keywords: existence; uniqueness; Hadamard fractional derivative; (ω, c)-periodic solution; impulsive; integro-differential equation

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1. Introduction

The invention and continuous progress of modern technologies have heightened interest in systems with impulsive automated trajectories, which are examples of discontinuous trajectory control and impulsive computer systems. These systems have gained prominence and are currently used to tackle a wide variety of technological issues in a range of fields, including medicine, biology, threshold phenomena, hypotheses of bursting rhythms, economics, pharmacokinetics, and optimal control models for frequency-modulated systems [1]. Short-term disruptions with short durations in relation to the entire process times are introduced into these processes (see [2–4]). As a result of the significant interest in understanding their behaviors and characteristics, there is a compelling urge to study the qualitative aspects of these impulsive system solutions. A significant number of mathematicians are currently engaged in active research on periodic function theory. In [5], the existence and uniqueness of (ω, c)-periodic solutions were investigated for the similar evolution equation $y' = \lambda y + f(t, y)$ in complex Banach spaces, where λ is a bounded linear operator. Wang et al. [6] presented novel linear noninstantaneous impulsive differential equations and obtained solution representations and asymptotic stability for nonlinear and linear problems.

Alvarez et al. [7,8] defined (ω, c)-periodic functions, including periodic, Bloch periodic functions, and antiperiodic functions, among others. This notion was driven by the Mathieu's equation $x''(t) + [a + 2q\cos 2t]x(t) = 0$. One class of (ω, c)-periodic time-varying impulsive differential equations exists and is unique. A continuous function $g : \mathbb{R} \to X$, where X is a complex Banach space, is (ω, c)-periodic if $g(x + \omega) = cg(x)$ holds for all $x \in \mathbb{R}$, where $\omega > 0$ and $c \in \mathbb{C} \setminus \{0\}$. Ren and Wang [9] established a required and sufficient criterion for (ω, c)-periodic solutions to impulsive fractional differential equations and investigated the existence and uniqueness of solutions via Caputo derivatives to impulsive fractional differential equations in Banach spaces. For further information on the existence and uniqueness of solutions to impulsive and regular fractional differential equations, see [10–14]. We also examined fractional oscillators, fractional dynamical systems, and the periodic solution of fractional oscillation equations in [15–17]. To our knowledge, there have not been any investigations into the existence of (ω, c)- periodic solutions for impulsive Hadamard fractional differential equations. This study expands on previous research on (ω, c)-periodic solutions for linear and semilinear problems with ordinary and fractional order derivatives. We concentrate on impulsive Hadamard fractional differential equations with boundary value constraints and set lower limits.

2. Preliminary

Consider impulsive fra. integro-diff. eqs. with lower limits, as illustrated below.

$$\begin{aligned} D^{\vartheta}p(t) &= \rho(t,p(t)) + \textstyle\int_{t_0}^{t} \kappa(t-s)\sigma(s,p(s))ds, \;\; \vartheta \in (0,1), \; t \neq t_n, \; t \in [t_0,\infty], \\ p(t_n^+) &= p(t_n^-) + \Delta_n, \;\; n \in \mathbb{N}, \end{aligned} \tag{1}$$

where $D^{\vartheta}p(t)$ is the Hadamard fra. derivative (where $\vartheta \in (0,1)$ via the lower limit at t_0) and for $n \in \mathbb{N}$, $t_n < t_{n+1}$, $lim_{n\to\infty} t_n = \infty$.

Next, we present some definitions and results, which are used in this study. To begin, consider the following definitions:

Definition 1 ([18])**.** *The Hadamard fra. integral of the $\vartheta > 0$ of a function $\rho(t) \in (C[x,y])$, $1 \leq x \leq y < \infty$, is defined as*

$$I^{\vartheta}\rho(t) = \frac{1}{\Gamma(\vartheta)} \int_x^t \left(\log \frac{t}{s}\right)^{\vartheta-1} \frac{\rho(s)}{s}\, ds$$

for $t \geq x$, assuming the integral exists.

Definition 2 ([18])**.** *Let $1 \leq x \leq y < \infty$, $\Delta = t\frac{d}{dt}$, and $AC^n_{\Delta}[x,y] = \{\rho : [x,y] \to \mathbb{R} : \Delta^{n-1}[\rho(t)] \in AC[x,y]\}$. The Hadamard fra. derivative of $\vartheta > 0$ for a function $\rho \in AC^n_{\Delta}[x,y]$ is defined by*

$$D^{\vartheta}\rho(t) = \frac{1}{\Gamma(n-\vartheta)} \left(t\,\frac{d}{dt}\right)^{(n)} \int_x^t \left(\log \frac{t}{s}\right)^{n-\vartheta-1} \frac{\rho(s)}{s}\, ds$$

for $t \geq x$ and $\vartheta \in (n-1, n)$, where $n = [\vartheta] + 1$, such that $[\vartheta]$ signifies the integer component of a real number ϑ and $\log(.) = \log_e(.)$.

Lemma 1. *Let $\rho : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous functions. A solution $p \in PC(\mathbb{R}, \mathbb{R}^n)$ of the impulsive frac. integro-diff. eq. is shown below with a fixed lower limit.*

$$\begin{aligned} D^{\vartheta}p(t) =& \rho(t,p(t)) + \int_{t_0}^{t} \kappa(t-s)\sigma(s,p(s))ds, \;\; \vartheta \in (0,1), \; t \neq t_n, \; t \in [t_0,\infty], \\ p(t_n^+) =& p(t_n^-) + \Delta_n, \;\; n \in \mathbb{N}, \\ p(t_0) =& p_{t_0}. \end{aligned} \tag{2}$$

is given by

$$\begin{aligned} p(t) \;\; = \;\; & \frac{1}{\Gamma(\vartheta)} \textstyle\int_{t_0}^{t} \left(\log \frac{t}{\xi}\right)^{\vartheta-1} \left(\rho(\xi,p(\xi)) + \int_{t_0}^{\xi} \kappa(\xi-s)\sigma(s,p(s))ds\right) \frac{d\xi}{\xi} \\ & + p(t_0) + \sum_{t_0<t_i<t} \Delta_i, \; \textit{for all } t \geq t_0. \end{aligned} \tag{3}$$

Proof. From [19] in Lemma 3.2, a solution p of Equation (2) is provided by

$$p(t) = \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t} \left(\log \frac{t}{\xi}\right)^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_{t_0}^{\xi} \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} + p(t_0) + \sum_{i=1}^{n} \Delta_i, \text{ for every } t \in (t_n, t_{n+1}]. \tag{4}$$

Using

$$\sum_{i=1}^{n} \Delta_i = \sum_{t_0<t_i<t} \Delta_i, \text{ for every } t \in (t_n, t_{n+1}],$$

we obtain that the Equation (4) is equivalent to

$$p(t) = \frac{1}{\Gamma(\vartheta)} \int_{t_0}^{t} \left(\log \frac{t}{\xi}\right)^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_{t_0}^{\xi} \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} + p(t_0) + \sum_{t_0<t_i<t} \Delta_i, \tag{5}$$

for $t \in (t_n, t_{n+1}]$. Using the arbitrariness of n, we find that Equation (5) holds for $\cup_{n=1}^{\infty}(t_n, t_{n+1}]$. Since Equation (5) is independent of n, we have that Equation (3) holds for $[t_0, \infty)$. □

Definition 3 ([7])**.** *Let $c \in \mathbb{C} \setminus \{0\}$, where $\omega > 0$ and X represents a complex Banach space with a norm $\| \cdot \|$. A function $\gamma : \mathbb{R} \to X$ is called (ω, c)-periodic if $\gamma(t + \omega) = c\gamma(t)\ \forall\ t \in \mathbb{R}$, where γ is continuous.*

Lemma 2 ([5])**.** *Let $\Phi_{\omega,c} := \{p : p \in PC(\mathbb{R}, \mathbb{R}^n)\}$, and $p(\cdot + \omega) = cp(\cdot)$. Then, $p \in \Phi_{\omega,c}$ if $p(\omega) = c\, p(t_0)$.*

3. The (ω, c)-Periodic Solution to Semilinear Integro Differential Equations

For $t_0 = 1$, we investigate the (ω, c)-periodic solution of impulsive fra. integro-diff. eqs. with determined lower limits.

$$\begin{aligned} D^{\vartheta} p(t) &= \rho(t, p(t)) + \int_1^t \kappa(t - s)\sigma(s, p(s))ds, \ \ \vartheta \in (0, 1),\ t \neq t_n,\ t \in [1, \infty], \\ p(t_n^+) &= p(t_n^-) + \Delta_n, \ \ n \in \mathbb{N}, \\ p(1) &= p_0. \end{aligned} \tag{6}$$

We present the following assumptions:

(I): There are continuous functions $\rho : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\rho(t + \omega, cp) = c\,\rho(t, p),\ \forall\ t \in \mathbb{R}, \text{ and } p \in \mathbb{R}^n$$

$$\sigma(t + \omega, cp) = c\,\sigma(t, p),\ \forall\ t \in \mathbb{R}, \text{ and } p \in \mathbb{R}^n$$

(II): There are constants $A_1, A_2 > 0$ and $B_1, B_2 > 0$, such that

$$\|\rho(t, p)\| \leq A_1\|p\| + B_1,\ \forall\ t \in \mathbb{R}, \text{ and } p \in \mathbb{R}^n$$

$$\|\sigma(t, p)\| \leq A_2\|p\| + B_2,\ \forall\ t \in \mathbb{R}, \text{ and } p \in \mathbb{R}^n$$

(III): There is a constant $\epsilon_1, \epsilon_2 > 0$, such that

$$\|\rho(t, p) - \rho(t, q)\| \leq \epsilon_1\|p - q\|,\ \forall\ t \in \mathbb{R}, \text{ and } p, q \in \mathbb{R}^n$$

$$\|\sigma(t, p) - \sigma(t, q)\| \leq \epsilon_2\|p - q\|,\ \forall\ t \in \mathbb{R}, \text{ and } p, q \in \mathbb{R}^n$$

(IV): $\Delta_n \in \mathbb{R}^n$ and $N \in \mathbb{N}$, such that $\omega = t_N$, $t_{n+N} = t_n + \omega$ and $\Delta_{n+N} = \Delta_n$ for any $n \in \mathbb{N}$

(V): The kernel map $\kappa : \mathbb{R} \to \mathbb{R}$ is locally integrable on $[1, \infty)$ and $\int_1^\omega \|k(s)\|\, ds = a_T > 0$.

Lemma 3. *Assume that $c \neq 1$ and conditions (I) and (IV) are satisfied. Then, the solution $p \in \Psi := PC([1, \omega], \mathbb{R}^n)$ of Equation (6) satisfying Lemma 2 is provided by*

$$\begin{aligned} p(t) =& \frac{1}{(c-1)\Gamma(\vartheta)} \int_1^\omega \left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \left[\rho(\xi, p(\xi)) + \int_1^\xi \kappa(\xi - s)\sigma(s, p(s))ds\right] \frac{d\xi}{\xi} \\ &+ \frac{1}{\Gamma(\vartheta)} \int_1^t \left(\log \frac{t}{\xi}\right)^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_1^\xi \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} \\ &+ (c-1)^{-1} \sum_{i=1}^n \Delta_i + \sum_{1<t_i<t} \Delta_i, \quad t \in [1, \omega]. \end{aligned} \tag{7}$$

Proof. The solution $p \in \Psi := PC([1, \omega], \mathbb{R}^n)$ is provided by Equation (3), such that

$$\begin{aligned} p(t) \quad = \quad & \frac{1}{\Gamma(\vartheta)} \int_1^t \left[\log \frac{t}{\xi}\right]^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_1^\xi \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} \\ & + p(1) + \sum_{t_0<t_i<t} \Delta_i, \text{ for all } t \in [1, \omega]. \end{aligned} \tag{8}$$

So, we have

$$\begin{aligned} p(\omega) \quad = \quad & \frac{1}{\Gamma(\vartheta)} \int_1^\omega \left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_1^\xi \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} \\ & + p(1) + \sum_{t_0<t_i<\omega} \Delta_i = c\, p(1), \end{aligned} \tag{9}$$

which is the same as

$$\begin{aligned} p(1) \quad = \quad & \frac{1}{(c-1)\Gamma(\vartheta)} \int_1^\omega \left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_1^\xi \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} \\ & + (c-1)^{-1} \sum_{t_0<t_i<\omega} \Delta_i. \end{aligned} \tag{10}$$

From Equations (8) and (10), we obtain

$$\begin{aligned} p(t) =& \frac{1}{|c-1|\Gamma(\vartheta)} \int_1^\omega \left[]\log \frac{\omega}{\xi}\right]^{\vartheta-1} \left[\rho(\xi, p(\xi)) + \int_1^\xi \kappa(\xi - s)\sigma(s, p(s))ds\right] \frac{d\xi}{\xi} \\ &+ \frac{1}{\Gamma(\vartheta)} \int_1^t \left[]\log \frac{t}{\xi}\right]^{\vartheta-1} \left[]\rho(\xi, p(\xi)) + \int_1^\xi \kappa(\xi - s)\sigma(s, p(s))ds\right] \frac{d\xi}{\xi} \\ &+ (c-1)^{-1} \sum_{i=1}^n \Delta_i + \sum_{1<t_i<t} \Delta_i. \end{aligned} \tag{11}$$

The proof is complete. □

Theorem 1. *Assume that $c \neq 1$ and (I), (III), and (IV) of the conditions are true if*

$$0 < \frac{[\epsilon_1 + \epsilon_2 a_T](\log \omega)^\vartheta (1 + |c-1|^{-1})}{\Gamma(\vartheta + 1)} < 1.$$

Then, Equation (6) has one unique (ω, c)-periodic solution $p \in \Phi_{\omega,c}$. In addition, we obtain

$$\|p\|_\infty \leq \frac{(\log \omega)^\vartheta (\delta_1 + \delta_2 a_T)(|c-1|^{-1} + 1) + \Gamma(\vartheta + 1)(|c-1|^{-1} + 1) \sum_{i=1}^n \|\Delta_i\|}{\Gamma(\vartheta + 1) - (\log \omega)^\vartheta (\epsilon_1 + \epsilon_2 a_T)(|c-1|^{-1} + 1)}.$$

where $\delta_1 = sup_{t\in[1,\omega]}\|\rho(t,0)\|$ *and* $\delta_2 = sup_{t\in[1,\omega]}\|\sigma(t,0)\|$.

Proof. We can deduce from (I) that for every $p \in \Phi_{\omega,c}$, the following are true:

$$\rho(t+\omega, p(t+\omega)) = \rho(t+\omega, c\,p(t)) = c\,\rho(t, p(t)), \text{ for all } t \in \mathbb{R},$$

and

$$\sigma(t+\omega, p(t+\omega)) = \sigma(t+\omega, c\,p(t)) = c\,\sigma(t, p(t)), \text{ for all } t \in \mathbb{R},$$

which imply that $\rho(\cdot, p(\cdot))$ and $\sigma(\cdot, p(\cdot)) \in \Phi_{\omega,c}$.

Now, we determine the operator $\Omega : \Psi \to \Psi$ as

$$\begin{aligned}(\Omega p)(t) =& \frac{1}{|c-1|\Gamma(\vartheta)}\int_1^\omega \left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\left(\rho(\xi,p(\xi)) + \int_1^\xi \kappa(\xi-s)\sigma(s,p(s))ds\right)\frac{d\xi}{\xi} \\ &+ \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\rho(\xi,p(\xi)) + \int_1^\xi \kappa(\xi-s)\sigma(s,p(s))ds\right)\frac{d\xi}{\xi} \\ &+ (c-1)^{-1}\sum_{i=1}^n \Delta_i + \sum_{1<t_i<t}\Delta_i.\end{aligned} \tag{12}$$

Lemmas 2 and 3 show that the fixed points of Ω define the (ω, c)-periodic solution of Equation (6). It is straightforward to find $\Omega(\Psi) \subseteq \Psi$. For all $p_1, p_2 \in \Psi$, we obtain

$$\begin{aligned}&\|(\Omega p_1)(t) - (\Omega p_2)(t)\| \\ &= \Bigg\|\frac{1}{(c-1)\Gamma(\vartheta)}\int_1^\omega \left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\left[\rho(\xi,p_1(\xi)) + \int_1^\xi \kappa(\xi-s)\sigma(s,p_1(s))ds\right]\frac{d\xi}{\xi} \\ &+ \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\rho(\xi,p_1(\xi)) + \int_1^\xi \kappa(\xi-s)\sigma(s,p_1(s))ds\right)\frac{d\xi}{\xi} \\ &- (c-1)^{-1}\frac{1}{\Gamma(\vartheta)}\int_1^\omega \left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\left[\rho(\xi,p_2(\xi)) + \int_1^\xi \kappa(\xi-s)\sigma(s,p_2(s))ds\right]\frac{d\xi}{\xi} \\ &- \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\rho(\xi,p_2(\xi)) + \int_1^\xi \kappa(\xi-s)\sigma(s,p_2(s))ds\right)\frac{d\xi}{\xi}\Bigg\|\end{aligned}$$

$$\begin{aligned}&\leq (c-1)^{-1}\frac{1}{\Gamma(\vartheta)}\int_1^\omega \left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\Big(\|\rho(\xi,p_1(\xi)) - \rho(\xi,p_2(\xi))\| \\ &+ \int_1^\xi \|\kappa(\xi-s)\|\|\sigma(s,p_1(s)) - \sigma(s,p_2(s))\|\,ds\frac{d\xi}{\xi}\Big) \\ &+ \frac{1}{\Gamma(\vartheta)}\int_1^t \left(\log\frac{t}{\xi}\right)^{\vartheta-1}\Big(\|\rho(\xi,p_1(\xi)) - \rho(\xi,p_2(\xi))\| \\ &+ \int_1^\xi \|\kappa(\xi-s)\|\|\sigma(s,p_1(s)) - \sigma(s,p_2(s))\|\,ds\frac{d\xi}{\xi}\Big)\end{aligned}$$

$$\begin{aligned}&\leq |c-1|^{-1}\frac{\epsilon_1+\epsilon_2 a_T}{\Gamma(\vartheta)}\int_1^\omega \left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\|p_1(\xi)-p_2(\xi)\|\frac{d\xi}{\xi} \\ &+ \frac{\epsilon_1+\epsilon_2 a_T}{\Gamma(\vartheta)}\int_1^t \left(\log\frac{t}{\xi}\right)^{\vartheta-1}\|p_1(\xi)-p_2(\xi)\|\frac{d\xi}{\xi}\end{aligned}$$

$$\leq \frac{\epsilon_1+\epsilon_2 a_T}{\Gamma(\vartheta)}\|p_1-p_2\|_\infty\left(|c-1|^{-1}\int_1^\omega \left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi} + \int_1^t \left(\log\frac{t}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi}\right)$$

$$\leq \frac{[\epsilon_1 + \epsilon_2 a_T](\log \omega)^{\vartheta}\left(|c-1|^{-1}+1\right)}{\Gamma(\vartheta+1)}\|p_1 - p_2\|_\infty,$$

and hence, we have

$$\|(\Omega p_1)(t) - (\Omega p_2)(t)\|_\infty \leq \frac{[\epsilon_1 + \epsilon_2 a_T](\log \omega)^{\vartheta}\left(1+|c-1|^{-1}\right)}{\Gamma(\vartheta+1)}\|p_1 - p_2\|_\infty.$$

From the assumption

$$0 < \frac{[\epsilon_1 + \epsilon_2 a_T](\log \omega)^{\vartheta}\left(1+|c-1|^{-1}\right)}{\Gamma(\vartheta+1)} < 1.$$

Now, we find that Ω is a contraction mapping. Therefore, p is a unique fixed point of Equation (12) that satisfies $p(\omega) = c\,p(1)$. Then, we obtain from Lemma 2 that $p \in \Phi_{\omega,c}$. Hence, Equation (6) has a (ω, c)-periodic $p \in \Phi_{\omega,c}$, which is a unique. In addition, we have

$$\begin{aligned}
&\|p(t)\| \\
&\leq |c-1|^{-1}\frac{1}{\Gamma(\vartheta)}\int_1^{\omega}\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\Big(\|\rho(\xi,p(\xi))-\rho(\xi,0)\| \\
&\quad + \int_1^{\xi}\|\kappa(\xi-s)\|\|\sigma(s,p(s))-\sigma(s,0)\|\,ds\Big)\frac{d\xi}{\xi} \\
&+\frac{1}{|c-1|\Gamma(\vartheta)}\int_1^{\omega}\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\left[\|\rho(\xi,0)\| + \int_1^{\xi}\|\kappa(\xi-s)\|\|\sigma(s,0)\|\,ds\right]\frac{d\xi}{\xi} \\
&+\frac{1}{\Gamma(\vartheta)}\int_1^{t}\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\|\rho(\xi,p(\xi))-\rho(\xi,0)\| + \int_1^{\xi}\|\kappa(\xi-s)\|\|\sigma(s,p(s))-\sigma(s,0))\|\,ds\right)\frac{d\xi}{\xi} \\
&+\frac{1}{\Gamma(\vartheta)}\int_1^{t}\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\|\rho(\xi,0)\| + \int_1^{\xi}\|\kappa(\xi-s)\|\|\sigma(s,0)\|\,ds\right)\frac{d\xi}{\xi} \\
&+|c-1|^{-1}\sum_{i=1}^{n}\|\Delta_i\| + \sum_{1<t_i<t}\|\Delta_i\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\epsilon_1+\epsilon_2 a_T}{|c-1|\Gamma(\vartheta)}\int_1^{\omega}\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\|p(\xi)\|\frac{d\xi}{\xi} + |c-1|^{-1}\frac{\delta_1+\delta_2 a_T}{\Gamma(\vartheta)}\int_1^{\omega}\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi} \\
&+\frac{\epsilon_1+\epsilon_2 a_T}{\Gamma(\vartheta)}\int_1^{t}\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\|p(\xi)\|\frac{d\xi}{\xi} + \frac{\delta_1+\delta_2 a_T}{\Gamma(\vartheta)}\int_1^{t}\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi} \\
&+\left(|c-1|^{-1}+1\right)\sum_{i=1}^{n}\|\Delta_i\|
\end{aligned}$$

$$\begin{aligned}
\|p(t)\| \leq& \frac{(\log\omega)^{\vartheta}(\epsilon_1+\epsilon_2 a_T)\left(|c-1|^{-1}+1\right)}{\Gamma(\vartheta+1)}\|p\|_\infty + \frac{(\log\omega)^{\vartheta}(\delta_1+\delta_2 a_T)\left(|c-1|^{-1}+1\right)}{\Gamma(\vartheta+1)} \\
&+\left(|c-1|^{-1}+1x\right)\sum_{i=1}^{n}\|\Delta_i\|,
\end{aligned}$$

which implies that

$$\|p\|_\infty \leq \frac{(\log\omega)^{\vartheta}(\delta_1+\delta_2 a_T)\left(|c-1|^{-1}+1\right) + \Gamma(\vartheta+1)\left(|c-1|^{-1}+1\right)\sum_{i=1}^{n}\|\Delta_i\|}{\Gamma(\vartheta+1) - (\log\omega)^{\vartheta}(\epsilon_1+\epsilon_2 a_T)\left(|c-1|^{-1}+1\right)}.$$

The proof is complete. □

Theorem 2. *Assume $c \neq 1$ and the conditions (I), (II), and (IV) are satisfied. If*

$$\Gamma(\vartheta+1) > (\log\omega)^{\vartheta}(A_1 + A_2 a_T)\left(1+|c-1|^{-1}\right),$$

then there is at least one (ω, c)-periodic solution to the impulsive fra. integro-diff. Equation (6), such that $p \in \Phi_{\omega,c}$.

Proof. Let $\mathbb{B}_\lambda = \{p \in \Psi : \|p\| \leq \lambda\}$, where

$$\lambda \geq \frac{(\log \omega)^{\vartheta}(B_1 + B_2 a_T)\left(1 + |c-1|^{-1}\right) + \Gamma(\vartheta+1)\left(1 + |c-1|^{-1}\right)\sum_{i=1}^{n}\|\Delta_i\|}{\Gamma(\vartheta+1) - (\log \omega)^{\vartheta}(A_1 + A_2 a_T)(|c-1|^{-1}+1)}.$$

We assume the Ω operator given in Equation (12) for $\mathbb{B}_\lambda$. In any case where $t \in [1, \omega]$ and $p \in \mathbb{B}_\lambda$, we have

$$\begin{aligned}
&\|(\Omega p)(t)\| \\
\leq &\frac{1}{|c-1|\Gamma(\vartheta)} \int_1^{\omega} \left[\log \frac{\omega}{\xi}\right]^{\vartheta-1} \left[\|\rho(\xi, p(\xi))\| + \int_1^{\xi} \|\kappa(\xi - s)\| \|\sigma(s, p(s)) ds\|\right] \frac{d\xi}{\xi} \\
+ &\frac{1}{\Gamma(\vartheta)} \int_1^{t} \left(\log \frac{t}{\xi}\right)^{\vartheta-1} \left(\|\rho(\xi, p(\xi))\| + \int_1^{\xi} \|\kappa(\xi - s)\| \|\sigma(s, p(s)) ds\right)\| \frac{d\xi}{\xi} \qquad (13) \\
+ &|c-1|^{-1} \sum_{i=1}^{n} \|\Delta_i\| + \sum_{1<t_i<t} \|\Delta_i\|
\end{aligned}$$

$$\begin{aligned}
\leq &\frac{1}{|c-1|\Gamma(\vartheta)} \int_1^{\omega} \left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} [(A_1 + A_2 a_T)\|p(\xi)\| + (B_1 + B_2 a_T)] \frac{d\xi}{\xi} \\
+ &\frac{1}{\Gamma(\vartheta)} \int_1^{t} \left(\log \frac{t}{\xi}\right)^{\vartheta-1} [(A_1 + A_2 a_T)\|p(\xi)\| + (B_1 + B_2 a_T)]\| \frac{d\xi}{\xi} \\
+ &|c-1|^{-1} \sum_{i=1}^{n} \|\Delta_i\| + \sum_{1<t_i<t} \|\Delta_i\|
\end{aligned}$$

$$\begin{aligned}
\leq &\frac{(\log \omega)^{\vartheta}(A_1 + A_2 a_T)\left(1 + |c-1|^{-1}\right)}{\Gamma(\vartheta+1)}\|p\|_\infty + \frac{(\log \omega)^{\vartheta}(B_1 + B_2 a_T)\left(1 + |c-1|^{-1}\right)}{\Gamma(\vartheta+1)} \\
+ &\left(1 + |c-1|^{-1}\right) \sum_{i=1}^{n} \|\Delta_i\|.
\end{aligned}$$

Therefore,

$$\|\Omega p\|_\infty \leq \frac{(\log \omega)^{\vartheta}(B_1 + B_2 a_T)\left(|c-1|^{-1} + 1\right) + \Gamma(\vartheta+1)\left(|c-1|^{-1} + 1\right)\sum_{i=1}^{n}\|\Delta_i\|}{\Gamma(\vartheta+1) - (\log \omega)^{\vartheta}(A_1 + A_2 a_T)(|c-1|^{-1}+1)} \leq \lambda,$$

which implies that $\|\Omega p\|_\infty \leq \lambda$. So, $\Omega(\mathbb{B}_\lambda) \subseteq \mathbb{B}_\lambda$.

Now, we show that Ω is continuous for $\mathbb{B}_\lambda$.

Let $\{p_i\}_{i \geq 1} \subseteq \mathbb{B}_\lambda$ and $p_i \to p$ for $\mathbb{B}_\lambda$ as $i \to \infty$. By the continuity of ρ and σ, we obtain $\rho(\xi, p_i(\xi)) \to \rho(\xi, p(\xi))$ and $\sigma(\xi, p_i(\xi)) \to \sigma(\xi, p(\xi))$ as $i \to \infty$. As a result, we have

$$\left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \rho(\xi, p_i(\xi)) \to \left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \rho(\xi, p(\xi))$$

as $i \to \infty$.

$$\left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \left(\int_1^{\xi} \kappa(\xi - s)\sigma(s, p_i(s)) ds\right) \to \left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \left(\int_1^{\xi} \kappa(\xi - s)\sigma(s, p(s)) ds\right)$$

as $i \to \infty$.

$$\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\rho(\xi,p_i(\xi))\to\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\rho(\xi,p(\xi))$$
$$\text{as } i\to\infty.$$
$$\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p_i(s))ds\right)\to\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p(s))ds\right)$$
$$\text{as } i\to\infty.$$

Using assumption (II), we find that for any $1\le i\le t\le\omega$,

(i)

$$\begin{aligned}&\int_1^{\omega}\left\|\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\rho(\xi,p_i(\xi))-\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\rho(\xi,p(\xi))\right\|\frac{d\xi}{\xi}\\&\le 2(A_1\|p\|_\infty+B_1)\int_1^{\omega}\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi}=2(\lambda A_1+B_1)\int_1^{\omega}\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi}\\&\le 2(\lambda A_1+B_1)\frac{(\log\omega)^{\vartheta}}{\vartheta}<\infty,\end{aligned}$$

(ii)

$$\begin{aligned}&\int_1^{\omega}\left\|\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\left[\int_1^{\xi}\kappa(\xi-s)\sigma(s,p_i(s))ds\right]-\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\left[\int_1^{\xi}\kappa(\xi-s)\sigma(s,p(s))ds\right]\right\|\frac{d\xi}{\xi}\\&\le 2a_T(A_2\|p\|_\infty+B_2)\int_1^{\omega}\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi}=2a_T(\lambda A_2+B_2)\int_1^{\omega}\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi}\\&\le 2a_T(\lambda A_2+B_2)\frac{(\log\omega)^{\vartheta}}{\vartheta}<\infty,\end{aligned}$$

(iii)

$$\begin{aligned}&\int_1^{t}\left\|\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\rho(\xi,p_i(\xi))-\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\rho(\xi,p(\xi))\right\|\frac{d\xi}{\xi}\\&\le 2(A_1\|p\|_\infty+B_1)\int_1^{t}\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\frac{d\xi}{\xi}=2(\lambda A_1+B_1)\int_1^{t}\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\frac{d\xi}{\xi}\\&\le 2(\lambda A_1+B_1)\frac{(\log\omega)^{\vartheta}}{\vartheta}<\infty,\end{aligned}$$

(iv)

$$\begin{aligned}&\int_1^{t}\left\|\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\left[\int_1^{\xi}\kappa(\xi-s)\sigma(s,p_i(s))ds\right]-\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p(s))ds\right)\right\|\frac{d\xi}{\xi}\\&\le 2a_T(A_2\|p\|_\infty+B_2)\int_1^{t}\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi}=2a_T(\lambda A_2+B_2)\int_1^{t}\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\frac{d\xi}{\xi}\\&\le 2a_T(\lambda A_2+B_2)\frac{(\log\omega)^{\vartheta}}{\vartheta}<\infty.\end{aligned}$$

Then, by applying the theorem of Lebesgue dominated convergence, we obtain

(i)

$$\int_1^{\omega}\left\|\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\rho(\xi,p_i(\xi))-\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\rho(\xi,p(\xi))\right\|\frac{d\xi}{\xi}\to 0 \text{ as } i\to\infty,$$

(ii)

$$\int_1^{\omega}\left\|\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p_i(s))ds\right)\right.$$
$$\left.-\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p(s))ds\right)\right\|\frac{d\xi}{\xi}\to 0\,\text{as}\,i\to\infty$$

(iii)

$$\int_1^{t}\left\|\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\rho(\xi,p_i(\xi))-\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\rho(\xi,p(\xi))\right\|\frac{d\xi}{\xi}\to 0\ \text{as}\,i\to\infty,$$

(iv)

$$\int_1^{t}\left\|\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p_i(s))ds\right)\right.$$
$$\left.-\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p(s))ds\right)\right\|\frac{d\xi}{\xi}\to 0\ \text{ as}\,i\to\infty.$$

Hence, for every $t\in[1,\omega]$, we obtain

$$\begin{aligned}
&\|(\Omega p_i)(t)-(\Omega p)(t)\|\\
\leq&\frac{|c-1|^{-1}}{\Gamma(\vartheta)}\int_1^{\omega}\left\|\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\rho(\xi,p_i(\xi))-\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\rho(\xi,p(\xi))\right\|\frac{d\xi}{\xi}\\
&+\frac{1}{|c-1|\Gamma(\vartheta)}\int_1^{\omega}\left\|\left[\log\frac{\omega}{\xi}\right]^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p_i(s))ds\right)\right.\\
&\qquad\left.-\left(\log\frac{\omega}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p(s))ds\right)\right\|\frac{d\xi}{\xi}\\
&+\frac{1}{\Gamma(\vartheta)}\int_1^{t}\left\|\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p_i(s))ds\right)\right.\\
&\qquad\left.-\left(\log\frac{t}{\xi}\right)^{\vartheta-1}\left(\int_1^{\xi}\kappa(\xi-s)\sigma(s,p(s))ds\right)\right\|\frac{d\xi}{\xi}\\
&+\frac{1}{\Gamma(\vartheta)}\int_1^{t}\left\|\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\rho(\xi,p_i(\xi))-\left[\log\frac{t}{\xi}\right]^{\vartheta-1}\rho(\xi,p(\xi))\right\|\frac{d\xi}{\xi}\to 0\,\text{as}\,i\to\infty.
\end{aligned}$$

So, Ω is continuous for $\mathbb{B}_\lambda$. Now, we show that Ω is pre-compact.

For every $t_n\leq t\leq s\leq t_{n+1}$, where $n\in\mathbb{N}_0$, we obtain

$$\|\sum_{1<t_i<t}\Delta_i-\sum_{1<t_i<s}\Delta_i\|=\|\sum_{i=1}^{n}\Delta_i-\sum_{i=1}^{n}\Delta_i\|=0$$

which implies that

$$\|\sum_{1<t_i<t}\Delta_i-\sum_{1<t_i<s}\Delta_i\|\to 0,\ \text{as}\ t\to s.$$

So, for every $1\leq s_1<s_2\leq\omega$, where $p\in\mathbb{B}_\lambda$, and the operator $\Omega:\Psi\to\Psi$ is provided by

$$
\begin{aligned}
(\Omega p)(t) =& \frac{1}{|c-1|\Gamma(\vartheta)} \int_1^{\omega} \left(\log \frac{\omega}{\xi}\right)^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_1^{\xi} \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} \\
&+ \frac{1}{\Gamma(\vartheta)} \int_1^{t} \left(\log \frac{t}{\xi}\right)^{\vartheta-1} \left(\rho(\xi, p(\xi)) + \int_1^{\xi} \kappa(\xi - s)\sigma(s, p(s))ds\right) \frac{d\xi}{\xi} \\
&+ (c-1)^{-1} \sum_{i=1}^{n} \Delta_i - \sum_{1<t_i<t} \Delta_i,
\end{aligned}
$$

the following holds:

$$
\begin{aligned}
&\|(\Omega p)(s_1) - (\Omega p)(s_2)\| \\
&\quad \leq \frac{1}{\Gamma(\vartheta)} \int_1^{s_1} \left\| \left[\log \frac{s_1}{\xi}\right]^{\vartheta-1} \rho(\xi, p(\xi)) - \frac{1}{\Gamma(\vartheta)} \int_1^{s_2} \left[\log \frac{s_2}{\xi}\right]^{\vartheta-1} \rho(\xi, p(\xi)) \right\| \frac{d\xi}{\xi} \\
&\quad + \frac{1}{\Gamma(\vartheta)} \int_1^{s_1} \left\| \left(\log \frac{s_1}{\xi}\right)^{\vartheta-1} \left(\int_1^{\xi} \kappa(\xi - s)\sigma(s, p(s))ds\right) \right. \\
&\quad \left. - \frac{1}{\Gamma(\vartheta)} \int_1^{s_2} \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \left(\int_1^{\xi} \kappa(\xi - s)\sigma(s, p(s))ds\right) \right\| \frac{d\xi}{\xi} \\
&\quad + \left\| \sum_{1<t_i<s_1} \Delta_i - \sum_{1<t_i<s_2} \Delta_i \right\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\vartheta)} \int_1^{s_1} \left[\left(\log \frac{s_1}{\xi}\right)^{\vartheta-1} - \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \right] \|\rho(\xi, p(\xi))\| \frac{d\xi}{\xi} \\
&+ \frac{1}{\Gamma(\vartheta)} \int_1^{s_1} \left[\left(\log \frac{s_1}{\xi}\right)^{\vartheta-1} - \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \right] \int_1^{\xi} \|\kappa(\xi - s)\| \|\sigma(s, p(s))\| ds \frac{d\xi}{\xi} \\
&+ \left\| \sum_{1<t_i<s_1} \Delta_i - \sum_{1<t_i<s_2} \Delta_i \right\| - \frac{1}{\Gamma(\vartheta)} \int_{s_1}^{s_2} \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \|\rho(\xi, p(\xi))\| \frac{d\xi}{\xi} \\
&- \frac{1}{\Gamma(\vartheta)} \int_{s_1}^{s_2} \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \int_1^{\xi} \|\kappa(\xi - s)\| \|\sigma(s, p(s))\| ds \frac{d\xi}{\xi}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\lambda A_1 + B_1}{\Gamma(\vartheta)} \int_1^{s_1} \left[\left(\log \frac{s_1}{\xi}\right)^{\vartheta-1} - \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \right] \frac{d\xi}{\xi} \\
&+ \frac{a_{T(\lambda A_2 + B_2)}}{\Gamma(\vartheta)} \int_1^{s_1} \left[\left(\log \frac{s_1}{\xi}\right)^{\vartheta-1} - \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \right] \frac{d\xi}{\xi} \\
&+ \left\| \sum_{1<t_i<s_1} \Delta_i - \sum_{1<t_i<s_2} \Delta_i \right\| \\
&- \frac{\lambda A_1 + B_1}{\Gamma(\vartheta)} \int_{s_1}^{s_2} \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \frac{d\xi}{\xi} - \frac{a_{T(\lambda A_2 + B_2)}}{\Gamma(\vartheta)} \int_{s_1}^{s_2} \left(\log \frac{s_2}{\xi}\right)^{\vartheta-1} \frac{d\xi}{\xi}
\end{aligned}
$$

$$\leq \frac{\lambda A_1 + B_1}{\Gamma(\vartheta+1)}\left[(\log s_1)^{\vartheta} - (\log s_2)^{\vartheta} + 2\left(\log\frac{s_2}{s_1}\right)^{\vartheta}\right]$$
$$+\frac{a_{T(\lambda A_2+B_2)}}{\Gamma(\vartheta+1)}\left[(\log s_1)^{\vartheta} - (\log s_2)^{\vartheta} + 2\left(\log\frac{s_2}{s_1}\right)^{\vartheta}\right]$$
$$+\left\|\sum_{1<t_i<s_1}\Delta_i - \sum_{1<t_i<s_2}\Delta_i\right\|$$

$$\left\| (\Omega p)(s_1) - (\Omega p)(s_2)\right\| \leq \frac{[\lambda A_1 + B_1] + [a_{T(\lambda A_2+B_2)}]}{\Gamma(\vartheta+1)}\left[(\log s_1)^{\vartheta} - (\log s_2)^{\vartheta} + 2\left(\log\frac{s_2}{s_1}\right)^{\vartheta}\right]$$
$$+\left\|\sum_{1<t_i<s_1}\Delta_i - \sum_{1<t_i<s_2}\Delta_i\right\| \longrightarrow 0 \text{ as } s_1 \longrightarrow s_2.$$

So, $\Omega(\mathbb{B}_\lambda)$ is equicontinuous. From Equation (13), we obtain that $\Omega(\mathbb{B}_\lambda)$ is bounded uniformly.

Using the Arzelà–Ascoli theorem, we can show that $\Omega(\mathbb{B}_\lambda)$ is pre-compact. The impulsive fra. diff. arises from Schauder's fixed point theorem. At least one (ω, c)-periodic solution $p \in \Phi_{\omega,c}$ exists in Equation (6). The proof is complete. □

We end this section with two illustrative examples.

Example 1. *We investigate the following impulsive fra. diff. boundary value problem:*

$$D^{\frac{2}{3}}p(t) = \lambda\cos 2t\,\sin p(t) + \int_1^t \frac{\sin(s-t)}{5+\cos p(s)}\,ds,\ t \neq t_n,\ t \in [1,\infty),$$
$$p(t_n^+) = p(t_n^-) + \cos n\pi,\ \ n = 1, 2, \cdots, m,$$

where $\lambda \in \mathbb{R}$, $t_n = \frac{n\pi}{2}$, *and* $n = 1, 2, \cdots, m$. *Therefore,* $\Delta_n = \cos n\pi$ *and*

$$\rho(t, p(t)) = \lambda\cos 2t\,\sin p(t)$$

$$\sigma(t, p(t)) = \frac{\sin(s-t)}{5+\cos p(t)}$$

Let $c = -1$ *and* $\omega = \pi$. *It is clear that for any* $n \in \mathbb{N}$, $t_{n+2} = t_n + \pi$, $\Delta_{n+2} = \Delta_n$. *Hence, we have* $N = 2$ *and* (IV) *holds. For all* $t \in \mathbb{R}$ *and* $p(t) \in \mathbb{R}$, *we obtain*

$$\rho(t+\omega, cp(t)) = \rho(t+\pi, -p(t)) = -\lambda\cos 2t\,\sin p(t) = -\rho(t, p(t)) = c\rho(t, p(t)),$$

$$\sigma(t+\omega, cp(t)) = \sigma(t+\pi, -p(t)) = -\frac{\sin(s-t)}{5+\cos p(t)} = -\sigma(t, p(t)) = c\sigma(t, p(t))$$

which implies that (I) holds. Also, $\kappa(s) = 1$ *and* $a_T = \int_1^{\pi}\|1\|ds \leq \pi - 1$.

For all $t \in [1, \pi]$, *and* $p, q \in \mathbb{R}$, *we obtain*

$$\|\rho(t,p) - \rho(t,q)\| = \|\lambda\cos 2t\,\sin p(t) - \lambda\cos 2t\,\sin q(t)\|$$
$$\leq \|\lambda\cos 2t\,\sin p(t) - \lambda\cos 2t\,\sin q(t)\|$$
$$\leq |\lambda|\|p-q\| = \epsilon_1\|p-q\|.$$

$$\|\sigma(t,p) - \sigma(t,q)\| = \|\frac{\sin(s-t)}{5+\cos p(t)} - \frac{\sin(s-t)}{5+\cos q(t)}\|$$
$$\leq \frac{1}{25}\|p-q\| = \epsilon_2\|p-q\|.$$

So, (III) is satisfied for $\epsilon_1 = |\lambda|$ and $\epsilon_2 = \frac{1}{25}$. Note that

$$\frac{[\epsilon_1 + \epsilon_2 a_T](\log\omega)^{\vartheta}\left(|c-1|^{-1}+1\right)}{\Gamma(\vartheta+1)} = \frac{\frac{3}{2}\left[|\lambda| + \frac{\pi-1}{25}\right]\sqrt[3]{(\log\pi)^2}}{\Gamma(3/2)}.$$

Let $0 < \|\lambda\| < \frac{2\Gamma(3/2)}{3\sqrt[3]{(\log\pi)^2}} - \frac{a_T}{25}$ (where $a_T < \pi - 1$) and we obtain

$$0 < \frac{[\epsilon_1 + \epsilon_2 a_T](\log\omega)^{\vartheta}\left(1+|c-1|^{-1}\right)}{\Gamma(\vartheta+1)} = 0.456 < 1.$$

Hence, all assumptions in Theorem 1 hold and the eq. in Example 1 has a unique $(\pi,-1)$-periodic solution $p \in \Phi_{\pi,-1}$. In addition, we now obtain

$$\|p\|_\infty \leq \frac{(\log\omega)^{\vartheta}(\delta_1+\delta_2 a_T)\left(|c-1|^{-1}+1\right) + \Gamma(\vartheta+1)\left(|c-1|^{-1}+1\right)\sum_{i=1}^{N}\|\Delta_i\|}{\Gamma(\vartheta+1) - (\log\omega)^{\vartheta}(\epsilon_1+\epsilon_2 a_T)(|c-1|^{-1}+1)},$$

where $\delta_1 = sup_{t\in[1,\omega]}\|\rho(t,0)\| = 0$ and $\delta_2 = sup_{t\in[1,\omega]}\|\sigma(t,0)\| = \frac{1}{6}$. Furthermore, we have

$$\|p\|_\infty \leq \frac{\sqrt[3]{(\log\pi)^2}\left(\frac{\pi-1}{6}\right)\left(\frac{3}{2}\right) + \Gamma(3/2)\left(\frac{3}{2}\right)2}{\Gamma(3/2) - \sqrt[3]{(\log\pi)^2}\left(\|\lambda\| + \frac{\pi-1}{25}\right)\left(\frac{3}{2}\right)}.$$

Example 2. *We consider the following impulsive fra. diff. boundary value problem:*

$$D^{\frac{1}{3}}(t) = \lambda p(t)\cos\frac{p(t)}{e^t} + \int_1^t e^{-s}\sin\frac{p(s)}{e^s}ds,\ t \neq t_n,\ t \in [1,\infty),$$
$$p(t_n^+) = p(t_n^-) + 3,\ \ n = 1,2,\cdots,m,$$

where $\lambda \in \mathbb{R}$, $t_n = \frac{n\pi}{2}$, $n = 1,2,\cdots,m$, $\Delta_n = 3$, and

$$\rho(t,p(t)) = \lambda p(t)\cos\frac{p(t)}{e^t}$$

$$\sigma(t,p(t)) = e^{-t}\sin\frac{p(t)}{e^t}$$

Let $c = e^{\pi}$ and $\omega = \pi$. It is clear that for all $n \in \mathbb{N}$, $t_{n+2} = t_n + \pi$ and $\Delta_{n+2} = \Delta_n$. Hence, we have $N = 2$ and (IV) holds. For all $t \in \mathbb{R}$ and $p(t) \in \mathbb{R}$, we obtain

$$\rho(t+\omega, cp(t)) = \rho(t+\pi, e^{\pi}p(t)) = \lambda e^{\pi}p(t)\cos\frac{e^{\pi}p(t)}{e^{t+\pi}} = \lambda e^{\pi}p(t)\cos\frac{p(t)}{e^t} = e^{\pi}\rho(t,p(t))$$

$$\sigma(t+\omega, cp(t)) = \sigma(t+\pi, e^{\pi}p(t)) = e^{-t+\pi}\sin\frac{e^{\pi}p(t)}{e^{t+\pi}} = e^{\pi}e^{-t}\sin\frac{p(t)}{e^t} = e^{\pi}\sigma(t,p(t))$$

which implies that (I) holds. Also, $\kappa(s) = 1$ and $a_T = \int_0^1 \|1\|ds \leq 1$.

For any $t \in [1,\pi]$,and $p,q \in \mathbb{R}$, we have

$$\|\rho(t,p)\| = \|\lambda p(t)\cos\frac{p(t)}{e^t}\| = \|\lambda\|\|p(t)\|\|\cos\frac{p(t)}{e^t}\| \leq \|\lambda\|\|p(t)\|,$$

$$\|\sigma(t,p)\| = \|e^{-t}\sin\frac{p(t)}{e^t}\| = \|e^{-t}\|\|\sin\frac{p(t)}{e^t}\| \leq \|e^{-t}\| \leq \|e^{-1}\|,$$

which implies that $A_1 = \|\lambda\|$, $B_1 > 0$, $A_2 = e^{-1}$, *and* $B_2 > 0$ *and (II) holds.*

Letting $\|\lambda\| < \dfrac{e^{\pi-1}\,\Gamma(4/3)}{e^{\pi}\,\sqrt[3]{\log\pi}} - e^{-1}$, *we obtain* $\Gamma(4/3) > \sqrt[3]{\log\pi}\left(\|\lambda\| + e^{-1}\right)\left(\dfrac{e^{\pi}}{e^{\pi}-1}\right)$ *and*

$$\Gamma(\vartheta+1) > (\log\omega)^{\vartheta}(A_1 + A_2 a_T)\left(1 + |c-1|^{-1}\right).$$

Hence, all assumptions in Theorem 2 hold in Example 2. Hence, the eq. in Example 2 has at least one (π, e^{π})*-periodic solution* $p \in \Phi_{\pi,e^{\pi}}$.

4. Conclusions

This study demonstrated that (ω, c)-periodic solutions exist for impulsive Hadamard fractional differential equations with boundary value restrictions on Banach contractions. The study aimed to establish the existence and uniqueness of the (ω, c)-periodic solution of Equation (6), which applies the Banach contraction mapping concept. Furthermore, the paper demonstrated the existence of an (ω, c)-periodic solution to Equation (6) using Schaefer's fixed point theorem and the Arzelà-Ascoli theorem. We concluded the study with two examples demonstrating how the generated results could be used. In the future, we will conduct further investigations on the existence and uniqueness of fractional derivatives. There are also other possible lines of research on this topic, such as (ω, c)-periodic solutions for impulsive Hadamard fractional differential equations with varying lower limits. Moreover, mildly (ω, c)-periodic solutions to abstract semilinear Hadamard integro-differential equations and (ω, c)-almost periodic-type functions could also be studied and discussed.This work opens the door for other possible contributions to this topic by combining types of fractional derivatives and types of periodic solutions.

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Abbreviations

The following abbreviations are utilized in this document:

diff.	differential
eq.	equation
eqs.	equations
fra.	fractional

References

1. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. *Theory of Impulsive Differential Equations*; World Scientific Publishing Co., Pte. Ltd.: Singapore, 1989.
2. Bainov, D.; Simeonov, P. *Impulsive Differential Equations: Periodic Solutions and Applications*; Wiley: New York, NY, USA, 1993.
3. Bainov, D.; Simeonov, P. *Oscillation Theory of Impulsive Differential Equations*; International Publications: Philadelphia, PA, USA, 1998.
4. Stamov, G.T. *Almost Periodic Solutions of Impulsive Differential Equations*; Springer: Berlin, Germany, 2012.
5. Agaoglou, M.; Fečkan, M.; Panagiotidou, A.P. Existence and uniqueness of (ω, c)-periodic solutions of semilinear evolution equations. *Int. J. Dyn. Sys. Differ. Equ.* **2020**, *10*, 149–166. [CrossRef]
6. Wang, J.; Fečkan, M.; Tian, Y. Stability analysis for a general class of non-instantaneous impulsive differential equations. *Mediter. J. Math.* **2017**, *14*, 46. [CrossRef]
7. Alvarez, E.; Gómez, A.; Pinto, M. (ω, c)-periodic functions and mild solutions to abstract fractional integro-differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2018**, *16*, 1–8. [CrossRef]
8. Alvarez, E.X; Castillo, S.; Pinto, M. (ω, c)-Pseudo periodic functions, first order Cauchy problem and Lasota-Wazewska model with ergodic and unbounded oscillating production of red cells. *Bound. Value Probl.* **2019**, *2019*, 106. [CrossRef]

9. Ren, L.; Wang, J.R. (ω, c)-Periodic Solutions to Fractional Differential Equations with Impulses. *Axioms* **2022**, *11*, 83. [CrossRef]
10. Li, M.; Wang, J.; Fečkan, M. (ω, c)-periodic solutions for impulsive differential systems. *Commun. Math. Anal.* **2018**, *21*, 35–45.
11. Wang, J.; Ren, L.; Zhou, Y. (ω, c)-periodic solutions for time varying impulsive differential equations. *Adv. Differ. Equ.* **2019**, *2019*, 259. [CrossRef]
12. Khalladi, M.T.; Rahmani, A. (ω, c)-Pseudo almost periodic distributions. *Nonauton. Dyn. Syst.* **2020**, *7*, 237–248. [CrossRef]
13. Al-Omari, A.; Al-Saadi, H. (ω, ρ)-BVP solution of impulsive Hadamard fractional differential equations. *Mathematics* **2023**, *11*, 4370. [CrossRef]
14. Al-Omari, A.; Al-Saadi, H. Impulsive fractional order integrodifferential equation via fractional operators. *PLoS ONE* **2023**, *18*, e0282665. [CrossRef] [PubMed]
15. Duan, J. The periodic solution of fractional oscillation equation with periodic input. *Adv. Math. Phys.* **2013**, *2013*, 869484. [CrossRef]
16. Li, M.; Lim, S.C.; Chen, S. Exact solution of impulse response to a class of fractional oscillators and its stability. *Math. Probl. Eng.* **2011**, *2011*, 657839. [CrossRef]
17. El-Dib, Y.; Elgazery, N. Effect of fractional derivative properties on the periodic solution of the nonlinear oscillations. *Fractals* **2020**, *28*, 2050095. [CrossRef]
18. Kilbas, A.A. Hadamard-type fractional calculus. *Korean Math. Soc.* **2001**, *38*, 1191–1204.
19. Wang, J.R.; Fečkan, M.; Zhou, Y. A survey on impulsive fractional differential equations. *Frac. Calc. Appl. Anal.* **2016**, 19, 806–831. [CrossRef]