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Decay Properties for Transmission System with Infinite Memory and Distributed Delay

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Abstract: We consider a damped transmission problem in a bounded domain where the damping is effective in a neighborhood of a suitable subset of the boundary. Using the semigroup approach together with Hille–Yosida theorem, we prove the existence and uniqueness of global solution. Under suitable assumption on the geometrical conditions on the damping, we establish the exponential stability of the solution by introducing a suitable Lyapunov functional.

Keywords: Lyapunov functions; transmission problem; well-posedness; internal distributed delay; exponential stability; multiplier method; convexity; partial differential equations

MSC: 35B40; 35L56; 74F05; 93D15; 93D20

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1. Introduction and Position of Problem

Currently, most times, PDEs studies are migrating towards qualitative behaviors of new mathematical models, and for this it is important to pose the question of decay of solutions for evolutionary systems with different damping terms and show the asymptotic behaviors of solutions. For the extent to which different types of decay rate properties of transmission systems attract our attention to study the impact of two damping terms, it is largely requested, especially with the existence of infinite memory and distributed delay. While there is no novelty, as far as we know, in the idea of interaction between the problems of infinite memory and of distributed delay, nevertheless, there are few recent works in only one part of them. This research addresses the needs of mathematical physics interests for the transmission problem with infinite memory and distributed delay, which are acting at the same time. We consider the following transmission problem with past history and a distributed delay term:

$$\begin{cases} \partial_{tt}u - \alpha u_{xx} + \mu_1 \partial_t u + \displaystyle\int_0^\infty g(\sigma) u_{xx}(x, t-\sigma) d\sigma \\ \qquad\qquad + \eta(x) \displaystyle\int_{\tau_1}^{\tau_2} \mu(\tau) \partial_t u(x, t-\tau) d\tau = 0, & in\ \Sigma_1, \\ \partial_{tt} v - b v_{xx} = 0, & in\ \Sigma_2, \end{cases} \tag{1}$$

subject to the following initial conditions

$$u(x,t=0)=u_0(x),\quad \partial_t u(x,t=0)=u_1(x),\qquad x\in\Omega, \tag{2}$$
$$\partial_t u(x,-t)=f_0(x,-t),\qquad x\in\Omega,\ t\in(0,\tau_2), \tag{3}$$
$$v(x,t=0)=v_0(x),\quad \partial_t v(x,t=0)=v_1(x),\qquad x\in(\ell_1,\ell_2), \tag{4}$$

and the boundary transmission conditions

$$\begin{cases} u(x=0,t)=u(x=\ell_3,t)=0,\\ u(x=\ell_i,t)=v(x=\ell_i,t),\quad i=1,2,\\ \alpha u_x(x=\ell_i,t)-\displaystyle\int_0^\infty g(\tau)u_x(x=\ell_i,t-\tau)\mathrm{d}\tau=bv_x(x=\ell_i,t),\quad i=1,2,\end{cases} \tag{5}$$

where

$$0<\ell_1<\ell_2<\ell_3, \Omega=(0,\ell_1)\cup(\ell_2,\ell_3),\ \Sigma_1=\Omega\times\mathbb{R}_+^*,\ \Sigma_2=(\ell_1,\ell_2)\times\mathbb{R}_+^*,\ a,\ \mu_1,b>0.$$

The initial data $(u_0,\ u_1,\ v_0,\ v_1,\ f_0)$ belong to a suitable space. Moreover, $\mu\in([\tau_1,\tau_2],\mathbb{R})$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0\le\tau_1<\tau_2$.

Now, we mention some recent results regarding the stabilization of transmission problem. In [1], the authors considered one space dimension in the case $g\equiv 0$ and $\mu(s)=0$, $\forall s\in[\tau_1,\tau_2]$ and showed that the solution to the transmission problem (1)–(5) decays exponentially to zero as time tends to infinity. In [2], Raposo et al. treated the following transmission problem, where one small part of the beam is made of a viscoelastic material with Kelvin–Voigt damping:

$$\begin{cases} \partial_{tt}u-\alpha u_{xx}-\gamma u_{txx}=0 & in\quad (0,\ell_0)\times(0,\infty)\\ \partial_{tt}v-\beta v_{xx}=0 & in\quad (\ell_0,\ell)\times(0,\infty).\end{cases} \tag{6}$$

and boundary transmission conditions:

$$\begin{cases} u(\ell_0,t)=v(\ell_0,t), & t>0,\\ \alpha u_x(\ell_0,t)=\beta v_x(\ell_0,t), & t>0.\end{cases}$$

The authors proved that the energy of the system (6) is exponentially stable. When $g\equiv 0$ and $\eta(x)=1$, $\forall x\in\Omega$; the system (1)–(5) was studied in [3] by Gongwei Liu. The author demonstrated the existence and uniqueness of global solutions under appropriate assumptions on the weights of damping and distributed delay. Then, they obtained the exponential stability of the solution by introducing a suitable Lyapunov functional. In [4], Nicaise and Pignotti discussed the wave equation in two cases; the first case is when the delay is internally distributed with initial and mixed Dirichlet–Newman boundary conditions, and the second one is when the delay is distributed over part of the border. In both cases, the authors obtained the same result, namely, that the solution decreases exponentially under the following assumption:

$$\|a\|_\infty\int_{\tau_1}^{\tau_2}\mu_2(\tau)d\tau<\mu_1.$$

Li et al. [5] considered the following problem with history and delay:

$$\begin{cases} \partial_{tt}u-au_{xx}+\mu_1\partial_t u+\displaystyle\int_0^\infty g(\tau)u_{xx}(x,t-\tau)d\tau+\mu_2\partial_t u(x,t-\tau)=0,\\ \partial_{tt}v-bv_{xx}=0.\end{cases} \tag{7}$$

Under suitable assumptions on the delay term and the function g, an exponential stability result is proved for these two cases: $\mu_2<\mu_1$ and $\mu_2=\mu_1$. Also, system (1)–(5)

was studied by Bahri et al. [6]; the distributed delay is replaced with the fixed delay and $\mu_1 = 0$. Under certain appropriate hypotheses on the relaxation function and the weight of the delay, the authors proved the well-posedness by using the semigroup theory technique. Furthermore, they established a decay result by introducing a specific Lyapunov functional (see [7–9]).

Electronic devices are considered to be one of the largest areas of application for the transmission systems. On a daily basis, more and more people use these devices in multiple forms, ranging from smartphones to electronic tablets and connected watches. Despite the fact that the new electronics are miniaturized, they always require more energy to operate. This is one reason for which models with damped phenomenon are important, in order to increase control factors as much as possible. Transmission systems, when it comes in damping form such as our model, are revealed to be very interesting in applied real sciences. These increasingly complex systems with varied functions are fertile fields of research that require extensive qualitative studies (see [10,11]). The case considered in this paper, where there is an interaction between two damping terms, has not been tackled before. This makes the results obtained in this study very rare and useful.

This article is organized as follows: In Section 2, we give some preliminary results and we establish the well-posedness of the system (1), using the Hille–Yosida Theorem. Finally, in Section 3, we study the exponential stability of our main system.

2. Preliminary Results and Well-Posedness

First we recall and make use of the following assumptions on the function g.

(A1): We assume that the function $g \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$g(0) > 0, \quad \alpha - \int_0^{\infty} g(t)dt = \alpha - g_0 = l > 0. \tag{8}$$

(A2): There exists a positive constant δ such that

$$g'(\nu) \leq -\delta g(\nu), \quad \forall \nu \in (0, +\infty). \tag{9}$$

As in [12], we introduce the following new variable:

$$z(x,t,\varrho,s) = \partial_t u(x, t - \varrho s), \quad x \in \Omega, t \in \mathbb{R}_+^*, \varrho \in (0,1), s \in (\tau_1, \tau_2) = \Sigma_{1,\varrho,s},$$

which satisfies

$$s\partial_t z(x,t,\varrho,s) + z_\varrho(x,t,\varrho,s) = 0, \quad (x,t,\varrho,s) \in \Sigma_{1,\varrho,s}.$$

Using the idea in [13], if we set

$$\eta^t(x,t,\sigma) = u(x,t) - u(x,t-\sigma), \quad x \in \Omega, t \in \mathbb{R}_+, \sigma \in \mathbb{R}_+^* = \Sigma_{1,\sigma}, \tag{10}$$

then we obtain

$$\eta_t^t(x,t,\sigma) + \eta_\sigma^t(x,t,\sigma) = \partial_t u(x,t), \quad (x,t,\sigma) \in \Sigma_{1,\sigma}. \tag{11}$$

Now , the problem (1) is equivalent to

$$\begin{cases} \partial_{tt}u - lu_{xx} + \mu_1 \partial_t u - \int_0^{+\infty} g(\sigma)\eta_{xx}^t(\sigma)d\sigma + \eta(x)\int_{\tau_1}^{\tau_2} \mu(s)z(1,s)ds = 0, & in\, \Sigma_1, \\ \partial_{tt}v - bv_{xx} = 0, & in\, \Sigma_2, \\ s\partial_t z(\varrho,s) + z_\varrho(\varrho,s) = 0, & in\, \Sigma_{1,\varrho,s}, \\ \eta_t^t(\sigma) + \eta_\sigma^t(\sigma) = \partial_t u, & in\, \Sigma_{1,\sigma}. \end{cases} \tag{12}$$

and the system (12) is now subject to the initial conditions:

$$\begin{array}{lll} u(x,t=0)=u_0(x), & \partial_t u(x,t=0)=u_1(x), & x\in\Omega, \\ v(x,t=0)=v_0(x), & \partial_t v(x,t=0)=v_1(x), & x\in(\ell_1,\ell_2), \\ z(x,t,0,s)=\partial_t u(x,t), & & x\in\Omega, t\in(0,+\infty), s\in(\tau_1,\tau_2), \\ z(x,t=0,\varrho,s)=f_0(x,\varrho,s), & & x\in\Omega, \varrho\in(0,1), s\in(0,\tau_2), \end{array} \tag{13}$$

and the boundary transmission conditions (5) become

$$\begin{cases} u(x=0,t)=u(x=\ell_3,t)=0, \\ u(x=\ell_i,t)=v(x=\ell_i,t), \quad i=1,2,\ t\in]0,+\infty[, \\ lu_x(x=\ell_i,t)+\int_0^{+\infty} g(\sigma)\eta_x^t(x=\ell_i,\sigma)\mathrm{d}\sigma=bv_x(x=\ell_i,t), \quad i=1,2,\ t>0. \end{cases} \tag{14}$$

It is clear that

$$\begin{cases} \eta^t(x,t=0)=0, & x\in\Sigma_1, \\ \eta^t(x=0,\sigma)=\eta^t(\ell_3,\sigma)=0, & t\in\mathbb{R}_+, \sigma\in\mathbb{R}_+^*, \\ \eta^0(x,\sigma)=\eta_0(\sigma), & x\in\Omega, \sigma\in\mathbb{R}_+^*. \end{cases} \tag{15}$$

In order to transform the problem system (12)–(15) to an abstract problem on the Hilbert space $\mathcal{H}$ (see below), we introduce the vector function $U=(u,v,\partial_t u,\partial_t v,z,\eta^t)^T$, where $w=\partial_t u$ and $\varphi=\partial_t v$. Then, the system (12)–(15) can be rewritten as

$$\begin{cases} \partial_t U=\mathcal{A}U \\ U(0)=U_0, \end{cases} \tag{16}$$

where $U_0=(u_0,v_0,u_1,v_1,f_0,\eta_0)$ and $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\longrightarrow\mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \partial_t u \\ \partial_t v \\ z \\ \eta^t \end{pmatrix}=\begin{pmatrix} \partial_t u \\ \partial_t v \\ lu_{xx}-\mu_1\partial_t u-\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)z(1,s)\mathrm{d}s+\int_0^\infty g(\sigma)\eta_{xx}^t(\sigma)\mathrm{d}\sigma \\ bv_{xx} \\ -\frac{1}{s}z_\varrho(\varrho,s) \\ -\eta_\sigma^t(\sigma)+\partial_t u \end{pmatrix}. \tag{17}$$

We introduce the space

$$H_*=\left\{\begin{array}{l} (u,v)\in H^1(\Omega)\times H^1(\ell_1,\ell_2): u(x=0,t)=u(x=\ell_3,t)=0, \\ u(x=\ell_i,t)=v(x=\ell_i,t), lu_x(x=\ell_i,t) \\ +\int_0^\infty g(\sigma)\eta_x^t(x=\ell_i,\sigma)\mathrm{d}\sigma=bv_x(x=\ell_i,t),\ i=1,2 \end{array}\right\}, \tag{18}$$

and we set the energy space as

$$\mathcal{H}=H_*\times L^2(\Omega)\times L^2(\ell_1,\ell_2)\times L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)). \tag{19}$$

Then $\mathcal{H}$ is equipped with the inner product defined by

$$\begin{aligned} \langle V,\tilde{V}\rangle_{\mathcal{H}} &= \int_\Omega \varphi\tilde{\varphi}+lu_x\tilde{u}_x dx+\int_{\ell_1}^{\ell_2}(\omega\tilde{\omega}+bv_x\tilde{v}_x)dx \\ &+\int_\Omega\int_0^{+\infty} g(\sigma)w_x(\sigma)\tilde{w}_x(\sigma)d\sigma\mathrm{d}x \\ &+\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2}\int_0^1 s\mu(s)z(\varrho,s)\tilde{z}(\varrho,s)\mathrm{d}\varrho\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{20}$$

where

$$V = (u, v, \varphi, \varpi, z, w)^T, \quad \tilde{V} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\varpi}, \tilde{z}, \tilde{w})^T. \tag{21}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, v, \varphi, \varpi, z, w)^T \in \mathcal{H} : (u, v) \in \left(H^2(\Omega) \times H^2(\ell_1, \ell_2)\right) \cap H_*, \\ \varphi \in H^1(\Omega),\ \varpi \in H^1(\ell_1, \ell_2),\ w \in L^2_g\left(\mathbb{R}_+, H^2(\Omega) \cap H^1(\Omega)\right), \\ w_s \in \left(\mathbb{R}_+, H^1(\Omega)\right),\ z_\varrho \in L^2((0,1), L^2(\Omega)), \\ w(x, t=0) = 0,\ z(x, t=0) = \varphi(x) \end{array} \right\},$$

where $L^2_g(\mathbb{R}_+, H^1(\Omega))$ denotes the Hilbert space of H^1-valued functions on $\mathbb{R}_+$, endowed with the inner product

$$(\phi, \vartheta)_{L^2_g(\mathbb{R}_+, H^1(\Omega))} = \int_\Omega \int_0^{+\infty} g(s) \phi_x(s) \vartheta_x(s) \mathrm{d}s \mathrm{d}x. \tag{22}$$

We will show that $\mathcal{A}$ generates a C_0-semigroup on $\mathcal{H}$, under the assumption

$$\|\eta(x)\|_\infty \int_{\tau_1}^{\tau_2} |\mu(s)| \mathrm{d}s \le \mu_1. \tag{23}$$

Theorem 1. *Assume that **(A1)**,**(A2)** hold. For any $U_0 \in \mathcal{H}$, the problem (12) admits a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}). \tag{24}$$

Proof. We will use the semigroup approach together with Hille–Yosida theorem to prove the well-posedness of the system. First, we shall prove that the operator $\mathcal{A}$ is dissipative. Indeed, for $(u, v, \varphi, \varpi, z, w)^T \in D(\mathcal{A})$, we have

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \int_\Omega \Big(l u_{xx} - \mu_1 \varphi - \eta(x) \int_{\tau_1}^{\tau_2} \mu(s) z(1, s) \mathrm{d}s \Big) \varphi \, dx \\ &+ \int_\Omega \Big(\int_0^{+\infty} g(\sigma) w_{xx}(\sigma) \mathrm{d}\sigma \Big) \varphi \mathrm{d}x + b \int_{\ell_1}^{\ell_2} (v_{xx} \varpi + \varpi_x v_x) \mathrm{d}x \\ &+ l \int_\Omega \varphi_x u_x \mathrm{d}x + \int_\Omega \int_0^{+\infty} g(\sigma)(-w_{x\sigma}(\sigma) + \varphi_x) w_x(\sigma) d\sigma \mathrm{d}x \\ &- \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} \int_0^1 |\mu(s)| z_\varrho(\varrho, s) z(\varrho, s) \mathrm{d}\varrho \mathrm{d}s \mathrm{d}x. \end{aligned}$$

Then, by integration by parts, we obtain

$$\begin{aligned} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} \quad = \quad & \Big[l u_x \varphi + \Big(\int_0^{+\infty} g(\sigma) w_x(\sigma) \mathrm{d}\sigma \Big) \varphi \Big]_{\partial\Omega} - \mu_1 \int_\Omega \varphi^2 \mathrm{d}x \\ & + b [v_x \varpi]_{\ell_1}^{\ell_2} - \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} \mu(s) z(1, s) \varphi \mathrm{d}s \mathrm{d}x \\ & - \int_\Omega \int_0^{+\infty} g(\sigma) w_{x\sigma}(\sigma) w_x(\sigma) d\sigma \mathrm{d}x \\ & - \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} \int_0^1 |\mu(s)| z_\varrho(\varrho, s) z(\varrho, s) \mathrm{d}\varrho \mathrm{d}s \mathrm{d}x. \end{aligned} \tag{25}$$

For the last three terms of the RHS of the above equality, by noticing the fact $z(x, t = 0, t, s) = \varphi(x, t)$, $w(x, t = 0) = 0$ and $\varphi(x = \ell_i) = \varpi(x = \ell_i)$, $i = 1, 2$, we obtain

$$\begin{aligned}
&\int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} \int_0^1 |\mu(s)| z_\varrho(\varrho, s) z(\varrho, s) \mathrm{d}\varrho \mathrm{d}s \mathrm{d}x \\
&= \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} \int_0^1 |\mu(s)| \frac{d}{2d\varrho} z^2(\varrho, s) \mathrm{d}\varrho \mathrm{d}s \mathrm{d}x \\
&= \frac{1}{2} \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} |\mu(s)| \left[z^2(1, s) - \varphi^2 \right] \mathrm{d}s \mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \int_0^{+\infty} g(\sigma) w_{x\sigma}(\sigma) w_x(\sigma) d\sigma \mathrm{d}x \\
= \;& \frac{1}{2} \int_\Omega \left[g(\sigma) w_x^2(\sigma) \right]_0^{+\infty} \mathrm{d}x \\
&-\frac{1}{2} \int_\Omega \int_0^{+\infty} g'(\sigma) w_x^2(\sigma) d\sigma \mathrm{d}x.
\end{aligned}$$

By using the Cauchy–Schwartz and Young's inequalities, we obtain

$$\begin{aligned}
&\int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} \mu(s) z(1, s) \varphi \mathrm{d}s \mathrm{d}x \\
&\leq \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} |\mu(s)| |z(1, s)| |\varphi| \mathrm{d}s \mathrm{d}x \\
&\leq \int_\Omega \eta(x) |\varphi| \left(\int_{\tau_1}^{\tau_2} |\mu(s)| ds \right)^{\frac{1}{2}} \left(\int_{\tau_1}^{\tau_2} |\mu(s)| |z(1, s)|^2 ds \right)^{\frac{1}{2}} \mathrm{d}x \\
&\leq \frac{1}{2} \int_\Omega \eta(x) \left(\int_{\tau_1}^{\tau_2} |\mu(s)| ds \right) |\varphi|^2 dx + \frac{1}{2} \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} |\mu(s)| |z(1, s)|^2 ds \mathrm{d}x.
\end{aligned}$$

Substituting these estimations in (25) leads to

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} \leq\;& \left[l u_x \varphi + \left(\int_0^{+\infty} g(\sigma) w_x(\sigma) \mathrm{d}\sigma \right) \varphi \right]_{\partial\Omega} - \mu_1 \int_\Omega \varphi^2 \mathrm{d}x \\
&+ b[v_x \varpi]_{\ell_1}^{\ell_2} + \frac{1}{2} \int_\Omega \eta(x) \left(\int_{\tau_1}^{\tau_2} |\mu(s)| ds \right) |\varphi|^2 dx \\
&+ \frac{1}{2} \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} |\mu(s)| |z(1, s)|^2 ds \mathrm{d}x \\
&- \frac{1}{2} \int_\Omega \left[g(\sigma) w_x^2(\sigma) \right]_0^{+\infty} \mathrm{d}x + \frac{1}{2} \int_\Omega \int_0^{+\infty} g'(\sigma) w_x^2(\sigma) d\sigma \mathrm{d}x \\
&- \frac{1}{2} \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} |\mu(s)| \left[z^2(1, s) - \varphi^2 \right] \mathrm{d}s \mathrm{d}x \\
\leq\;& \left[l u_x \varphi + \left(\int_0^{+\infty} g(\sigma) w_x(\sigma) \mathrm{d}\sigma \right) \varphi \right]_{\partial\Omega} + b[v_x \varpi]_{\ell_1}^{\ell_2} \\
&- \frac{1}{2} \int_\Omega \left[g(\sigma) w_x^2(\sigma) \right]_0^{+\infty} \mathrm{d}x + \frac{1}{2} \int_\Omega \int_0^{+\infty} g'(\sigma) w_x^2(\sigma) d\sigma \mathrm{d}x \\
&- \mu_1 \int_\Omega \varphi^2 \mathrm{d}x + \left(\int_{\tau_1}^{\tau_2} |\mu(s)| ds \right) \int_\Omega \eta(x) |\varphi|^2 dx \\
\leq\;& \frac{1}{2} \int_\Omega \int_0^{+\infty} g'(\sigma) w_x^2(\sigma) d\sigma \mathrm{d}x \\
&- \left(\mu_1 - \|\eta(x)\|_\infty \int_{\tau_1}^{\tau_2} |\mu(s)| ds \right) \int_\Omega |\varphi|^2 dx,
\end{aligned} \tag{26}$$

which implies that $\mathcal{A}$ is dissipative.

Next, we prove that $-\mathcal{A}$ is maximal. It is sufficient to show that the operator $\lambda I - \mathcal{A}$ is surjective for a fixed $\lambda > 0$. Equivalently, we need to show that for a given

$$F = (h_1, h_2, h_3, h_4, h_5, h_6)^T \in \mathcal{H},$$

there exists

$$U = (u, v, \varphi, \varpi, z, w)^T \in D(\mathcal{A}),$$

satisfying

$$(\lambda I - \mathcal{A})U = F, \tag{27}$$

that is,

$$\begin{cases} \lambda u - \varphi = h_1, \\ \lambda v - \varpi = h_2, \\ \lambda \varphi - l u_{xx} - \mu_1 \varphi - \int_0^{+\infty} g(\sigma) w_{xx}(\sigma) d\sigma + \eta(x) \int_{\tau_1}^{\tau_2} \mu(s) z(x, t, 1, s) ds = h_3, \\ \lambda \varpi - b v_{xx} = h_4, \\ \lambda s z + \frac{1}{\tau} z_\varrho = s h_5, \\ \lambda w + w_s - \varphi = h_6. \end{cases} \tag{28}$$

Suppose that (u, v) is found with the appropriate regularity. Then, from the first and second equations of (28), we find that

$$\begin{cases} \varphi = \lambda u - h_1, \\ \varpi = \lambda v - h_2. \end{cases} \tag{29}$$

Then, $\varphi \in H^1(\Omega)$ and $\varpi \in H^1(\ell_1, \ell_2)$. Moreover, we obtain z for

$$z(0, s) = \varphi, \quad x \in \Omega.$$

Using the fifth equation of (28), we obtain

$$z(\varrho, s) = \varphi \exp(-\lambda \varrho s) + s \exp(-\lambda \varrho s) \int_0^\varrho h_5(x, \omega, s) \exp(\lambda \omega s) d\omega.$$

From (29), we obtain

$$z(\varrho, s) = \lambda u \exp(-\lambda \varrho s) - h_1 \exp(-\lambda \varrho s) + s \exp(-\lambda \varrho s) \int_0^\varrho h_5(x, \omega, s) \exp(\lambda \omega s) d\omega. \tag{30}$$

In particular,

$$z(1, s) = \lambda u \exp(-\lambda s) + z_0(x, t, s), \tag{31}$$

where

$$z_0(x, t, s) = -f_1 \exp(-\lambda s) + s \exp(\lambda s) \int_0^1 \exp(\lambda \omega s) h_5(x, \omega, s) d\omega.$$

It is not hard to see that the sixth equation of (28) with $w(x, t = 0) = 0$ has the unique solution

$$w(x, \sigma) = \Big(\int_0^\sigma \exp(\lambda y)(h_6(x, y) + \varphi(x)) dy \Big) \exp(-\lambda \sigma).$$

Thus, from (29), we have

$$w(x, \sigma) = \Big(\int_0^\sigma \exp(\lambda y)(h_6(x, y) + \lambda u(x) - h_1(x)) dy \Big) \exp(-\lambda \sigma). \tag{32}$$

From (28), (29), (31), and (32), we deduce that the functions u and v satisfy the equations

$$\begin{cases} \left(\lambda^2 - \lambda\mu_1 - \lambda\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)\exp(-\lambda s)\mathrm{d}s\right)u - \tilde{l}u_{xx} = \tilde{f}, \\ \lambda^2 v - bv_{xx} = h_4 + \lambda h_2, \end{cases} \tag{33}$$

where

$$\tilde{l} = l + \lambda\int_0^{+\infty} g(\sigma)\exp(-\lambda\sigma)\left(\int_0^{\sigma}\exp(\lambda y)dy\right)\mathrm{d}\sigma, \tag{34}$$

and

$$\begin{aligned} \tilde{f} = \; & \int_0^{+\infty} g(\sigma)\exp(-\lambda\sigma)\left(\int_0^{\sigma}\exp(\lambda y)(h_6(x,y) - h_1(x))\mathrm{d}y\right)_{xx}\mathrm{d}\sigma + (\lambda+\mu_1)f_1 + f_3 \\ & +\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)\left(f_1\exp(-\lambda s) - s\exp(\lambda s)\int_0^1\exp(\lambda\sigma s)h_5(x,\sigma,s)\mathrm{d}\sigma\right)\mathrm{d}s, \end{aligned}$$

which can be reformulated as follows:

$$\begin{cases} \int_{\Omega}\left(\lambda^2 u - \mu_1\lambda u - \lambda\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)\exp(-\lambda s)u\mathrm{d}s - \tilde{l}u_{xx}\right)\omega_1\mathrm{d}x = \int_{\Omega}\tilde{f}\omega_1\mathrm{d}x, \\ \int_{\ell_1}^{\ell_2}(\lambda^2 v - bv_{xx})\omega_2\mathrm{d}x = \int_{\ell_1}^{\ell_2}(h_4 + \lambda h_2)\omega_2\mathrm{d}x, \end{cases}$$

for any $(\omega_1, \omega_2) \in H_*$.

Now, by integrating by parts, we obtain the variational formulation corresponding to (33)

$$\Phi((u,v),(\omega_1,\omega_2)) = l(\omega_1,\omega_2), \tag{35}$$

where $\Phi \in C((H_*, H_*), \mathbb{R})$ is the coercive bilinear form, given by

$$\begin{aligned} \Phi((u,v),(\omega_1,\omega_2)) \; = & \int_{\Omega}\left[\left(\lambda^2 - \mu_1\lambda - \lambda\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)\exp(-\lambda s)\mathrm{d}s\right)u\omega_1 + \tilde{l}u_x\omega_{1x}\right]\mathrm{d}x \\ & + \int_{\ell_1}^{\ell_2}\left(\lambda^2 v\omega_2 + bv_x\omega_{2x}\right)\mathrm{d}x, \end{aligned}$$

and $l \in C(H_*, \mathbb{R})$ is the linear form defined by

$$l(\omega_1,\omega_2) = \int_{\Omega}\tilde{f}\omega_1\mathrm{d}x + \int_{\ell_1}^{\ell_2}(h_4 + \lambda h_2)\omega_2\mathrm{d}x.$$

From the Lax–Milgram Theorem, we deduce that there exists a unique solution $(u,v) \in \left(H^2(\Omega) \times H^2(\ell_1,\ell_2)\right) \cap H_*$ of the variational problem (35). Therefore, the operator $\lambda I - \mathcal{A}$ is subjective for any $\lambda > 0$ and so $\mathcal{A}$ is a maximal monotone operator. Then, following theorem 4.6 in [14], we obtain $\overline{D(\mathcal{A})} = \mathcal{H}$. Thus, according to Lumer–Philips Theorem (see [15] and Theorem 4.3 in [14]), the operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $\exp(t\mathcal{A})$. Hence, the solution of the evolution problem (16) admits the following representation:

$$U(t) = \exp(t\mathcal{A})U_0; t \geq 0,$$

which leads to the well-posedness of (16). □

3. Exponential Stability

We can now prove the stability result for the energy of the system (12)–(15), using the multiplied techniques together with Lyapunov function. The total energy associated with the system (12)–(15) is

$$\begin{aligned} E(t) = \ & \tfrac{1}{2}\Big[\int_\Omega \partial_t u^2 \mathrm{d}x + l\int_\Omega u_x^2 \mathrm{d}x + \int_{\ell_1}^{\ell_2} \partial_t v^2 \mathrm{d}x + b\int_{\ell_1}^{\ell_2} v_x^2 \mathrm{d}x \\ & + g\circ u_x + \int_\Omega \eta(x)\int_{\tau_1}^{\tau_2} s\mu(s)\int_0^1 z^2(\varrho,s)\mathrm{d}\varrho \mathrm{d}s \mathrm{d}x\Big], \end{aligned} \tag{36}$$

where

$$g\circ w = \int_\Omega \int_0^{+\infty} g(\sigma)(w(t)-w(t-\sigma))^2 \mathrm{d}\sigma \mathrm{d}x.$$

Theorem 2. *Let (u,v) be the solution of (1)–(5). Assume that **(A1)**, **(A2)** hold and that*

$$\frac{3}{2}\frac{\ell_3+3(\ell_2-\ell_1)}{\ell_1+\ell_3-\ell_2} l < \min\{\alpha, b\}. \tag{37}$$

Then, there exist two constants $\gamma_1,\gamma_2>0$ such that

$$E(t) \le \gamma_2 \exp(-\gamma_1 t), \quad \forall t\in\mathbb{R}_+. \tag{38}$$

For the proof of Theorem 2, we use the following Lemmas:

Lemma 1. *For any regular solution (u,v,z) of the problem (12)–(15), there exists a positive constant C such that*

$$E'(t) \le C\Big(\int_\Omega \int_0^{+\infty} g'(\sigma)(u_x - u_x(t-\sigma))^2 \mathrm{d}\sigma \mathrm{d}x - \int_\Omega \partial_t u^2 dx\Big), \tag{39}$$

where

$$C = \max\Big\{\frac{1}{2}, \mu_1 - \|\eta(x)\|_\infty \int_{\tau_1}^{\tau_2} |\mu(s)| \mathrm{d}s\Big\}.$$

Proof. By multiplying the first equation of (12) by $\partial_t u$, then integrating over Ω and using integration by parts, we obtain

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\Big[\int_\Omega \partial_t u^2 \mathrm{d}x + l\int_\Omega u_x^2 \mathrm{d}x + \int_\Omega\int_0^{+\infty} g(\sigma)\big(\eta_x^t(\sigma)\big)^2 \mathrm{d}\sigma \mathrm{d}x\Big] \\ & = -\mu_1\int_\Omega \partial_t u^2 \mathrm{d}x + l[u_x \partial_t u]_{\partial\Omega} + \frac{1}{2}\int_\Omega\int_0^{+\infty} g'(\sigma)\big(\eta_x^t(\sigma)\big)^2 \mathrm{d}\sigma \mathrm{d}x \\ & -\tfrac{1}{2}\int_\Omega \Big[g(\sigma)\big(\eta_x^t(\sigma)\big)^2\Big]_0^{+\infty} dx + \Big[\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\partial_t u d\sigma\Big]_{\partial\Omega} \\ & - \int_\Omega \Big(\eta(x)\int_{\tau_1}^{\tau_2} \mu(s) z(1,s)\mathrm{d}s\Big)\partial_t u \mathrm{d}x. \end{aligned} \tag{40}$$

Using Cauchy–Schwarz and Young's inequalities on the last term of the right-hand side of the above inequality, we see that

$$\begin{aligned} & -\int_\Omega \Big(\eta(x)\int_{\tau_1}^{\tau_2} \mu(s) z(1,s)\mathrm{d}s\Big)\partial_t u \mathrm{d}x \\ & \le \tfrac{1}{2}\|\eta(x)\|_\infty \Big(\int_{\tau_1}^{\tau_2} |\mu(s)| ds\Big)\int_\Omega \partial_t u^2 dx \\ & +\tfrac{1}{2}\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2} |\mu(s)| z^2(1,s)\mathrm{d}s \mathrm{d}x. \end{aligned} \tag{41}$$

As for the second equation of (12), we have

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega}\partial_t v^2 \mathrm{d}x + b\int_{\Omega} v_x^2 \mathrm{d}x\right] = b[v_x \partial_t v]_{\ell_1}^{\ell_2}. \tag{42}$$

Multiplying the third equation of (12) by $s\eta(x)|\mu(s)|z(\varrho,s)$, then integrating over $\Omega \times [\tau_1,\tau_2] \times [0,1]$ and using integration by parts, gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega}\int_{\tau_1}^{\tau_2}\int_0^1 s\eta(x)|\mu(s)|z^2(\varrho,s)d\varrho ds dx\right]\\ &\leq -\frac{1}{2}\int_{\Omega}\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)dsdx\\ &\quad+\frac{1}{2}\int_{\Omega}\eta(x)\left(\int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)\partial_t u^2 dx.\end{aligned} \tag{43}$$

By introducing (41) into (40), then summing (40)–(43), and using the transmission conditions (14), we conclude that

$$E^{'}(t) \leq C\left(\int_{\Omega}\int_0^{+\infty} g'(\sigma)(u_x - u_x(t-\sigma))^2 \mathrm{d}\sigma \mathrm{d}x - \int_{\Omega}\partial_t u^2 dx\right).$$

This completes the proof. □

Lemma 2. *The functional*

$$\mathcal{D}(t) = \int_{\Omega} u\partial_t u \mathrm{d}x + \int_{\ell_1}^{\ell_2} v\partial_t v \mathrm{d}x + \frac{\mu_1}{2}\int_{\Omega} u^2 \mathrm{d}x, \tag{44}$$

satisfies the following inequality

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t) \leq\ & \int_{\Omega}\partial_t u^2 \mathrm{d}x + \int_{\ell_1}^{\ell_2}\partial_t v^2 \mathrm{d}x - b\int_{\ell_1}^{\ell_2} v_x^2 \mathrm{d}x\\ &-\left(l - \epsilon\left(1 + C_0\|\eta(x)\|_{\infty}\int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)\right)\int_{\Omega} u_x^2 \mathrm{d}x\\ &+\frac{1}{4\epsilon}\int_{\Omega}\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)\mathrm{d}s\mathrm{d}x + \frac{g_0}{4\epsilon}\int_{\Omega}\int_0^{+\infty} g(\sigma)\left(\eta_x^t(\sigma)\right)^2 \mathrm{d}\sigma dx.\end{aligned} \tag{45}$$

Proof. Deriving $\mathcal{D}(t)$ with respect to t and integrating by parts while using (12), we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t) =\ & \int_{\Omega}\partial_t u^2 \mathrm{d}x - l\int_{\Omega} u_x^2 \mathrm{d}x + \int_{\ell_1}^{\ell_2}\partial_t v^2 \mathrm{d}x - b\int_{\ell_1}^{\ell_2} v_x^2 \mathrm{d}x\\ &+\left[\left(lu_x + \int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)u\right]_{\partial\Omega}\\ &+b[v_x v]_{\ell_1}^{\ell_2} - \int_{\Omega}\left(\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)u_x dx\\ &-\int_{\Omega}\left(\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)z(1,s)\mathrm{d}s\right)u\mathrm{d}x.\end{aligned} \tag{46}$$

Now, from the boundary conditions of (14), one can see that

$$u^2(x,t) = \left(\int_0^x u_x(x,t)\mathrm{d}x\right)^2 \leq \ell_1\int_0^{\ell_1} u_x^2(x,t)\mathrm{d}x,\ x \in [0,\ell_1], \tag{47}$$

and

$$u^2(x,t) \leq \left(\ell_3 - \ell_2\right)\int_{\ell_2}^{\ell_3} u_x^2(x,t)\mathrm{d}x,\ x \in [\ell_2,\ell_3], \tag{48}$$

which imply the following Poincaré's inequalities:

$$\int_\Omega u^2(x,t)\mathrm{d}x \le C_0 \int_\Omega u_x^2(x,t)\mathrm{d}x,\ x\in\Omega, \tag{49}$$

where $C_0 = \max\left\{L_1^2, (\ell_3-\ell_2)^2\right\}$.

From (14), we have

$$\left[\left(lu_x + \int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)u\right]_{\partial\Omega} = -b[v_x v]_{\ell_1}^{\ell_2}. \tag{50}$$

We can estimate the two last terms from the right-hand side of (46) as follows: Using Cauchy–Schwarz, Yong's, and Poincare's inequalities, we obtain

$$\begin{aligned}
&-\int_\Omega\left(\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)z(1,s)\right)u\mathrm{d}s\mathrm{d}x\\
&\le \int_\Omega \eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)||u||z(1,s)|\mathrm{d}s\mathrm{d}x\\
&\le \int_\Omega \eta(x)|u|\left(\int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)^{\frac{1}{2}}\left(\int_{\tau_1}^{\tau_2}|\mu(s)|\left|z^2(1,s)\right|\mathrm{d}s\right)^{\frac{1}{2}}\mathrm{d}x\\
&\le \epsilon\left(\int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)\int_\Omega|\eta(x)|u^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega|\eta(x)|\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)\mathrm{d}s\mathrm{d}x\\
&\le \epsilon C_0\|\eta(x)\|_\infty\left(\int_{\tau_1}^{\tau_2}|\mu(s)|ds\right)\int_\Omega u_x^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)\mathrm{d}s\mathrm{d}x,
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
&-\int_\Omega\left(\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)u_x dx\\
&\le \int_\Omega|u_x|\left(\int_0^{+\infty}g(\sigma)\left|\eta_x^t(\sigma)\right|\mathrm{d}\sigma\right)dx\\
&\le \epsilon\int_\Omega u_x^2 dx + \frac{1}{4\epsilon}\int_\Omega\left(\int_0^{+\infty}g(\sigma)\left|\eta_x^t(\sigma)\right|\mathrm{d}\sigma\right)^2 dx\\
&\le \tfrac{1}{4\epsilon}\int_\Omega\left[\left(\int_0^{+\infty}g(\sigma)\mathrm{d}\sigma\right)^{\frac{1}{2}}\left(\int_0^{+\infty}g(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma\right)^{\frac{1}{2}}\right]^2 dx + \epsilon\int_\Omega u_x^2 dx\\
&\le \epsilon\int_\Omega u_x^2 dx + \frac{g_0}{4\epsilon}\int_\Omega\int_0^{+\infty}g(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma dx.
\end{aligned} \tag{52}$$

By substituting (51) and (52) into (46), we obtain (45), as desired. □

As in [16], we define the function $\mathcal{I}(t)$ as follows:

$$\mathcal{I}(t) = \int_\Omega \eta(x)\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-\varrho s)|\mu(s)|z^2(\varrho,s)\mathrm{d}s\mathrm{d}\varrho\mathrm{d}x. \tag{53}$$

Lemma 3. *The functional $\mathcal{I}(t)$ satisfies the following:*

$$\begin{aligned}
\tfrac{d}{dt}\mathcal{I}(t) \le& -\exp(-\tau_2)\int_\Omega\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)\mathrm{d}s\mathrm{d}x\\
&+\left(\int_{\tau_1}^{\tau_2}|\mu(s)|\mathrm{d}s\right)\int_\Omega\eta(x)\partial_t u^2\mathrm{d}x - \exp(-\tau_2)\int_\Omega\eta(x)\int_0^1\int_{\tau_1}^{\tau_2}s|\mu(s)|z^2(\varrho,s)\mathrm{d}s\mathrm{d}\varrho\mathrm{d}x.
\end{aligned} \tag{54}$$

Proof. By differentiating $\mathcal{I}(t)$ and using the third equation in (12), we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{I}(t) &= -\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2}\int_0^1 \exp(-\varrho s)|\mu(s)|\frac{d}{d\varrho}z^2(\varrho,s)\mathrm{d}\varrho\mathrm{d}s\mathrm{d}x \\
&= -\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|\int_0^1 \frac{d}{d\varrho}\Big(\exp(-\varrho s)z^2(\varrho,s)\Big)\mathrm{d}\varrho\mathrm{d}s\mathrm{d}x \\
&\quad -\int_\Omega \eta(x)\int_0^1\int_{\tau_1}^{\tau_2} s|\mu(s)|\exp(-\varrho s)z^2(\varrho,s)\mathrm{d}\varrho\mathrm{d}s\mathrm{d}x \\
&= -\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2}\exp(-s)|\mu(s)|z^2(1,s)\mathrm{d}s\mathrm{d}x + \left(\int_{\tau_1}^{\tau_2}|\mu(s)|\mathrm{d}s\right)\int_\Omega \eta(x)\partial_t u^2\mathrm{d}x \\
&\quad -\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2} s|\mu(s)|\int_0^1 \exp(-\varrho s)z^2(\varrho,s)\mathrm{d}\varrho\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{55}
$$

Since $\exp(-s)\le\exp(-\varrho s)\le 1$, for all $\varrho\in[0,1]$ and $-\exp(-s)\le-\exp(-\tau_2)$, for all $s\in[\tau_1,\tau_2]$, the result (54) is achieved. □

We are now ready to introduce the functional

$$
p(x)=\begin{cases}
\frac{x}{2}-\frac{\ell_1}{4}, & x\in[0,\ell_1]\\
\frac{\ell_1}{4}+\frac{\ell_1-\ell_2+\ell_3}{4(\ell_1-\ell_2)}(x-\ell_1), & x\in[\ell_1,\ell_2]\\
\frac{x}{2}-\frac{\ell_2+\ell_3}{4}, & x\in[\ell_2,\ell_3].
\end{cases} \tag{56}
$$

It is easy to see that there exists a constant M such that

$$
\forall x\in[0,\ell_3];\ |p(x)|\le M=\frac{1}{4}\max\{\ell_1,\ell_3-\ell_2\}.
$$

Lemma 4. *Let (u,v,η^t,z) be the solution of (12). Then the functional ϖ defined by*

$$
\varpi(t):=-\int_\Omega p(x)\partial_t u\left(lu+\int_0^{+\infty}g(\sigma)\eta^t(\sigma)\mathrm{d}\sigma\right)_x\mathrm{d}x, \tag{57}
$$

satisfies the following inequality:

$$
\begin{aligned}
\frac{d}{dt}\varpi(t) &\le C_0'\int_\Omega \partial_t u^2\mathrm{d}x + C_1 l^2\int_\Omega u_x^2\mathrm{d}x + g_0C_1\int_\Omega\int_0^{+\infty} g(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}x \\
&+\frac{M^2}{2\delta_1}\int_\Omega \eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)dsdx-\frac{\alpha}{8}\Big[\ell_1\partial_t u^2(\ell_1)+(\ell_3-\ell_2)\partial_t u^2(\ell_2)\Big] \\
&+\frac{b^2}{2}\left[p(x)v_x^2\right]_{\ell_1}^{\ell_2}-\frac{g(0)M^2}{4\delta_3}\int_\Omega\int_0^{+\infty} g'(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}x,
\end{aligned} \tag{58}
$$

where $C_0'=\delta_3+\frac{\alpha}{4}+\frac{(M\mu_1)^2}{2\delta_2}$ and $C_1=\frac{1}{2}+\delta_2+\delta_1\|\eta(x)\|_\infty\int_{\tau_1}^{\tau_2}|\mu(s)|ds$.

Proof. Multiplying the first equation in (12) by $p(x)\left(lu+\int_0^{+\infty}g(\sigma)\eta^t(\sigma)\mathrm{d}\sigma\right)_x$ and integrating over Ω, implies that

$$
\begin{aligned}
&\int_\Omega p(x)\partial_{tt}u\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x-\frac{1}{2}\left[p(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\right]_{\partial\Omega} \\
&+\frac{1}{4}\int_\Omega\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\mathrm{d}x+\mu_1\int_\Omega \partial_t u p(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x \\
&+\int_\Omega\left(\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)z(1,s)\mathrm{d}s\right)p(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x=0.
\end{aligned} \tag{59}
$$

On the other hand, by using the forth equation in (12), we find

$$\begin{aligned}&\int_{\Omega}\partial_{tt}up(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x\\&=-\tfrac{d}{dt}\varpi(t)-\int_{\Omega}\partial_t up(x)\left(l\partial_t u_x+\int_0^{+\infty}g(\sigma)\left(\partial_t u_x-\eta_{x\sigma}^t(\sigma)\right)\mathrm{d}\sigma\right)\mathrm{d}x.\end{aligned}$$

Integrating by parts with respect to σ and using the fact that $\lim_{\sigma\longrightarrow\infty}g(\sigma)\eta^t(t,\sigma)=0$ and $\eta^t(t,0)=0$ yields

$$\begin{aligned}&\int_{\Omega}\partial_{tt}up(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x\\&=-\tfrac{d}{dt}\varpi(t)-\tfrac{\alpha}{2}\left[p(x)\partial_t u^2\right]_{\partial\Omega}+\tfrac{\alpha}{4}\int_{\Omega}\partial_t u^2dx\\&-\int_{\Omega}\partial_t up(x)\left(\int_0^{+\infty}g'(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x.\end{aligned}\tag{60}$$

By replacing (60) into (59) and integrating by parts, we have

$$\begin{aligned}\tfrac{d}{dt}\varpi(t)=&-\tfrac{\alpha}{2}\left[p(x)\partial_t u^2\right]_{\partial\Omega}-\tfrac{1}{2}\left[p(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\right]_{\partial\Omega}\\&+\tfrac{\alpha}{4}\int_{\Omega}\partial_t u^2dx-\int_{\Omega}\partial_t up(x)\left(\int_0^{+\infty}g'(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x\\&+\tfrac{1}{4}\int_{\Omega}\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\mathrm{d}x\\&+\mu_1\int_{\Omega}\partial_t up(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x\\&+\int_{\Omega}\left(\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)z(1,s)\mathrm{d}s\right)p(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x.\end{aligned}\tag{61}$$

If we use the transmission conditions, we find

$$\left[p(x)\partial_t u^2\right]_{\partial\Omega}=\tfrac{\ell_1}{4}\partial_t u^2(\ell_1)+\tfrac{\ell_3-\ell_2}{4}\partial_t u^2(\ell_2),\tag{62}$$

and

$$\begin{aligned}&\left[p(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\right]_{\partial\Omega}\\&=-\left[p(x)b^2v_x^2\right]_{\ell_1}^{\ell_2}+\tfrac{\ell_1}{4}\left(lu_x(0)+\int_0^{+\infty}g(\sigma)\eta_x^t(0,\sigma)\mathrm{d}\sigma\right)^2\\&\quad+\tfrac{\ell_3-\ell_2}{4}\left(lu_x(\ell_3)+\int_0^{+\infty}g(\sigma)\eta_x^t(\ell_3,\sigma)\mathrm{d}\sigma\right)^2.\end{aligned}$$

This means that

$$-\tfrac{1}{2}\left[p(x)\left(lu_x+\int_0^{+\infty}g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\right]_{\partial\Omega}\le\tfrac{b^2}{2}\left[p(x)v_x^2\right]_{\ell_1}^{\ell_2}.\tag{63}$$

By Cauchy–Schwarz and Young's inequalities, we conclude that

$$\begin{aligned}
&-\int_\Omega p(x)\partial_t u\left(\int_0^{+\infty} g'(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x \\
&\leq M\int_\Omega|\partial_t u|\left|\int_0^{+\infty} -g'(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right|\mathrm{d}x \\
&\leq \tfrac{M^2}{4\delta_3}\int_\Omega\left(\int_0^{+\infty} -g'(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\mathrm{d}x+\delta_3\int_\Omega \partial_t u^2 dx \\
&\leq -\tfrac{g(0)M^2}{4\delta_3}\int_\Omega\int_0^{+\infty} g'(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}x+\delta_3\int_\Omega \partial_t u^2 dx.
\end{aligned} \tag{64}$$

Similarly, we have

$$\begin{aligned}
&\tfrac{1}{4}\int_\Omega\left(lu_x+\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\mathrm{d}x \\
&\leq \int_\Omega \frac{l^2}{2}u_x^2+\frac{1}{2}\left(\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)^2\mathrm{d}x \\
&\leq \tfrac{l^2}{2}\int_\Omega u_x^2 dx+\frac{g_0}{2}\int_\Omega\int_0^{+\infty} g(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}x.
\end{aligned} \tag{65}$$

For the last two integrals, using Young, Minkowski, and Cauchy–Schwarz's inequalities, we obtain

$$\begin{aligned}
&\mu_1\int_\Omega p(x)\partial_t u\left(lu_x+\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x \\
&\qquad\leq \mu_1 M\int_\Omega|\partial_t u|\left|lu_x+\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right|\mathrm{d}x \\
&\qquad\leq \tfrac{(\mu_1 M)^2}{2\delta_2}\int_\Omega \partial_t u^2 dx+\frac{\delta_2}{2}\int_\Omega\left|lu_x+\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right|^2\mathrm{d}x \\
&\qquad\leq \tfrac{(\mu_1 M)^2}{2\delta_2}\int_\Omega \partial_t u^2 dx+\delta_2 l^2\int_\Omega u_x^2 dx+\delta_2 g_0\int_\Omega\int_0^{+\infty} g(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}x,
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
&\int_\Omega p(x)\left(\eta(x)\int_{\tau_1}^{\tau_2}\mu(s)z(1,s)\mathrm{d}s\right)\left(lu_x+\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\mathrm{d}x \\
&\leq M\int_\Omega\left(\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)||z(1,s)|\mathrm{d}s\right)\left|\left(lu_x+\int_0^{+\infty} g(\sigma)\eta_x^t(\sigma)\mathrm{d}\sigma\right)\right|\mathrm{d}x \\
&\leq M\int_\Omega\left(\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|\mathrm{d}s\right)^{\frac{1}{2}}\left(\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)\mathrm{d}s\right)^{\frac{1}{2}}\times\left(l|u_x|+\int_0^{+\infty} g(\sigma)\left|\eta_x^t(\sigma)\right|\mathrm{d}\sigma\right)\mathrm{d}x \\
&\leq \tfrac{\delta_1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu(s)|\mathrm{d}s\right)\int_\Omega\eta(x)\left(l|u_x|+\int_0^{+\infty} g(\sigma)\left|\eta_x^t(\sigma)\right|\mathrm{d}\sigma\right)^2\mathrm{d}x \\
&\quad+\tfrac{M^2}{2\delta_1}\int_\Omega\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)\mathrm{d}s\mathrm{d}x \\
&\leq \delta_1\left(\int_{\tau_1}^{\tau_2}|\mu(s)|\mathrm{d}s\right)\|\eta(x)\|_\infty\left(l^2\int_\Omega u_x^2 dx+g_0\int_\Omega\int_0^{+\infty} g(\sigma)\left(\eta_x^t(\sigma)\right)^2\mathrm{d}\sigma\mathrm{d}x\right) \\
&\quad+\tfrac{M^2}{2\delta_1}\int_\Omega\eta(x)\int_{\tau_1}^{\tau_2}|\mu(s)|z^2(1,s)\mathrm{d}s\mathrm{d}x.
\end{aligned} \tag{67}$$

Inserting the estimates (62) and (63) into (61) and using Young and Poincaré's inequalities leads to the desired estimate. □

Now, we define the functional

$$\mathcal{F}(t) = -\int_{\ell_1}^{\ell_2} p(x)\partial_t v v_x \mathrm{d}x,$$

Lemma 5. *The functional $\mathcal{F}(t)$ satisfies*

$$\begin{aligned} \tfrac{d}{dt}\mathcal{F}(t) &= -\beta\int_{\ell_1}^{\ell_2} \partial_t v^2 \mathrm{d}x - b\beta\int_{\ell_1}^{\ell_2} v_x^2 \mathrm{d}x + \frac{\ell_1}{8}\partial_t v^2(\ell_1) + \frac{\ell_3-\ell_2}{8}\partial_t v^2(\ell_2) \\ &\quad + \tfrac{b\ell_1}{8} v_x^2(\ell_1) + \tfrac{b(\ell_3-\ell_2)}{8} v_x^2(\ell_2), \end{aligned} \tag{68}$$

where $\beta = \frac{\ell_1-\ell_2+\ell_3}{8(\ell_2-\ell_1)} > 0$.

Proof. By multiplying the second equation of (12) by $p(x)v_x$ and integrating over $[\ell_1, \ell_2]$, we obtain

$$\tfrac{d}{dt}\mathcal{F}(t) = -\tfrac{1}{2}\int_{\ell_1}^{\ell_2} p(x)\frac{d}{dx}\partial_t v^2 \mathrm{d}x - \frac{b}{2}\int_{\ell_1}^{\ell_2} p(x)\frac{d}{dx} v_x^2 \mathrm{d}x, \tag{69}$$

which, integrated by parts, leads to

$$\begin{aligned} \tfrac{d}{dt}\mathcal{F}(t) &= -\tfrac{\ell_1-\ell_2+\ell_3}{8(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_2} \partial_t v^2 \mathrm{d}x - b\frac{\ell_1-\ell_2+\ell_3}{8(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_2} v_x^2 \mathrm{d}x \\ &\quad -\tfrac{1}{2}\left[p(x)\partial_t v^2\right]_{\ell_1}^{\ell_2} - \tfrac{b}{2}\left[p(x)v_x^2\right]_{\ell_1}^{\ell_2}. \end{aligned} \tag{70}$$

This completes the proof. □

We are now in a position to define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the total energy functional E.

Lemma 6. *For sufficiently large N, the functional defined by*

$$\mathcal{L}(t) = NE(t) + N_1\mathcal{D}(t) + \varpi(t) + N_2\mathcal{F}(t) + N_3\mathcal{I}(t), \tag{71}$$

where N_1, N_2, and N_3 are positive real numbers to be chosen appropriately later on, satisfies

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t), \tag{72}$$

for two positive constants β_1 and β_2.

Proof. For $\mathcal{D}(t)$, using Young and Poincare's inequalities, we have

$$\begin{aligned} |\mathcal{D}(t)| &\le \int_\Omega |u||\partial_t u|\mathrm{d}x + \int_{\ell_1}^{\ell_2} |v||\partial_t v|\mathrm{d}x + \frac{\mu_1}{2}\int_\Omega u^2 \mathrm{d}x \\ &\le \int_\Omega c_0 u_x^2 + c_1\partial_t u^2 dx + \int_{\ell_1}^{\ell_2} c_2 v_x^2 + c_3 \partial_t v^2 dx \\ &\le \kappa_0 E(t), \end{aligned} \tag{73}$$

where $\kappa_0 = 2\max\{\frac{c_0}{l}, c_1, \frac{c_2}{b}, c_3\}$ and

$$\begin{aligned} |\mathcal{I}(t)| &\le \tau_2\int_\Omega \eta(x)\int_0^1\int_{\tau_1}^{\tau_2} s\exp(-\varrho s)|\mu(s)|z^2(\varrho, s)\mathrm{d}s\mathrm{d}\varrho\mathrm{d}x \\ &\le 2\tau_2 E(t). \end{aligned} \tag{74}$$

For $\varpi(t)$, using Young, Minkowski, and Cauchy–Schwarz's inequalities, we have

$$\begin{aligned} |\varpi(t)| &\leq \int_\Omega |p(x)||\partial_t u|\left(l|u_x| + \int_0^{+\infty} g(\sigma)|\eta_x^t(\sigma)|\mathrm{d}\sigma\right)\mathrm{d}x \\ &\leq a_0 \int_\Omega \partial_t u^2 dx + a_1 \int_\Omega u_x^2 dx + a_2 \int_\Omega \int_0^{+\infty} g(\sigma)\left(\eta_x^t(\sigma)\right)^2 \mathrm{d}\sigma\mathrm{d}x \\ &\leq \kappa_1 E(t), \end{aligned} \tag{75}$$

where $\kappa_1 = 2\max\{a_0, \frac{a_1}{l}, a_2\}$.

For $\mathcal{F}(t)$, using Young's inequality, we have

$$\begin{aligned} |\mathcal{F}(t)| &\leq \int_{\ell_1}^{\ell_2} |p(x)||\partial_t v||v_x|\mathrm{d}x \\ &\leq b_0 \int_{\ell_1}^{\ell_2} \partial_t v^2 \mathrm{d}x + b_1 \int_{\ell_1}^{\ell_2} v_x \mathrm{d}x \\ &\leq \kappa_2 E(t), \end{aligned} \tag{76}$$

where $\kappa_2 = 2\max\left\{b_0, \frac{b_1}{b}\right\}$.

By these estimates we deduce that

$$\begin{aligned} |\mathcal{L}(t) - NE(t)| &\leq N_1|\mathcal{D}(t)| + |\varpi(t)| + N_2|\mathcal{F}(t)| + N_3|\mathcal{I}(t)| \\ &\leq CE(t). \end{aligned} \tag{77}$$

□

Proof of Theorem 2. Taking the derivative of (71) with respect to t and making use of (36), (39), (45), (53), (58), and (68), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) \leq &-\left[NC - N_1 - \delta_3 - \frac{\alpha}{4} - \frac{(M\mu_1)^2}{2\delta_2} - N_3 C_{\eta,\mu}\right] \int_\Omega \partial_t u^2 \mathrm{d}x - [N_2\beta - N_1] \int_{\ell_1}^{\ell_2} \partial_t v^2 \mathrm{d}x \\ &- \left[N_1\left(l - \epsilon\left(1 + C_0 C_{\eta,\mu}\right)\right) - C_1 l^2\right] \int_\Omega u_x^2 \mathrm{d}x - b[N_1 + N_2\beta] \int_{\ell_1}^{\ell_2} v_x^2 \mathrm{d}x \\ &+ \left[\frac{N}{2} - \frac{g(0)M^2}{4\delta_3}\right](g' \circ u_x) - \left[N_3 \exp(-\tau_2) - \frac{N_3}{4\epsilon} - \frac{M^2}{2\delta_3}\right] \int_\Omega \eta(x) \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(1,s) \mathrm{d}s\mathrm{d}x \\ &- \frac{\alpha - N_2}{8}\left[\ell_1 \partial_t u^2(\ell_1) + (\ell_3 - \ell_2)\partial_t u^2(\ell_2)\right] - \frac{b}{8}(b - N_2)\left[\ell_1 v_x^2(\ell_1) + (\ell_3 - \ell_2) v_x^2(\ell_2)\right] \\ &- N_3 \exp(-\tau_2) \int_\Omega \eta(x) \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu(s)| z^2(\varrho, s) \mathrm{d}s\mathrm{d}\varrho\mathrm{d}x + g_0\left[\frac{N_1}{4\epsilon} + C_1\right](g \circ u_x), \end{aligned} \tag{78}$$

where $C_{\eta,\mu} = \|\eta(x)\|_\infty \int_{\tau_1}^{\tau_2} |\mu(s)|\mathrm{d}s$.

We can then find N_1, N_2, and ϵ such that

$$N_1 = \frac{3}{2}C_1 l, \quad N_2 > \frac{3}{2}\frac{\ell_3 + 3(\ell_2 - \ell_1)}{\ell_1 + \ell_3 - \ell_2} l \quad \text{and} \quad \epsilon < \frac{l}{6\left(1 + C_0 C_{\eta,\mu}\right)}.$$

We may take N large enough such that

$$N_3 \geq \left(\frac{N_1}{4\varepsilon} + \frac{M^2}{2\delta_1}\right)\exp(-\tau_2).$$

Thus, we conclude that there exists a positive constant γ such that (78) yields

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leq -\gamma\left[\int_\Omega \partial_t u^2 + u_x^2 \mathrm{d}x + \int_{\ell_1}^{\ell_2} \partial_t v^2 + v_x^2 \mathrm{d}x \right.\\ &\left. + \int_\Omega \eta(x) \int_0^1 \int_{\tau_1}^{\tau_2} s z^2(\varrho, s) \mathrm{d}s \mathrm{d}\varrho \mathrm{d}x \right] + \xi g \circ u_x,\end{aligned} \tag{79}$$

where $\xi = g_0\left[\frac{N_1}{4\varepsilon} + C_1\right]$.

Using (36), which implies

$$\frac{d}{dt}\mathcal{L}(t) \leq -CE(t) + \xi g \circ u_x,$$

and by multiplying this equality by δ while using (9) and (39), we obtain

$$\delta \frac{d}{dt}\mathcal{L}(t) \leq -\delta C E(t) - \lambda \frac{d}{dt} E(t). \tag{80}$$

That is,

$$\frac{d}{dt}\big(\delta \mathcal{L}(t) + \lambda E(t)\big) \leq -\delta C E(t), \tag{81}$$

where $\lambda > 0$. Denote $\mathcal{E}(t) = \delta \mathcal{L}(t) + \lambda E(t)$, then it is easy to see that there exist two positive constants, β_1, β_2, such that

$$\beta_1 \, E(t) \leq \mathcal{E}(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{82}$$

Combining (81) and (82), we deduce that there exists $\gamma_1 > 0$ for which the following estimate holds:

$$\frac{d\mathcal{E}(t)}{dt} \leq -\gamma_1 \mathcal{E}(t), \quad \forall t \geq 0. \tag{83}$$

Now a simple integration of this inequality on $]t_0, t[$ leads to

$$\mathcal{E}(t) \leq \mathcal{E}(t_0) \exp(-\gamma_1(t - t_0)), \quad \forall t \geq t_0. \tag{84}$$

This estimate is also true for $t \in [0, t_0]$ by virtue of the continuity and boundedness of $\mathcal{E}(t)$. The proof of Theorem 2 is, hence, completed. □

4. Conclusions and Discussion

One of the main achievements of our research is to associate some physical processes (damping terms) with the transmission system and develop techniques to ensure the existence of a unique solution. Furthermore, we succeeded in relaxing the complicated standard requirement on the transmission conditions existing in previous works in the literature. A variety of techniques are known to achieve the desired result, including the semigroups techniques which are used in our work. We also obtained new results related to the qualitative aspect of the problem, namely, the exponential stability. This field of research is very important for researchers who are interested in modern science; engineers and new physical principles. In many practical applications, it is possible to formulate extremely important problems, the solution of which requires newly derived methods such as problems that contain fractional derivative in the transmission conditions, with a variable time delay (see [17–19]).

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