Some Properties of Normalized Tails of Maclaurin Power Series Expansions of Sine and Cosine

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1. Motivations and Preliminaries

It is well known ([1], p. 649) that

\[ \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots, \quad x \in \mathbb{R} \]  

(1)

and

\[ \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \cdots, \quad x \in \mathbb{R}. \]  

(2)

For our own convenience, we denote the tails, or say, the remainders, of the power series expansions (1) and (2) by

\[ \text{SR}_n(x) = \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = (-1)^n x^{2n+1} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2n + 2k + 1)!} \]  

(3)
and
\[ CR_n(x) = \cos x - \sum_{k=0}^{n-1} \frac{(-1)^k x^{2k}}{(2k)!} = (-1)^n x^{2n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2n+2k)!} x^{2k} \] (4)

for \( n \geq 1 \) and \( x \in \mathbb{R} \), respectively. By virtue of (Theorem 7.6 of [2]) or in view of the results at the site https://math.stackexchange.com/a/477549 (accessed on 8 March 2024), we acquire the integral representations

\[ SR_n(x) = \frac{(-1)^n}{(2n-1)!} \int_0^x (x-t)^{2n-1} \sin t \, dt \] (5)
and
\[ CR_n(x) = \frac{(-1)^n}{(2n-2)!} \int_0^x (x-t)^{2n-2} \sin t \, dt \] (6)

for \( n \geq 1 \) and \( x \in \mathbb{R} \).

It is known (Chapter XIII of [3]) that

\[ \Omega_{\mu}(f(x); y) = \frac{1}{\Gamma(\mu)} \int_0^y f(x)(y-x)^{\mu-1} \, dx \] (7)

is called the Riemann–Liouville fractional integral of order \( \mu \), where the classical Euler gamma function \( \Gamma(\mu) \) is defined (Chapter 3 of [4]) by

\[ \Gamma(\mu) = \lim_{n \to \infty} \frac{n! n^\mu}{\prod_{k=0}^{n} (\mu + k)} , \quad \mu \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} , \]

whose reciprocal \( \frac{1}{\Gamma(\mu)} \) is an entire function on the complex plane \( \mathbb{C} \); see also (Section 1 of [5]).

In (Theorem 1.1 and Corollary 1.5 of [6]), Koumandos proved that the function

\[ F_\lambda(x) = \Gamma(\lambda + 1) \Omega_{\lambda+1}(\sin t; x) = \int_0^x (x-t)^{\lambda} \sin t \, dt \] (8)

is logarithmically concave in \( x \in (0, \infty) \) if and only if \( \lambda \geq 2 \) and that for all \( \mu \geq 1 \) the inequality

\[ \left(1 + \frac{1}{\mu}\right) F_\mu(x) - F_{\mu-1}(x) F_{\mu+1}(x) \geq 0 \] (9)

is valid for all \( x > 0 \) and that the equality in (9) occurs only when \( \mu = 1 \) and \( \tan \frac{x}{2} = \frac{x}{2} \).

It is well known ([7], pp. 322 and 326) that, for \( n \geq 1 \) and \( x \in \mathbb{R} \),

\[ |SR_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!} , \quad |CR_n(x)| \leq \frac{|x|^{2n}}{(2n)!} \] (10)

and

\[ (-1)^n x SR_n(x) \geq 0 , \quad (-1)^n CR_n(x) \geq 0 . \] (11)

In (Corollaries 1.3 and 1.4 of [6]), Koumandos gave that the functions \((-1)^{n+1} SR_{n+1}(x)\) and \((-1)^{n+1} CR_{n+1}(x)\) for \( n \in \mathbb{N} \) are positive, increasing, logarithmically concave, and convex on \((0, \infty)\) and that the ratios

\[ \frac{CR_{n+1}(x)}{SR_{n+1}(x)} , \quad \frac{CR_n(x)}{CR_{n+1}(x)} , \quad \frac{SR_n(x)}{SR_{n+1}(x)} \]

for \( n \in \mathbb{N} \) are decreasing on \((0, \infty)\).

In the papers [8–10], Qi and his coauthors considered the following functions and problems:

1. What are the Maclaurin power series expansions of the logarithmic functions
Fractal Fract. 2024, 8, 257

\[ F(x) = \begin{cases} \ln \frac{2(1 - \cos x)}{x^2}, & 0 < |x| < 2\pi \\ 0, & x = 0 \end{cases} \quad (12) \]

and

\[ Q(x) = \begin{cases} \ln \frac{6(x - \sin x)}{x^3}, & 0 < |x| < \infty \\ 0, & x = 0 \end{cases} \quad (13) \]

around \( x = 0 \)?

2. Are the ratios

\[ R(x) = \begin{cases} \ln \frac{2(1 - \cos x)}{\ln \cos x}, & 0 < |x| < \frac{\pi}{2} \\ \frac{1}{6}, & x = 0 \\ 0, & x = \pm \frac{\pi}{2} \end{cases} \quad (14) \]

and

\[ T(x) = \begin{cases} \ln \frac{6(x - \sin x)}{\ln \sin x}, & 0 < |x| < \pi \\ \frac{3}{10}, & x = 0 \\ 0, & x = \pm \pi \end{cases} \quad (15) \]

decreasing on the close intervals \([0, \frac{\pi}{2}]\) and \([0, \pi]\) respectively?

Stimulated by main results in the papers \([6,8–10]\) and motivated by the four functions in (12)–(15), we now introduce two new functions \( \text{SinR}_n(x) \) and \( \text{CosR}_n(x) \) by

\[ \text{SinR}_n(x) = \begin{cases} (-1)^n \frac{(2n + 1)!}{x^{2n+1}} \left[ \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right], & x \neq 0 \\ 1, & x = 0 \end{cases} \]

and

\[ \text{CosR}_n(x) = \begin{cases} (-1)^n \frac{(2n)!}{x^{2n}} \left[ \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} \right], & x \neq 0 \\ 1, & x = 0 \end{cases} \]

for \( n \geq 1 \) and \( x \in \mathbb{R} \). We call the quantities \( \text{SinR}_n(x) \) and \( \text{CosR}_n(x) \) for \( n \geq 1 \) the \( n \)th normalized tails or the normalized remainders of the Maclaurin power series expansions (1) and (2).

Making use of the series expansions (3) and (4), we derive

\[ \text{SinR}_n(x) = \begin{cases} (-1)^n \frac{(2n + 1)!}{x^{2n+1}} \text{SR}_n(x), & x \neq 0 \\ 1, & x = 0 \end{cases} \]

\[ = (2n + 1)! \sum_{k=0}^{\infty} \frac{(-1)^k}{(2n + 2k + 1)!} x^{2k} \quad (18) \]

and

\[ \text{CosR}_n(x) = \begin{cases} (-1)^n \frac{(2n)!}{x^{2n}} \text{CR}_n(x), & x \neq 0 \\ 1, & x = 0 \end{cases} \]

\[ = (2n)! \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2n + 2k)!} \quad (19) \]
for \( n \geq 1 \) and \( x \in \mathbb{R} \). Utilizing the integral representations (5) and (6), we acquire

\[
\text{SinR}_n(x) = \begin{cases} 
2n(2n+1) \int_0^1 (1-u)^{2n-1} \frac{\sin(xu)}{x} \, du, & x \neq 0 \\
1, & x = 0
\end{cases}
\]

(20)

and

\[
\text{CosR}_n(x) = \begin{cases} 
2n(2n-1) \int_0^1 (1-u)^{2n-2} \frac{\sin(xu)}{x} \, du, & x \neq 0 \\
1, & x = 0
\end{cases}
\]

(21)

for \( n \geq 1 \) and \( x \in \mathbb{R} \).

It is obvious that both of the normalized tails \( \text{SinR}_n(x) \) and \( \text{CosR}_n(x) \) for \( n \geq 1 \) are even functions in \( x \in (-\infty, \infty) \).

Combining two inequalities in (11) with the first equalities in (18) and (19) reveals that both of the normalized tails \( \text{SinR}_n(x) \) and \( \text{CosR}_n(x) \) for \( n \geq 1 \) are nonnegative in \( x \in (0, \infty) \).

In this paper, among other things, we will mainly prove that the normalized tail \( \text{SinR}_n(x) \) for \( n \geq 1 \) and the normalized tail \( \text{CosR}_n(x) \) for \( n \geq 2 \) are decreasing in \( x \in (0, \infty) \) and concave in \( x \in (0, \pi) \).

2. Decreasing Property and Concavity of \( \text{SinR}_n(x) \)

In this section, we will prove the decreasing property and concavity of the even function \( \text{SinR}_n(x) \) for \( n \geq 1 \).

**Theorem 1.** For \( n \geq 1 \), the normalized remainder \( \text{SinR}_n(x) \) is decreasing and positive in \( x \in (0, \infty) \), while it is concave in \( x \in (0, \pi) \).

**Proof.** Employing the first expression in (18) and directly differentiating yield

\[
\text{SinR}'_n(x) = \frac{x \text{SR}'_n(x) - (2n+1) \text{SR}_n(x)}{x^{2n+2}} = (-1)^n \frac{(2n+1)!}{x^{2n+2}} S_n(x)
\]

for \( n \geq 1 \) and \( x \neq 0 \), where

\[
S_n(x) = x \cos x - (2n+1) \sin x + \sum_{k=0}^{n-1} (-1)^k \frac{2(n-k)}{(2k+1)!} x^{2k+1}
\]

for \( n \geq 1 \) and \( x \in \mathbb{R} \). A straightforward differentiation gives

\[
S^{(j)}_n(x) = x \cos \left( x + \frac{j\pi}{2} \right) - (2n-j+1) \sin \left( x + \frac{j\pi}{2} \right) + \sum_{k=0}^{n-1} (-1)^k \frac{2(n-k)}{(2k+1)!} (2k+1)_j x^{2k-j+1}
\]

for \( j \geq 0 \), where

\[
\langle z \rangle_n = \frac{\Gamma(z+1)}{\Gamma(z-n+1)} = \prod_{k=0}^{n-1} (z-k) = \begin{cases} 
z(z-1) \cdots (z-n+1), & n \in \mathbb{N} \\
1, & n = 0
\end{cases}
\]
is known ([11], p. 6) as the $n$th falling factorial of $z \in \mathbb{C}$. In particular,

$$S_n^{(2n-2)}(x) = (-1)^{n+1}(x \cos x + 2x - 3\sin x) = (-1)^{n+1}3x \left( \frac{2 + \cos x}{3} - \frac{\sin x}{x} \right) \tag{22}$$

for $n \geq 1$. Since the inequality

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad x \in (0, \infty) \supset (0, \pi) \tag{23}$$

is valid, see ([12], Lemma 4), we acquire

$$(-1)^{n+1}S_n^{(2n-2)}(x) = 3x \left( \frac{2 + \cos x}{3} - \frac{\sin x}{x} \right) > 0$$

for $x > 0$ and $n \geq 1$. It is not difficult to see that $S_n^{(j)}(0) = 0$ for $0 \leq j \leq 2n - 2$. Therefore, it follows that $(-1)^{n+1}S_n^{(j)}(x) > 0$ for $0 \leq j \leq 2n - 2$ and $x > 0$. Hence, the derivative $\sin R_n'(x) < 0$ for $n \geq 1$ on $(0, \infty)$. Accordingly, the even function $\sin R_n(x)$ for $n \geq 1$ is decreasing on $(0, \infty)$.

From the first expression in (20), we deduce the limit

$$\lim_{x \to \infty} \sin R_n(x) = 0, \quad n \geq 1. \tag{24}$$

Combining this with the decreasing property of $\sin R_n(x)$ for $n \geq 1$ on $(0, \infty)$, we conclude that the function $\sin R_n(x)$ for $n \geq 1$ is positive on $(0, \infty)$.

Straightforward computation gives

$$\sin R_n'(x) = (-1)^n(2n+1)\sin R_n'(x) - (2n+1) \sin R_n'(x)$$

$$\sin R_n'(x) = (-1)^n(2n+1) \frac{\sin R_n(x)}{x^{2n+2}},$$

$$\sin R_n'(x) = (-1)^n(2n+1) \frac{\Phi_n(x)}{x^{2n+3}},$$

where

$$\Phi_n(x) = xS_n'(x) - (2n+2)S_n(x). \tag{25}$$

Then

$$\Phi_n^{(k)}(x) = xS_n^{(k+1)}(x) - (2n+2-k)S_n^{(k)}(x)$$

for $k \in \mathbb{N}$. Further making use of the equalities in (22), we have

$$(-1)^{n+1}\Phi_n^{(2n-2)}(x) = (-1)^{n+1}\left[xS_n^{(2n-1)}(x) - 4S_n^{(2n-2)}(x)\right]$$

$$= x(x \cos x + 2x - 3\sin x)' - 4(x \cos x + 2x - 3\sin x)$$

$$= 12\sin x - x^2 \sin x - 6x - 6x \cos x.$$

A direct computation yields

$$\frac{(-1)^{n+1}\Phi_n^{(2n-2)}(x)}{\sin x} = 12 - x^2 - 6x \frac{1 + \cos x}{\sin x}$$

and

$$\left[\frac{(-1)^{n+1}\Phi_n^{(2n-2)}(x)}{\sin x}\right]' = 6\frac{x(\cos x + 1)}{\sin^2 x} \left( \frac{2 + \cos x}{3} - \frac{\sin x}{x} \right) > 0$$

for $x > 0$. Accordingly, we obtain
which implies that
\[ (-1)^{n+1} \frac{\Phi_n^{(2n-2)}(x)}{\sin x} > 0, \quad x \in (0, \pi) \]
and
\[ (-1)^{n+1} \frac{\Phi_n^{(2n-2)}(x)}{\sin x} < 0, \quad x \in (\pi, 2\pi). \]
Since \( \Phi_n^{(k)}(0) = 0 \) for \( 0 \leq k \leq 2n - 2 \), it follows that \( (-1)^{n+1} \Phi_n^{(k)}(x) > 0 \) for \( x \in (0, \pi) \) for \( 0 \leq k \leq 2n - 2 \). This implies that \( \text{SinR}_n''(x) < 0 \) for \( x \in (0, \pi) \) and \( n \geq 1 \). The first proof of Theorem 1 is thus complete.

**Second proof of concavity in Theorem 1.** Using the second integral representation in (20) and differentiating twice yield

\[
\text{SinR}_n''(x) = -(2n + 1) \int_0^1 u^2 (1 - u)^{2n} \cos(ux) \, du.
\]

Differentiation again gives

\[
\text{SinR}_n'''(x) = (2n + 1) \int_0^1 u^3 (1 - u)^{2n} \sin(ux) \, du > 0
\]
for \( x \in (0, \pi) \). Then

\[
\text{SinR}_n''(x) < \text{SinR}_n''(\pi) = -(2n + 1) \int_0^1 u^2 (1 - u)^{2n} \cos(\pi u) \, du, \quad x \in (0, \pi).
\]

Let

\[
P_1(n) = \int_0^{1/2} u^2 (1 - u)^{2n} \cos(\pi u) \, du
\]
and

\[
P_2(n) = \int_{1/2}^1 u^2 (1 - u)^{2n} \cos(\pi u) \, du.
\]

Making a change of variables \( u = 1 - v \) gives

\[
P_2(n) = -\int_0^{1/2} (1 - v)^2 v^{2n} \cos(\pi v) \, dv.
\]

Accordingly, since

\[
(1 - t)^\alpha - t^\alpha \begin{cases} 0, & \alpha = 0 \\ > 0, & \alpha > 0 \end{cases}
\]
for \( t \in (0, \frac{1}{2}) \), we derive

\[
P_1(n) + P_2(n) = \int_0^{1/2} \left[ (t^2(1 - t)^{2n} - (1 - t)^2 t^{2n}) \cos(\pi t) \right] \, dt
\]
\[
= \int_0^{1/2} \left[ (1 - t)^{2n-2} - t^{2n-2} \right] t^2 (1 - t)^2 \cos(\pi t) \, dt
\]
\[
\begin{cases} = 0, & n = 1; \\ > 0, & n \geq 2. \end{cases}
\]

This implies that \( \text{SinR}_n'''(x) < 0 \) for \( x \in (0, \pi) \) and \( n \in \mathbb{N} \). The normalized remainder \( \text{SinR}_n(x) \) for \( n \in \mathbb{N} \) is thus concave on \( (0, \pi) \). \( \square \)
Remark 1. From the decreasing property in Theorem 1, the limit (24), and the definition (16) of the function \( \text{SinR}_n(x) \) for \( n \geq 1 \), we immediately deduce the inequality
\[
0 < (-1)^n \frac{(2n+1)!}{x^{2n}} \left( \frac{\sin x}{x} - \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!} x^{2k} \right) < 1
\]
for \( n \geq 1 \) and \( x \in (0, \infty) \). Consequently, the first inequalities in (10) and (11) are recovered and the double inequality
\[
0 < (-1)^n \frac{\sin x}{x} - \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)!} x^{2k} < \frac{x^{2n}}{(2n+1)!}
\]
(26) for \( n \geq 1 \) and \( x \in \mathbb{R} \setminus \{0\} \) of the sinc function \( \frac{\sin x}{x} \) is deduced. For more information on the sinc function, please refer to closely related texts and references in the paper [13].

Remark 2. The integral representation (5) can be reformulated as
\[
\text{SR}_n(x) = (-1)^n \frac{x^{2n}}{(2n-1)!} \int_0^1 (1-t)^{2n-1} \sin(x t) \, dt
\]
(27) for \( n \geq 1 \) and \( x \in \mathbb{R} \). Accordingly, substituting (27) into the first equality (18) leads to
\[
\text{SinR}_n(x) = \begin{cases} 
(2n+1)! \int_0^1 (1-t)^{2n-1} \frac{\sin(x t)}{x t} \, dt, & x \neq 0 \\
1, & x = 0 
\end{cases}
\]
and directly differentiating shows
\[
\text{SinR}_n''(x) = \frac{(2n+1)!}{(2n-1)!} \int_0^1 (1-t)^{2n-1} \frac{d^2}{dx^2} \left[ \frac{\sin(x t)}{x t} \right] \, dt
\]
for \( n \geq 1 \) and \( x > 0 \). Straightforward differentiation and computation give
\[
\left( \frac{\sin u}{u} \right)'' = -\frac{(u^2 - 2) \sin u + 2u \cos u}{u^3}
\]
(28) and
\[
\left( u^2 \cos u \right)' = u^2 \cos u.
\]
The function \( u^2 \cos u \) is positive on \( (0, \frac{\pi}{2}) \) and negative on \( (\frac{\pi}{2}, \pi) \). Since
\[
\lim_{u \to \pi/2} \left( (u^2 - 2) \sin u + 2u \cos u \right) = \frac{\pi^2}{4} - 2 = 0.467 \ldots
\]
and
\[
\lim_{u \to \pi} \left( (u^2 - 2) \sin u + 2u \cos u \right) = -2\pi,
\]
there exists a unique point \( u_0 \in (\frac{\pi}{2}, \pi) \) such that
\[
(u^2 - 2) \sin u + 2u \cos u > 0, \quad u \in (0, u_0).
\]
Hence, the second derivative \( \text{SinR}_n''(x) \) for \( n \geq 1 \) is negative, and then the normalized remainder \( \text{SinR}_n(x) \) is concave, on the interval \( (0, x_0) \), where
\[
x_0 = \inf \left\{ \frac{u_0}{t}, t \in (0, 1) \right\} = u_0.
\]
Remark 3. Direct computation gives

\[-1^{n+1}\Phi_n^{(2n-3)}(x) = x^2 \cos x - 20 \cos x - 8x \sin x - 3x^2 + 20,\]
\[-1^{n+1}\Phi_n^{(2n-3)}(\pi) = 4(10 - \pi^2)\]
\[> 0,\]
\[-1^{n+1}\Phi_n^{(2n-3)}(2\pi) = -8\pi^2,\]
\[-1^{n+1}\Phi_n^{(2n-4)}(x) = 20x - 30 \sin x + x^2 \sin x + 10x \cos x - x^3,\]
\[-1^{n+1}\Phi_n^{(2n-4)}\left(\frac{4\pi}{3}\right) = 20\pi - \frac{8}{9}\sqrt{3}\pi^2 - \frac{64}{27}\pi^3 + 15\sqrt{3}\]
\[= 0.121\ldots,\]
\[-1^{n+1}\Phi_n^{(2n-4)}(2\pi) = -4\pi(2\pi^2 - 15)\]
\[< 0,\]
\[-1^{n+1}\Phi_n^{(2n-5)}(x) = 42 \cos x - x^2 \cos x + 12x \sin x + 10x^2 - \frac{1}{4}x^4 - 42,\]
\[-1^{n+1}\Phi_n^{(2n-5)}\left(\frac{3\pi}{2}\right) = \frac{45}{2}\pi^2 - 18\pi - \frac{81}{64}\pi^4 - 42\]
\[= 0.234\ldots,\]
\[-1^{n+1}\Phi_n^{(2n-5)}(2\pi) = -4\pi^2(\pi^2 - 9)\]
\[< 0,\]
\[-1^{n+1}\Phi_n^{(2n-6)}(x) = 56 \sin x - 42x - x^2 \sin x - 14x \cos x + \frac{10}{3}x^3 - \frac{1}{20}x^5,\]
\[-1^{n+1}\Phi_n^{(2n-6)}\left(\frac{5\pi}{3}\right) = \frac{25}{18}\sqrt{3}\pi^2 - \frac{245}{3}\pi + \frac{1250}{81}\pi^3 - \frac{625}{972}\pi^5 - 28\sqrt{3}\]
\[= 0.401\ldots,\]
\[-1^{n+1}\Phi_n^{(2n-6)}(2\pi) = -4\pi^2(\pi^2 - 9)\]
\[< 0,\]
\[-1^{n+1}\Phi_n^{(2n-7)}(x) = x^2 \cos x - 72 \cos x - 16x \sin x - 21x^2\]
\[+ \frac{5}{6}x^4 - \frac{1}{120}x^6 + 72,\]
\[-1^{n+1}\Phi_n^{(2n-7)}\left(\frac{7\pi}{4}\right) = 14\sqrt{2}\pi + \frac{49}{32}\sqrt{2}\pi^2 - \frac{1029}{16}\pi^3 + \frac{12005}{1536}\pi^4\]
\[+ \frac{117649}{491520}\pi^6 - 36\sqrt{2} + 72\]
\[= 1.132\ldots,\]
\[-1^{n+1}\Phi_n^{(2n-7)}(2\pi) = \frac{40}{3}\pi^4 - 80\pi^2 - \frac{8}{15}\pi^6\]
\[= -3.521\ldots,\]
\[-1^{n+1}\Phi_n^{(2n-8)}(x) = 72x - 90 \sin x + x^2 \sin x + 18x \cos x - 7x^3 + 3x^5 - \frac{x^7}{840},\]
\[-1^{n+1}\Phi_n^{(2n-8)}(2\pi) = 180\pi - 56\pi^3 + \frac{16}{3}\pi^5 - \frac{16}{105}\pi^7\]
\[= 1.005\ldots.\]

Consequently, we conclude that
1. when \(n \geq 2\), the second derivative \(\text{SinR}_n''(x)\) is negative on \((0, \frac{4\pi}{3})\);
2. when \(n \geq 3\), the second derivative \(\text{SinR}_n''(x)\) is negative on \((0, \frac{2\pi}{3})\);
3. when \(n \geq 4\), the second derivative \(\text{SinR}_n''(x)\) is negative on \((0, 2\pi)\).
In other words,
1. when \( n \geq 2 \), the normalized remainder \( \text{SinR}_n(x) \) is concave on \( (0, \frac{4\pi}{3}) \);
2. when \( n \geq 3 \), the normalized remainder \( \text{SinR}_n(x) \) is concave on \( (0, \frac{3\pi}{2}) \);
3. when \( n \geq 4 \), the normalized remainder \( \text{SinR}_n(x) \) is concave on \( (0, 2\pi) \).

**Remark 4.** Using the second integral representation in (20) and differentiating under integration \( k \geq 0 \) times consecutively result in

\[
\text{SinR}_n^{(k)}(x) = (2n+1) \int_0^1 (1-u)^{2n} u^k \cos\left( xu + \frac{k\pi}{2} \right) \, du
\]

\[
= \begin{cases} 
(-1)^{m-1}(2n+1) \int_0^1 (1-u)^{2n} u^{2m-2} \cos(xu) \, du, & k = 2m - 2 \\
(-1)^m(2n+1) \int_0^1 (1-u)^{2n} u^{2m-1} \sin(xu) \, du, & k = 2m - 1
\end{cases}
\]

for \( m, n \geq 1 \) and \( x \in \mathbb{R} \). This means that

\[
(-1)^m \text{SinR}_n^{(2m)}(x) = (2n+1) \int_0^1 (1-u)^{2n} u^{2m} \cos(xu) \, du > 0
\]

for \( m \geq 0 \) and \( x \in (0, \frac{\pi}{2}) \) and that

\[
(-1)^{m+1} \text{SinR}_n^{(2m+1)}(x) = (2n+1) \int_0^1 (1-u)^{2n} u^{2m+1} \sin(xu) \, du > 0
\]

for \( m \geq 0 \) and \( x \in (0, \pi) \). These imply that the functions

\[
(-1)^m \text{SinR}_n^{(2m-1)}(x) = (2n+1) \int_0^1 (1-u)^{2n} u^{2m-1} \sin(xu) \, du
\]

for \( m, n \geq 1 \) are increasing in \( x \in (0, \frac{\pi}{2}) \) and that the functions

\[
(-1)^m \text{SinR}_n^{(2m)}(x) = (2n+1) \int_0^1 (1-u)^{2n} u^{2m} \cos(xu) \, du
\]

for \( m \geq 0 \) and \( n \geq 1 \) are decreasing in \( x \in (0, \pi) \). Consequently, we obtain

1. for \( m, n \geq 1 \) and \( x \in (0, \frac{\pi}{2}) \)

\[
0 < (-1)^m \text{SinR}_n^{(2m-1)}(x) < (2n+1) \int_0^1 (1-u)^{2n} u^{2m-1} \sin\left( \frac{u\pi}{2} \right) \, du;
\]

2. for \( m \geq 0, n \geq 1, \) and \( x \in (0, \pi) \)

\[
(2n+1) \int_0^1 (1-u)^{2n} u^{2m} \cos(u\pi) \, du < (-1)^m \text{SinR}_n^{(2m)}(x)
\]

\[
< (2n+1)B(2m+1,2n+1),
\]

where the classical Euler beta function \( B(p, q) \) can be defined ([14], p. 258) by

\[
B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1} \, dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}
\]

for \( \Re(p), \Re(q) > 0 \).
Remark 5. The function $F_\lambda(x)$ in (8) can be reformulated as

$$F_\lambda(x) = x^{\lambda+1} \int_0^1 (1-u)^\lambda \sin(xu) \, du.$$  

Hence, we deduce the relation

$$\left[ F_\lambda(x) \right]^{(k)} = \int_0^1 (1-u)^\lambda u^k \sin\left(xu + \frac{k\pi}{2}\right) \, du$$

for $\lambda \in \mathbb{R}$ and $k \geq 0$. This reveals that the function $\left[ F_\lambda(x) \right]^{(k)}$ is different from the function

$$\frac{\text{SinR}_n^{(k)}(x)}{2n+1} = \int_0^1 (1-u)^{2n} u^k \cos\left(xu + \frac{k\pi}{2}\right) \, du$$

for $k \geq 0$ and $n \geq 1$, but they are siblings and special cases of the Riemann–Liouville (fractional) integral defined in (7).

Remark 6. How about the convexity or concavity of the remainder $\text{SinR}_n(x)$ for $n \geq 1$ on the whole infinite interval $(0, \infty)$?

3. Decreasing Property and Concavity of $\text{CosR}_n(x)$

In this section, we prove the decreasing property and concavity of the even function $\text{CosR}_n(x)$ for $n \geq 1$ on $(0, \infty)$.

Theorem 2. The normalized remainder $\text{CosR}_1(x)$ is nonnegative on $(0, \infty)$, is decreasing on $[0, 2\pi]$, and is concave on $(0, x_0)$, where $x_0 \in (\frac{\pi}{2}, \pi)$ is the first positive zero of the equation

$$(x^2 - 2) \sin x + 2x \cos x = 0.$$  

(29)

For $n \geq 2$, the normalized remainder $\text{CosR}_n(x)$ is positive and decreasing on $(0, \infty)$ and is concave on $(0, \pi)$.

Proof. It is immediate that the even function

$$\text{CosR}_1(x) = \begin{cases} \frac{2(1-\cos x)}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is positive on the set

$$\bigcup_{k=0}^{\infty} (2k\pi, 2k\pi + 2\pi)$$

and equals zero at the points $2k\pi$ for $k = 1, 2, \ldots$.

Straightforward differentiation gives

$$[x^3 \text{CosR}_1'(x)]'' = -x \sin x \begin{cases} < 0, & x \in (0, \pi); \\ 0, & x = 0, \pi, 2\pi; \\ > 0, & x \in (\pi, 2\pi). \end{cases}$$

Hence, the function

$$[x^3 \text{CosR}_1'(x)]' = x \cos x - \sin x$$
has a unique minimum $-\pi$ at $x = \pi$ on $[0, 2\pi]$, is equal to 0 at $x = 0$, and is equal to $2\pi$ at $x = 2\pi$. Accordingly, the function
\[ x^3 \cos n'(x) = x \sin x + 2 \cos x - 2 \]
has a unique minimum at some point $x_0 \in (\pi, 2\pi)$ on $[0, 2\pi]$ and is apparently equal to 0 at the points $x = 0, 2\pi$. Thus, the first derivative
\[ \cos n'(x) = \frac{x \sin x + 2 \cos x - 2}{x^3} \]
is negative on $(0, 2\pi)$. As a result, the normalized remainder $\cos n(x)$ is decreasing on $[0, 2\pi]$.

Utilizing the expression (19) and straightforwardly differentiating give
\[ \cos n''(x) = (-1)^n \frac{x \cos n(x) - 2n \cos_n(x)}{x^{2n+1}} = (-1)^n \frac{(2n)!}{x^{2n+1}} T_n(x) \]
for $n \geq 2$, where
\[ T_n(x) = -x \sin x + 2n(1 - \cos x) + \sum_{k=1}^{n-1} (-1)^k \frac{2n - 2k}{(2k)!} x^{2k} \]
and
\[ T_n^{(j)}(x) = (-1)^{n-j} 3x \left( \frac{\sin x}{x} - 2 + \cos x \right) < 0 \]
for $j \geq 0$ and $n \geq 2$. In particular,
\[ T_n^{(2n-3)}(x) = (-1)^n 3x \left( \frac{\sin x}{x} - 2 + \cos x \right) < 0 \]
for $n \geq 2$ on $(0, \infty)$, where we used the inequality (23) for $x > 0$. This means the negativity
\[ (-1)^n T_n^{(2n-3)}(x) < 0 \]
for $n \geq 2$ on $(0, \infty)$. It is not difficult to verify that $(-1)^n T_n^{(j)}(0) = 0$ for $0 \leq j \leq 2n - 3$ and $n \geq 2$. Accordingly, the functions $(-1)^n T_n^{(j)}(x)$ for $0 \leq j \leq 2n - 3$ and $n \geq 2$ are negative on $(0, \infty)$. This leads to that the derivative $\cos n''(x)$ for $n \geq 2$ is negative on $(0, \infty)$, and then the function $\cos n(x)$ for $n \geq 2$ is decreasing on $(0, \infty)$.

Taking $x \to \infty$ on both sides of the first equality in (21) results in the limit
\[ \lim_{x \to \infty} \cos n(x) = 0, \quad n \geq 1. \]

Consequently, the function $\cos n(x)$ for $n \geq 2$ is positive on $(0, \infty)$.

The integral representation (6) can be rearranged as
\[ \cos_n(x) = (-1)^n \frac{x^{2n-1}}{(2n-2)!} \int_0^1 (1-t)^{2n-2} \sin(xt) \, dt \]
for $n \geq 1$ and $x \in \mathbb{R}$. Thus, substituting (32) into the left equality in (19) gives
\[ \cos n(x) = \frac{(2n)!}{(2n-2)!} \int_0^1 t(1-t)^{2n-2} \frac{\sin(xt)}{xt} \, dt \]
and straightforwardly differentiating shows
\[ \cos n''(x) = \frac{(2n)!}{(2n-2)!} \int_0^1 t(1-t)^{2n-2} \frac{d^2}{dx^2} \left[ \frac{\sin(xt)}{xt} \right] \, dt \]
for $n \geq 1$ and $x > 0$. Making use of the negativity of the second derivative in (28), we find that the second derivative $\operatorname{CosR}_n''(x)$ for $n \geq 1$ is negative, and then the normalized remainder $\operatorname{CosR}_n(x)$ is concave, on the interval $(0, x_0)$, where $x_0 \in \left(\frac{\pi}{2} , \pi\right)$.

Differentiation gives

$$\operatorname{CosR}_n''(x) = (-1)^n(2n)! \frac{\Psi_n(x)}{x^{2n+2}} ,$$

where

$$\Psi_n(x) = xT_n'(x) - (2n + 1)T_n(x)$$

and

$$\Psi_n^{(k)}(x) = x\Psi_n^{(k+1)}(x) - (2n + 1 - k)\Psi_n^{(k)}(x)$$

for $k \in \mathbb{N}$. The Equations (22) and (30) means

$$T_n(2n-3)(x) = \delta_n(2n-2)(x) = (-1)^{n+1}(x \cos x + 2x - 3 \sin x) , \tag{33}$$

so we have

$$\Psi_n(2n-3)(x) = xT_n(2n-2)(x) - 4T_n(2n-3)(x) = \Phi_n(2n-2)(x) ,$$

where $\Phi_n(x)$ is given by (25). Using the same techniques as used in the proof of Theorem 1, we acquire

$$(-1)^{n+1}\Psi_n(2n-3)(x) > 0 , \quad x \in (0, \pi)$$

and

$$(-1)^{n+1}\Psi_n(2n-3)(x) < 0 , \quad x \in (\pi, 2\pi) .$$

It then follows that $\operatorname{CosR}_n''(x) < 0$ for $x \in (0, \pi)$ and $n \geq 2$. The proof of Theorem 2 is complete. \hfill \Box

Remark 7. From the decreasing property of $\operatorname{CosR}_n(x)$ for $n \geq 2$ in Theorem 2, the limit (31), and the definition (17) of the function $\operatorname{CosR}_n(x)$ for $n \geq 1$, we derive the inequality

$$0 < (-1)^n(2n)! \frac{\cos x - \sum_{k=0}^{n-1}(-1)^k \frac{x^{2k}}{(2k)!}}{x^{2n}} < 1 \tag{34}$$

for $n \geq 2$ and $x \in (0, \infty)$. Consequently, the second inequalities for $n \geq 2$ in (10) and (11) are recovered and the double inequality

$$0 < (-1)^n\left[\cos x - \sum_{k=0}^{n-1}(-1)^k \frac{x^{2k}}{(2k)!}\right] < \frac{x^{2n}}{(2n)!} \tag{35}$$

for $n \geq 2$ and $x \in \mathbb{R} \setminus \{0\}$ is obtained.

In light of the nonnegativity of $\operatorname{CosR}_n(x)$ in Theorem 2, as long as replacing the left strict inequalities $0 < \ldots$ by non-strict inequalities $0 \leq \ldots$, both of the inequalities (34) and (35) are also valid for $n = 1$ in $x \in (0, \infty)$.

Remark 8. Combining the limits in (24) and (31) with the second integral representations in (20) and (21) leads to the limit

$$\lim_{x \to \infty} \int_0^1 (1 - u)^k \cos(xu) \, du = 0, \quad k \in \mathbb{N} . \tag{36}$$

Remark 9. The inequalities in (10), (11), (26) and (35) imply that the sine and cosine functions are enveloped by their Maclaurin series expansions. Various related results on enveloping series were given in [15,16].
Remark 10. Combining Remark 3 with the relation (33), we find that,
1. when \( n \geq 3 \), the normalized remainder \( \cos R_n(x) \) is concave on \( (0, \frac{3\pi}{4}) \);
2. when \( n \geq 4 \), the normalized remainder \( \cos R_n(x) \) is concave on \( (0, \frac{7\pi}{4}) \);
3. when \( n \geq 5 \), the normalized remainder \( \cos R_n(x) \) is concave on \( (0, 2\pi) \).

Remark 11. For \( n \geq 1 \), how about the convexity or concavity of the normalized remainder \( \cos R_n(x) \) on the whole infinite interval \( (0, \infty) \)?

4. An Open Problem and Several Special Values

Stimulated by integral representations in (20) and (21), we now consider the function \( I(x, y) \) defined by

\[
I(x, y) = \begin{cases} 
\int_0^1 (1 - u)^y \sin xu \frac{dx}{x}, & x \neq 0 \\
1 & x = 0
\end{cases} 
\]

(37)

for \( x \in \mathbb{R} \) and \( y > -1 \). It is clear that

\[
\lim_{y \to \infty} I(x, y) = 0, \quad x \in (0, \infty)
\]

and

\[
\lim_{x \to \infty} \int_0^1 (1 - u)^{y+1} \cos xu \, du = 0, \quad y > -1.
\]

(38)

The limit (38) is a generalization of the limit (36). It is easy to see that, for \( n \geq 1 \) and \( x \in \mathbb{R} \),

\[
I(x, 2n-1) = \frac{\sin R_n(x)}{2n(2n+1)} \quad \text{and} \quad I(x, 2n-2) = \frac{\cos R_n(x)}{2n(2n-1)}.
\]

(39)

By virtue of two relations in (39), in view of Theorems 1 and 2, we can derive the following corollary immediately.

Corollary 1. The function \( I(x, 0) \) is nonnegative on \( (0, \infty) \), is decreasing on \( [0, 2\pi) \), and is concave on \( (0, x_0) \), where \( x_0 \subset (\frac{\pi}{2}, \pi) \) is the first positive zero of Equation (29).

For \( n \geq 1 \), the function \( I(x, n) \) is decreasing and positive on \( (0, \infty) \).

For \( n \geq 1 \), the function \( I(x, n) \) is concave in \( x \in (0, \pi) \).

How about the nonnegativity, monotonicity, and concavity of the function \( I(x, y) \) defined by (37) in the real variables \( x \in (0, \infty) \) and \( y > -1 \)? By instinct and intuition, we list the following four problems.

Open Problem 1. For \( y > -1 \), the function \( I(x, y) \) has the following properties.
1. For fixed \( y \geq 1 \), the function \( I(x, y) \) is decreasing and positive in \( x \in (0, \infty) \).
2. For fixed \( y \in (0, 1) \), the function \( I(x, y) \) is positive but not monotonic in \( x \in (0, \infty) \).
3. For fixed \( y < -1 \), the function \( I(x, y) \) oscillates in \( x \in (0, \infty) \).
4. For fixed \( y > -1 \), the function \( I(x, y) \) is concave if and only if \( x \in (0, x_0) \), where \( x_0 \subset (\frac{\pi}{2}, \pi) \) is the first positive zero of Equation (29).
Remark 12. For $\alpha_i \in \mathbb{C}$, $\beta_i \in \mathbb{C} \setminus \{0, -1, -2, \ldots \}$, $p, q \in \mathbb{N} = \{1, 2, \ldots \}$, and $z \in \mathbb{C}$, in terms of the rising factorial, or say, the Pochhammer symbol,

\[
(z)_n = \frac{\Gamma(z + n)}{\Gamma(z)} = \prod_{\ell=0}^{n-1} (z + \ell),
\]

the generalized hypergeometric series is defined ([17], p. 1020) by

\[
pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{n=1}^{p} (\alpha_k)_n z^n}{\prod_{n=1}^{q} (\beta_k)_n n!}.
\]

In particular, the function $2F_1(\alpha_1, \alpha_2; \beta_1; z)$ is called the Gauss hypergeometric function, the function $1F_1(\alpha; \beta; z)$ is called the confluent hypergeometric function of the first kind, the function $\varphi_F(\beta; z)$ is called the confluent hypergeometric limit function, and the like.

In ([17], pp. 443–444), there are the following three integral formulas.

1. For $a > 0$ and $\Re(v) > -1$,

\[
\int_0^1 (1 - x)^v \sin(ax) \, dx = \frac{1}{a} - \frac{\Gamma(v + 1)}{a^{v+1}} C_v(a) = \frac{s_{v+1/2, 1/2}(a)}{a^{v+1/2}}, \tag{40}
\]

where $C_v(a)$ is the Young function given in ([17], p. 443) by

\[
C_v(a) = \frac{a^v}{2\Gamma(v + 1)} \left[ 1F_1(1; v + 1; a i) + 1F_1(1; v + 1; -a i) \right]
= \sum_{n=0}^{\infty} (-1)^n \frac{a^{v+2n}}{\Gamma(v + 2n + 1)},
\]

the notation $i = \sqrt{-1}$ is the imaginary unit, and $s_{\mu, \nu}(z)$ is the Lommel function defined in ([17], p. 954) and (Chapter 11 of [18]) by

\[
s_{\mu, \nu}(z) = \sum_{m=0}^{\infty} (-1)^m \frac{z^{\mu+1+2m}}{\prod_{\ell=0}^{m} ((\mu + 2\ell + 1)^2 - \nu^2)}
= z^{\mu-1} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma\left(\frac{1}{2} \mu - \frac{1}{2} \nu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} \mu + \frac{1}{2} \nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} \mu - \frac{1}{2} \nu + m + \frac{1}{2}\right) \Gamma\left(\frac{1}{2} \mu + \frac{1}{2} \nu + m + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{2m+2}
\]

for $\mu \pm \nu$ being a not negative odd integer. In [19], Koumandos investigated the Lommel functions.

2. For $a > 0$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, and $\nu \neq 0$,

\[
\int_0^u x^{\nu-1} (u - x)^{\mu-1} \sin(ax) \, dx
= \frac{u^{\mu+\nu-1}}{2i} B(\mu, \nu) [1F_1(v; \mu + v; au i) - 1F_1(v; \mu + v; -au i)]. \tag{41}
\]

3. For $a > 0$, $\Re(\mu) > 0$, $\Re(\nu) > -1$, and $\nu \neq 0$,

\[
\int_0^1 x^{\nu-1} (1 - x)^{\mu-1} \sin(ax) \, dx
= \frac{i}{2} B(\mu, \nu) [1F_1(v; \nu + \mu; a i) - 1F_1(v; \nu + \mu; -a i)]. \tag{42}
\]

Combining the first integral representation in (37) with the Formula (40) results in that

\[
I(x, y) = \frac{1}{x^2} \left[ 1 - \Gamma(y + 1) \frac{C_v(x)}{x^y} \right] = \frac{s_{v+1/2, 1/2}(x)}{x^{v+3/2}} \tag{43}
\]
for $x > 0$ and $y > -1$. Combining this with Corollary 7, we can derive some monotonicity results of the Young and Lommel functions $C_n(x)$ and $s_{n+1/2,1/2}(x)$ for $n \geq 0$ in $x \in (0, \infty)$. 

Taking $\nu = 1$ and replacing $\mu$ by $y + 1$ in (42) leads to

$$I(x, y) = -\frac{i}{2(y+1)} \frac{1}{x} \left( F_1(1; y + 2; x i) - F_1(1; y + 2; -x i) \right)$$

for $x > 0$ and $y > -1$. Letting $\nu = 1$ and $\mu = y + 1$ in (41), changing the variable of integration in (41), and simplifying arrive at the Formula (44) once again. The Formula (44) can be reformulated as

$$I(x, y) = \frac{1}{2(y+1)} \frac{1}{x} \int_{-x i}^{x i} \frac{\partial [F_1(1; y + 2; u)]}{\partial u} \, du$$

$\Rightarrow$ $\Rightarrow$

$$= \frac{1}{2(y+1)(y+2)} \frac{1}{x i} \int_{-x i}^{x i} F_1(2; y + 3; u) \, du$$

$$= \frac{1}{2(y+1)(y+2)} \int_{-1}^{1} F_1(2; y + 3; xu i) \, du$$

for $x > 0$ and $y > -1$, where we used the relation

$$\frac{\partial [F_1(a; b; u)]}{\partial u} = a \frac{1}{b+1} F_1(a+1; b+1; u)$$

in ([14], p. 507, Entry 13.4.8).

**Remark 13.** In (Theorem 1.2 of [6]), Koumandos proved that the inequality

$$[s_{a,1/2}(x)]^2 - s_{a-1,1/2}(x)s_{a+1,1/2}(x) \geq \frac{1}{2 - a} [s_{a,1/2}(x)]^2$$

holds true for all $x > 0$ when $a \geq \frac{3}{2}$ and that the inequality (45) fails to hold for appropriate $x > 0$ when $-\frac{1}{2} < a < \frac{3}{2}$ and $a \neq \frac{1}{2}$.

**Remark 14.** Setting $y = 2n - 1$ and $y = 2n - 2$ in (43) and considering the relations in (39), we derive

$$\frac{\sin R_n(x)}{2n(2n + 1)} = \frac{1}{x^2} \left\{ 1 - \Gamma(2n) \frac{C_{2n-1}(x)}{x^{2n-2}} \right\} = \frac{s_{2n-1,1/2,1/2}(x)}{x^{2n+1/2}}$$

and

$$\frac{\cos R_n(x)}{2n(2n + 1)} = \frac{1}{x^2} \left\{ 1 - \Gamma(2n - 1) \frac{C_{2n-2}(x)}{x^{2n-2}} \right\} = \frac{s_{2n-3,1/2,1/2}(x)}{x^{2n-1/2}}$$

for $n \geq 1$ and $x > 0$. From the Equations (46) and (47), we conclude the closed-form expressions

$$C_{2n-1}(x) = (-1)^{n+1} \left[ \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right],$$

$$C_{2n-2}(x) = (-1)^{n+1} \left[ \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} \right],$$

$$s_{2n-1,1/2,1/2}(x) = \frac{(-1)^n (2n - 1)!}{x^{1/2}} \left[ \sin x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \right],$$

$$s_{2n-3,1/2,1/2}(x) = \frac{(-1)^n (2n - 2)!}{x^{1/2}} \left[ \cos x - \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k}}{(2k)!} \right]$$

for $n \geq 1$ and $x > 0$ of the Young function $C_r(x)$ and the Lommel function $s_{\mu,\nu}(z)$. 
1. Introduced two notions, the normalized tails $\text{SinR}_n(x)$ and $\text{CosR}_n(x)$ in (16) and (17) of the Maclaurin power series expansions (1) and (2), derived four integral representations in (20) and (21) of the normalized tails $\text{SinR}_n(x)$ and $\text{CosR}_n(x)$.

2. Acquired the positivity, decreasing property, and concavity of the normalized remainder $\text{SinR}_n(x)$. See Theorem 1.

3. Discovered the nonnegativity, positivity, decreasing property, and concavity of the normalized remainder $\text{CosR}_n(x)$. See Theorem 2.

4. Computed several special values of the Young function $C_n(x)$, see (48) and (49), of the Lommel function $s_{\mu,\nu}(z)$, see (50) and (51), and of the hypergeometric function $\text{F}_2(\alpha; \beta, \gamma; z)$, see (52) and (53).

5. Recovered the inequalities in (10) and (11). See the inequalities (26) and (35).

6. Proposed three open problems about the positivity, nonnegativity, decreasing property, and concavity of the newly introduced function $I(x, y)$ which is a generalization of the normalized tails $\text{SinR}_n(x)$ and $\text{CosR}_n(x)$. See Remarks 6 and 11 and Open Problem 1.

Author Contributions: Writing original draft, T.Z., Z.-H.Y., F.Q. and W.-S.D.; writing—review and editing, T.Z., Z.-H.Y., F.Q. and W.-S.D. All authors contributed equally to the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: Tao Zhang was partially supported by the National Nature Science Foundation of China (Grant No. 12001472). Wei-Shih Du was partially supported by Grant No. NSTC 112-2115-M-017-002 of the National Science and Technology Council of the Republic of China.

Data Availability Statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.
Acknowledgments: The authors are grateful to anonymous referees for their helpful suggestions and valuable comments on the original version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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