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Applications of Fractional Order Logistic Grey Models for Carbon Emission Forecasting

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Abstract: In recent years, global attention to carbon emissions has increased, becoming one of the main drivers of global climate change. Accurate prediction of carbon emission trends in small and medium-sized countries and scientific regulation of carbon emissions can provide theoretical support and policy references for the effective and rational use of energy and the promotion of the coordinated development of energy, environment, and economy. This paper establishes a grey prediction model using the classical Logistic mathematical model in a determined environment to investigate the carbon emission system. At the same time, we use the basic principle of fractional-order accumulation to establish a grey prediction model with fractional-order Logistic and obtain the parameter estimation and time-response equation of the new model by solving the model through the theory related to fractional-order operators. The particle swarm optimization algorithm is used to complete the optimization process of the order of the fractional order grey prediction model and obtain the optimal model order. Then, the new model is applied to predict carbon emissions in five medium-emission countries: Ethiopia, Djibouti, Ghana, Belgium, and Austria. The new model shows better advantages in the validity analysis process, and the simulation results indicate that the new model proposed in this paper has stronger stability and better simulation and prediction accuracy than other comparative models, proving the model's validity. Finally, the model is used to forecast the carbon emissions of these five countries for the five years of 2021–2025, and the results are analyzed, and relevant policy recommendations are made.



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1. Introduction

Global carbon emissions are the main culprit of the greenhouse effect, which continues to affect the functioning of human society, with far-reaching impacts on the Earth's climate and environment. In 2020, 25 scientists from the World Climate Research Program (WCRP) warned of a predicted increase in the Earth's temperature of 2.6 °C to 3.9 °C if the amount of greenhouse gases in the atmosphere is doubled compared to the pre-industrial revolution level, triggering such disasters as glacier meltdowns and sea level rises. This global warming prediction has led to more urgent concerns about greenhouse gas emissions. One of the main components of greenhouse gases is carbon dioxide, and the sources of its emissions involve energy combustion, industrial manufacturing, transportation, and other aspects that are indispensable to the normal functioning of human society. Although these activities support social development, they are a major source of environmental threats. Against this background, addressing carbon emissions on a global scale has become an urgent task.

Concerning global climate change, there are common interests among countries that need to work together to face the challenges posed by climate change due to carbon

emissions. The emission reduction efforts of large countries such as the United States, China, and Russia have attracted much attention. These countries are actively promoting the development of renewable energy, enhancing industrial upgrading, and promoting innovations in green transportation, among other things, to reduce carbon emissions. However, small and medium-sized countries are also indispensable in global emissions reduction goals. Although they do not emit as much as larger countries, their efforts to reduce emissions are equally important to realizing the global emissions reduction target. While these countries may face different challenges, their mitigation experiences and measures are also worthy of in-depth study to provide a more comprehensive perspective on global mitigation strategies.

Therefore, an in-depth study of the emission reduction actions of small and medium-sized countries is of great significance to realizing global emission reduction goals. Accurately predicting the trend of carbon emissions in small and medium-sized countries and scientifically regulating carbon emissions provide theoretical support and policy reference for the effective and rational use of energy and the promotion of the coordinated development of energy, the environment, and the economy, and by understanding the specific emission reduction initiatives, policy implementation and results achieved by these countries, we can gain a more comprehensive understanding of the actual situation of global emission reduction. This provides an opportunity for countries to draw on and learn from each other and helps form more effective and sustainable emission reduction strategies. Small and medium-sized countries may face unique challenges in terms of limited resources and relatively low levels of technology. However, it is through studying these challenges and responses that more comprehensive solutions can be provided for global emissions reduction. In jointly addressing the issue of carbon emissions, countries should work together to promote scientific and technological innovation and sustainable development to create a more sustainable and healthier environment for the planet's future.

2. Literature Review

2.1. The Research on Carbon Emissions

The rapid growth of carbon emissions has people's concerns simultaneously. However, it also spawned more related discussions and research on how to inhibit or reduce carbon emissions, how scientific and effective cooperation manage carbon emissions, and other issues that need to be resolved urgently. Many scholars on carbon emissions from various perspectives carry out a series of research, which has resulted in many valuable research results.

Among the studies on carbon quota allocation for individual countries or specific enterprises, Dong et al. [1] used a zero-sum game data envelopment model (ZSG-DEA) to discuss the initial allocation of regional quotas and the optimal carbon quota scheme based on the Chinese government's expected carbon emission targets and economic and social development indicators. The study's conclusions are important policy guidance for establishing a fair and effective carbon quota allocation mechanism, realizing the national total carbon emission control target, stimulating the vitality of the unified carbon market, and promoting regional coordinated emission reduction. Yuan et al. [2] proposed a two-tier collaborative demand-side management framework that allows users to participate in demand response for distributed energy systems. This ensures that the system's carbon emissions do not exceed the quota while minimizing the impact on thermal comfort. This study is of great significance in promoting user participation in distributed energy systems for energy conservation and carbon reduction.

Regarding the carbon price prediction research in each carbon trading market, Ding et al. [3] proposed a new hybrid prediction model to predict the carbon price in China's unified carbon market by combining the improved complete ensemble empirical mode decomposition with adaptive noise (iCEEMDAN), the variational mode decomposition (VMD), and the three-stage algorithm of fine-to-coarse (REC) data reconstruction. The proposed hybrid prediction model can more effectively help enterprises make de-

cisions in the face of nonlinear, non-stationary, and irregular carbon prices, which is conducive to implementing the government's energy-saving and emission-reduction policies. Wang et al. [4] developed a novel ensemble prediction system for point and interval carbon price forecasting based on data decomposition, feature selection, optimal sub-model determination, and improvement through multi-objective optimization. This system aims to provide a scientific and comprehensive reference for the carbon trading market. Considering the alignment between prediction models and carbon price markets, Cao et al. [5] designed a decomposed ensemble carbon price forecasting method incorporating a reinforcement learning model fusion. The model was validated in seven carbon trading markets, and experimental results demonstrated superior and stable predictive performance compared to other methods. The proposed model can serve as a feasible tool for the assessment and management of carbon price market prices.

Among the studies on energy saving and emission reduction in various industries, Zhang et al. [6] conducted a study using carbon emission rights trading pilot companies listed in the non-financial sector of the A-share market in China as samples. They employed a difference analysis method to examine the impact of carbon emission rights trading on the financial asset allocation of these companies. The analysis revealed a positive correlation between carbon emission rights trading and corporate financialization, which was more pronounced in non-state-owned enterprises, carbon-intensive enterprises, and companies operating in regions with developed financial markets or significant fluctuations in the carbon market. Ou et al. [7] used Guangdong Province, China, as an example to assess the land use carbon emissions (LUCE) at the city level in emerging economies through the extended STIRPAT model. The results of the study showed that the importance of construction land use planning and management in achieving emission reduction targets is increasing daily. Ma et al. [8] investigated the impact of the commercial building industry on carbon emissions, conducting case studies using various urban clusters. Their research made significant contributions to the analysis of carbon emission peaks in the Chinese commercial building industry.

2.2. The Research on Carbon Emissions Forecasting

Carbon emission is an important indicator to judge the overall trend, but also the focus of attention of countries. A timely and effective grasp of the changes in carbon emissions will help governments adjust the emission reduction measures promptly, so many scholars have researched the prediction of carbon emissions and have obtained some results. Generally, the prediction methods are divided into the following categories: statistical models, nonlinear intelligent prediction models, grey prediction models, and so on.

In terms of statistical modeling, Senatro et al. [9] used regression analysis to predict the energy demand in Italy and achieved better results. Son et al. [10] used statistical downscaling analysis to analyze the climate situation of the Korean Peninsula. Finally, they obtained the relationship between temperature increase and carbon dioxide emission, which can effectively predict Earth's future climate. In terms of nonlinear intelligent modeling, Ahmadi et al. [11] used an artificial neural network (ANN) method, known as the grouping method of data processing (GMDH), to determine the emissions of the most important greenhouse gas, carbon dioxide, based on the proportion of various energy sources in the primary energy supply and gross domestic product (GDP), which is an indicator of economic activity, and the proposed method effectively improves the accuracy in predicting carbon dioxide emissions. Ye et al. [12] proposed a novel carbon emission prediction framework combining linear and machine learning models by integrating the autoregressive integrated moving average (ARIMA) and support vector regression (SVR) models, which take into account both temporal dynamics and external influences. Simulation results show that the proposed model has a stronger prediction performance than the benchmark model. Khajavi et al. [13] used a hybrid Random Forest, support vector regression, and response surface methodology to predict the CO₂ emissions of 30 major cities in China.

Jeong et al. [14] proposed a statistical forecasting model for predicting carbon emissions in the South Korean construction industry based on three models, including regression analysis. The model presented by them demonstrated high accuracy. Huang et al. [15] utilized principal component analysis to extract four major carbon emission influencing factors. They then employed Long Short-Term Memory (LSTM) to forecast carbon emissions in China, showcasing the effectiveness of principal component analysis and LSTM in carbon emission prediction. Ruiz et al. [16] introduced a time series clustering method for predicting energy consumption in public buildings, achieving favorable results.

Among the statistical models, a regression model can analyze the correlation between factors, which requires a large amount of valid data for simulation and verification. The time series model considers the impact of historical data on the current data, which requires the data to be sensitive to changes in time; nonlinear intelligent computing models such as neural networks, a research hotspot in recent years, which can learn the characteristics of the sample data by changing the network architecture and parameter settings. However, it also requires a large amount of data for determining the parameters. There may be a case of overfitting, which results in high accuracy of model simulation, but the actual predictive effect is not good.

To summarize, there are two main issues in the research of carbon emission prediction. On the one hand, the relevant data on carbon emission are only within the last 20 years, and the data samples are relatively small, so the model training results of the nonlinear intelligent model may not be satisfactory. On the other hand, carbon emission is primarily affected by external factors, such as socio-economic development, population growth, and others. Hence, the classical statistical model with a simpler structure makes it difficult to accurately describe the correlation. Therefore, there is a need for a model suitable for small-sample cases, which can consider the important influencing factors relatively completely and make more accurate predictions of carbon emissions. The grey prediction model happens to have such a nature.

2.3. The Research on Grey Prediction Model and Carbon Emission Forecasting

Deng proposed a “small sample, poor information” uncertain system as the object of study of the grey system theory in 1982 [17] and gave the theory based on the grey prediction model of the basic modeling steps and mechanisms, which has a high accuracy and stability. The grey theory includes the grey prediction model [18,19], the grey correlation model [20], and other theoretical parts. The grey prediction model can be relatively simple in structure, easy to calculate, and has good simulation and prediction performance [21]. Its results have been generally recognized by scholars at home and abroad and are widely used in many fields, such as energy [22–24], economy [25,26], transportation [27–30], and so on. Since carbon emissions are mainly based on small-sample data, in recent years, due to the excellent performance of the grey model in small-sample prediction, many scholars have considered using the grey prediction model for modeling and calculation and have achieved better results [31–39]. For example, Wu et al. [31] proposed a new grey prediction model and effectively applied it to predict carbon dioxide emissions in the BRIC countries. Javed et al. [32] improved the existing grey prediction model GM(1,1) by proposing an optimized model. They applied this model to forecast the consumption of biofuels in major CO₂-emitting countries. Gao et al. [33] considered Gompertz’s law of slowing down the growth of carbon emissions to establish Gompertz’s differential equation, which was transformed into a fractional cumulative grey Gompertz model and effectively applied it to the prediction of carbon emissions in the US industry. Cheng et al. [34] established a combined Markov model applied to the prediction accuracy of carbon emissions from the transportation sector in China. They put forward some policy recommendations to the relevant departments to reduce the intensity of carbon emissions in the transportation sector. Zeng et al. [35] proposed an improved grey system model and applied it to predict the production of coalbed methane in China. Nie et al. [36] established a novel multivariate grey differential dynamic prediction model for analyzing and predicting China’s carbon

emissions. Şahin et al. [37] combined the linear and nonlinear metabolic grey models with optimization techniques to propose an optimized nonlinear metabolic grey model to predict GHG emissions in Turkey. Zhao et al. [38] proposed a grey prediction model that effectively predicts the carbon emissions of 30 Chinese provinces by combining manta ray foraging optimization algorithms to find the optimal heterogeneity coefficients. Wang et al. [39] introduced a novel multivariate grey prediction model and applied it to forecast energy consumption in seven regions of China.

The accumulation technique is an important part of the modeling mechanism of the grey prediction model, and the above grey prediction model is built on the basis of first-order accumulation and gets better results. The fractional order theory has been applied to various fields in recent years. The grey system theory also introduces the fractional order theory, which is mainly put forward by Wu [40] under the influence of the “in-between” idea of the fractional order cumulant theory, which is to transform the original irregular fluctuation sequence into an exponential form through the cumulant operator to weaken the randomness of the data and show the regularity of the data, which is the most important component of the modeling mechanism. Many scholars are also studying the grey prediction model of fractional order [27–33]. For example, Wang et al. [41] proposed a reverse cumulative fractional order grey prediction model with a time power term. They applied it to the power generation of five clean energy sources in China: biomass, wind, nuclear, natural gas, and hydroelectricity to achieve better results. Wang et al. [42] proposed a new grey prediction model with a fractional-order grey Eulerian model with dynamic cumulative order and accurately predicted China’s hydroelectric power generation, China’s natural gas production, and China’s total primary energy production. Chen et al. [43] proposed a grey model with fractional Hausdorff derivatives, which has higher prediction accuracy than the traditional fractional and grey models. He et al. [44] introduced an enhanced fractional cumulative grey model with order optimization constraints through matrix decomposition, functional analysis, numerical simulation, and other perspectives. Zhang et al. [45] proposed a novel fractional discrete dynamic multivariate grey model to forecast food demand. The model is based on the discrete modeling idea, which considers the influence of the changing trend of the correlation factor series and improves the prediction accuracy and robustness. Yang et al. [46] proposed a novel structurally adaptive discrete fractional-order grey Bernoulli model, which uses a well-performing differential evolutionary algorithm to optimize the model in terms of the order, and better prediction performance can be obtained. Yan et al. [47] established a fractional-order multivariable time-lag discrete grey model to predict the trend of short-term online public opinion, and compared with the related grey model, the new grey model has higher prediction accuracy.

Accumulation and subtraction of information processing, as the main method of the grey system, occupies a very important position in the theory of the grey system. At the same time, the fractional-order cumulative generating operator is the basis for establishing the fractional-order grey prediction model. The above studies are some of the results of the fractional-order grey prediction model in recent years. The research methods of these results and the nature of the fractional-order grey cumulative operator help to enrich the theory of the fractional-order grey operator and lay the theoretical foundation for the fractional-order grey prediction model.

The fractional-order grey prediction model is currently a focal point in grey prediction model research, and our study also addresses this prominent issue. While there are existing studies on the application of grey prediction models to carbon emission forecasting, there are limited studies specifically focusing on small countries. The primary objective of our study is to apply the grey prediction model to carbon emissions in small and medium-sized countries and, subsequently, to summarize the experiences related to carbon emissions in these nations. Therefore, we introduce a novel approach by combining the classical Logistic model with fractional-order operator theory to create a fractional-order grey Logistic model for carbon emission prediction. The key contributions include:

1. Integration of the classical Logistic model with the grey prediction mechanism to establish a grey Logistic prediction model. Introduction of the fractional-order cumulative operator theory to formulate a new fractional-order Logistic grey prediction model. Utilization of the particle swarm optimization algorithm to optimize the fractional-order grey prediction model, determining the optimal model order;
2. In-depth examination of CO₂ emissions in five medium-emission countries (Ethiopia, Djibouti, Ghana, Belgium, and Austria) using four grey models. Through thorough analysis and comparison of results, the proposed fractional grey model demonstrates superior stability, simulation, and prediction accuracy compared to the other three models, affirming its effectiveness;
3. Effective application of the new fractional-order logistic grey prediction model in forecasting the five-year CO₂ emissions for the mentioned countries from 2021 to 2025. We also conclude with policy recommendations based on the analysis of the prediction results.

The remainder of the paper is structured as follows: Section 3 develops a fractional order logistic grey prediction model, solves the model through the theory related to fractional order operators, and proves and discusses the nature of the model and the modeling mechanism. Section 4 applies the model to the CO₂ emission examples of five medium-emission countries, namely, Ethiopia, Djibouti, Ghana, Belgium, and Austria; Section 5 describes the five-year CO₂ emissions of the five countries for the period 2021–2025 for an effective forecast; and Section 6 concludes the paper.

3. Fractional Order Logistic Grey Prediction Model

This Section introduces the establishment of the logistic grey prediction model, the theory related to the fractional order operator, and the establishment of the fractional order logistic grey prediction model, as well as investigates the new model's parameter estimation and time correspondence equation.

3.1. Logistic Grey Prediction Model

The logistic model is a classical mathematical model that studies a given population within a defined environment. Assuming that the individuals of the population do not discriminate between sizes, that they are uniformly distributed within the environment, that there is no out-migration or in-migration, and that the supply of resources within the environment always remains a constant and is equally distributed to each individual.

For ease of reference, we have organized the main notations into Table 1.

Table 1. Main Notations.

Notation	Definition	Notation	Definition
N	CO ₂ emission intensity	r	Growth rate
$N^{(0)}$	The sequence of carbon emissions	$N^{(1)}$	Accumulative generating sequence of carbon emissions
$Z^{(1)}$	The mean-generating sequence of $N^{(1)}$	$N^{(r)}(1)$	The first-order cumulative generator operator
$\Gamma(n)$	The Gamma function for n	$\hat{p} = [a, b]^T$	The parameter vector of the FLGM(1,1) model

The differential equations of the logistic model are as follows:

$$\begin{cases} \frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \\ N_0 = N|_{t=0} \end{cases} \quad (1)$$

Assuming CO₂ emissions at a given time, N is the CO₂ emission intensity at time t , and r is the growth rate that can be achieved.

Define the sequence of carbon emissions as follows:

$$N^{(0)} = \{N^{(0)}(1), N^{(0)}(2), \dots, N^{(0)}(n)\} \quad (2)$$

For an accumulative generating sequence,

$$\begin{aligned} N^{(1)} &= \{N^{(1)}(1), N^{(1)}(2), \dots, N^{(1)}(n)\} \\ N^{(1)}(k) &= \sum_{m=1}^k N^{(0)}(m) \end{aligned} \quad (3)$$

where the logistic equation is deformed to the following:

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right) = -\frac{r}{K} N^2 + rN$$

Set $-\frac{r}{K} = a, r = b$; thus, the differential equation of carbon emission $N(t)$ versus time t can be expressed as follows:

$$\frac{dN^{(1)}(t)}{dt} = aN^{(1)}(t)^2 + bN^{(1)}(t) \quad (4)$$

Replace the differential equation at the left end of Equation (4) with a first-order difference equation. Then, at $t = k$, we have the following:

$$\frac{dN^{(1)}(t)}{dt} \Big|_{t=k} \approx \frac{\Delta N^{(1)}(t)}{\Delta t} \Big|_{t=k} = \frac{N^{(1)}(k) - N^{(1)}(k-1)}{k - (k-1)} = N^{(1)}(k) - N^{(1)}(k-1) = N^{(0)}(k)$$

This leads to the definition of the following grey prediction model.

Definition 1. Let the sequence $N^{(0)}, N^{(1)}$, as shown in Equations (2) and (3), then the sequence $Z^{(1)}$ is said to be the mean generating sequence of $N^{(1)}$:

$$Z^{(1)} = \{z^{(1)}(1), z^{(1)}(2), \dots, z^{(1)}(n)\} \quad (5)$$

$$\text{where } z^{(1)}(k) = \frac{N^{(1)}(k) + N^{(1)}(k-1)}{2}.$$

Definition 2. Let the sequence $N^{(0)}, N^{(1)}$ and $Z^{(1)}$, as shown in Equations (2), (3) and (5), then

$$N^{(0)}(k) = az^{(1)}(k)^2 + bz^{(1)}(k) \quad (6)$$

Equation (6) is called the Logistic Grey model (abbreviated as LGM(1,1) model), and differential equation Equation (4) is called the whitening equation of the model.

The parameters of LGM(1,1) can be estimated by the least squares method of the grey prediction model, and the following theorem can be obtained by using the method of solving differential equations in the literature [17]:

Theorem 1. Let $Z^{(1)} = (z^{(1)}(2), z^{(1)}(3), \dots, z^{(1)}(n))$ be carbon emission $N^{(0)}$ one time cumulative sequence $N^{(1)}$ of the neighboring mean generating sequence,

$$B = \begin{bmatrix} (z^{(1)}(2))^2 & z^{(1)}(2) \\ (z^{(1)}(3))^2 & z^{(1)}(3) \\ \vdots & \vdots \\ (z^{(1)}(n))^2 & z^{(1)}(n) \end{bmatrix}, Y = \begin{bmatrix} N^{(0)}(2) \\ N^{(0)}(3) \\ \vdots \\ N^{(0)}(n) \end{bmatrix}, P = \begin{bmatrix} a \\ b \end{bmatrix} \quad (7)$$

Then, the parameter estimates a, b are obtained by least squares as follows:

$$\hat{P} = [a, b]^T = (B^T B)^{-1} B^T Y. \quad (8)$$

Theorem 2. Following Theorem 1, the time response sequence of the LGM(1,1) model $N^{(0)}(k) = az^{(1)}(k)^2 + bz^{(1)}(k)$ is as follows:

$$\hat{N}^{(1)}(k) = \frac{1}{\left(\frac{1}{N^{(0)}(1)} + \frac{a}{b}\right)e^{-r(k-1)} - \frac{a}{b}}, k = 1, 2, \dots, n \quad (9)$$

where the initial value condition is as follows: when time $t = 1$, there is $N^{(1)}(t) = N^{(0)}(1)$.

3.2. Fractional Order Operator Theory

Definition 3. Let $N^{(r)}(1) = (N^{(r)}(1), N^{(r)}(2), \dots, N^{(r)}(n))$. By the definition of the first-order cumulative generator operator, where

$$N^{(r)}(k) = \sum_{i=1}^k N^{(r-1)}(i) = \sum_{i=1}^k \sum_{j=1}^i N^{(r-2)}(j), r \in R^+, k = 1, 2, \dots, n \quad (10)$$

therefore:

$$N^{(r)}(k) = \sum_{i=1}^k \frac{(k-i+1)(k-i+2) \cdots (k-i+r-1)}{(r-1)!} x^{(0)}(i), r \in Z^+, k = 1, 2, \dots, n. \quad (11)$$

The definition and properties of the Gamma function are given below.

Definition 4. Define $\Gamma(n)$ to be the Gamma function for $n, n \in R$ and $n \notin \{0, -1, -2, -3, \dots\}$

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt \quad (12)$$

There is $\Gamma(n+1) = \int_0^\infty e^{-t} t^n dt = n!$ when $n \in N$, then there are $\Gamma(n+1) = n\Gamma(n)$,
Therefore:

$$N^{(r)}(k) = \sum_{i=1}^k \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} N^{(0)}(i), r \in Z^+, k = 1, 2, \dots, n \quad (13)$$

In particular, when $r \in Z^+$, the expansion factor of $N^{(r)}$ is

$$a_k = \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} = \frac{(r+k-i-1)!}{(k-i)!(r-1)!} \quad (14)$$

3.3. Fractional Order LGM(1,1) Model

This Section introduces the fractional order LGM(1,1) model, i.e., the FLGM(1,1) model, and analyzes its essential properties. The fractional order FLGM(1,1) model is first defined using the operator theory of Section 3.2.

Definition 5. Let $N^{(0)}$ be the original sequence and $N^{(r)}$ be the r -order cumulative generated sequence of $N^{(0)}$, given by Definition 3.

$$N^{(r-1)}(k) = aZ^{(r)}(k)^2 + bZ^{(r)}(k) \quad (15)$$

Equation (15) is referred to as the FLGM(1,1) model, where $N^{(r)}(k)$ is given by Equation (10)

$$\begin{aligned} N^{(r-1)}(k) &= N^{(r)}(k) - N^{(r)}(k-1) \\ Z^{(r)}(k) &= \frac{1}{2} \left(N^{(r)}(k) + N^{(r)}(k-1) \right) \end{aligned} \quad (16)$$

Specifically, when $r = 1$, Equation (10) becomes $N^{(0)}(k) = aZ^{(1)}(k)^2 + bZ^{(1)}(k)$, which is the original form of the LGM(1,1) model.

According to the definition of the model FLGM(1,1) model, the following properties can be obtained:

Theorem 3. Let $\hat{p} = [a, b]^T$ be the parameter vector of the FLGM(1,1) model. After using the least squares method, we have the following:

$$\hat{p} = (B^T B)^{-1} B^T Y \quad (17)$$

where B, Y

$$\begin{aligned} Y &= \begin{bmatrix} \sum_{i=1}^2 \frac{\Gamma(r+2-i)}{\Gamma(2-i+1)\Gamma(r)} N^{(0)}(i) - \sum_{i=1}^1 \frac{\Gamma(r+2-i-1)}{\Gamma(2-i)\Gamma(r)} N^{(0)}(i) \\ \sum_{i=1}^3 \frac{\Gamma(r+3-i)}{\Gamma(3-i+1)\Gamma(r)} N^{(0)}(i) - \sum_{i=1}^2 \frac{\Gamma(r+3-i-1)}{\Gamma(3-i)\Gamma(r)} N^{(0)}(i) \\ \vdots \\ \sum_{i=1}^n \frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)} N^{(0)}(i) - \sum_{i=1}^{k-1} \frac{\Gamma(r+n-i-1)}{\Gamma(n-i)\Gamma(r)} N^{(0)}(i) \end{bmatrix} \\ &= \begin{bmatrix} (r-1)N^{(0)}(1) + N^{(0)}(2) \\ \frac{r(r-1)}{2}N^{(0)}(1) + (r-1)N^{(0)}(2) + N^{(0)}(3) \\ \vdots \\ \sum_{i=1}^n \frac{\Gamma(r+n-i)}{\Gamma(n-i+1)\Gamma(r)} N^{(0)}(i) - \sum_{i=1}^{k-1} \frac{\Gamma(r+n-i-1)}{\Gamma(n-i)\Gamma(r)} N^{(0)}(i) \end{bmatrix} \\ B &= \begin{bmatrix} Z^{(r)}(2)^2 & Z^{(r)}(2) \\ Z^{(r)}(3)^2 & Z^{(r)}(3) \\ \vdots & \vdots \\ Z^{(r)}(n)^2 & Z^{(r)}(n) \end{bmatrix} = \begin{bmatrix} \left(\frac{N^{(r)}(1)+N^{(r)}(2)}{2} \right)^2 & \frac{N^{(r)}(1)+N^{(r)}(2)}{2} \\ \left(\frac{N^{(r)}(2)+N^{(r)}(3)}{2} \right)^2 & \frac{N^{(r)}(2)+N^{(r)}(3)}{2} \\ \vdots & \vdots \\ \left(\frac{N^{(r)}(n-1)+N^{(r)}(n)}{2} \right)^2 & \frac{N^{(r)}(n-1)+N^{(r)}(n)}{2} \end{bmatrix} \end{aligned}$$

Thus:

$$\begin{aligned} N^{(r-1)}(k) &= N^{(r)}(k) - N^{(r)}(k-1) \\ &= \sum_{i=1}^k \frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)} N^{(0)}(i) - \sum_{i=1}^{k-1} \frac{\Gamma(r+k-i-1)}{\Gamma(k-i)\Gamma(r)} N^{(0)}(i), \quad k = 1, 2, \dots, n, \\ Z^{(r)}(k) &= \frac{1}{2} \left(N^{(r)}(k) + N^{(r)}(k-1) \right) \\ &= \frac{\sum_{i=1}^k (\Gamma(r+k-i)/\Gamma(r-i+1)\Gamma(r)) N^{(0)}(i) + \sum_{i=1}^{k-1} (\Gamma(r+k-i)/\Gamma(k-i+1)\Gamma(r)) N^{(0)}(i)}{2} \end{aligned}$$

Definition 6.

$$\frac{dN^{(r)}(t)}{dt} = aN^{(r)}(t)^2 + bN^{(r)}(t) \quad (18)$$

Define Equation (18) as the differential whitening equation for the FLGM(1,1) model $N^{(r-1)}(k) = aZ^{(r)}(k)^2 + bZ^{(r)}(k)$.

Theorem 4. Define the time response function for the FLGM(1,1) model as

$$\hat{N}^{(r-1)}(k) = \frac{1}{\left(\frac{1}{N^{(r)}(k)} + \frac{a}{b}\right)e^{-b(k-1)} - \frac{a}{b}}, k = 1, 2, \dots, n \quad (19)$$

This leads to the cumulative reduction value $\hat{X}^{(0)} = \hat{N}^{(r)}(-r)(k), k = 1, 2, \dots, n$, where $\hat{N}^{(0)}(1) = N^{(0)}(1)$.

3.4. Modeling Steps for the FLGM(1,1) Model

In this Section, the principle of the particle swarm optimization algorithm is given. Then, the optimization process of the fractional-order grey prediction model is completed, and the optimal model order is solved to establish the optimal fractional-order FLGM(1,1) model, which can be referred to in the literature [48].

The specific steps of the PONGM(1,N) model are described below as follows:

- (1) Collect the primary sequence of the original data $N^{(0)}$;
- (2) $Z^{(r)}$ Calculate the $r - AGO$ sequence $N^{(r)}$ and the sequence of immediate neighboring means $Z^{(r)}$ using Equation (5);
- (3) The resulting data are used to construct the B, Y matrix and substituted into Equation (17) of Theorem 3, and the coefficients a, b are calculated using the least squares method;
- (4) Using Equation (19) of Theorem 4 to calculate the reduction value, the particle swarm algorithm is used to find the optimal fractional order r , and to calculate the APEs and MAPEs, MAPEp, and CMRPE, which are defined as follows:

$$APE = \left(\frac{x^{(0)}(k) - \hat{x}^{(0)}(k)}{x^{(0)}(k)} \right) \times 100\%, k = 1, 2, 3 \dots n$$

$$MAPEs = \frac{1}{m} \left(\sum_{i=1}^n \left| \frac{x^{(0)}(k) - \hat{x}^{(0)}(k)}{x^{(0)}(k)} \right| \right) \times 100\%, k = 1, 2, 3, \dots, m$$

$$MAPEp = \sum_{i=m+1}^{m+p} \frac{1}{p} \left| \frac{x^{(0)}(k) - \hat{x}^{(0)}(k)}{x^{(0)}(k)} \right| \times 100\%, i = m+1, m+2, \dots, m+p$$

$$CMRPE = \frac{MAPEs + MAPEp}{2}$$

4. Applications and Analyzing

Global climate change is one of the most critical environmental issues today, with far-reaching implications for the future of the global economy and the natural environment, and carbon emissions are one of the main drivers of global climate change. Many countries are now making efforts to reduce CO₂ emissions. Despite the increasingly urgent global need to reduce CO₂ emissions, most researchers still focus on those countries with the largest emissions, such as the United States, China, and Russia. While the emission reduction efforts and achievements of these large countries are essential, the emission reduction actions of small and medium-sized countries should not be ignored, and their emission reduction trends and challenges are worthy of our in-depth discussion. Although small and medium-sized countries do not emit as much carbon dioxide as large countries, their

efforts are equally crucial to realizing global emission reduction targets. By investigating the emission reduction actions of these medium-emitting countries, we can gain a more comprehensive understanding of the actual situation of global emission reduction and draw lessons from them to formulate more effective emission reduction strategies.

4.1. Simulation Setup

- (1) **Dataset:** Our study selected five medium-emitting countries, Ethiopia, Djibouti, Ghana, Belgium, and Austria, to provide insight into their efforts and challenges in reducing CO₂ emissions. The raw data on CO₂ emissions for these countries are derived from World Bank data <https://data.worldbank.org.cn/> (accessed on 15 October 2023), where CO₂ emissions are expressed in metric tons per capita. The raw data on CO₂ emissions comprise the emission year and its corresponding carbon intensity;
- (2) **Data preprocessing:** The raw data on CO₂ emissions from World Bank data adhere to standardization, exhibiting no instances of missing values or anomalies, thereby obviating the need for specialized processing. Since the CO₂ emissions in the World Bank data are only updated until 2020, data from 2006 to 2020 were selected for the calculations in our manuscript. This is because the grey prediction model operates under the premise of “small sample, poor information,” hence the selection of these 15 data points from the years 2016 to 2020. These data were divided into two groups: the training and the testing group. The first 10 groups of data from 2006 to 2015, as the training set, were used to construct different types of models. The remaining data from 2016 to 2020, as the test set, were used to compare the performance of the models;
- (3) **Simulation parameters:** the APEs and MAPEs, MAPEp, and CMRPE are described at the end of Section 3;
- (4) **Comparisons model:** Comparisons were made with the existing grey prediction models LGM(1,1), WGM(1,1) [49], and CCRGM(1,1) [50] to validate the performance of the model in practical applications. The CO₂ emissions of these five countries were predicted for 2021–2025 using each of the four models.

4.2. Carbon Emissions in Ethiopia

In this Section, CO₂ emissions in Ethiopia are discussed through a grey prediction model. The PSO algorithm was first used to search for the MAPE_{total} and the corresponding optimal value of r for the raw data for 2006–2015, which corresponded to 15.6520% and 0.1907, respectively, as shown in Table 2. Once the optimal fractional order has been determined, it is possible to establish a grey prediction model directly and predict the data for 2016–2020. Applying the grey system theory and actual data, we establish four prediction models: FLGM(1,1), LGM(1,1), WGM(1,1), and CCRGM(1,1), and the corresponding results are shown in Table 3. The comparison of APE (%) of Ethiopia’s CO₂ emissions from four grey models is shown in Table 4. The results of the Mean Absolute Percentage Error (MAPE), the Composite Mean Relative Percentage Error (CMRPE), and the Root Mean Square Error (RMSPE) of four grey models for CO₂ emissions in Ethiopia are presented in Table 5.

Table 2. Minimum MAPE_{total} and its corresponding optimal fractional order for FLGM(1,1) model based on the PSO algorithm.

	Ethiopia	Djibouti	Ghana	Belgium	Austria
MAPE _{total} (%)	15.6520	10.8330	11.5987	5.3956	5.9175
r	0.1907	0.3048	0.1515	0.0210	0.0152

Table 3. Prediction results of four grey models for carbon emissions in Ethiopia.

Year	Data	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.0678	0.0678	0.0678	0.0678	0.0678
2007	0.0727	0.0595	0.0295	0.0727	0.0460
2008	0.0769	0.0629	0.0336	0.0567	0.0639
2009	0.0753	0.0682	0.0437	0.0864	0.0811
2010	0.0725	0.0747	0.0558	0.0710	0.0998
2011	0.0822	0.0822	0.0697	0.1013	0.1227
2012	0.0906	0.0907	0.0843	0.0866	0.1539
2013	0.1050	0.1000	0.0985	0.1175	0.2015
2014	0.1252	0.1100	0.1103	0.1036	0.2837
2015	0.1273	0.1207	0.1179	0.1353	0.4509
2016	0.1448	0.1321	0.1199	0.1221	0.8957
2017	0.1468	0.1439	0.1159	0.1547	3.0774
2018	0.1532	0.1562	0.1066	0.1424	−21.9126
2019	0.1552	0.1687	0.0937	0.1758	13.5661
2020	0.1544	0.1814	0.0792	0.1645	1.5477
2021	—	0.1939	0.0647	0.1989	0.6713
2022	—	0.2063	0.0514	0.1886	0.4319
2023	—	0.2184	0.0399	0.2242	0.3519
2024	—	0.2299	0.0305	0.2150	0.3440
2025	—	0.2409	0.0230	0.2517	0.3980

Table 4. Comparison of APE (%) of Ethiopia's CO₂ emissions from four grey models.

Year	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.0000	0.0000	0.0000	0.0000
2007	18.1708	59.4791	0.0000	36.7412
2008	18.1684	56.3629	26.2244	16.9641
2009	9.4648	41.9862	14.6744	7.7053
2010	2.9904	23.0291	2.1271	37.6197
2011	0.0000	15.3110	23.1332	49.1385
2012	0.0741	6.9320	4.4778	69.7993
2013	4.7739	6.1564	11.9697	91.9701
2014	12.1579	11.8892	17.3034	126.5709
2015	5.1722	7.3774	6.2613	254.1263
2016	8.7897	17.2062	15.6782	518.5261
2017	1.9414	21.0747	5.3686	1996.4352
2018	1.9366	30.4403	7.0962	14,400.4225
2019	8.7312	39.6027	13.3165	8642.8008
2020	17.4321	48.7296	6.4985	902.1936

Table 5. Error indicators (%) of four grey models for CO₂ emissions in Ethiopia.

	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
MAPEsim	7.8859	25.3915	13.2714	76.7373
MAPEpre	7.7662	31.4107	9.5916	5292.0756
CMRPE	7.8260	28.4011	11.4315	2684.4064
RMSPE	9.7333	31.7633	12.8821	4375.9815

According to Table 5, it can be seen that the MAPEsim, MAPEpre, CMRPE, and RMSPE of the new model are 7.8859%, 7.7662%, 7.8260, and 9.7333%, respectively, which are much lower than those of the other three models. The new model almost achieves an unbiased simulation, with the best prediction accuracy among the four models predicting the data in 2011. The LGM(1,1) model has an error of more than 10% in both simulation and prediction, which is much larger than the FLGM(1,1) model. The WGM(1,1) model shows high prediction accuracy in the data prediction part. However, the MAPE value in the prediction part is significantly larger than that of the proposed model, with an error of

up to 13.2714%. In addition, the CCRGM(1,1) model shows the worst results in both the simulation and prediction part, where the new model has the best values for the fitting and prediction error metrics. In summary, the FLGM(1,1) model proposed in this paper has high prediction accuracy, adaptability, and applicability. The result illustrates that the fractional-order grey model has higher accuracy than the integer-order grey model, which shows that the fractional-order accumulation improves the model's accuracy and obtains satisfactory results. In order to further analyze the simulation and prediction effect of the model, Figure 1 is plotted.

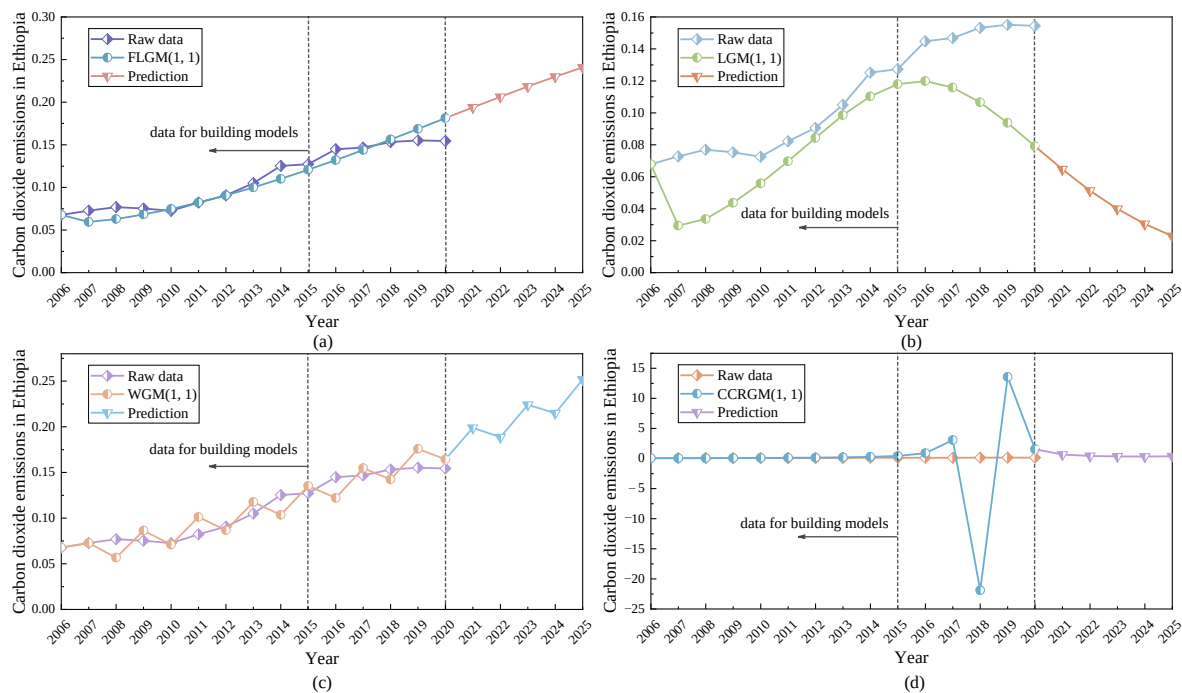


Figure 1. Analysis of CO₂ emissions in Ethiopia by four grey models. (a) by FLGM(1,1), (b) by LGM(1,1), (c) by WGM(1,1), (d) by CCRGM(1,1).

As shown in Figure 1, the LGM(1,1) model and CCRGM(1,1) model underestimated the trend of CO₂ emissions in Ethiopia. In contrast, the FLGM(1,1) model and the WGM(1,1) model succeeded in identifying the trend. However, compared with the raw data, the WGM(1,1) model's trend fluctuates too much and is not the same as the raw data. In contrast, the FLGM(1,1) model accurately captures the slow upward trend of the data set. This shows that the FLGM(1,1) model has the best prediction performance, the WGM(1,1) model is the second, and the CCRGM(1,1) model is the worst, which indicates that the new model successfully captures the trend of CO₂ emissions in Ethiopia.

4.3. Carbon Emissions in Djibouti

This Section performs the validity analysis using Djibouti's CO₂ emissions by applying the PSO algorithm to determine the MAPE_{total} and the corresponding optimal fractional order r . The corresponding results are 10.8330% and 0.3048, respectively, as shown in Table 2. Based on the modeling mechanism of the new model, the raw data from 2006–2015 were used to predict the data for the following five years. The simulation results are presented in Tables 6–8.

Table 6. Prediction results of four grey models for carbon emissions in Djibouti.

Year	Data	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.5108	0.5108	0.5108	0.5108	0.5108
2007	0.5435	0.5435	−0.9241	0.5435	−0.4750
2008	0.5372	0.5557	−0.2396	0.5681	−0.4842
2009	0.5166	0.5603	−0.4016	0.5192	−0.4939
2010	0.5647	0.5502	−0.7154	0.5442	−0.5042
2011	0.5069	0.5311	−1.4236	0.4957	−0.5151
2012	0.5162	0.5086	−3.5804	0.5211	−0.5268
2013	0.5750	0.4861	−18.9550	0.4729	−0.5392
2014	0.4066	0.4653	56.4032	0.4987	−0.5524
2015	0.4438	0.4467	−18.1057	0.4509	−0.5666
2016	0.3865	0.4303	−3.5126	0.4770	−0.5820
2017	0.3878	0.4159	−1.4051	0.4296	−0.5986
2018	0.3925	0.4032	−0.7079	0.4561	−0.6167
2019	0.3988	0.3919	−0.3979	0.4090	−0.6366
2020	0.3923	0.3819	−0.2376	0.4358	−0.6585
2021	—	0.3729	−0.1469	0.3891	−0.6828
2022	—	0.3647	−0.0929	0.4162	−0.7100
2023	—	0.3573	−0.0595	0.3698	−0.7407
2024	—	0.3505	−0.0385	0.3972	−0.7758
2025	—	0.3442	−0.0250	0.3511	−0.8164

Table 7. Comparison of APE (%) of Djibouti's CO₂ emissions from four grey models.

Year	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.0000	0.0000	0.0000	0.0000
2007	0.0012	270.0452	0.0000	187.4076
2008	3.4433	144.5936	5.7478	190.1253
2009	8.4529	177.7419	0.4991	195.6055
2010	2.5795	226.6762	3.6362	189.2819
2011	4.7649	380.8308	2.2236	201.6199
2012	1.4695	793.6136	0.9457	202.0493
2013	15.4578	3396.3248	17.7588	193.7609
2014	14.4419	13,771.6740	22.6486	235.8566
2015	0.6608	4179.9607	1.6071	227.6843
2016	11.3343	1008.8613	23.4339	250.5827
2017	7.2365	462.3211	10.7812	254.3599
2018	2.7280	280.3728	16.2173	257.1463
2019	1.7185	199.7909	2.5668	259.6380
2020	2.6630	160.5497	11.0869	267.8357

Table 8. Error indicators (%) of four grey models for CO₂ emissions in Djibouti.

	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
MAPEsim	5.6969	2593.4956	6.8833	202.5990
MAPEpre	5.1361	422.3792	12.8172	257.9125
CMRPE	5.4165	1507.9374	9.8503	230.2558
RMSPE	7.1232	3838.3469	11.3759	216.7345

As shown in Tables 6–8, the LGM(1,1) and the CCRGM(1,1) models have negative values in their poor prediction performance, indicating that the two models do not accurately capture Djibouti's CO₂ emissions. Regarding simulation and prediction errors, the MAPEsim, MAPEpre, CMRPE, and RMSPE of these two models are well over 10%. Meanwhile, the FLGM(1,1) and WGM(1,1) models are better in the simulation part, with their performances of 5.6969% and 6.8833%, respectively. The simulation error of the new model is still smaller than that of the WGM(1,1) model by 1.1864%. It exhibits a MAPEpre of 5.1361% in the prediction part, which is much smaller than that of the WGM(1,1) model

in the prediction part. The CMRPE and RMSPE of the new model are 5.4165% and 7.1232%, respectively, which are also the best among the four models. Then, we plot Figure 2 to compare these four models visually.

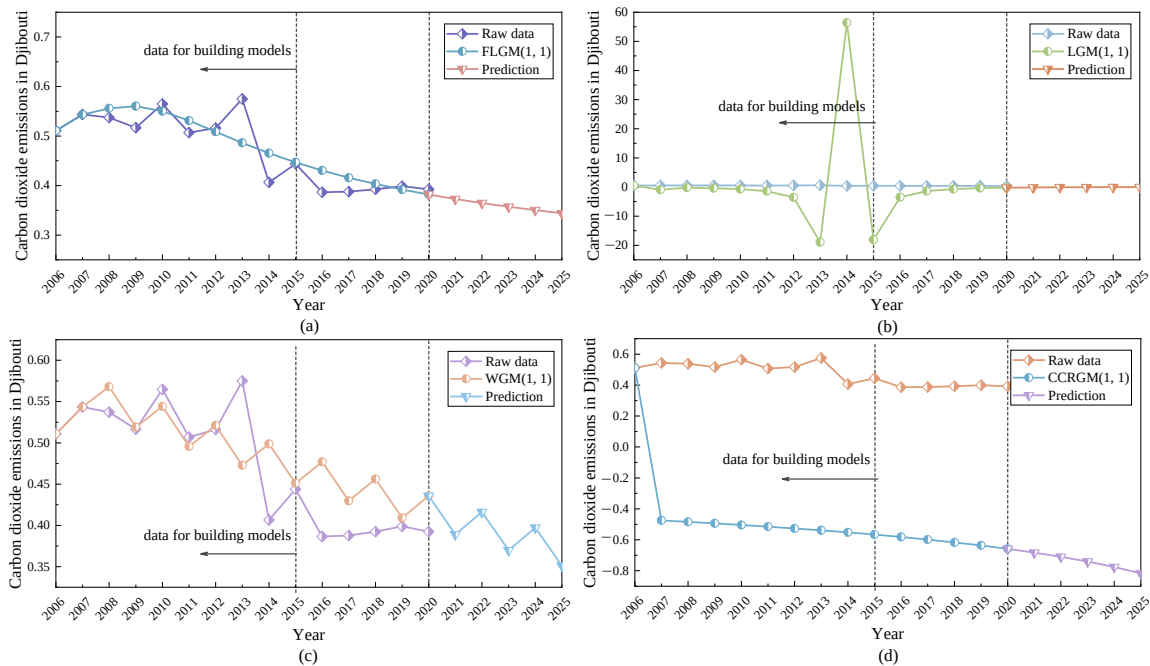


Figure 2. Analysis of CO₂ emissions in Djibouti by four grey models. (a) by FLGM (1,1), (b) by LGM(1,1), (c) by WGM(1,1), (d) by CCRGM(1,1).

The simulation and prediction fitting curves of the FLGM (1,1) model, LGM (1,1) model, WGM (1,1) model, and CCRGM (1,1) model are plotted in Figure 2, respectively. From Figure 2, it can be seen that the predicted trends of the LGM (1,1) model and the CCRGM (1,1) model are too far from the trends of the raw data, and the simulated trends are completely inconsistent with the trends of the actual CO₂ emissions. Although the WGM(1,1) model fits the raw data trend relatively well before 2012, it performs poorly in the later part of the prediction. It does not recognize the trend of the data, which results in unsatisfactory predictions. The new model matches the trend of the raw data, fits the raw data better in the simulation stage, and gets the same reflection in the prediction stage. This indicates that the proposed model has good prediction performance and is suitable for calculating and predicting CO₂ emission trends in Djibouti.

4.4. Carbon Emissions in Ghana

The data in this Section are the CO₂ emissions of Ghana. Using the historical data from 2006–2015 for modeling. The optimal fractional order of the new model was derived as 0.1515 with the PSO algorithm, which corresponds to a MAPE_{total} value of 11.5987%, as shown in Table 2. Based on the modeling mechanism of the new model, the simulation results are presented in Tables 9–11.

Table 9. Prediction results of four grey models for carbon emissions in Ghana.

Year	Data	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.3376	0.3376	0.3376	0.3376	0.3376
2007	0.3551	0.3017	−0.9393	0.3551	−0.2725
2008	0.3275	0.3276	−0.3147	0.3331	−0.2862
2009	0.3735	0.3562	−0.5172	0.3984	−0.3008
2010	0.4060	0.3850	−0.9137	0.3779	−0.3158
2011	0.4122	0.4122	−1.8414	0.4447	−0.3317
2012	0.4794	0.4367	−4.9393	0.4259	−0.3493
2013	0.5050	0.4579	−39.7761	0.4944	−0.3695
2014	0.4741	0.4753	73.2871	0.4773	−0.3933
2015	0.4960	0.4891	−12.3933	0.5476	−0.4224
2016	0.4922	0.4996	−3.1123	0.5324	−0.4589
2017	0.5008	0.5070	−1.3481	0.6046	−0.5068
2018	0.5434	0.5118	−0.7139	0.5913	−0.5738
2019	0.5723	0.5145	−0.4183	0.6656	−0.6777
2020	0.6029	0.5155	−0.2598	0.6545	−0.8737
2021	—	0.5152	−0.1673	0.7310	−1.6403
2022	—	0.5138	−0.1102	0.7222	−16.2417
2023	—	0.5117	−0.0736	0.8010	18.7774
2024	—	0.5091	−0.0497	0.7946	−2.9788
2025	—	0.5061	−0.0338	0.8760	−3.6027

Table 10. Comparison of APE (%) of Ghana's CO₂ emissions from four grey models.

Year	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.0000	0.0000	0.0000	0.0000
2007	15.0286	364.5265	0.0000	176.7517
2008	0.0189	196.0896	1.7156	187.3766
2009	4.6460	238.4466	6.6466	180.5297
2010	5.1626	325.0637	6.9084	177.7934
2011	0.0013	546.6922	7.8850	180.4626
2012	8.8997	1130.2720	11.1599	172.8563
2013	9.3289	7976.8576	2.0923	173.1672
2014	0.2659	15,359.5419	0.6866	182.9733
2015	1.3887	2598.5238	10.3984	185.1505
2016	1.5051	732.3804	8.1705	193.2351
2017	1.2349	369.1937	20.7255	201.1990
2018	5.8126	231.3730	8.8227	205.6047
2019	10.0919	173.1041	16.3115	218.4235
2020	14.4935	143.0996	8.5603	244.9274

Table 11. Error indicators (%) of four grey models for CO₂ emissions in Ghana.

	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
MAPEsim	4.9712	3192.8904	5.9366	179.6735
MAPEpre	6.6276	329.8302	12.5181	212.6779
CMRPE	5.7994	1761.3603	9.2274	196.1757
RMSPE	7.2655	4538.6122	9.3382	185.9279

Tables 9–11 show that the two models, the LGM(1,1) model and the CCRGM(1,1) model, have negative values in their predictions and are less effective, indicating that the two models do not accurately capture Ghana's carbon emissions and their errors are much larger than those of the FLGM(1,1) and WGM(1,1) models. The MAPEsim, MAPEpre, CMRPE, and RMSPE of the new model performed 4.9712%, 6.6276%, 5.7994%, and 7.2655%, while the MAPEsim, MAPEpre, CMRPE, and RMSPE of the WGM(1,1) model performed 5.9366%, 12.5158%, 9.2274%, and 9.3382%. The four error evaluation metrics of the new model are all below 8%, and the simulation of the WGM(1,1) model, although second only

to the new model, performs poorly in the prediction part with an error higher than 10%. It can be seen that the new model has the best performance in both modeling and prediction. Then, we plotted Figure 3 to compare these four models visually.

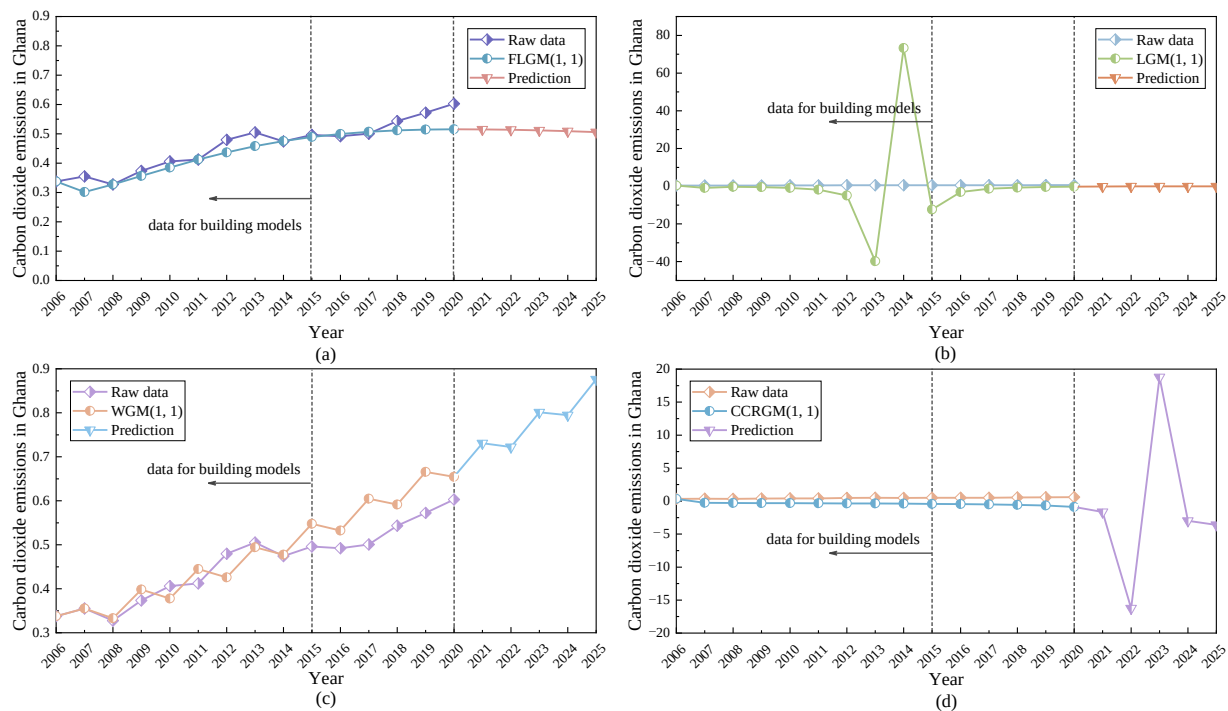


Figure 3. Analysis of CO₂ emissions in Ghana by four grey models. (a) by FLGM(1,1), (b) by LGM(1,1), (c) by WGM(1,1), (d) by CCRGM(1,1).

The modeling and prediction fitting curves for the FLGM (1,1) model, LGM (1,1) model, WGM (1,1) model, and CCRGM (1,1) model are plotted in Figure 3, respectively, from which it can be seen that the predicted trend of the LGM (1,1) model and the CCRGM (1,1) model is too far away from the trend of the raw data and that the LGM (1,1) model showed a steep downward and upward trend, and the CCRGM (1,1) model also shows the trend in the prediction stage, which is not consistent with the trend of CO₂ emissions in Ghana. The WGM (1,1) model shows a slightly fluctuating trend, but it is not consistent with the fluctuating pattern of the raw data, and it gradually deviates from the prediction part in the later stage, which leads to a gradual increase in the prediction error. The new model can fit the trend of CO₂ emissions in Ghana very well, indicating that the model has a good prediction performance. It can be seen that the new model has the best fitting performance among all the models and is suitable for simulating and predicting the trend of CO₂ emissions in Ghana.

4.5. Carbon Emissions in Belgium

This Section investigates CO₂ emissions in Belgium through different grey models. First, the optimal fractional order is derived using the PSO algorithm and the raw data for 2006–2015. The results are shown in Table 2. Four grey prediction models, FLGM(1,1), LGM(1,1), WGM(1,1), and CCRGM(1,1), were developed using the optimal value of r and the raw data of Belgium. The simulation results are presented in Tables 12–14.

Table 12. Prediction results of four grey models for carbon emissions in Belgium.

Year	Data	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	10.2467	10.2467	10.2467	10.2467	10.2467
2007	9.7787	9.9739	−10.2607	9.7787	8.8228
2008	9.9641	9.7109	−0.0069	10.0487	8.5950
2009	9.2534	9.4911	−0.0103	9.2466	8.3840
2010	9.8088	9.2948	−0.0154	9.5276	8.1878
2011	8.7412	9.1141	−0.0229	8.7365	8.0048
2012	8.5780	8.9452	−0.0342	9.0281	7.8335
2013	8.6519	8.7855	−0.0511	8.2473	7.6728
2014	8.0616	8.6336	−0.0764	8.5492	7.5216
2015	8.4348	8.4885	−0.1142	7.7784	7.3790
2016	8.3107	8.3493	−0.1708	8.0900	7.2442
2017	8.1394	8.2155	−0.2559	7.3288	7.1165
2018	8.1844	8.0865	−0.3840	7.6498	6.9953
2019	8.0938	7.9620	−0.5782	6.8978	6.8802
2020	7.3981	7.8416	−0.8743	7.2278	6.7705
2021	—	7.7250	−1.3311	6.4846	6.6659
2022	—	7.6121	−2.0476	6.8232	6.5660
2023	—	7.5025	−3.2003	6.0884	6.4705
2024	—	7.3962	−5.1285	6.4352	6.3790
2025	—	7.2929	−8.5568	5.7085	6.2913

Table 13. Comparison of APE (%) of Belgium's CO₂ emissions from four grey models.

Year	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.0000	0.0000	0.0000	0.0000
2007	1.9962	204.9287	0.0000	9.7755
2008	2.5411	100.0692	0.8484	13.7404
2009	2.5685	100.1112	0.0735	9.3956
2010	5.2406	100.1566	2.8666	16.5259
2011	4.2658	100.2623	0.0548	8.4249
2012	4.2798	100.3991	5.2469	8.6791
2013	1.5436	100.5909	4.6764	11.3169
2014	7.0965	100.9474	6.0491	6.6979
2015	0.6363	101.3534	7.7822	12.5179
2016	0.4652	102.0551	2.6546	12.8328
2017	0.9341	103.1434	9.9589	12.5678
2018	1.1960	104.6923	6.5310	14.5281
2019	1.6289	107.1432	14.7765	14.9946
2020	5.9938	111.8179	2.3024	8.4837

Table 14. Error indicators (%) of four grey models for CO₂ emissions in Belgium.

	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
MAPEsim	3.3520	112.0910	3.4497	10.7860
MAPEpre	2.0436	105.7704	7.2447	12.6814
CMRPE	2.6978	108.9307	5.3472	11.7337
RMSPE	3.4182	109.5741	5.9331	11.4082

As can be seen from Table 12, there are negative values in the predicted values of the LGM(1,1) model, and its error is much larger than that of the FLGM(1,1) model, the WGM(1,1) model, and the CCRGM(1,1) model. Therefore, the LGM(1,1) model has the lowest prediction accuracy. Following Tables 13 and 14, in terms of the prediction accuracy in the modeling part, the new model has a MAPEsim of 3.3520%, which indicates a higher prediction accuracy in the 10 years modeled. The WGM(1,1) model has a MAPEsim of 3.4497%, the second-highest simulation accuracy of the new model. However, the new model has a prediction error of only 2.0436% in the prediction part, and the WGM(1,1)

model has a prediction error of 7.2447%, significantly larger than the proposed new model. The CMRPE and RMSPE of the new model perform 2.6978% and 3.4182%, which are much smaller than the other three models. It can be seen that the new model shows higher accuracy in both simulation and prediction parts. Then, we plotted Figure 4 to compare these four models visually.

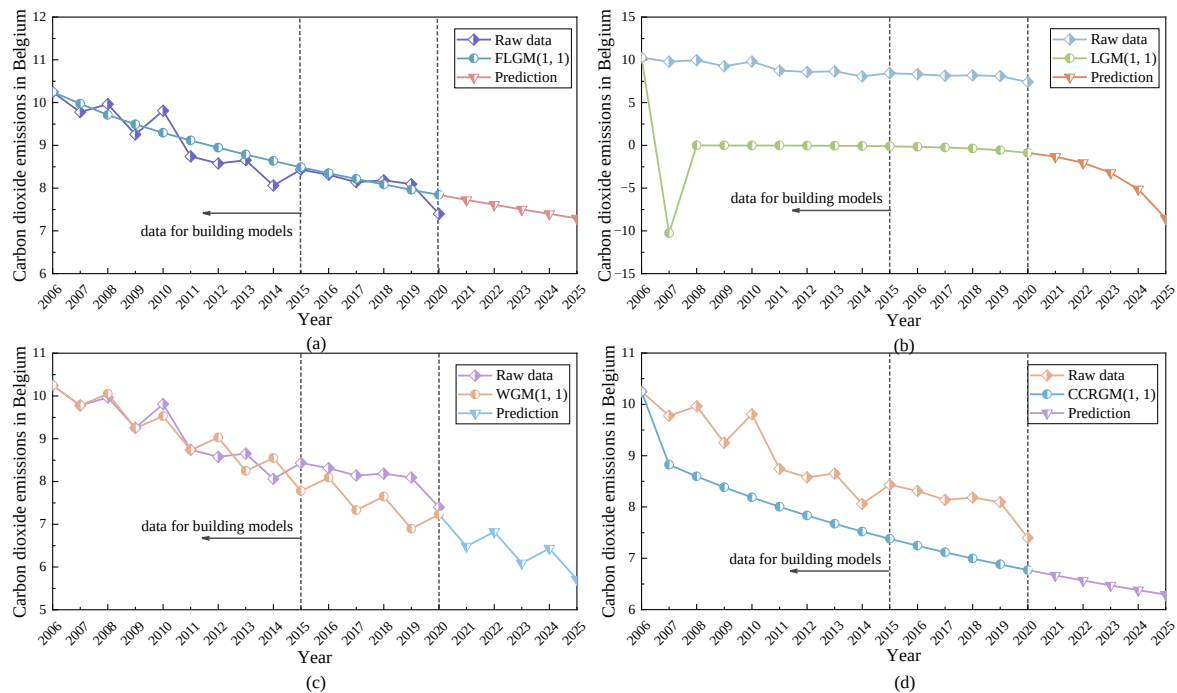


Figure 4. Analysis of CO₂ emissions in Belgium by four grey models. (a) by FLGM(1,1), (b) by LGM(1,1), (c) by WGM(1,1), (d) by CCRGM(1,1).

Figure 4 plots the degree of data fit of the FLGM(1,1) model, the LGM(1,1) model, the WGM(1,1) model, and the CCRGM(1,1) model to the raw CO₂ emissions. It can be seen that both the LGM(1,1) and CCRGM(1,1) models underestimate the trend of CO₂ emissions in Belgium to varying degrees, with the LGM(1,1) model having the most severe deviation from the trend. In contrast, the predicted values of the new model reflect the overall trend of CO₂ emissions. Although the WGM(1,1) model performs better in the simulation phase, it does not match the actual values of CO₂ emissions in the prediction part of the model. The direction of the trend of the model does not match the actual value, falling too fast and fluctuating too much. The new model almost matches the real-time CO₂ emissions in the prediction part, closer to the observed values. Therefore, the model has the best fitting performance among all models.

4.6. Carbon Emissions in Austria

Based on similar arguments in the previous sections, CO₂ emissions in Austria are also investigated through a grey prediction model. The PSO algorithm was utilized to search for MAPE_{total} and the corresponding optimal fractional order r for the raw data from 2006–2015, and the optimal values of the parameters found are shown in Table 2. According to the modeling mechanism of the grey prediction model, four grey prediction models, FLGM(1,1), LGM(1,1), WGM(1,1), and CCRGM(1,1), were established, respectively. The simulation results are presented in Tables 15–17.

Table 15. Prediction results of four grey models for carbon emissions in Austria.

Year	Data	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	8.9621	8.9621	8.9621	8.9621	8.9621
2007	8.5888	8.9766	−8.9779	8.5888	10.7352
2008	8.5002	8.7334	−0.0077	8.3624	10.6173
2009	7.7212	8.5236	−0.0114	8.2657	10.5032
2010	8.3656	8.3331	−0.0170	8.0442	10.3927
2011	8.1362	8.1563	−0.0253	7.9523	10.2857
2012	7.7239	7.9904	−0.0376	7.7354	10.1818
2013	7.7540	7.8335	−0.0560	7.6481	10.0811
2014	7.2609	7.6844	−0.0834	7.4358	9.9833
2015	7.3189	7.5423	−0.1242	7.3530	9.8883
2016	7.2908	7.4064	−0.1854	7.1451	9.7959
2017	7.4870	7.2762	−0.2770	7.0666	9.7061
2018	7.1411	7.1513	−0.4148	6.8630	9.6187
2019	7.2633	7.0312	−0.6236	6.7887	9.5336
2020	6.6326	6.9156	−0.9424	6.5893	9.4508
2021	—	6.8043	−1.4359	6.5191	9.3700
2022	—	6.6969	−2.2153	6.3237	9.2913
2023	—	6.5933	−3.4844	6.2575	9.2146
2024	—	6.4932	−5.6503	6.0660	9.1397
2025	—	6.3964	−9.6321	6.0036	9.0666

Table 16. Comparison of APE (%) of Austria's CO₂ emissions from four grey models.

Year.	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
2006	0.0000	0.0000	0.0000	0.0000
2007	4.5156	204.5304	0.0000	24.9912
2008	2.7433	100.0904	1.6212	24.9060
2009	10.3917	100.1480	7.0519	36.0305
2010	0.3890	100.2031	3.8420	24.2314
2011	0.2474	100.3107	2.2607	26.4187
2012	3.4506	100.4870	0.1500	31.8233
2013	1.0247	100.7220	1.3660	30.0111
2014	5.8321	101.1482	2.4087	37.4930
2015	3.0524	101.6976	0.4660	35.1064
2016	1.5856	102.5423	1.9980	34.3601
2017	2.8160	103.6991	5.6157	29.6384
2018	0.1418	105.8090	3.8947	34.6939
2019	3.1961	108.5850	6.5344	31.2567
2020	4.2666	114.2084	0.6537	42.4884

Table 17. Error indicators (%) of four grey models for CO₂ emissions in Austria.

	FLGM(1,1)	LGM(1,1)	WGM(1,1)	CCRGM(1,1)
MAPEsim	3.5163	112.1486	2.3958	30.1124
MAPEpre	2.4012	106.9687	3.7393	34.4875
CMRPE	2.9588	109.5587	3.0676	32.3000
RMSPE	3.9267	189.8143	3.4055	31.0104

By observing Tables 15–17, the LGM(1,1) model has the lowest prediction accuracy in both the simulation and prediction parts. The FLGM(1,1) model has higher accuracy in the simulation part with its MAPEsim and RMSPE values of 3.5163% and 3.9267%, second only to the WGM(1,1) model. However, its MAPEpre and CMRPE are 2.4012% and 2.9588%, respectively. In comparison, the MAPEpre and CMRPE of the WGM(1,1) model are 3.7393% and 3.0676%, respectively, indicating that the accuracy of the new model in the prediction stage is higher than that of the WGM(1,1) model. The errors of both simulation and prediction of the CCRGM(1,1) model are above 10%, much larger than the new model.

In summary, the FLGM(1,1) model proposed in this paper has high prediction accuracy and good applicability for predicting CO₂ emissions in Austria. The trend comparison of the four models is plotted in Figure 5.

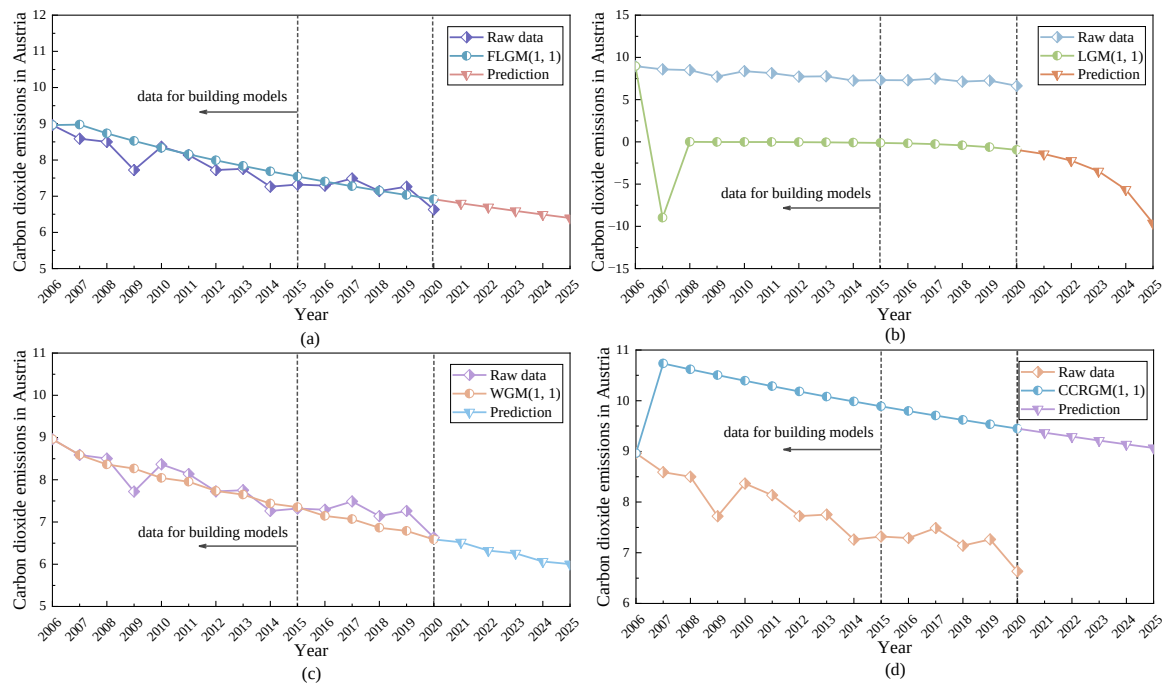


Figure 5. Analysis of CO₂ emissions in Austria by four grey models. (a) by FLGM(1,1), (b) by LGM(1,1), (c) by WGM(1,1), (d) by CCRGM(1,1).

Figure 5 shows the simulation and prediction fitting curves of the FLGM(1,1) model, LGM(1,1) model, WGM(1,1) model, and CCRGM(1,1) model. The whole-order LGM(1,1) model underestimates the trend of CO₂ emissions in Austria and is poorly simulated and predicted, whereas the fractional-order grey model successfully identifies the trend of CO₂ emissions. The CCRGM(1,1) model overestimates the trend of CO₂ emissions in Austria with a large bias. The FLGM(1,1) model and the WGM(1,1) model in the simulation phase both better fit the trend of CO₂ emissions in the region. However, in the prediction stage, the WGM(1,1) model does not perform as well as the new model, compared to the new model proposed in this paper, which has a higher prediction accuracy, higher flexibility, and better fitting accuracy.

5. Carbon Emission Predictions for the Next Five Years for Five Countries and Policy Recommendations

In Sections 4.2–4.6, we have conducted an in-depth study on the CO₂ emissions of five countries through four different grey models: FLGM(1,1), LGM(1,1), WGM(1,1), and CCRGM(1,1). After detailed analysis and comparison, we can see from the simulation results that the two whole-order grey models, LGM(1,1) and CCRGM(1,1), have low-performance indexes in the five cases, which means they are ineffective in predicting carbon emissions. They cannot provide decision-makers with an accurate and reliable basis. On the other hand, although the WGM(1,1) model performs best in predicting the fifth country, the newly developed fractional grey model FLGM(1,1) exhibits the best performance index when predicting CO₂ emissions in the first four countries. This demonstrates the significant advantage of fractional order accumulation in improving the accuracy of model prediction and verifies the applicability and reliability of the FLGM(1,1) model in making effective predictions of CO₂ emissions for the next five years for the five countries. Figure 6 shows the predicted values of CO₂ emissions for the next five years (2021–2025).

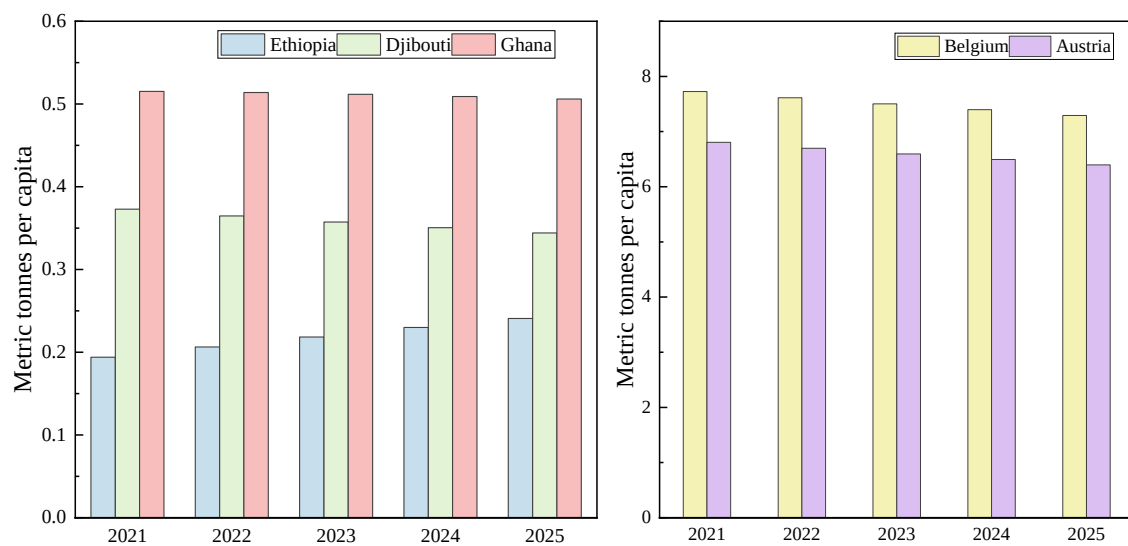


Figure 6. FLGM(1,1) model predictions of CO₂ emissions for 2021–2025 with five countries.

From Figure 6, it can be seen that Ethiopia's CO₂ emissions show an upward trend from 2021–2025. It is projected to increase from 0.1939 metric tons per capita in 2021 to 0.2409 metric tons per capita by 2025. This change is closely related to Ethiopia's rapid economic growth in recent years. As a fundamental country in the Central African region, Ethiopia has experienced strong economic growth and is regarded as one of the fastest-growing economies in the world. However, this rapid growth has also brought about a sharp increase in energy demand, which has led to a rise in carbon emissions. The country's transportation sector is the primary source of emissions from oil use, and other fossil energy sources are also proliferating under the development of infrastructure projects. At the same time, Ethiopia's high economic growth, accompanied by a massive demand for energy, also implies the potential for high emissions.

Ghana's CO₂ emissions have shown a steady downward trend. Although the annual decline has not been significant, overall, it has remained relatively low, with per capita emissions stabilizing at around 0.5 metric tons. This phenomenon is closely related to the industrial model on which the Ghanaian region mainly relies. Ghana's region mainly relies on industries such as agriculture, mining, and fisheries, with agriculture being one of the most important economic pillars of the Ghanaian region. While promoting economic development, the government also pays attention to environmental protection and social welfare, strives to realize the coordinated development of the economy, society, and the environment, and promotes the adoption of environmental protection measures by enterprises to maintain the stability of carbon emissions.

Djibouti's carbon emissions are gradually decreasing over the next five years. Emissions per capita are expected to stabilize at 0.3442 metric tons by 2025, a slight decrease of 0.0287 metric tons compared to 2021. Behind this trend is a reflection of Djibouti's efforts and commitment to sustainable development. It is well known that Djibouti is not rich in natural resources and has relatively little industrial activity. Despite this, its CO₂ emissions are still lower than those of industry-dominated countries. In conclusion, the Government of Djibouti should take comprehensive measures to promote appropriate economic instruments, balance economic and environmental benefits, and promote sustainable regional development.

CO₂ emissions in the Belgian region will gradually decrease by a small margin over the next five years, from 7.7250 metric tons per capita in 2021 to 7.2929 metric tons per capita in 2025, a decrease of about 0.4321 metric tons per capita. This shows that the Belgian government has achieved some success in its emission reduction efforts. However, Belgium still faces several challenges to achieve a more substantial reduction in emissions. The country is still highly dependent on fossil fuels, especially coal and natural gas, which

leads to high carbon emissions. In order to meet these challenges, Belgium will need to continue to strengthen the implementation of its emission reduction measures and increase its international cooperation and assistance to meet the challenges posed by climate change in the future.

Austria's carbon emissions show a decreasing trend from 2021 to 2025, from 6.8043 metric tons per capita in 2021 to 6.3964 metric tons per capita in 2025, a decrease of about 0.4079 metric tons per capita. Austria is a relatively industrially developed country, primarily known for its heavy steel and machine-building industries. In recent years, Austria has made many efforts to develop renewable energy and research new energy conservation and emission reduction technologies, especially in renewable energy, which has won the world's praise. As a result, the country has increased its efforts in recent years in the research, development, and application of renewable energy and new technologies for energy conservation and emission reduction, and it has further increased its support for renewable energy and clean energy.

6. Conclusions

This paper proposes a logistic grey prediction model with fractional order using the characteristics of the carbon emission system. The model is solved by the theory related to the fractional order operator. The specific modeling and calculation steps are given, and then the parameter estimation and time response equation of the new model FLGM(1,1) are studied. The model combines with the principle of particle swarm optimization algorithm to complete the optimization process of the order of the fractional-order grey prediction model, solves the optimal model order, and finally establishes the optimal fractional-order FLGM(1,1) model.

The newly developed FLGM(1,1) model to predict carbon dioxide emissions in five medium-emission countries: Ethiopia, Djibouti, Ghana, Belgium, and Austria. In the analysis of the effectiveness of the FLGM(1,1) model, its performance is compared with three different grey models, namely LGM(1,1), WGM(1,1), and CCRGM(1,1). Historical data on carbon dioxide emissions from the years 2006 to 2015 for these five countries were selected. The first 10 sets of data from 2006 to 2015 were used for modeling, and the data from 2016 to 2020 were employed for prediction. The results consistently indicate that the new FLGM(1,1) model exhibits superior performance. The detailed comparative results for the five countries are presented below:

Through the analysis of Ethiopia, the results indicate that the LGM(1,1), CCRGM(1,1), and WGM(1,1) models underestimated the carbon dioxide emission trend of the country, while the FLGM(1,1) model successfully captured this trend. Specifically, the WGM(1,1) model exhibited significant fluctuations, deviating from the actual trend, while the FLGM(1,1) model accurately reflected the gradual upward trend, performing optimally. In the analysis of Djibouti, it was observed that the LGM(1,1) and CCRGM(1,1) models had significant prediction gaps, the WGM(1,1) model performed poorly in later stages, whereas the FLGM(1,1) model successfully simulated the actual trend, demonstrating excellent performance. The study on Ghana revealed that the LGM(1,1) and CCRGM(1,1) models were unable to accurately predict the trend, and the WGM(1,1) model exhibited significant fluctuations. In contrast, the FLGM(1,1) model successfully fitted Ghana's CO₂ emission trend, achieving the best results. The analysis for Belgium showed that the LGM(1,1) and CCRGM(1,1) models underestimated the emission trend, the WGM(1,1) model had inaccuracies in later predictions, while the FLGM(1,1) model accurately reflected the overall trend, possessing the best fitting performance. Finally, the study on Austria demonstrated that the LGM(1,1) model had poor simulation and prediction performance, the CCRGM(1,1) model overestimated the emission trend, the WGM(1,1) model had inaccuracies in the prediction phase, while the FLGM(1,1) model exhibited excellent performance in both simulation and prediction stages.

Through detailed analysis and comparison, it is evident from the simulation results that both the LGM(1,1) and CCRGM(1,1) whole-order grey models performed poorly in

all five cases. This implies that these models are not ideal for predicting carbon dioxide emissions and cannot provide decision-makers with accurate and reliable guidance.

On the other hand, although the WGM(1,1) model performed optimally in predicting emissions for the fifth country, the new FLGM(1,1) demonstrated the best performance metrics when forecasting carbon dioxide emissions for the first four countries. This not only underscores the significant advantages of fractional-order accumulation in enhancing model prediction accuracy but also validates the applicability and reliability of the LFGM(1,1) model. Consequently, we employ the proposed FLGM(1,1) model in this study to effectively forecast the carbon dioxide emissions for the next five years (2021–2025) in these five countries, providing policy recommendations based on the predictions. For Ethiopia, the forecast for the period 2021–2025 indicates an upward trend in carbon dioxide emissions, rising from 0.1939 metric tons per capita to 0.2409 metric tons per capita. This is associated with its rapid economic growth but also signifies a sharp increase in energy demand and emissions from the transportation sector. Ghana's emissions show a steady declining trend, maintaining at relatively low levels. This is primarily attributed to an agricultural-dominant industrial model, along with the government's emphasis on environmental conservation and social welfare. Djibouti's emissions are expected to gradually decrease, reflecting its efforts in sustainable development. Belgium is projected to experience a slight reduction in emissions, indicating governmental success in emission reduction efforts. However, the country still faces challenges related to fossil fuel dependence and high carbon emissions. Austria has achieved a reduction in emissions in recent years, acknowledging its efforts in developing renewable energy and researching new technologies. Nevertheless, there is a need to continue strengthening emission reduction measures, especially in reducing dependence on fossil fuels.

Carbon emissions are a complex system affected by various factors, so the prediction of carbon emissions also needs to consider various factors. This paper, from the optimization of the model structure, the establishment of a fractional-order Logistic grey prediction model, the model combined with the principle of particle swarm optimization algorithm, completed the optimization process of the order of fractional-order grey prediction model, and ultimately the establishment of the optimal fractional-order FLGM(1,1) model, and also proved the effectiveness and stability of the model in the actual case of carbon emissions, but there are still some problems that can be improved. On the one hand, considering the complexity of the carbon emission system itself, there is room for further optimization in the screening and cleaning of data, selecting relevant factors, etc. On the other hand, the model has a new structure, and it is also necessary to think about it from the perspective of the modeling mechanism and optimize and improve it in the calculation of the background value and the estimation of parameters. Moreover, carbon emissions are intricately linked to carbon trading prices, internal relationships within the energy structure, and economic growth. Therefore, predicting carbon emissions in such a scenario becomes more complex and challenging. How to establish grey prediction models that incorporate multiple factors and apply them to carbon emission forecasting in this context will be a further area of research. Consequently, in future work, due consideration will be given to the current challenges and limitations, paving the way for more in-depth research.

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