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The Existence of Solutions for w -Weighted ψ -Hilfer Fractional Differential Inclusions of Order $\mu \in (1, 2)$ with Non-Instantaneous Impulses in Banach Spaces

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Abstract: In this research, we obtain the sufficient conditions that guarantee that the set of solutions for an impulsive fractional differential inclusion involving a w -weighted ψ -Hilfer fractional derivative, $D_{0,t}^{\sigma,v,\psi,w}$, of order $\mu \in (1, 2)$, in infinite dimensional Banach spaces that are not empty and compact. We demonstrate the exact relation between a differential equation involving $D_{0,t}^{\sigma,v,\psi,w}$ of order $\mu \in (1, 2)$ in the presence of non-instantaneous impulses and its corresponding fractional integral equation. Then, we derive the formula for the solution for the considered problem. The desired results are achieved using the properties of the w -weighted ψ -Hilfer fractional derivative and appropriate fixed-point theorems for multivalued functions. Since the operator $D_{0,t}^{\sigma,v,\psi,w}$ includes many types of well-known fractional differential operators, our results generalize several results recently published in the literature. We give an example that illustrates and supports our theoretical results.

Keywords: differential inclusions; non-instantaneous impulses; w -weighted ψ -Hilfer fractional derivative; measure of non-compactness



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1. Introduction

A differential inclusion is a generalization of the concept of a differential equation where the right-hand side is a multivalued function. Differential inclusions arise in many situations, including differential variational inequalities, optimization problems, and certain non-smooth mechanical systems, such as stick–slip oscillations. Moreover, it can be used to understand and interpret discontinuous ordinary differential equations, such as the functions that model Coulomb friction in mechanical systems and ideal switches in power electronics. For applications of differential inclusions, we refer the reader to [1,2].

Differential equations and differential inclusions, whose order is fractional, have many applications in many branches, such as medicine, engineering, physics, fluid mechanics, and polymer science [3–10].

Impulsive differential problems are appropriate models for describing processes which at certain moments change their state rapidly and which cannot be described using classical differential problems. If this change occurs instantaneously, it is called an instantaneous impulse, and if the change continues for a period of time, it is called a non-instantaneous impulse. One of the examples is the motion of an elastic ball bouncing vertically on a surface. The moments of the impulses are the times when the ball touches the surface and its velocity changes rapidly. Differential equations and differential inclusions with non-instantaneous impulses have many applications in physics and applied mathematics. For more applications, we refer the reader to [11–13]. For papers concerning the existence

of solutions or mild solutions for different kinds of impulsive differential equations and inclusions, we refer the reader to [14–24]. In the literature, there are two types of problems that contain non-instantaneous impulses. The first one keeps the lower limit of the fractional derivative at zero. The second one switches it at the impulsive points. In this work, we will adopt the second type.

There is more than one concept to define fractional differentiation, and some of them generalize the others. Therefore, it is useful to study fractional differential equations and fractional differential inclusions that contain a fractional differential operator that generalizes many other fractional differential operators. This is our goal in this work. In fact, in this work, we establish conditions which guarantee that the set of solutions to differential inclusions containing the w -weighted ψ -Hilfer fractional derivative, $D_{0,t}^{\sigma,v,\psi,w}$ (Definition 3, below), in the presence of non-instantaneous impulses is not empty and compact. We consider a general infinite dimensional Banach space. Note that differential inclusions are a generalization of differential equations. It is acknowledged that the w -weighted ψ -Hilfer fractional derivative, $D_{0,t}^{\sigma,v,\psi,w}$, generalizes the following well-known fractional derivatives:

- i—The ψ -Hilfer derivative ($w(t) = 1; t \geq 0$);
- ii—The Hilfer-Katugampola derivative ($w(t) = 1; t \geq 0, \psi(t) = t^\rho, \rho \geq 1$);
- iii—The Hilfer derivative ($w(t) = 1; t \geq 0, \psi(t) = t$);
- iv—The Hilfer-Hadamard derivative ($w(t) = 1; t \geq 0, \psi(t) = \log t, t > 0$);
- v—The Katugampola fractional derivative ($w(t) = 1; t \geq 0, \psi(t) = t^\rho, \rho \geq 1, v = 1$);
- vi—The ψ -Caputo derivative ($w(t) = 1; t \geq 0, v = 1$);
- vii—The ψ -Riemann-Liouville derivative ($w(t) = 1; t \geq 0, v = 0$);
- viii—The Hadamard derivative ($w(t) = 1; t \geq 0, \psi(t) = \log t, t > 0, v = 1$);
- ix—The Caputo derivative ($w(t) = 1; t \geq 0, \psi(t) = t, v = 1$);
- x—The Riemann-Liouville derivative ($w(t) = 1; t \geq 0, \psi(t) = t, v = 0$).

To formulate the problem that we study in this paper, we assume that $\mathcal{E} = [0, b]$, E is a Banach space; $\sigma \in (1, 2)$, $v \in [0, 1]$, $\gamma = \sigma + 2v - \sigma v$, $w : \mathcal{E} \rightarrow]0, \infty[, w^{-1}(\varrho) = \frac{1}{w(\varrho)}$, $\psi : \mathcal{E} \rightarrow \mathbb{R}$ is a strictly increasing continuously differentiable function where $\psi'(s) \neq 0$; for any $s \in \mathcal{E}$, $F : \mathcal{E} \times E \rightarrow P_{ck}(E)$ (the family of non-empty convex and compact subsets of E), $m \in \mathbb{N}$ is fixed; $L_0 = \{0, 1, 2, \dots, m\}$, $L_1 = \{1, 2, \dots, m\}$, $s_0 = 0 < \kappa_1 \leq s_1 < \kappa_2 \leq s_2 < \dots < \kappa_m \leq s_m < \kappa_{m+1} = b$ and $g_i, g_i^* : [\kappa_i, s_i] \times E \rightarrow E; i \in L_1$ are two continuous functions; and $x_0, x_1 \in E$ are two fixed points.

Here, we derive the formula of solutions for the below-mentioned w -weighted ψ -Hilfer differential inclusion of the order σ and of the type v in the existence of non-instantaneous impulses:

$$\begin{cases} D_{s_i, \varrho}^{\sigma,v,\psi,w} x(\varrho) \in F(\varrho, x(\varrho)), a.e., \varrho \in (s_i, \kappa_{i+1}], i \in L_0, \\ x(\kappa_i^+) = g_i(\kappa_i, x(\kappa_i^-)), i \in L_1, \\ x(\varrho) = g_i(\varrho, x(\kappa_i^-)), \varrho \in (\kappa_i, s_i], i \in L_1, \\ \lim_{\varrho \rightarrow 0^+} I_{0, \varrho}^{2-\gamma, \psi, w} w(\varrho) x(\varrho) = x_0, \lim_{\varrho \rightarrow 0^+} \frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} (I_{0, \varrho}^{2-\gamma, \psi, w} w(\varrho) x(\varrho)) = x_1, \\ \lim_{\varrho \rightarrow s_i^-} (w(\varrho) I_{0, \varrho}^{2-\gamma} x(\varrho)) = g_i(s_i, x(s_i^-)), i \in L_1, \\ \lim_{\varrho \rightarrow s_i^-} \frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} (w(\varrho) I_{0, \varrho}^{2-\gamma} x(\varrho)) = g_i^*(s_i, x(s_i^-)), i \in L_1, \end{cases} \quad (1)$$

Then, we prove the compactness of the solution set and that it is not empty, where $D_{s_i, \varrho}^{\sigma,v,\psi,w}$ is the w -weighted ψ -Hilfer derivative whose order is $\sigma \in (1, 2)$ and type is $v \in [0, 1]$ and the lower limit at s_i and $I_{s_i, \varrho}^{2-\sigma, \psi, w}$ is the w -weighted ψ -integral of the order $2 - \sigma$ with a lower limit of s_i .

In order to further demonstrate the importance of this work, we will mention some recently published works that are concerned with studying differential equations or differential inclusions containing fractional differential operators that are special cases of $D_{0, \varrho}^{\sigma,v,\psi,w}$.

For works on differential equations with Riemann–Liouville derivatives, we refer the reader to [25]; for those with Caputo derivatives, we refer the reader to [21]; for those with Katugampola derivatives, we refer the reader to [26]; and for those with Hilfer fractional derivatives, we refer the reader to [27–30].

Differential equations containing Hilfer–Katugampola fractional derivatives are considered in [31–35].

Asawasamrit et al. [36] studied non-instantaneous impulsive differential equations involving ψ -Caputo fractional derivatives of the order $\alpha \in (0, 1)$ and with Riemann–Stieltjes fractional integral boundary conditions.

Differential equations concerning the ψ -Hilfer fractional derivative $D_{0,\varrho}^{\sigma,v,\psi}$, $\sigma \in (0, 1)$, $v \in [0, 1]$, without impulses are studied in [37–40], with instantaneous impulses in [41] and with non-instantaneous impulses in [41–45]. Also, Sitho [46] studied implicit fractional integro-differential equations with the ψ -Hilfer fractional derivative $D_{0,\varrho}^{\sigma,v,\psi}$ without impulses and $\sigma \in (1, 2)$, $v \in [0, 1]$. Dhayal et al. [47] studied the stability and controllability of impulsive ψ -Hilfer fractional integro-differential equation. Afsfari et al. [48] studied the boundary value problems via ψ -Hilfer fractional derivatives on b -metric spaces and showed that the solution set exists and it is positive.

Weighted differential equations with Atangana–Baleanu fractional operators are studied in [49], while weighted differential equations with Caputo–Fabrizio fractional derivatives are studied in [50].

Very recently, Alshammari et al. [51] studied differential equations involving weighted ψ -Caputo–Fabrizio fractional derivatives, and Benial et al. [52] considered a w -weighted ψ -Riemann–Liouville fractional derivative of the order $h(\xi)$, where $h(\xi) : [0, b] \rightarrow (1, 2]$.

In all these works, except [45,47], the problems were investigated in finite-dimensional Banach spaces.

The following summarizes the main contributions of this paper.

i—A new type of differential inclusion (Problem (1)) involving the w -weighted ψ -Hilfer differential operator $D_{s_i,\varrho}^{\sigma,v,\psi,w}$ of the order $\sigma \in (1, 2)$ and of the type v in the existence of non-instantaneous impulses and in infinite dimensional Banach spaces is formulated.

ii—The relationship between Problem (1) and the integral equation correlated with it is derived (Lemma 5).

iii—The formula of solutions for Problem (1) is formulated (Definition 5).

iv—The conditions to ensure that the set of solutions for Problem (1) is not empty and compact are obtained (Theorem 1).

v—An example is given to illustrate that our results are applicable (Example 1).

vi—Our method can help researchers to generalize the majority of the aforementioned works to the case where the nonlinear term is multifunctional in the existence of impulsive effects and the dimension of the space is infinite.

vii—Since it is possible to obtain a wide class of fractional differential operators from $D_{s_i,\varrho}^{\sigma,v,\psi,w}$ (as we cited above), therefore, one can generalize all the above-cited papers to the case where the considered fractional differential operator is replaced by $D_{s_i,\varrho}^{\sigma,v,\psi,w}$ and the dimension of the space is infinite, and these can be considered as suggestions for future research work as a result of this paper.

viii—One can derive a broad class of fractional differential equations and inclusions as a particular case of Problem (1).

The paper is divided into the following sections. The second section presents some of the concepts that will be used to obtain the main results. We end this section by obtaining the exact relationship between Problem (1) and the corresponding fractional integral equation and then derive the representation of the solution for Problem (1). In the third section, we prove the existence of the solution and the compactness of the set of solutions. Finally, we give an example to illustrate that our results are applicable.

2. Preliminaries and Notations

Hereafter, we use the following notations:

— $P_{cc}(E) := \{Z \subseteq E : Z \text{ is not empty, closed, or convex}\}.$

— $P_{ck}(E) := \{Z \subseteq E : Z \text{ is not empty, compact, or convex}\}.$

— $P_b(E) := \{Z \subseteq E : Z \text{ is not empty or bounded}\}.$

— $\mathbb{N}$ is a set of natural numbers.

— $\gamma = \sigma + 2\nu - \sigma\nu.$

— $\mathcal{E}_k = (s_k, \kappa_{k+1}], \bar{\mathcal{E}}_k = [s_k, \kappa_{k+1}]$ ($k \in L_0$), $T_k = (\kappa_i, s_i]$ and $\bar{T}_i = [\kappa_i, s_i]$ ($i \in L_1$).

— $AC(\mathcal{E}, E)$ is the Banach space of absolutely continuous functions from $\mathcal{E} = [0, b]$ to E .

— $L_w^{p,\psi}((0, b), E)$, $p \geq 1$ is the Banach space of all Lebesgue measurable functions f such that $fw(\psi')^{\frac{1}{p}} \in L^p((a, b), E)$ when $p \in [1, \infty)$ and $fw \in L^\infty((0, b), E)$ when $p = \infty$, where

$$\|f\|_{L_w^{p,\psi}((0,b),E)} := \begin{cases} (\int_0^b \|f(x)w(x)\|^p \psi'(x)dx)^{\frac{1}{p}}, & \text{if } p \in [1, \infty), \\ \|fw\|_{L^\infty((0,b),E)}, & \text{if } p = \infty. \end{cases}$$

— $C_w([a, b], E) := \{x \in C((0, b], E) : wx \in C([a, b], E)\}.$

— $C_{2-\gamma,\psi,w}([a, b], E) = \{x \in C((a, b], E) : (\psi(\cdot) - \psi(a))^{2-\gamma}x(\cdot) \in C_w([a, b], E)\}.$

—

$$\begin{aligned} PC_{2-\gamma,\psi,w}(\mathcal{E}, E) &:= \{x : [0, b] \rightarrow E, (\psi(\cdot) - \psi(s_k))^{2-\gamma}x(\cdot) \in C_w(\mathcal{E}_k, E), \\ x|_{T_i} &\in C(T_i, E), \lim_{t \rightarrow s_k^+} (\psi(t) - \psi(s_k))^{2-\gamma}x(t) \text{ and} \\ \lim_{t \rightarrow t_i^+} x(t) \text{ exist}, k &\in L_0, i \in i \in L_1\}. \end{aligned}$$

—For any function $f : [0, b] \rightarrow E$, define

$$D_t^{k,\psi,w}f(t) := D_t^{1,\psi,w}D_t^{k-1,\psi,w}f(t) = w^{-1}(t)\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k(f(t)w(t)), k = \mathbb{N} - \{1\},$$

where $w^{-1}(t) = \frac{1}{w(t)}$; $t \in [0, b]$ and

$$f_{k,\psi,w}(t) := w(t)D_t^{k,\psi,w}f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^k(f(t)w(t)), k \in \mathbb{N}.$$

— $AC^{1,w}(\mathcal{E}, E) := \{f : [a, b] \rightarrow E, fw \in AC(\mathcal{E}, E)\}.$

—For any $n \in \mathbb{N}$, let,

$$AC^{n,\psi,w}(\mathcal{E}, E) := \{f : \mathcal{E} \rightarrow E, D_t^{n-1,\psi,w}f \in AC^{1,w}(\mathcal{E}, E), D_t^{n,\psi,w}f \in L_w^{1,\psi}((0, b), E)\}.$$

— $C_{2-\gamma,\psi,w}^1(\mathcal{E}, E) := \{x \in C_{2-\gamma,\psi,w}(\mathcal{E}, E) : D_t^{1,\psi,w}x \in C_{2-\gamma,\psi,w}(\mathcal{E}, E)\}.$

—For any $n \in \mathbb{N} - \{1\}$, let

$$C_{2-\gamma,\psi,w}^n(\mathcal{E}, E) := \{x \in C_{2-\gamma,\psi,w}^{n-1}(\mathcal{E}, E) : D_t^{n,\psi,w}x \in C_{2-\gamma,\psi,w}(\mathcal{E}, E)\}.$$

— $C_{2-\gamma,\psi,w}^\gamma(\mathcal{E}, E) := \{x \in C_{2-\gamma,\psi,w}(\mathcal{E}, E), D_{0,t}^{\gamma,\psi,w}x \in C_{2-\gamma,\psi,w}(\mathcal{E}, E)\}.$

— $C_{2-\gamma,\psi,w}^{\sigma,\nu}(\mathcal{E}, E) := \{x \in C_{2-\gamma,\psi,w}(\mathcal{E}, E), D_{0,t}^{\sigma,\nu,\psi,w}x \in C_{2-\sigma,\psi}(\mathcal{E}, E)\}.$

—

$$PC_{2-\gamma,\psi,w}^\gamma(\mathcal{E}, E) := \{x \in PC_{2-\gamma,\psi,w}(\mathcal{E}, E), D_{s_k^+, t}^{\gamma,\psi}x|_{\bar{\mathcal{E}}_k} \in C_{2-\sigma,\psi,w}(\bar{\mathcal{E}}_k, E), k \in L_0\}.$$

—

$$PC_{2-\sigma,\psi,w}^{\theta,\nu}(\mathcal{E}, E) := \{x \in PC_{2-\sigma,\psi,w}(\mathcal{E}, E), D_{s_k^+, t}^{\theta,\nu,\psi}x|_{\bar{\mathcal{E}}_k} \in C_{2-\gamma,\psi}(\bar{\mathcal{E}}_k, E), k \in L_0\}.$$

The spaces $C_w(\mathcal{L}, E)$, $C_{2-\gamma, \psi, w}(\mathcal{L}, E)$, $PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$ and $C_{2-\gamma, \psi, w}^n(\mathcal{L}, E)$ are Banach spaces with the norms

$$\|x\|_{C_w(\mathcal{L}, E)} := \sup_{\varrho \in \mathcal{L}} \|w(\varrho)x(\varrho)\|,$$

$$\|x\|_{C_{2-\gamma, \psi, w}(\mathcal{L}, E)} := \sup_{\varrho \in \mathcal{L}} \|(\psi(\varrho) - \psi(0))^{2-\gamma} w(\varrho)x(\varrho)\|,$$

$$\|x\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} := \max\left\{\sup_{\substack{\varrho \in \mathcal{L}_k \\ k \in \mathbb{N}_0}} (\psi(\varrho) - \psi(s_k))^{2-\gamma} \|w(\varrho)x(\varrho)\|_E, \sup_{\substack{\varrho \in \overline{T}_k \\ k \in \mathbb{N}_1}} \|x(\varrho)\|_E\right\},$$

$$\|x\|_{C_{2-\gamma, \psi, w}^n(\mathcal{L}, E)} := \sum_{k=1}^{n-1} \|D_{\varrho}^{k, \psi, w} x\|_{C_{2-\gamma, \psi, w}^k(\mathcal{L}, E)} + \|D_{\varrho}^{n, \psi, w} x\|_{C_{2-\gamma, \psi, w}(\mathcal{L}, E)},$$

and

$$\|x\|_{AC^{n, \psi, w}(\mathcal{L}, E)} := \|D^{n-1, \psi, w} x w\|_{AC(\mathcal{L}, E)}.$$

Definition 1 ([53]). Let $\alpha > 0$. The w -weighted Riemann–Liouville fractional integral whose order is α and which has a lower limit of 0 for a function $f \in L_w^{p, \psi}(\mathcal{L}, E)$ in regard to ψ is given by

$$(I_{0, \varrho}^{\alpha, \psi, w} f)(\varrho) := \frac{w^{-1}(\varrho)}{\Gamma(\alpha)} \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\alpha-1} \psi'(s) f(s) w(s) ds.$$

Lemma 1 ([53], Theorem 2.4). If $f \in L_w^{p, \psi}(\mathcal{L}, E)$, $1 \leq p \leq \infty$, $\alpha > 0$ and $\beta > 0$, then $I_{0, \varrho}^{\alpha, \psi, w} I_{0, \varrho}^{\beta, \psi, w} f = I_{0, \varrho}^{\alpha+\beta, \psi, w} f$.

Definition 2 ([53]). Let $n \in \mathbb{N}$ and $\alpha \in (n-1, n)$. The w -weighted Riemann–Liouville fractional derivative whose order is α and which has a lower limit of a for a function f with respect to ψ is defined by:

$$\begin{aligned} (D_{a, \varrho}^{\alpha, \psi, w} f)(\varrho) &: = D_{\varrho}^{n, \psi, w} (I_{a, \varrho}^{n-\alpha, \psi, w} f)(\varrho) \\ &= \frac{w^{-1}(\varrho)}{\Gamma(n-\alpha)} D_{\varrho}^{n, \psi, w} \left(\int_a^{\varrho} (\psi(\varrho) - \psi(s))^{n-\alpha-1} \psi'(s) f(s) w(s) ds, \varrho \geq a \right) \end{aligned}$$

given that the right-hand side is well-defined.

Lemma 2 ([53], Proposition 1.3). i—If $\alpha > 0$ and $\beta > 0$, then

$$\begin{aligned} I_{a, \varrho}^{\alpha, \psi, w} (w^{-1}(\varrho)(\psi(\varrho) - \psi(a))^{\beta-1}) \\ = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} w^{-1}(\varrho)(\psi(\varrho) - \psi(a))^{\beta+\alpha-1}; \forall \varrho \in [a, b]. \end{aligned}$$

ii—If $\beta > 0$ and $\alpha < \beta$, then

$$\begin{aligned} D_{a, \varrho}^{\alpha, \psi, w} (w^{-1}(\varrho)(\psi(\varrho) - \psi(a))^{\beta-1}) \\ = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} w^{-1}(\varrho)(\psi(\varrho) - \psi(a))^{\beta-\alpha-1}; \forall \varrho \in [a, b]. \end{aligned}$$

Lemma 3. If $f \in C_{2-\gamma, \psi, w}([a, b], E)$, then $\lim_{\varrho \rightarrow a} \|w(\varrho) I_{a, \varrho}^{1-v(2-\sigma), \psi, w} f(\varrho)\| = 0$.

Proof. From the assumption that $f \in C_{2-\gamma, \psi, w}(\mathcal{L}, E)$, we have that there is a positive real number M satisfying the following:

$$\|w(\varrho)f(\varrho)\| \leq M(\psi(\varrho) - \psi(a))^{\gamma-2}, \forall \varrho \in \mathcal{L}.$$

Thus, from Lemma 2, we obtain

$$\begin{aligned}
 & \lim_{\varrho \rightarrow a} \|w(\varrho) I_{a,\varrho}^{1-v(2-\sigma),\psi,w} f(\varrho)\| \\
 &= \lim_{\varrho \rightarrow a} \frac{M}{\Gamma(1-v(2-\sigma))} \int_a^\varrho (\psi(\varrho) - \psi(s))^{v(2-\sigma)} \psi'(s) (\psi(\varrho) - \psi(a))^{\gamma-2} \psi'(s) ds \\
 &= M \lim_{\varrho \rightarrow a} |I_{a,\varrho}^{1-v(2-\sigma),\psi,w} w^{-1}(\varrho) (\psi(\varrho) - \psi(a))^{\gamma-2}(\varrho)| \\
 &= M \lim_{\varrho \rightarrow a} \left| \frac{\Gamma(\gamma-1)}{\Gamma(\gamma-v(2-\sigma))} w^{-1}(\varrho) (\psi(\varrho) - \psi(a))^{1+\gamma-v(2-\sigma)}(\varrho) \right| \\
 &= 0.
 \end{aligned}$$

□

The proof of the next lemma can be derived using the steps used in the proof of the case when $E = \mathbb{R}$ ([53], Theorems 3.3–3.6).

Lemma 4. Let $f \in L_w^{p,\psi}([a, b], E)$.

i—If $\alpha > \beta > 0$, then, for any $\varrho \in [a, b]$,

$$D_{a,\varrho}^{\beta,\psi,w} I_{a,\varrho}^{\alpha,\psi,w} f(\varrho) = I_{a,\varrho}^{\alpha-\beta,\psi,w} f(\varrho).$$

ii— $D_{a,\varrho}^{\alpha,\psi,w} I_{a,\varrho}^{\alpha,\psi,w} f(\varrho) = f(\varrho)$, $\forall \varrho \in [a, b]$.

iii—If $\alpha \in (n-1, n)$ and $I_{a,\varrho}^{n-\alpha,\psi,w} f \in AC^{n,\psi,w}([a, b], E)$, then, for any $\varrho \in [a, b]$,

$$\begin{aligned}
 & I_{a,\varrho}^{\alpha,\psi,w} (D_{a,\varrho}^{\alpha,\psi,w} f)(\varrho) \\
 &= f(\varrho) - w^{-1}(\varrho) \sum_{k=1}^n \frac{(\psi(\varrho) - \psi(a))^{\alpha-k}}{\Gamma(\alpha-k+1)} \lim_{\varrho \rightarrow a} \left(\frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} \right)^{n-k} (w(\varrho) I^{n-\alpha} f(\varrho)). \quad (2)
 \end{aligned}$$

Definition 3 ([53]). Let $n \in \mathbb{N}$, $\alpha \in]n-1, n[$. The w -weighted ψ -Caputo fractional derivative whose order is α and which has a lower limit of a for a function f with respect to ψ is defined as:

$$({}^c D_{a,\varrho}^{\alpha,\psi,w} f)(\varrho) := D_{a,\varrho}^{\alpha,\psi,w} (f(\varrho) - \sum_{k=0}^{n-1} \frac{(\psi(\varrho) - \psi(a))^k}{k!} f_{k,\psi,w}(0)),$$

given that the right-hand side is well-defined.

In the following, we recall some properties for $I_{a,\varrho}^{\alpha,\psi,w}$, $D_{a,\varrho}^{\alpha,\psi,w}$ and ${}^c D_{a,\varrho}^{\alpha,\psi,w}$.

Lemma 5 ([53], Theorems 4.2, 4.3, 4.5). Let $n \in \mathbb{N}$ and $\alpha \in (n-1, n)$

i—If $f \in AC^{n,\psi,w}([a, b], E)$, then, for a.e.,

$$\begin{aligned}
 ({}^c D_{a,\varrho}^{\alpha,\psi,w} f)(\varrho) &= I_{a,\varrho}^{n-\alpha,\psi,w} (D_{a,\varrho}^{n,\psi,w} f)(\varrho) \\
 &= \frac{w^{-1}(\varrho)}{\Gamma(n-\alpha)} \left(\int_a^\varrho (\psi(\varrho) - \psi(s))^{n-\alpha-1} \psi'(s) w(s) D_{a,\varrho}^{n,\psi,w} f(s) ds \right).
 \end{aligned}$$

ii—

$$({}^c D_{a,\varrho}^{\alpha,\psi,w} (w^{-1}(\varrho) (\psi(\varrho) - \psi(a))^k)(\varrho) = 0; \forall k \in \{0, 1, \dots, n-1\}. \quad (3)$$

Definition 4 ([53]). The w -weighted ψ -Hilfer derivative whose order is σ and is of the type v and has a lower limit of a for a function $f : [a, b] \rightarrow E$ is defined as

$$\begin{aligned} D_{a,\varrho}^{\sigma,v,\psi,w} f(\varrho) &: = I_{a,\varrho}^{v(2-\sigma)} D_{\varrho}^{2,\psi,w} I_{a,\varrho}^{(2-\sigma)(1-v),\psi,w} f(\varrho) \\ &= I_{a,\varrho}^{v(2-\sigma)} D_{\varrho}^{2,\psi,w} I_{a,\varrho}^{2-\gamma,\psi,w} f(\varrho) \\ &= I_{a,\varrho}^{v(2-\sigma)} D_{a,\varrho}^{\gamma,\psi,w} f(\varrho), \end{aligned} \quad (4)$$

given that the right-hand side is well-defined.

Remark 1. If $f \in C_{2-\gamma,\psi,w}^{\gamma}([a, b], E)$, then $D_{a,\varrho}^{\gamma,\psi,w} f(\varrho)$ exists, and consequently, by (4), $D_{a,\varrho}^{\sigma,v,\psi,w} f(\varrho)$ exists. Therefore, $C_{2-\gamma,\psi,w}^{\gamma}([a, b], E) \subseteq C_{2-\gamma,\psi,w}^{\sigma,\nu}([a, b], E)$. Likewise, $PC_{2-\gamma,\psi,w}^{\gamma}([a, b], E) \subseteq PC_{2-\gamma,\psi,w}^{\sigma,\nu}([a, b], E)$.

Definition 5. A function $x \in PC_{2-\gamma,\psi,w}(\mathcal{L}, E)$ is a solution for Problem (1) if $x(\varrho) = g_i(\varrho, x(\kappa_i^-))$, $\forall \varrho \in (\kappa_i, s_i]; i \in L_1$, $D_{s_k,\varrho}^{\sigma,v,\psi,w} x(\varrho)$ exists for $\varrho \in (s_k, \kappa_{k+1}]; k \in L_0$ and x verifies (1).

Lemma 6. i—Assume that $x : \mathcal{L} \rightarrow E$, $x \in PC_{2-\gamma,\psi,w}^2(\mathcal{L}, E)$ and $f \in L_w^{p,\psi}(\mathcal{L}, E)$, $p \geq 1$. If x and f satisfy the following w -weighted ψ -differential equation of the order σ and the type v :

$$\left\{ \begin{array}{l} D_{s_i,\varrho}^{\sigma,v,\psi,w} x(\varrho) = f(\varrho), \varrho \in (s_i, \kappa_{i+1}], i \in L_0, \\ x(\kappa_i^+) = g_i(\kappa_i, x(\kappa_i^-)), i \in L_1, \\ x(\varrho) = g_i(\varrho, x(\kappa_i^-)), \varrho \in (\kappa_i, s_i], i \in L_1, \\ \lim_{\varrho \rightarrow 0^+} I_{0,\varrho}^{2-\gamma,\psi,w} w(\varrho) x(\varrho) = x_0, \lim_{\varrho \rightarrow 0^+} \frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} (I_{0,\varrho}^{2-\gamma,\psi,w} w(\varrho) x(\varrho)) = x_1, \\ \lim_{\varrho \rightarrow s_i^-} (w(\varrho) I^{2-\gamma} x(\varrho)) = g_i(s_i, x(s_i^-)), i \in L_1, \\ \lim_{\varrho \rightarrow s_i^-} \frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} (w(\varrho) I_{0,\varrho}^{2-\gamma} x(\varrho)) = g_i^*(s_i, x(s_i^-)), i \in L_1, \end{array} \right. \quad (5)$$

Then,

$$x(\varrho) = \left\{ \begin{array}{l} w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 \\ + I_{s_0,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (s_0, \kappa_1], \\ g_i(\varrho, x(\kappa_i^-)), \varrho \in (\kappa_i, s_i], i \in L_1, \\ w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, x(s_i^-)) \\ + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, x(s_i^-)) + I_{s_i,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (s_i, \kappa_{i+1}], i \in L_1. \end{array} \right. \quad (6)$$

ii—If $f \in PC_{2-\gamma,\psi,w}^{v(2-\sigma)}(\mathcal{L}, E)$, then the function x , defined in (6) belongs to $PC_{2-\gamma,\psi,w}^{\gamma}(\mathcal{L}, E)$.

$D_{s_k,\varrho}^{\sigma,v,\psi,w} x(\varrho)$ exists for $\varrho \in (s_k, \kappa_{k+1}], \forall k \in L_0$, and it is a solution of (5).

Proof. i—Suppose that $x \in PC_{2-\gamma,\psi,w}^2(\mathcal{L}, E)$, $f \in L_w^{p,\psi}(\mathcal{L}, E)$; $p \geq 1$. Also suppose that (5) is verified and let $i = 0$. Then,

$$D_{0,\varrho}^{\sigma,v,\psi,w} x(\varrho) = f(\varrho), \varrho \in \mathcal{L}_0. \quad (7)$$

Since $C_{2-\gamma,\psi,w}^2(\bar{E}_0, E) \subseteq AC^{2,\psi,w}(\bar{E}_0, E)$, then $I_{0,\varrho}^{2-\gamma,\psi,w} x \in AC^{2,\psi,w}(\bar{E}_0, E)$. Therefore, by applying $I_{0,\varrho}^{\sigma,\psi,w}$ to (7) and using Lemma (1), (2) and (4), we obtain, for $\varrho \in (0, \kappa_1]$,

$$\begin{aligned} I_{0,\varrho}^{\sigma,\psi,w} f(\varrho) &= I_{0,\varrho}^{\sigma,\psi,w} (D_{s_i,\varrho}^{\sigma,\psi,w} x)(\varrho) = I_{0,\varrho}^{\sigma,\psi,w} I_{s_i,\varrho}^{v(2-\sigma),\psi,w} D_{0,\varrho}^{\gamma,\psi,w} x(\varrho) \\ &= I_{0,\varrho}^{\gamma,\psi,w} D_{0,\varrho}^{\gamma,\psi,w} x(\varrho) \\ &= x(\varrho) - w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} \lim_{\varrho \rightarrow 0} \left(\frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} \right) (w(\varrho) I^{2-\gamma} x(\varrho)) \\ &\quad - w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} \lim_{\varrho \rightarrow 0} (w(\varrho) I^{2-\gamma} x(\varrho)). \end{aligned}$$

Therefore, for $\varrho \in (0, \kappa_1]$,

$$\begin{aligned} x(\varrho) &= w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 \\ &\quad + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_1 + I_{0,\varrho}^{\sigma,\psi,w} f(\varrho). \end{aligned} \quad (8)$$

Let $i \geq 2$. Then,

$$D_{s_i,\varrho}^{\sigma,\psi,w} x(\varrho) = f(\varrho).$$

Since $C_{2-\gamma,\psi,w}^2(\bar{E}_i, E) \subseteq AC^{2,\psi,w}(\bar{E}_i, E)$, then, $I_{s_i,\varrho}^{2-\gamma,\psi,w} x \in AC^{2,\psi,w}(\bar{E}_i, E)$. Therefore, by applying $I_{s_i,\varrho}^{\sigma,\psi,w}$ to (7) and using Lemma (1), (2) and (4), we obtain, for $\varrho \in (s_i, \kappa_{i+1}]$, $i \in L_1$,

$$\begin{aligned} x(\varrho) &= w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} \lim_{\varrho \rightarrow s_i} (w(\varrho) I^{2-\gamma} x(\varrho)) \\ &\quad + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} \lim_{\varrho \rightarrow s_i} (w(\varrho) I^{2-\gamma} x(\varrho)) + I_{s_i,\varrho}^{\sigma,\psi,w} f(\varrho) \\ &= w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, x(s_i^-)) \\ &\quad + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i^*(s_i, x(s_i^-)) + I_{s_i,\varrho}^{\sigma,\psi,w} f(\varrho). \end{aligned}$$

Hence, (6) is satisfied.

ii—Assume that (6) is verified and let $\varrho \in (0, \kappa_1]$. Then,

$$\begin{aligned} x(\varrho) &= w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 \\ &\quad + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 + I_{s_0,\varrho}^{\sigma,\psi,w} f(\varrho). \end{aligned} \quad (9)$$

Note that $1 < \sigma < \gamma \leq 2$ and $v(2-\sigma) < 1$. Applying $D_{s_0,\varrho}^{\gamma,\psi,w}$ on both sides of (9) and considering (3) and Lemma (1) and (ii) of Lemma (4), it follows, for $\varrho \in \mathcal{E}_0 = (0, \kappa_1]$, that

$$\begin{aligned} D_{0,\varrho}^{\gamma,\psi,w} x(\varrho) &= D_{0,\varrho}^{\gamma,\psi,w} I_{0,\varrho}^{\sigma,\psi,w} f(\varrho) \\ &= D_{0,\varrho}^{2,\psi,w} I_{0,\varrho}^{2-\gamma,\psi,w} I_{0,\varrho}^{\sigma,\psi,w} f(\varrho) \\ &= D_{0,\varrho}^{2,\psi,w} I_{0,\varrho}^{2-v(2-\sigma),\psi,w} f(\varrho) \\ &= D_{0,\varrho}^{1,\psi,w} (D_{0,\varrho}^{1,\psi,w} I_{0,\varrho}^{1-\psi,w} I_{0,\varrho}^{1-v(2-\sigma),\psi,w}) f(\varrho) \\ &= D_{0,\varrho}^{1,\psi,w} I_{0,\varrho}^{1-v(2-\sigma),\psi,w} f(\varrho) \\ &= D_{0,\varrho}^{v(2-\sigma),\psi,w} f(\varrho). \end{aligned}$$

Since $f \in PC_{2-\gamma,\psi,w}^{v(2-\sigma)}(\mathcal{E}, E)$, $D_{0,\varrho}^{v(2-\sigma),\psi,w} f|_{\mathcal{E}_0} \in C_{2-\gamma,\psi,w}(\bar{\mathcal{E}}_0, E)$, hence, $D_{0,\varrho}^{\gamma,\psi,w} x|_{\mathcal{E}_0} \in C_{2-\gamma,\psi,w}(\bar{\mathcal{E}}_0, E)$. This implies that $x|_{\bar{\mathcal{E}}_0} \in C_{2-\gamma,\psi,w}^{\gamma}(\bar{\mathcal{E}}_0, E)$. Therefore, by Remark (2), $D_{0,\varrho}^{\sigma,\psi,w} x(\varrho)$ exists for $\varrho \in \bar{\mathcal{E}}_0$. We will show that $D_{0,\varrho}^{\sigma,\psi,w} x(\varrho) = f(\varrho)$, $\forall \varrho \in \mathcal{E}_0$. Let $\varrho \in \mathcal{E}_0$. Since $f \in PC_{2-\gamma,\psi,w}^{v(2-\sigma)}(\mathcal{E}, E)$, it follows that $D_{s_0,\varrho}^{v(2-\sigma),\psi,w} f|_{\bar{\mathcal{E}}_0} \in C_{2-\gamma,\psi,w}(\bar{\mathcal{E}}_0, E)$, and then

$D_{0,\varrho}^{1,\psi,w} I_{0,\varrho}^{1-v(2-\sigma),\psi,w} f|_{\mathcal{E}_0} \in C_{2-\gamma,\psi,w}(\overline{\mathcal{E}_0}, E)$, and hence, $I_{s_0,\varrho}^{1-v(2-\sigma),\psi,w} f|_{\overline{\mathcal{E}_0}} \in C_{2-\gamma,\psi,w}^1(\overline{\mathcal{E}_0}, E)$. Therefore, as a result of (2), (4) and (9), we obtain, for $\varrho \in \mathcal{E}_0$,

$$\begin{aligned} D_{0,\varrho}^{\sigma,v,\psi,w} x(\varrho) &= I_{0,\varrho}^{v(2-\sigma)} D_{0,\varrho}^{\gamma,\psi,w} x(\varrho) = I_{0,\varrho}^{v(2-\sigma)} D_{0,\varrho}^{v(2-\sigma),\psi,w} f(\varrho) \\ &= f(\varrho) - w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{v(2-\sigma)-1}}{\Gamma(v(2-\sigma))} \lim_{\varrho \rightarrow 0} (w(\varrho) I^{1-v(2-\sigma)} f(\varrho)). \end{aligned}$$

Using Lemma (3), we obtain $\lim_{\varrho \rightarrow 0} (w(\varrho) I^{1-v(2-\sigma)} f(\varrho)) = 0$. Thus, $D_{0,\varrho}^{\sigma,v,\psi,w} x(\varrho) = f(\varrho)$, for any $\varrho \in \mathcal{E}_0$. Likewise, it can be shown that $D_{s_1,\varrho}^{\sigma,v,\psi,w} x(\varrho)$ exists for $\varrho \in \mathcal{E}_i, i \in L_1$ and $D_{s_i,\varrho}^{\sigma,v,\psi,w} x(\varrho) = f(\varrho), \varrho \in (s_i, \kappa_{i+1}]$. $\square$

As a result of Definition (5) and Lemma (6), we drive the next corollary.

Corollary 1. The function $x : \mathcal{E} \rightarrow E$ is defined by:

$$x(\varrho) = \begin{cases} w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 \\ + I_{s_0,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (s_0, \kappa_1], \\ g_i(\varrho, x(\kappa_i^-)), \varrho \in (\kappa_i, s_i], i \in L_1, \\ w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, x(s_i^-)) \\ + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, x(s_i^-)) + I_{s_i,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (s_i, \kappa_{i+1}], i \in L_1, \end{cases} \quad (10)$$

Here, $f \in C_{2-\gamma,\psi,w}^{v(2-\sigma)}(\mathcal{E}, E)$ with $f(\varrho) \in F(\varrho, x(\varrho)), a.e.,$ is a solution for Problem (1).

Remark 2. The solution function x defined in (10) has these properties:

- i— $x \in PC_{2-\gamma,\psi,w}(\mathcal{E}, E)$.
- ii— x is not continuous at $s_i, \forall i \in L_1$.
- iii—If $w(\varrho) = 1; \varrho \in \mathcal{E}$ and $v = 1$, then $\gamma = 2, D_{s_i,\varrho}^{\sigma,v,\psi,w} x(\varrho) =^c D_{s_i,\varrho}^{\sigma,\psi,w} x(\varrho); \varrho \in (s_i, \kappa_{i+1}], i \in L_1$, and x becomes continuous at $s_i, \forall i \in L_1$.

To verify the third property in Remark (3), we give the following corollary, which is a direct consequence from Corollary (1):

Corollary 2. Suppose that $w(\varrho) = 1; \varrho \in \mathcal{E}, v = 1$ and there is $f \in C_{2-\gamma,\psi,w}^{v(2-\sigma)}(\mathcal{E}, E)$ with $f(\varrho) \in F(\varrho, x(\varrho)), a.e..$ Then, the function

$$x(\varrho) = \begin{cases} x_0 + \frac{\psi(\varrho) - \psi(0)}{\Gamma(\gamma)} x_1 \\ + I_{s_0,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (s_0, \kappa_1], \\ g_i(\varrho, x(\kappa_i^-)), \kappa \in (\kappa_i, s_i], i \in L_1, \\ g_i(s_i, x(s_i^-)) \\ + \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, x(s_i^-)) + I_{s_i,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (s_i, \kappa_{i+1}], i \in L_1, \end{cases}$$

is a solution for the next ψ -Caputo differential inclusion in the presence of non-instantaneous impulses:

$$\begin{cases} {}^c D_{s_i, \varrho}^{\sigma, \psi, w} x(\varrho) \in F(\varrho, x(\varrho)), a.e. \varrho \in (s_i, \kappa_{i+1}], i \in L_0, \\ x(\kappa_i^+) = g_i(\kappa_i, x(\kappa_i^-)), i \in L_1 \\ x(\varrho) = g_i(\varrho, x(\kappa_i^-)), t \in (\kappa_i, s_i], i \in L_1, \\ x(0) = x_0, \lim_{\varrho \rightarrow 0^+} \frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} x(\varrho) = x_1, \\ \lim_{\varrho \rightarrow s_i^-} \frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} x(\varrho) = g_i^*(s_i, x(s_i^-)), i \in L_1. \end{cases}$$

Definition 6 ([54]). A function $\gamma_E : P_b(E) \rightarrow [0, \infty)$ is called a measure of non-compactness (MNC) on E if, for $\Omega \in P_b(E)$,

$$\gamma_E(\Omega) = \gamma_E(\overline{\text{co}}\Omega),$$

where $\overline{\text{co}}\Omega$ is the convex hull of Ω .

The MNC γ_E is said to be the following:

i—Monotone if $\Omega_1, \Omega_2 \in P_b(E), \Omega_1 \subseteq \Omega_2 \implies \gamma_E(\Omega_1) \leq \gamma_E(\Omega_2)$.

ii—Non-singular if $\gamma_E(\delta\Omega) = \delta\gamma_E(\overline{\text{co}}\Omega), \delta \in [0, \infty)$.

iii—Regular if $\gamma_E(\Omega) = 0 \iff \Omega$ is relatively compact.

One of the most important examples of MNC is the Hausdorff MNC, χ_E , which is defined by:

$$\chi_E(\Omega) = \inf \{ \epsilon > 0 : \Omega \text{ can be covered by a finite number of open balls with radius } \leq \epsilon \}.$$

χ_E is monotone, non-singular, and regular.

Definition 7 ([55]). Let γ_E be an MNC on E and $U \in P_{cc}(E)$. A multivalued function $Y : U \rightarrow P_{ck}(X)$ is called γ_E -condensing if there is $\beta \in (0, 1)$ such that

$$\gamma_E(Y(\Omega)) < \beta\gamma_E(\Omega), \forall \Omega \in P_b(E).$$

For more about the measure of non-compactness, we refer the reader to [56,57].

Now, the function $\chi_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} : P_b(PC_{2-\gamma, \psi, w}(\mathcal{L}, E)) \rightarrow [0, \infty)$, defined by

$$\chi_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)}(Z) := \max \{ \max_{k \in L_0} \chi_{C(\overline{\mathcal{L}_k}, E)}(Z|_{\overline{\mathcal{L}_k}}), \max_{i \in L_1} \chi_{C(\overline{\mathcal{T}_i}, E)}(Z|_{\overline{\mathcal{T}_i}}) \}$$

is a Hausdorff measure of non-compactness on $PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$, where $\chi_{C(\overline{\mathcal{L}_k}, E)}$ is the Hausdorff MNC on $C(\overline{\mathcal{L}_k}, E)$,

$$\begin{aligned} Z|_{\overline{\mathcal{L}_k}} &: = \{ z^* \in C(\overline{\mathcal{L}_k}, E) : z^*(\varrho) = (\psi(\varrho) - \psi(s_k))^{2-\gamma} z(\varrho), \varrho \in \mathcal{L}_k, \\ z^*(s_k) &= \lim_{\varrho \rightarrow s_k^+} z^*(\varrho), z \in Z \}, \end{aligned}$$

and

$$Z|_{\overline{\mathcal{T}_i}} := \{ z^* \in C(\overline{\mathcal{T}_i}, E) : z^*(\varrho) = z(\varrho), \varrho \in \mathcal{T}_i, z^*(\kappa_i) = z(\kappa_i^+), z \in Z \}.$$

The following next lemmas describe fixed points. We will need them to derive our results.

Lemma 7 ([55], Corollary 3.3.1). Let $U \in P_{cc}(E)$ and $Y : U \rightarrow P_{ck}(X)$ be χ -condensing with a closed graph, where χ is a non-singular measure of non-compactness defined on subsets of U . Then, Y has a fixed point.

Lemma 8 ([55], Prop.3.5.1). Let $U \in P_{cc}(E)$ and $Y : U \rightarrow P_{ck}(X)$ be χ -condensing on all bounded subsets of U , where χ is a monotone measure of non-compactness defined on E . If Y has a closed graph, and the set of fixed points for Y , $\text{Fix}(Y)$ is a bounded subset of E , then it is compact.

3. Existence of Solutions for Problem (1)

This section is devoted to demonstrating the existence of solutions for Problem (1).

Let $p > \frac{1}{2-\sigma} > 1$ be a fixed real number and, for any $z \in PC_{2-\gamma, \psi, w}$, let

$$S_{F(\cdot, z)}^p := \{f \in L_w^{p, \psi}(\mathcal{L}, E) : f(q) \in F(q, z(q)), \text{ a.e. for } q \in \mathcal{L}\},$$

and

$$S_{F(\cdot, z)}^{v(2-\sigma), p} := \{f \in S_{F(\cdot, z)}^p, f \in PC_{2-\gamma, \psi, w}^{v(2-\sigma)}(\mathcal{L}, E)\}.$$

Theorem 1. Suppose that $F : \mathcal{L} \times E \rightarrow P_{ck}(E)$ and $g_i, g_i^* : [\kappa_i, s_i] \times E \rightarrow E; i \in L_1$. We assume the following conditions:

(F₁) For any $z \in PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$, the multivalued function $q \rightarrow F(q, z(q))$ is measurable, and the set $S_{F(\cdot, z)}^{v(2-\sigma), p}$ is not empty.

(F₂) There is a $\varphi \in L_w^{p, \psi}(\mathcal{L}, \mathbb{R}^+)$ such that, for any $z \in PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$,

$$\|F(q, z(q))\| = \sup_{y \in F(q, z(q))} \|y\| \leq \varphi(q)(1 + \|z(q)\|), \text{ and for almost } q \in \cup_{k \in L_1} (s_i, \kappa_{i+1}].$$

(F₃) If $z_n \in PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$, $z_n \rightarrow z \in PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$, $f_n \in S_{F(\cdot, z_n)}^{v(2-\sigma), p}$ and $f_n \rightarrow f \in L_w^{p, \psi}(\mathcal{L}, E)$, then $f \in S_{F(\cdot, z)}^{v(2-\sigma), p}$.

(F₄) There is a $\varsigma \in L_w^{1, \psi}(\mathcal{L}, \mathbb{R}^+)$ such that, for any bounded subset $D \subseteq PC_{2-\gamma, \psi, w}$ and any $k \in L_0$, we obtain

$$\chi_E(F(q, z(q)) : z \in D) \leq (\psi(q) - \psi(s_k))^{2-\gamma} \varsigma(q) \chi_E\{(z(q)) : z \in D\}, \text{ a.e. } q \in \mathcal{L}_k,$$

and

$$\frac{(\psi(b) - \psi(0))^{\sigma-\gamma+1}}{\Gamma(\sigma)} \|\varsigma\|_{L_w^{1, \psi}(\mathcal{L}, \mathbb{R}^+)} < 1, \quad (11)$$

where χ is the Hausdorff measure of non-compactness on E .

(H₁) For any $i \in L_1$, $g_i : [\kappa_i, s_i] \times E \rightarrow E$ has uniform continuity on bounded sets, and, for any $q \in \mathcal{L}$, $g_i(q, \cdot)$ maps every bounded set to a relatively compact subset. We thus have $h_1 > 0$ with

$$\|g_i(q, u)\| \leq h_i \|u\|, \forall q \in (\kappa_i, s_i], \text{ and } \forall u \in E.$$

(H₂) For any $i \in L_1$, $g_i^* : [\kappa_i, s_i] \times E \rightarrow E$ is given such that it maps bounded sets to relatively compact sets, and we thus have $h_i^* > 0$ with

$$\|g_i^*(q, u)\| \leq h_i^* \|u\|, \forall q \in (\kappa_i, s_i], \text{ and } \forall u \in E.$$

Then, the set of solutions to Problem (1) is non-empty and relatively compact in $PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$ provided that

$$h_1 + \frac{h_1}{\Gamma(\gamma-1)} + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} h_2 + \frac{(\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1, \psi}}}{\Gamma(\sigma)} < 1, \quad (12)$$

where $h_1 = \sum_{i=0}^{i=n} h_i$ and $h_2 = \sum_{i=0}^{i=n} h_i^*$.

Proof. Due to (F_1) , for any $z \in PC_{2-\gamma,\psi,w}$, the set $S_{F(\cdot,z)}^{v(2-\sigma),p}$ is not empty. Therefore, a multivalued function $\Theta : PC_{2-\gamma,\psi,w} \rightarrow P_{cc}(PC_{2-\gamma,\psi,w})$ can be defined as follows: $x \in \Theta(z)$ if and only if

$$x(\varrho) = \begin{cases} w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 + w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 \\ + I_{s_0,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (0, \kappa_1], \\ g_i(\varrho, z(\kappa_i^-)), \varrho \in (\kappa_i, s_i], i \in L_1, \\ w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, z(s_i^-)) \\ + w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, z(s_i^-)) + I_{s_i,\varrho}^{\sigma,\psi,w} f(\varrho), \varrho \in (s_i, \kappa_{i+1}], i \in L_1, \end{cases} \quad (13)$$

where $f \in S_{F(\cdot,z)}^{v(2-\sigma),p}$.

Note that, as a result of Lemma (5) and Corollary 2, the function x given by (13), which belongs to $PC_{2-\gamma,\psi,w}^\gamma(\mathcal{E}, E) \subseteq PC_{2-\gamma,\psi,w}(\mathcal{E}, E)$, $D_{s_k,\varrho}^{\sigma,\psi,w} x(\varrho)$, exists for $\varrho \in (s_k, \kappa_{k+1}]$, $\forall k \in L_0$, and it is a solution to Problem (1). Next, by applying Lemma 6, we show that Θ has a fixed point. Because the values of F are in $P_{cc}(E)$, one can prove that the set of values of Θ is convex. For every $n \in \mathbb{N}$, let $D_n = \{x \in PC_{2-\gamma,\psi,w}(\mathcal{E}, E) : \|x\|_{PC_{2-\gamma,\psi,w}(\mathcal{E}, E)} \leq n\}$.

Step 1. We illustrate that there exists an $n_0 \in \mathbb{N}$ with $\Theta(D_{n_0}) \subseteq D_{n_0}$. Assume that, for every $n \in \mathbb{N}$, there exist $x_n, z_n \in PC_{2-\gamma,\psi,w}(\mathcal{E}, E)$ with $x_n \in \Theta(z_n)$, $\|z_n\|_{PC_{2-\gamma,\psi,w}} \leq n$ and $\|x_n\|_{PC_{2-\gamma,\psi,w}} > n$. According to the definition of Θ , we can find $f_n \in S_{F(\cdot,z_n)}^{v(2-\sigma),p}$; $n \in \mathbb{N}$ such that

$$x_n(\varrho) = \begin{cases} w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 + w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 \\ + I_{s_0,\varrho}^{\sigma,\psi,w} f_n(\varrho), \varrho \in (0, \kappa_1], \\ g_i(\varrho, z_n(\kappa_i^-)), \varrho \in (\kappa_i, s_i], i \in L_1, \\ w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, z_n(s_i^-)) \\ + w^{-1}(\varrho) \frac{(\psi(\varrho)-\psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, z_n(s_i^-)) + I_{s_i,\varrho}^{\sigma,\psi,w} f_n(\varrho), \varrho \in (s_i, \kappa_{i+1}], i \in L_1. \end{cases} \quad (14)$$

From (F_2) , it follows that

$$\|f_n(\varrho)\| \leq \varphi(\varrho)(1+n), a.e. \quad (15)$$

If $\varrho \in (s_0, \kappa_1]$, we obtain from (14) and (15) that

$$\begin{aligned} & (\psi(\varrho) - \psi(0))^{2-\gamma} \|x_n(\varrho)\| \\ & \leq \frac{w^{-1}(\varrho)}{\Gamma(\gamma-1)} \|x_0\| + w^{-1}(\varrho) \frac{\psi(\varrho)-\psi(0)}{\Gamma(\gamma)} \|x_1\| \\ & \quad + \frac{w^{-1}(\varrho)(1+n)}{\Gamma(\sigma)} \int_0^\varrho (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) \varphi(s) w(s) ds \\ & \leq \frac{w^{-1}(\varrho)}{\Gamma(\gamma-1)} \|x_0\| + w^{-1}(\varrho) \frac{\psi(\varrho)-\psi(0)}{\Gamma(\gamma)} \|x_1\| \\ & \quad + \frac{w^{-1}(\varrho)(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \int_0^\varrho \psi'(s) \varphi(s) w(s) ds \\ & \leq \frac{w^{-1}(\varrho)}{\Gamma(\gamma-1)} \|x_0\| + w^{-1}(\varrho) \frac{\psi(b)-\psi(0)}{\Gamma(\gamma)} \|x_1\| \\ & \quad + \frac{w^{-1}(\varrho)(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1,\psi}}. \end{aligned}$$

Then,

$$\begin{aligned} & (\psi(\varrho) - \psi(0))^{2-\gamma} \|w(\varrho)x_n(\varrho)\| \\ & \leq \frac{\|x_0\|}{\Gamma(\gamma-1)} \|x_0\| + w^{-1}(\varrho) \frac{\psi(b)-\psi(0)}{\Gamma(\gamma)} \|x_1\| \\ & \quad + \frac{(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1,\psi}((0,b), \mathbb{R}^+)}. \end{aligned} \quad (16)$$

Let $q \in (\kappa_i, s_i], i \in L_1$. Hence, from (H_1) , we obtain

$$\|x_n(q)\| = \|g_i(q, z_n(\kappa_i^-))\| \leq h_i \|z_n(\kappa_i^-)\| \leq h_1 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)}. \quad (17)$$

If $q \in (s_i, \kappa_{i+1}], i \in L_1$, it yields from (H_1) and (H_2) that

$$\begin{aligned} & (\psi(q) - \psi(s_i))^{2-\gamma} \|x_n(q)\| \\ & \leq \frac{w^{-1}(q)}{\Gamma(\gamma-1)} h_1 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} + w^{-1}(q) \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} h_2 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} \\ & \quad + \frac{w^{-1}(q)(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1, \psi}((0, b), \mathbb{R}^+)}, \end{aligned}$$

which gives us

$$\begin{aligned} & (\psi(q) - \psi(s_i))^{2-\gamma} \|w(q)x_n(q)\| \\ & \leq \frac{1}{\Gamma(\gamma-1)} h_1 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} h_2 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} \\ & \quad + \frac{(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1, \psi}((0, b), \mathbb{R}^+)}. \end{aligned} \quad (18)$$

The inequalities (16)–(18) imply that

$$\begin{aligned} \|x_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} & \leq \frac{\|x_0\|}{\Gamma(\gamma-1)} \|x_0\| + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} \|x_1\| + h_1 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} \\ & \quad + \frac{1}{\Gamma(\gamma-1)} h_1 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} \\ & \quad + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} h_2 \|z_n\|_{PC_{2-\gamma, \psi, w}(\mathcal{L}, E)} \\ & \quad + \frac{(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1, \psi}((0, b), \mathbb{R}^+)} \\ & \leq \frac{\|x_0\|}{\Gamma(\gamma-1)} \|x_0\| + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} \|x_1\| + nh_1 + \frac{nh_1}{\Gamma(\gamma-1)} \\ & \quad + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} nh_2 + \frac{(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1, \psi}((0, b), \mathbb{R}^+)}. \end{aligned}$$

Thus,

$$\begin{aligned} n & < \frac{\|x_0\|}{\Gamma(\gamma-1)} \|x_0\| + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} \|x_1\| + nh_1 + \frac{nh_1}{\Gamma(\gamma-1)} \\ & \quad + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} nh_2 + \frac{(1+n)}{\Gamma(\sigma)} (\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1, \psi}((0, b), \mathbb{R}^+)}. \end{aligned}$$

Dividing this inequality by n and taking the limit as $n \rightarrow \infty$, we have

$$1 < h_1 + \frac{h_1}{\Gamma(\gamma-1)} + \frac{\psi(b) - \psi(0)}{\Gamma(\gamma)} h_2 + \frac{(\psi(b) - \psi(0))^{\sigma-1} \|\varphi\|_{L_w^{1, \psi}((0, b), \mathbb{R}^+)}}{\Gamma(\sigma)}.$$

This contradicts (12). Therefore, there exists a natural number n_0 such that $\Theta(D_{n_0}) \subseteq \Theta(D_{n_0})$.

Step 2. The graph of Θ is closed on D_{n_0} .

Let $(x_n)_{n \geq 1}, (z_n)_{n \geq 1}$ be two sequences in D_{n_0} with $x_n \rightarrow x, z_n \rightarrow z$ in D_{n_0} and $x_n \in \Theta(z_n); n \geq 1$. Then, we have $f_n \in S_{F(\cdot, z_n)}^{v(2-\sigma), p}$ such that (14) is satisfied. Using (15), we obtain $\|f_n(q)\| \leq \varphi(q)(1+n_0)$, and consequently, $\{f_n\}_{n \geq 1}$ is weakly compact in $L_w^{p, \psi}(E, \mathbb{R}^+)$. Applying Mazur's lemma, one can find a subsequence, $\{f_{n_k}\}_{k \geq 1}$, converging strongly to $f \in L_w^{1, \psi}(E, \mathbb{R}^+)$. Using the Lebesgue convergence theorem, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} I_{s_0, q}^{\sigma, \psi, w} f_n(q) & = \lim_{n \rightarrow \infty} \frac{w^{-1}(q)}{\Gamma(\sigma)} \int_0^q (\psi(q) - \psi(s))^{\sigma-1} \psi'(s) f_n(s) w(s) ds \\ & = I_{s_0, q}^{\sigma, \psi, w} f(q), \forall q \in \mathcal{L}. \end{aligned}$$

Moreover, Conditions (H_1) and (H_2) imply $\lim_{n \rightarrow \infty} g_i(\varrho, z_n(\kappa_i^-)) = g_i(\varrho, z(\varrho))$ and $\lim_{n \rightarrow \infty} k_i(s_i, z_n(\kappa_i^-)) = k_i(s_i, z(\varrho)), \forall \varrho \in \mathcal{L}$. Therefore, $\lim_{n \rightarrow \infty} x_n(\varrho) = x(\varrho)$, where

$$x(\varrho) = \begin{cases} w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 \\ + I_{s_0, \varrho}^{\sigma, \psi, w} f(\varrho), \varrho \in (0, \kappa_1], \\ g_i(\varrho, z(\kappa_i^-)), \varrho \in (\kappa_i, s_i], i \in L_1, \\ w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, z(s_i^-)) \\ + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, z(s_i^-)) \\ + I_{s_i, \varrho}^{\sigma, \psi, w} f(\varrho), \varrho \in (s_i, \kappa_{i+1}], i \in L_1. \end{cases}$$

Note that Assumption (F_3) implies $f(\varrho) \in F(\varrho, z(\varrho)), a.e.$, and hence, that $x \in \Theta(z)$.

Step 3. For every $k \in L_0$ and $i \in L_1$, we set

$$\begin{aligned} \Pi_{|\overline{\mathcal{L}}_k} &= \{y : y : \overline{\mathcal{L}}_k \rightarrow E, y(\varrho) = (\psi(\varrho) - \psi(s_k))^{2-\gamma} w(\varrho) x(\varrho), \varrho \in \mathcal{L}_k, \\ y(s_k) &= \lim_{\varrho \rightarrow s_k^+} y(\varrho), x \in \Theta(z), z \in D_{n_0}\}, \end{aligned}$$

and

$$\begin{aligned} \Pi_{|\overline{\mathcal{T}}_i} &= \{x^* \in C(\overline{\mathcal{T}}_i, E) : y(t) = x(t), t \in (\kappa_i, s_i], \\ x^*(\kappa_i) &= x(\kappa_i^+), x \in \Theta(z), z \in D_{n_0}\}. \end{aligned}$$

Here, we show the equicontinuity of $\{\Pi_{|\overline{\mathcal{L}}_k}\}; k \in L_0$ and $\Pi_{|\overline{\mathcal{T}}_i}; i \in L_1$ in the Banach space $PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$.

Case 1. Let $y \in \Pi_{|\overline{\mathcal{L}}_0}$. Then, there exist $z \in D_{n_0}$ and $x \in \Phi(z)$ such that, for $\varrho \in (0, \kappa_1]$,

$$\begin{aligned} y(\varrho) &= (\psi(\varrho) - \psi(0))^{2-\gamma} w(\varrho) [w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 \\ &\quad + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 + I_{s_0, \varrho}^{\sigma, \psi, w} f(\varrho)] \\ &= \frac{x_0}{\Gamma(\gamma-1)} + \frac{(\psi(\varrho) - \psi(0))}{\Gamma(\gamma)} x_1 + (\psi(\varrho) - \psi(0))^{2-\gamma} w(\varrho) I_{s_0, \varrho}^{\sigma, \psi, w} f(\varrho), \end{aligned}$$

where $f \in S_{F(\cdot, z)}^{v(2-\sigma)}$ and $y(0) = \lim_{\varrho \rightarrow 0^+} y(\varrho)$. Therefore,

$$\begin{aligned} y(\varrho) &= \frac{x_0}{\Gamma(\gamma-1)} + \frac{(\psi(\varrho) - \psi(0))}{\Gamma(\gamma)} x_1 \\ &\quad + \frac{(\psi(\varrho) - \psi(0))^{2-\gamma}}{\Gamma(\sigma)} \int_0^\varrho (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) f(s) w(s) ds. \end{aligned}$$

Let $\varrho \in (0, \kappa_1]$ and $\delta > 0$ with $\varrho + \delta \in (0, \kappa_1]$. We thus have

$$\begin{aligned} &\|y(\varrho + \delta) - y(\varrho)\| \\ &\leq \left| \frac{(\psi(\varrho + \delta) - \psi(0))}{\Gamma(\gamma)} x_1 \right| \\ &\quad + \left| \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\sigma)} \int_0^{\varrho + \delta} (\psi(\varrho + \delta) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right. \\ &\quad \left. - \frac{(\psi(\varrho) - \psi(0))^{2-\gamma}}{\Gamma(\sigma)} \int_0^\varrho (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right| \\ &\leq \left| \frac{(\psi(\varrho + \delta) - \psi(0))}{\Gamma(\gamma)} x_1 \right| \\ &\quad + \left| \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\gamma)} \int_\varrho^{\varrho + \delta} (\psi(\varrho + \delta) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right| \\ &\quad + \left| \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\gamma)} \int_0^\varrho (\psi(\varrho + \delta) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right. \\ &\quad \left. - \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\gamma)} \int_0^\varrho (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right| \\ &\quad + \left| \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\gamma)} \int_0^\varrho (\psi(t) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right. \\ &\quad \left. - \frac{(\psi(\varrho) - \psi(0))^{2-\gamma}}{\Gamma(\sigma)} \int_0^\varrho (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right|. \end{aligned}$$

Then,

$$\begin{aligned}
& \|y(\varrho + \delta) - y(\varrho)\| \\
& \leq \frac{|\psi(\varrho + \delta) - \psi(\varrho)|}{\Gamma(\gamma)} x_1 \\
& + \left| \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\gamma)} \int_{\varrho}^{\varrho + \delta} (\psi(\varrho + \delta) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right| \\
& + \left| \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\gamma)} \right| \times \\
& \left| \int_0^{\varrho} (\psi(\varrho + \delta) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right. \\
& \left. - \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right| \\
& + \left| \frac{(\psi(\varrho + \delta) - \psi(0))^{2-\gamma}}{\Gamma(\gamma)} - \frac{(\psi(\varrho) - \psi(0))^{2-\gamma}}{\Gamma(\sigma)} \right| \times \\
& \left| \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) w(s) f(s) ds \right|.
\end{aligned} \tag{19}$$

Since $\sigma > 1$, ψ is continuous and $\|f(s)\| \leq \varphi(s)(1 + n_0)$, Relation (19) implies that $\lim_{\delta \rightarrow 0} \|y(t + \delta) - y(t)\| = 0$ independently of y .

Case 2. Let $x^* \in T_i; i \in L_1$ and $\varrho + \delta, \varrho \in (\kappa_i, s_i]$. Then, since g_i is uniformly continuous on bounded sets, we have

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \|x^*(\varrho + \delta) - x^*(\varrho)\| &= \lim_{\delta \rightarrow 0} \|x(\varrho + \delta) - x(\varrho)\| \\
&= \lim_{\delta \rightarrow 0} \|g_i(\varrho + \delta, x(\kappa_i^-)) - g_i(\varrho + \delta, x(\kappa_i^-))\| = 0,
\end{aligned}$$

and

$$\lim_{\delta \rightarrow 0+} \|x^*(\kappa_i + \delta) - x^*(\kappa_i)\| = \lim_{\delta \rightarrow 0+} \lim_{\lambda \rightarrow \kappa_i} \|x(\kappa_i + \delta) - x(\lambda)\| = 0.$$

Case 3. Let $y \in \Pi_{\overline{E}_1}$. Then, there are $z \in D_{n_0}$ and $x \in \Phi(z)$ such that, for $t \in (s_1, \kappa_2]$,

$$\begin{aligned}
y(\varrho) &= (\psi(\varrho) - \psi(0))^{2-\gamma} w(\varrho) [w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, z(s_i^-)) \\
&+ w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, z(s_i^-)) + I_{s_0, \varrho}^{\sigma, \psi, w} f(\varrho)] \\
&= g_i(s_i, z(s_i^-)) + \frac{(\psi(\varrho) - \psi(s_i))}{\Gamma(\gamma)} g_i^*(s_i, z(s_i^-)) \\
&+ (\psi(\varrho) - \psi(0))^{2-\gamma} w(\varrho) I_{s_0, \varrho}^{\sigma, \psi, w} f(\varrho),
\end{aligned}$$

where $f \in S_{F(\cdot, z)}^{v(2-\sigma)}$ and $y(0) = \lim_{\varrho \rightarrow 0+} y(\varrho)$. Therefore,

$$\begin{aligned}
y(\varrho) &= g_i(s_i, z(s_i^-)) + \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, z(s_i^-)) \\
&+ \frac{(\psi(\varrho) - \psi(0))^{2-\gamma}}{\Gamma(\sigma)} \int_0^{\varrho} (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) f(s) w(s) ds.
\end{aligned}$$

Because $\sigma > 1$, $\gamma > 1$, and ψ is continuous, using a similar argument as in Case (1), we can show that $\lim_{\delta \rightarrow 0} \|y(\varrho + \delta) - y(\varrho)\| = 0$ independently of y .

Finally, by repeating the same arguments, it can be proved that $\lim_{\delta \rightarrow 0} \|y(\varrho + \delta) - y(\varrho)\| = 0$ for any $y \in \Pi_{\overline{E}_k}, k = 2, \dots, m$.

Step 4. Set $K_0 = D_{n_0}, K_n = \Theta(K_{n-1}); n \geq 1$. In this step, we will show that

$$\lim_{n \rightarrow \infty} \chi_{PC_{2-\gamma, \psi, w}(\mathcal{E}, E)}(K_n) = 0, \tag{20}$$

where $\chi_{PC_{2-\gamma, \psi, w}(\mathcal{E}, E)}(K_n)$ is the measure of noncompactness on $PC_{2-\gamma, \psi, w}(\mathcal{E}, E)$. Let $n \in \mathbb{N}$ be fixed and let $\epsilon > 0$ be arbitrary small. Applying Lemma 5 in [58], there exists a sequence $(x_r)_{r \geq 1}$ in $K_n = \Theta(K_{n-1})$ such that

$$\begin{aligned}
\chi_{PC_{2-\gamma, \psi, w}(\mathcal{E}, E)}(K_n) &\leq 2\{\chi_{PC_{2-\gamma, \psi, w}(\mathcal{E}, E)} x_r : r \geq 1\} + \epsilon \\
&= \max_{k \in L_0} \{\max_{\overline{E}_k} \chi_{C_w(\overline{E}_k, E)}(Z_{\overline{E}_k}), \max_{i \in L_1} \chi_{C(\overline{T}_i, E)}(Z_{\overline{T}_i})\} + \epsilon,
\end{aligned} \tag{21}$$

where $Z = \{x_r : r \geq 1\}$,

$$\begin{aligned} Z|_{\bar{E}_k} &: = \{h^* \in C(\bar{E}_k, E) : h^*(\varrho) = (\psi(\varrho) - \psi(s_k))^{2-\gamma} w(\varrho) h(\varrho), \varrho \in E_k, \\ h^*(s_k) &= \lim_{\varrho \rightarrow s_k^+} h^*(\varrho), h \in K_n\}, \end{aligned}$$

and

$$Z|_{\bar{T}_i} := \{h^* \in C(\bar{T}_i, E) : h^*(\varrho) = h(\varrho), \varrho \in T_i, h^*(\kappa_i) = h(\kappa_i^+), h \in K_n\}.$$

Now, from Step (3), the sets $Z|_{\bar{E}_k}, k \in L_0, Z|_{\bar{T}_i}, i \in L_1$ are equicontinuous, and, consequently, (21) becomes

$$\chi_{PC_{2-\gamma, \psi, w}(E, E)}(K_n) \leq \max\{Y_1, Y_2\} + \varepsilon, \quad (22)$$

where

$$\begin{aligned} Y_1 &= \max_{k \in L_0} \max_{\varrho \in \bar{E}_k} \chi\{(\psi(\varrho) - \psi(s_k))^{2-\gamma} w(\varrho) x_r(\varrho) : r \geq 1\}, \\ Y_2 &= \max_{i \in L_1} \max_{\varrho \in \bar{T}_i} \chi\{x_r(\varrho) : r \geq 1\}, \end{aligned}$$

and χ is the measure of compactness in E .

Now, for any $r \in \mathbb{N}$, let $z_r \in K_{n-1}$ with $x_r \in \Theta(z_r)$. According to the definition of Θ , there are $f_r \in S_{F(\cdot, z_r)}^{v(2-\sigma), p}$; we thus have $r \in \mathbb{N}$ such that

$$x_r(\varrho) = \begin{cases} w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-2}}{\Gamma(\gamma-1)} x_0 + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_1 \\ + I_{s_0, \varrho}^{\sigma, \psi, w} f_r(\varrho), \varrho \in E_0, \\ g_i(\varrho, z_r(\kappa_i^-)), \varrho \in T_i, i \in L_1, \\ w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, z_r(s_i^-)) \\ + w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-1}}{\Gamma(\gamma)} g_i^*(s_i, z_r(s_i^-)) \\ + I_{s_i, \varrho}^{\sigma, \psi, w} f_r(\varrho), \varrho \in E_i, i \in L_1. \end{cases} \quad (23)$$

Let $\varrho \in E_0$. Set

$$\begin{aligned} \Omega_1(\varrho) &= (\psi(\varrho) - \psi(0))^{2-\gamma} w(\varrho) \chi\{I_{0, \varrho}^{\sigma, \psi, w} f_r(\varrho) : r \in \mathbb{N}\} \\ &= \frac{(\psi(\varrho) - \psi(0))^{2-\gamma}}{\Gamma(\sigma)} \Omega_2(\varrho), \end{aligned}$$

where

$$\Omega_2(\varrho) = \int_0^\varrho (\psi(\varrho) - \psi(s))^{\sigma-1} \psi'(s) w(s) \chi\{f_r(s) ds : r \in \mathbb{N}\}.$$

Due to (F_4) , for $s \in (0, \varrho]$, the following holds:

$$\begin{aligned} \chi\{f_r(s) : r \geq 1\} &\leq \chi_E(\cup_{r \in \mathbb{N}} F(s, z_r(s))) \\ &\leq \varsigma(s) (\psi(s) - \psi(0))^{2-\gamma} w(s) \chi\{z_r(s) : r \in \mathbb{N}\} \\ &\leq \varsigma(s) \chi\{(\psi(s) - \psi(0))^{2-\gamma} w(s) z_r(s) : r \in \mathbb{N}\} \\ &\leq \varsigma(s) \chi_{PC_{2-\gamma, \psi, w}(E, E)}(K_{n-1}). \end{aligned}$$

Since $\sigma > 1$, this yields

$$\begin{aligned} \Omega_2(\varrho) &\leq \psi(\varrho)^{\sigma-1} \int_0^\varrho \psi'(s) w(s) \chi\{f_r(s) ds : r \in \mathbb{N}\} \\ &\leq \psi(b)^{\sigma-1} \chi_{PC_{2-\gamma, \psi, w}(E, E)}(K_{n-1}) \int_0^\varrho \psi'(s) \varsigma(s) w(s) ds \\ &= \psi(b)^{\sigma-1} \chi_{PC_{2-\gamma, \psi, w}(E, E)}(K_{n-1}) \|\varsigma\|_{L_w^{1, \psi}(E, \mathbb{R}^+)}. \end{aligned}$$

Therefore,

$$\Omega_1(\varrho) \leq \frac{\psi(b)^{1-\gamma+\sigma}}{\Gamma(\sigma)} \chi_{PC_{2-\gamma,\psi,w}(\mathcal{E},E)}(K_{n-1}) \|\varsigma\|_{L_w^{1,\psi}(\mathcal{E},\mathbb{R}^+)}. \quad (24)$$

Next, since, for any $\varrho \in \mathcal{E}$, $g_i(\varrho)$ and $g_i^*(\varrho)$ map bounded sets to relatively compact sets, we obtain $Y_2 = 0$. For $\varrho \in \mathcal{E}_i, i \in L_1$,

$$\chi\{w^{-1}(\varrho) \frac{(\psi(\varrho) - \psi(s_i))^{\gamma-2}}{\Gamma(\gamma-1)} g_i(s_i, z_r(s_i^-)) : r \geq 1\} = 0, \quad (25)$$

Moreover, using the same arguments as when $\varrho \in \mathcal{E}_0$, one can show, for $\varrho \in \mathcal{E}_k, k \in L_1$, that

$$\begin{aligned} (\psi(\varrho) - \psi(s_k))^{2-\gamma} w(\varrho) \chi\{I_{s_k,\varrho}^{\sigma,\psi,w} f_r(\varrho) : r \in \mathbb{N}\} \\ \leq \frac{\psi(b)^{1-\gamma+\sigma}}{\Gamma(\sigma)} \chi_{PC_{2-\gamma,\psi,w}(\mathcal{E},E)}(K_{n-1}) \|\varsigma\|_{L_w^{1,\psi}(\mathcal{E},\mathbb{R}^+)}. \end{aligned} \quad (26)$$

From (22)–(26), one has

$$\begin{aligned} & \chi_{PC_{2-\gamma,\psi,w}(\mathcal{E},E)}(K_n) \\ & \leq \frac{\psi(b)^{1-\gamma+\sigma}}{\Gamma(\sigma)} \chi_{PC_{2-\gamma,\psi,w}(\mathcal{E},E)}(K_{n-1}) \|\varsigma\|_{L_w^{1,\psi}(\mathcal{E},\mathbb{R}^+)}, \forall n \in \mathbb{N}. \end{aligned}$$

Then,

$$\begin{aligned} & \chi_{PC_{2-\gamma,\psi,w}(\mathcal{E},E)}(K_n) \\ & \leq \left(\frac{\psi(b)^{1-\gamma+\sigma}}{\Gamma(\sigma)} \|\varsigma\|_{L_w^{1,\psi}(\mathcal{E},\mathbb{R}^+)} \right)^{n-1} \chi_{PC(\mathcal{E},E)}(K_1), \forall n \geq 1. \end{aligned} \quad (27)$$

Equations (11) and (27) imply that (20) is achieved.

Step 5. By applying the Cantor intersection property, the set $K = \bigcap_{n \in \mathbb{N}} K_n$ is not empty and compact. Then, the multivalued function $\Theta : K \rightarrow K$ satisfies the assumptions of Lemma 6, and consequently, Θ has a fixed point $x \in K$. By using a similar arguments as in Step 1, one can demonstrate that the set of fixed points of Θ is bounded. Then, by Lemma 7, the set of solutions for Problem (1) is compact in $PC_{2-\gamma,\psi,w}(\mathcal{E}, E)$. $\square$

Remark 3. If there is $h \in PC_{2-\gamma,\psi,w}(\mathcal{E}, E)$ such that $f(\varrho) = I_{s_i+\varrho}^{v(1-\sigma),\psi} h(\varrho), \forall \varrho \in \mathcal{E}_k, k \in L_0$, then

$$D_{s_i+\varrho}^{v(2-\sigma),\psi,w} f(\varrho) = D_{s_i+\varrho}^{v(2-\sigma),\psi,w} I_{s_i+\varrho}^{v(2-\sigma),\psi,w} h(\varrho) = h(\varrho), \forall \varrho \in \mathcal{E}_k, k \in L_0,$$

and this implies that $f \in PC_{2-\gamma,\psi,w}^{v(2-\sigma)}(\mathcal{E}, E)$. Therefore, Condition (F_1) will be satisfied if the following assumption is verified:

$(F_1)^*$ For any $z \in PC_{2-\gamma,\psi,w}(\mathcal{E}, E)$, the multivalued function $\varrho \rightarrow F(\varrho, z(\varrho))$ is measurable, and there is $f \in L_w^{p,\psi}(I, E)$ and $h \in PC_{2-\gamma,\psi,w}(\mathcal{E}, E)$ with $f(\varrho) \in F(\varrho, z(\varrho)), a.e.$ and $f(\varrho) = I_{s_k,\varrho}^{v(2-\sigma),\psi,w} h(\varrho), \forall \varrho \in \mathcal{E}_k, k \in L_0$.

Indeed,

$$D_{s_k,\varrho}^{v(2-\sigma),\psi,w} f(\varrho) = D_{s_k,\varrho}^{v(2-\sigma),\psi,w} I_{s_k,\varrho}^{v(2-\sigma),\psi,w} h(\varrho) = h(\varrho).$$

Then, $f \in S_{F(\cdot,z)}^{v(2-\sigma),p}$.

4. Example

In this section, we give an example to show that our results are applicable.

Example 1. Let E be a Hilbert space, let Ω be a non-empty convex compact subset of $E, b = 3, \mathcal{E} = [0, 3], m = 1, s_0 = 0, \kappa_1 = 1, s_1 = 2, \kappa_2 = 3, \sigma = \frac{3}{2}, v = \frac{1}{4}$. Then, $v(2-\sigma) = \frac{1}{8}, \gamma = \sigma + 2v - \sigma v = \frac{13}{8}$ and let $2-\gamma = \frac{3}{8}, w : \mathcal{E} \rightarrow]0, \infty[, \psi : \mathcal{E} \rightarrow \mathbb{R}$ be a strictly increasing

continuously differentiable function with $\psi'(s) \neq 0$ for any $s \in \mathcal{L}$ and $\lambda = \sup\{\|x\| : x \in \Omega\}$. We define a multifunction $F : \mathcal{L} \times E \rightarrow P_{ck}(E)$ such that, for any $z \in PC_{\frac{3}{8}, \psi, w}(\mathcal{L}, E)$,

$$F(\varrho, z(\varrho)) = \begin{cases} \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{4}{8}} \|z(0)\| \Omega, \varrho \in [0, 1], \\ \eta \Omega, \varrho \in (1, 2], \\ \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{4}{8}} \|z(0)\| \Omega, \varrho \in (2, 3], \end{cases} \quad (28)$$

where $\eta > 0$. Let $z \in PC_{\frac{3}{8}, \psi, w}(\mathcal{L}, E)$. Obviously, the multivalued function $\varrho \rightarrow F(\cdot, z(\cdot))$ is measurable. Moreover, we define $f_z, h_z : \mathcal{L} \rightarrow E$ by

$$f_z(\varrho) = \begin{cases} \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{4}{8}} \|z(\varrho)\| \omega, \varrho \in [0, 1], \\ \eta \omega, \varrho \in (1, 2], \\ \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{4}{8}} \|z(\varrho)\| \omega, \varrho \in (2, 3], \end{cases}$$

and

$$h_u(\varrho) = \begin{cases} \frac{\eta \Gamma(\frac{12}{8})}{\Gamma(\frac{11}{8})} w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{3}{8}} \|z(\varrho)\| \omega, \varrho \in [0, 1], \\ \eta \omega, \varrho \in (1, 2], \\ \frac{\eta \Gamma(\frac{12}{8})}{\Gamma(\frac{11}{8})} w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{3}{8}} \|u\| \omega \in (2, 3], \end{cases}$$

where $\omega \in \Omega$ is a fixed point. Obviously, $f(\varrho) \in F(\varrho, z(\varrho))$, a.e., and, by the first assertion of Lemma 2, we obtain

$$\begin{aligned} I_{0, \varrho}^{\frac{1}{8}, \psi, w} h_u(\varrho) &= \frac{\eta \|u\| \omega \Gamma(\frac{12}{8})}{\Gamma(\frac{11}{8})} I_{0, \varrho}^{\frac{1}{8}, \psi, w} (w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{3}{8}}) \\ &= \eta \|u\| \omega (w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{4}{8}}); \varrho \in [0, 1], \end{aligned}$$

and

$$\begin{aligned} I_{2, \varrho}^{\frac{1}{8}, \psi, w} h(\varrho) &= \frac{\eta \|u\| \omega \Gamma(\frac{12}{8})}{\Gamma(\frac{11}{8})} I_{0, \varrho}^{\frac{1}{8}, \psi, w} (w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{3}{8}}) \\ &= \eta \|u\| \omega (w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{4}{8}}); \varrho \in (2, 3]. \end{aligned}$$

Therefore, $f_z(\varrho) = I_{s_k, \varrho}^{\nu(2-\sigma), \psi, w} h_u(\varrho)$, $\forall \varrho \in \mathcal{L}_k, k \in \{0, 1\}$, and hence, by Remark 3, (F_1) is satisfied. In addition, for any $y \in F(\varrho, z(\varrho))$, one has

$$\|y(\varrho)\| \leq \begin{cases} \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{3}{8}} \lambda \|z(0)\|, \varrho \in [0, 1], \\ \eta \lambda, \varrho \in (1, 2], \\ \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{3}{8}} \lambda \|z(0)\|, \varrho \in (2, 3]. \end{cases}$$

Then, (F_2) is verified with

$$\varphi(\varrho) = \begin{cases} \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{3}{8}} \lambda, \varrho \in [0, 1], \\ \eta \lambda, \varrho \in (1, 2], \\ \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{3}{8}} \lambda, \varrho \in (2, 3]. \end{cases} \quad (29)$$

Note that

$$\begin{aligned} \|\varphi\|_{L_w^{1, \psi}(\mathcal{L}, R^+)} &= \int_0^{\varrho} \varphi(s) w(s) \psi'(s) ds \\ &= \eta \lambda \left[\int_0^1 (\psi(\varrho) - \psi(0))^{\frac{3}{8}} \psi'(s) ds + \int_1^2 w(s) \psi'(s) ds \right. \\ &\quad \left. + \int_2^3 (\psi(\varrho) - \psi(2))^{\frac{3}{8}} \psi'(s) ds \right] \\ &= \eta \lambda \left[\frac{8}{11} (\psi(1) - \psi(0))^{\frac{11}{8}} + \int_1^2 w(s) \psi'(s) ds \right. \\ &\quad \left. + \frac{8}{11} (\psi(3) - \psi(2))^{\frac{11}{8}} \right] \end{aligned} \quad (30)$$

Now, let D be a bounded subset of $PC_{2-\gamma, \psi, w}(\mathcal{L}, E)$, $z_1, z_2 \in D$, $\varrho \in (0, 1]$, and $x \in F(\varrho, z_1(\varrho))$. Then,

$$x = \begin{cases} \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{5}{8}} \|z_1(0)\| e^*, & \varrho \in [0, 1], \\ \eta \omega, & \varrho \in (1, 2], \\ \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{5}{8}} \|z_1(0)\| e^*, & \varrho \in (2, 3], \end{cases} \quad (31)$$

where $e^* \in \Omega$. Set

$$y = \begin{cases} \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{3}{8}} \|z_2(0)\| e^*, & \varrho \in [0, 1], \\ \eta \omega, & \varrho \in (1, 2], \\ \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{3}{8}} \|z_2(0)\| e^*, & \varrho \in (2, 3]. \end{cases} \quad (32)$$

From (31) and (32), it yields that $y \in F(\varrho, z_2(\varrho))$, and if $\varrho \in [0, 1] \cup (2, 3]$, then

$$\|x - y\| \leq \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(0))^{\frac{3}{8}} \|e^*\| [|\|z_1(\varrho)\| - \|z_2(\varrho)\||], \text{ if } \varrho \in (0, 1]$$

and

$$\|x - y\| \leq \eta w^{-1}(\varrho)(\psi(\varrho) - \psi(2))^{\frac{3}{8}} \|e^*\| [|\|z_1(\varrho)\| - \|z_2(\varrho)\||], \text{ if } \varrho \in (2, 3].$$

As a result,

$$h(F(\varrho, z_1(\varrho)), F(\varrho, z_2(\varrho))) \leq \eta \lambda \|z_1 - z_2\|_{PC_{2-\gamma, \psi, w}}, \quad (33)$$

and for almost $\varrho \in \mathcal{L}_k, k = 0, 1$,

$$\begin{aligned} \chi_E(F(\varrho, z(\varrho))) &: z \in D) \\ &\leq (\psi(\varrho) - \psi(s_k))^{2-\gamma} \varsigma(\varrho) \chi_E\{z(\varrho) : z \in D\}. \end{aligned} \quad (34)$$

Therefore, both (F_3) and (F_4) hold, where $\varsigma(\varrho) = \eta \lambda; \varrho \in \mathcal{L}$. Next, let $g_1 : [1, 2] \times E \rightarrow E$, $g_1^* : [1, 2] \times E \rightarrow E$ as follows:

$$g_1(\varrho, x) = \eta Y(x), g_1^*(\varrho, x) = \eta Y^*(x), \quad (35)$$

where Y and Y^* are the projection operators on Ω and Y . Note that $\|g_1(\varrho, x)\| \leq \lambda \eta \|x\|$ and $\|g_1^*(\varrho, x)\| \leq \lambda \eta \|x\|, \forall (\varrho, x) \in \mathcal{L} \times E$. Thus, Conditions (H_1) and (H_1) are satisfied when $h = h = \lambda \eta$.

As a consequence of Theorem (1), Problem (1) has a solution where $\sigma = \frac{3}{2}, \nu = \frac{1}{4}, F, g$ and g^* are given by (28) and (35) provided that

$$\frac{\psi(3)^{\frac{7}{8}}}{\Gamma(\frac{3}{2})} \|\varsigma\|_{L_w^{1, \psi}(\mathcal{L}, \mathbb{R}^+)} < 1, \quad (36)$$

and

$$\lambda \eta + \frac{\lambda \eta}{\Gamma(\frac{5}{8})} + \frac{\psi(b)}{\Gamma(\frac{13}{8})} \lambda \eta + \frac{\psi(3)^{\frac{1}{2}} \|\varphi\|_{L_w^{1, \psi}}}{\Gamma(\frac{3}{2})} < 1, \quad (37)$$

where $\varsigma(\varrho) = \lambda \eta$ and $\|\varphi\|_{L_w^{1, \psi}}$ is given by (30). By choosing values of λ, ρ and η small enough, Inequalities (36) and (37) will be verified.

5. Conclusions

It is known that the w -weighted ψ -Hilfer fractional derivative, $D_{s_i, \varrho}^{\sigma, \nu, \psi, w}$, is a generalization of several fractional differential operators. In this work, we were able to construct sufficient conditions that make the solution set of the differential inclusion (the nonlinear

part is a multivalued function) includes $D_{s_i, \varrho}^{\sigma, v, \psi, w}$ of the order $\mu \in (1, 2)$ in the presence of non-instantaneous impulses in Banach spaces of infinite dimensions is not empty and compact. To achieve this goal, we found first the relationship between a differential equation involving $D_{s_i, \varrho}^{\sigma, v, \psi, w}$ of the order $\mu \in (1, 2)$ and its corresponding integral equation. Then, using a suitable fixed-point theorem for multivalued functions, the existence of the solution for the considered problem was proven. Moreover, our technique can be used to generalize all the works mentioned in the introduction section to the case where the considered fractional differential operator is replaced by $D_{s_i, \varrho}^{\sigma, v, \psi, w}$ and the dimension of the space is infinite.

We suggest the following direction for future research.

I—Study the existence of S -asymptotically w -periodic solutions for Problem (1).

II—Study the topological properties of the set of solutions to Problem (1).

III—Extend the recent work conducted in [21,25–37] when the considered fractional differential operator is replaced by $D_{s_i, \varrho}^{\sigma, v, \psi, w}$ and the dimension of the space is infinite.

IV—Generalize this work to the case where the right-hand side contains the infinitesimal generator of a strongly continuous cosine family and the nonlinear part.

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References

- Bernard, B.; Aneel, T. Dynamical Systems Coupled with Monotone Set-Valued Operators: Formalisms, Applications, Well-Posedness, and Stability. *Siam Rev.* **2020**, *62*, 3–129.
- Smirnov, G.V. *Introduction to the Theory of Differential Inclusions*; University of Porto, Porto, Portugal Product Code: GSM/41; American Mathematical Society: Lowcountry, RI, USA, 2002.
- Etemad, S.; Avci, I.; Kumar, P.; Baleanu, D.; Rezapour, S. Some novel mathematical analysis on the fractal-fractional model of the AH1N1/09 virus and its generalized Caputo-type version. *Chaos Solitons Fractals* **2022**, *162*, 112511. [[CrossRef](#)]
- Khana, H.; Alam, K.; Gulzar, H.; Etemad, S.; Rezapour, S. A case study of fractal-fractional tuberculosis model in China: Existence and stability theories along with numerical simulations. *Math. Comput. Simul.* **2022**, *198*, 455–473. [[CrossRef](#)]
- Zhou, Y. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2014.
- Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000.
- Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science: Amsterdam, The Netherlands, 2006.
- Baleanu, D.; Lopes, A.M. *Handbook of Fractional Calculus with Applications: Applications in Engineering, Life and Social Sciences, Part A*; De Gruyter Reference: De Gruyter, Berlin, 2019; Volume 7.
- Lazopoulos, K.-A.; Lazopoulos, A.K. Fractional vector calculus and fluid mechanics. *Mech. Behav. Mater.* **2017**, *26*, 43–54. [[CrossRef](#)]
- Douglas, J.F. Some applications of fractional calculus to polymer science. *Adv. Chem. Phys.* **1997**, *102*, 121–191.
- Agur, Z.; Cojocaru, L.; Mazaur, G.; Anderson, R.M.; Danon, Y.L. Pulse mass measles vaccination across age shorts. *Proc. Natl. Acad. Sci. USA* **1993**, *90*, 11698–11702. [[CrossRef](#)]
- Ballinger, G.; Liu, X. Boundedness for impulsive delay differential equations and applications in populations growth models. *Nonlinear Anal. TMA* **2003**, *53*, 1041–1062. [[CrossRef](#)]
- Kevin, E.M. Church, Applications of Impulsive Differential Equations to the Control of Malaria Outbreaks and Introduction to Impulse Extension Equations: A General Framework to Study the Validity of Ordinary Differential Equation Models with Discontinuities in State, Corpus ID: 125227011, Mathematics, Medicine, Environmental Science. Ph.D. Thesis, Université d'Ottawa/University of Ottawa, Ottawa, ON, Canada, 2014. [[CrossRef](#)]
- Al Nuwairan, M.; Ibrahim, A.G. Nonlocal impulsive differential equations and inclusions involving Atangana-Baleanu fractional derivative in infinite dimensional spaces. *AIMS Math.* **2023**, *8*, 11752–11780. [[CrossRef](#)]

15. Benchohra, M.; Karapınar, E.; Lazreg, J.E.; Salim, A. Fractional Differential Equations with Instantaneous Impulses. In *Advanced Topics in Fractional Differential Equations*; Springer: Cham, Switzerland, 2023; pp. 77–111.
16. Alsheekhussain, Z.; Ibrahim, A.G.; Jawarneh, Y. Properties of solution sets for ψ -Caputo fractional non-instantaneous impulsive semi-linear differential inclusions with infinite delay. *Fractal Fract.* **2023**, *7*, 545. [\[CrossRef\]](#)
17. Agarwal, R.P.; Hristova, S.; O'Regan, D. *Non-Instantaneous Impulses in Differential Equations*; Springer: New York, NY, USA, 2017.
18. Terzieva, R. Some phenomena for non-instantaneous impulsive differential equations. *Int. J. Pure Appl. Math.* **2018**, *119*, 483–490.
19. Xu, H.F.; Zhu, Q.X.; Zheng, W.X. Exponential stability of stochastic nonlinear delay systems subject to multiple periodic impulses. *IEEE Trans. Autom. Control* **2023**. [\[CrossRef\]](#)
20. Ibrahim, A.G. Differential Equations and inclusions of fractional order with impulse effect in Banach spaces. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 69–109. [\[CrossRef\]](#)
21. Wang, J.; Ibrahim, A.G.; O'Regan, D. Nonempties and compactness of the solution set for fractional evolution inclusions with non-instantaneous impulses. *Electron. J. Differ. Equ.* **2019**, *2019*, 37.
22. Wang, J.R.; Ibrahim, A.G.; O'Regan, D.; Almandouh, A.A. Nonlocal fractional semilinear differential inclusions with noninstantaneous impulses of order $\alpha \in (1, 2)$. *Int. J. Nonlinear Sci. Numer. Simul.* **2021**, *22*, 593–603. [\[CrossRef\]](#)
23. Agarwal, R.; Hristova, S.; O'Regan, D. Noninstantaneous impulses in Caputo fractional differential equations and practical stability via Lyapunov functions. *J. Frankl. Inst.* **2017**, *354*, 3097–3119. [\[CrossRef\]](#)
24. Liu, K. Stability analysis for (w, c) -periodic non-instantaneous impulsive differential equations. *AIMS Math.* **2022**, *7*, 1758–1774. [\[CrossRef\]](#)
25. Ziane, M. On the Solution Set for weighted fractional differential equations in Banach spaces. *Differ. Equ. Dyn. Syst.* **2020**, *28*, 419–430. [\[CrossRef\]](#)
26. Mahmudov, N.I.; Emin, S. Fractional-order boundary value problems with Katugampola fractional integral conditions. *Adv. Differ. Equ.* **2018**, *81*, 1–17. [\[CrossRef\]](#)
27. Sousa, J.; Vanterler, D.C.; Oliveira, D.S.; Frederico, G.S.F.; Delfim, F.M.; Torres, D.F.M. Existence, uniqueness and controllability for Hilfer differential equations on times scales. *Math. Meth. Appl. Sci.* **2023**, *46*, 12378–12401. [\[CrossRef\]](#)
28. Salim, A.; Benchohra, M.; Graef, J.R.; Lazreg, J.E. Boundary value problem for fractional order generalized Hilfer-type fractional derivative with non-instantaneous impulses. *Fractal Fract.* **2021**, *5*, 1. [\[CrossRef\]](#)
29. Wang, J.R.; Ibrahim, A.G.; O'Regan, D. Global Attracting Solutions to Hilfer fractional noninstantaneous impulsive semilinear differential inclusions of Sobolev type and with nonlocal conditions. *Nonlinear Anal. Model. Control* **2019**, *24*, 775–803.
30. Wang, J.R.; Ibrahim, A.G.; O'Regan, D. Hilfer type fractional differential switched inclusions with noninstantaneous impulsive and nonlocal conditions. *Nonlinear Anal. Model. Control* **2018**, *23*, 921–941. [\[CrossRef\]](#)
31. Gou, H.; Li, Y. Study on Hilfer-Katugampola fractional implicit differential equations with nonlocal conditions. *Bull. Sci. Math. Matiques* **2021**, *167*, 102944. [\[CrossRef\]](#)
32. Oliveira, D.; Capelas De Oliveira, E. Hilfer-Katugampola fractional derivative. *Comput. Appl. Math.* **2018**, *37*, 672–690. [\[CrossRef\]](#)
33. Berhail, A.; Tabouche, N.; Alzabut, J.; Samei, M.S. Using the Hilfer–Katugampola fractional derivative in initial-value Mathieu fractional differential equations with application to a particle in the plane. *Adv. Contin. Discret. Model.* **2022**, *2022*, 44. [\[CrossRef\]](#)
34. Bhairat, S.P.; Samei, M.E. Nonexistence of global solutions for a Hilfer–Katugampola fractional differential problem. *Partial. Equ. Appl. Math.* **2023**, *7*, 100495. [\[CrossRef\]](#)
35. Elsayed, A.M.; Harikrishnan, S.; Kanagarajan, K. On the existence and stability of boundary value problem for differential equation with Hilfer-Katugampola fractional derivative. *Acta Math. Sci.* **2019**, *39*, 1568–1578. [\[CrossRef\]](#)
36. Asawasamrit, S.; Thadang, Y.; Ntouyas, S.K.; Tariboon, J. Non-instantaneous impulsive boundary value problems containing Caputo fractional derivative of a function with respect to another function and Riemann-Stieltjes fractional integral boundary conditions. *Axioms* **2021**, *10*, 130. [\[CrossRef\]](#)
37. Sousa, J.V.C.; Capelas de Oliveira, E. On the ψ -Hilfer fractional derivative. *Commun. Nonlinear Sci. Numer. Simulat.* **2018**, *91*, 72. [\[CrossRef\]](#)
38. Abdo, M.S.; Pancha, S.K. Fractional Integro-Differential Equations Involving ψ -Hilfer Fractional Derivative. *Adv. Appl. Math. Mech.* **2019**, *11*, 338–359.
39. Thaiprayoon, C.; Sudsutad, W.; Ntouyas, S.K. Mixed nonlocal boundary value problem for implicit fractional integro-differential equations via ψ -Hilfer fractional derivative. *Adv. Differ. Equ.* **2021**, *2021*, 50. [\[CrossRef\]](#)
40. Sudsutad, W.; Thaiprayoon, C.; Ntouyas, S.K. Existence and stability results for ψ -Hilfer fractional integro-differential equation with mixed nonlocal boundary conditions. *Aims Math.* **2021**, *6*, 4119–4141. [\[CrossRef\]](#)
41. Kucchea, K.D.; Jyoti, P.; Kharadea, J.P.; Sousab, J.; Vanterler, D.C. On the Nonlinear Impulsive ψ -Hilfer Fractional differential Equations. *Math. Model. Anal.* **2020**, *25*, 642–660. [\[CrossRef\]](#)
42. Abbas, M.I. Non-instantaneous impulsive fractional integro-differential equations with proportional fractional derivatives with respect to another function. *Stud. Univ. Babeş-Bolyai Math.* **2023**, *68*, 543–562. [\[CrossRef\]](#)
43. Arul, R.; Karthikeyan, P.; Karthikeyan, K.; Geetha, P.; Alruwaily, Y.; Almaghami, L.; El-hady, E. On ψ -Hilfer fractional integro-differential equations with non-instantaneous impulsive conditions. *Fractal Fract.* **2022**, *6*, 732. [\[CrossRef\]](#)
44. Ngo, V.H.; O'Regan, D. A remark on ψ -Hilfer fractional differential equations with noninstantaneous impulses. *Math. Meth. Appl. Sci.* **2020**, 1–15.

45. Ibrahim, A.G.; Elmandouh, A.A. Existence and stability of solutions of ψ -Hilfer fractional functional differential inclusions with non-instantaneous impulses. *AIMS Math.* **2021**, *6*, 10802–10832. [[CrossRef](#)]
46. Sitho, S.; Ntouyas, S.K.; Samadi, A.; Tariboon, J. Boundary value problems for ψ -Hilfer type sequential fractional differential equations and inclusions with integral multi-point boundary conditions. *Mathematics* **2021**, *9*, 1001. [[CrossRef](#)]
47. Dhayal, R.; Zhu, Q. Stability and controllability results of ψ -Hilfer fractional integro-differential systems under the influence of impulses. *Chaos Solitons Fractals* **2023**, *168*, 113105. [[CrossRef](#)]
48. Afshari, H.; Karapinar, E. A discussion on the existence of positive solutions of the boundary value problems via ψ -Hilfer fractional derivative on b-metric spaces. *Adv. Differ. Equ.* **2020**, *2020*, 616. [[CrossRef](#)]
49. Al-Refai, M. On weighted Atangana–Baleanu fractional operators. *Adv. Differ. Equ.* **2020**, *2020*, 3. [[CrossRef](#)]
50. Al-Refai, M.; Jarrah, A.M. Fundamental results on weighted Caputo–Fabrizio fractional derivative. *Chaos Solitons Fractals* **2019**, *126*, 7–11. [[CrossRef](#)]
51. Alshammari, M.; Alshammari, S.; Abdo, M.S. Existence theorems for hybrid fractional differential equations with ψ -weighted Caputo–Fabrizio derivatives. *J. Math.* **2023**, *2023*, 8843470. [[CrossRef](#)]
52. Benial, K.; Souid, M.S.; Jarad, F.; Alqudah, M.A.; Abdeljawad, T. Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order. *J. Inequalities Appl.* **2023**, *2023*, 127. [[CrossRef](#)]
53. Jarad, F.; Abdeljawad, T.; Shah, K. On the weighted fractional operators of a function with respect to another function. *Fractals* **2020**, *28*, 2040011. [[CrossRef](#)]
54. Banas, J.; Mursaleen, M. *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*; Springer: New Delhi, India, 2014.
55. Kamenskii, M.; Obukhowskii, V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*; Walter de Gruyter: Berlin, Germany, 2001.
56. Gabeleh, M.; Eberhard Malkowsky, E.; Mursaleen, M.; Rakocevic, V. A New Survey of Measures of Noncompactness and Their Applications. *Axioms* **2022**, *11*, 299. [[CrossRef](#)]
57. Chen, X.; Cheng, L. Representation of measures of noncompactness and its applications related to an initial value problem in Banach spaces. *Sci. China Math.* **2023**, *66*, 745–776. [[CrossRef](#)]
58. Bothe, D. Multivalued perturbation of m-accretive differential inclusions. *Israel J. Math.* **1998**, *108*, 109–138. [[CrossRef](#)]

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