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Article

Noether's Theorem of Herglotz Type for Fractional Lagrange System with Nonholonomic Constraints

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Abstract: This research aims to investigate the Noether symmetry and conserved quantity for the fractional Lagrange system with nonholonomic constraints, which are based on the Herglotz principle. Firstly, the fractional-order Herglotz principle is given, and the Herglotz-type fractional-order differential equations of motion for the fractional Lagrange system with nonholonomic constraints are derived. Secondly, by introducing infinitesimal generating functions of space and time, the Noether symmetry of the Herglotz type is defined, along with its criteria, and the conserved quantity of the Herglotz type is given. Finally, to demonstrate how to use this method, two examples are provided.

Keywords: fractional Lagrange system; nonholonomic constraint; fractional Noether's theorem; Herglotz principle

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1. Introduction

Since fractional calculus enables a more accurate and straightforward description of physical and mechanical systems with the ability to remember the past and spatially nonlocal correlations, it is an effective mathematical tool for solving several issues in a variety of engineering and scientific domains. Fractional calculus is frequently utilized in physics, mechanics, viscoelastic materials, biomedicine, control theory, robotics, and signal processing [1–4]. The use of fractional calculus in modeling nonconservative mechanics began with Riewe's work [5,6]. Agrawal [7,8] introduced the fractional Lagrange variational issue and the most basic fractional variational problem, and the Euler–Lagrange equations that arise have the structure resembling that of the equations produced for classical integer-order variational problem. Atanacković et al. [9] derived the fractional Noether theorem under the classical definition of conserved quantity, which reveals the inherent connection between Noether symmetry transformations and fractional-order conserved quantities. In recent years, the study of conserved quantities and symmetries in fractional mechanics using variational methods has made some headway [10–18].

The classical variational principle does not apply to nonconservative mechanics because it can no longer be expressed as the extremum of some functional being equal to zero. Herglotz introduced a generalized variational principle, where the functional is described through a differential equation [19], and it is applicable in nonconservative mechanics. Unlike the classical Hamilton's principle, the Herglotz principle can solve the problem of both conservative and nonconservative systems. Georgieva and Gueuther et al. [20] obtained Noether's theorem on the basis of the Herglotz principle. Santos et al. [21,22] derived the Noether theorems of the Herglotz type for the Lagrangians with higher order derivatives and time delay. Almeida et al. [23] proposed the fractional Herglotz principle by extending the Herglotz principle to fractional models. Zhang et al. [24–26] extended the Herglotz principle and its Noether theorems to Birkhoffian systems, nonconservative nonholonomic systems, etc. However, as far as we know, no research has been conducted on the Herglotz principle and its conserved quantities for nonholonomic systems based on fractional models.

A nonholonomic system is one that has at least one nonholonomic constraint, which is a nonintegrable differential constraint [27,28]. Almost all ice skate, roller, and general chain systems have nonholonomic constraints. Consequently, a substantial amount of work has been performed on the basis of nonholonomic mechanics in the domains of bicycle, motorcycle and other wheel systems [29–31], motor systems [32,33], robot dynamics [34–36], and other fields. In recent years, the research on nonholonomic mechanics has been extended to fractional-order models [37–39]. The main topic of this article is to investigate Herglotz's Noether symmetry for nonholonomic systems, as well as the corresponding conserved quantity under fractional models. We provide a criterion for Herglotz fractional Noether symmetry with nonholonomic constraints and prove the fractional Noether theorem of the Herglotz type.

The article structure is as follows: In Section 2, we will briefly introduce fractional derivatives and their basic properties. In Section 3, we present the fractional-order Herglotz principle and derive the equations of motion. In Section 4, we investigate the Herglotz-type Noether symmetry. In Section 5, we derive the Herglotz-type Noether theorem. In Section 6, we provide two examples of how to apply the results. Finally, Section 7 gives our conclusions.

2. Fractional Derivative

The fractional derivative is briefly introduced in this section. Specific proofs and discussions can be found in the literature [1,3].

If function $\varpi(t)$ is both integrable and continuous at $t \in [A, B]$, then the Riemann–Liouville fractional derivatives are specified as

$$ {}_{A}D_{t}^{\alpha}\varpi(t) = \frac{1}{\Gamma(j-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j}\int_{A}^{t}(t-\eta)^{j-\alpha-1}\varpi(\eta)\mathrm{d}\eta \tag{1} $$

$$ {}_{t}D_{B}^{\alpha}\varpi(t) = \frac{1}{\Gamma(j-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j}\int_{t}^{B}(\eta-t)^{j-\alpha-1}\varpi(\eta)\mathrm{d}\eta \tag{2} $$

The Caputo fractional derivatives are specified as

$$ {}_{A}^{C}D_{t}^{\alpha}\varpi(t) = \frac{1}{\Gamma(j-\alpha)}\int_{A}^{t}(t-\eta)^{j-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^{j}\varpi(\eta)\mathrm{d}\eta \tag{3} $$

$$ {}_{t}^{C}D_{B}^{\alpha}\varpi(t) = \frac{1}{\Gamma(j-\alpha)}\int_{t}^{B}(\eta-t)^{j-\alpha-1}\left(-\frac{\mathrm{d}}{\mathrm{d}\eta}\right)^{j}\varpi(\eta)\mathrm{d}\eta \tag{4} $$

where $j \in \mathrm{N}$, $j-1 \le \alpha < j$, and $\Gamma(*)$ is the Gamma function, and α is the derivative's order. The definition above becomes an integer-order derivative if α is an integer, with

$$ \begin{cases} {}_{A}D_{t}^{\alpha}\varpi(t) = {}_{A}^{C}D_{t}^{\alpha}\varpi(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}\varpi(t) \\ {}_{t}D_{B}^{\alpha}\varpi(t) = {}_{t}^{C}D_{B}^{\alpha}\varpi(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}\varpi(t) \end{cases} \tag{5} $$

Suppose $\kappa(t)$ and $\varpi(t)$ are smooth functions in the interval $[A, B]$, then the fractional-order integration-by-parts formulas under the Caputo derivative are [40,41]

$$ \int_{A}^{B}\varpi(t)\,{}_{A}^{C}D_{t}^{\alpha}\kappa(t)\mathrm{d}t = \int_{A}^{B}\kappa(t)\,{}_{t}D_{B}^{\alpha}\varpi(t)\mathrm{d}t + \sum_{k=0}^{j-1}D^{k}\kappa(t)\,{}_{t}D_{B}^{\alpha-1-k}\varpi(t)\Big|_{A}^{B} \tag{6} $$

and

$$\int_A^B \varpi(t)\, {}_t^C D_B^{\alpha} \kappa(t) \mathrm{d}t = \int_A^B \kappa(t)\, {}_A D_t^{\alpha} \varpi(t) \mathrm{d}t - \sum_{k=0}^{j-1} (-D)^k \kappa(t)\, {}_A D_t^{\alpha-1-k} \varpi(t) \Big|_A^B \tag{7}$$

For $0 < \alpha < 1$ and $\kappa'(A) = 0$, then there is

$$\frac{\mathrm{d}}{\mathrm{d}t} {}_A^C D_t^{\alpha} \kappa(t) = {}_A^C D_t^{\alpha} \kappa'(t) \tag{8}$$

3. Equations of Motion

The Herglotz principle can be elaborated as follows.

We assume a fractional Lagrange system, which is described by n generalized coordinates $q_s(s = 1, 2, \cdots, n)$. Determine the trajectory $q_s(t) \in C^2([A, B], \mathrm{R})$ such that $z(B)$ reaches an extremum, namely,

$$z(B) \to \text{extr.} \tag{9}$$

where $z(t)$ is determined by the differential equation

$$\dot{z}(t) = L\Big(t, q_s(t), \dot{q}_s(t), {}_A^C D_t^{\alpha} q_s(t), z(t)\Big), \alpha \in (0, 1) \tag{10}$$

with boundary conditions

$$q_s(t)|_{t=A} = q_A, q_s(t)|_{t=B} = q_B \tag{11}$$

and the initial condition

$$z(t)|_{t=A} = z(A) \tag{12}$$

where $L\big(t, q_s(t), \dot{q}_s(t), {}_A^C D_t^{\alpha} q_s(t), z(t)\big)$ is the Herglotz-type fractional Lagrangian, and $\alpha \in (0, 1)$.

The functional z is known as the Hamilton–Herglotz action [24].

Let the system have g nonholonomic constraints

$$\dot{q}_{\varepsilon+\beta} = \varphi_\beta(t, q_s, \dot{q}_\sigma), (\beta = 1, 2, \cdots, g; \sigma = 1, 2, \cdots, \varepsilon; \varepsilon = n - g) \tag{13}$$

The virtual displacements satisfy the Appell–Chetaev condition [27,28]

$$\delta q_{\varepsilon+\beta} = \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} \delta q_\sigma \tag{14}$$

For nonholonomic systems, variational and differential operations are generally not commutative [28], and according to the Hölder definition of commutativity [28,42,43], we have

$$\delta \dot{q}_s = \frac{\mathrm{d}}{\mathrm{d}t} \delta q_s, \;\; \delta\, {}_a^C D_t^{\alpha} q_s = {}_a^C D_t^{\alpha} \delta q_s, \;\; (s = 1, 2, \cdots, n) \tag{15}$$

Taking the simultaneous variation of Equation (10), we obtain

$$\delta \dot{z} = \frac{\mathrm{d}}{\mathrm{d}t} \delta z = \frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s + \frac{\partial L}{\partial\, {}_A^C D_t^{\alpha} q_s} \delta\, {}_A^C D_t^{\alpha} q_s + \frac{\partial L}{\partial z} \delta z \tag{16}$$

Equation (16) has a solution

$$\delta z(t) \exp\left(-\int_A^{\tau} \frac{\partial L}{\partial z} \mathrm{d}\theta\right) - \delta z(A) =$$

$$\int_A^t \exp\left(-\int_A^{\tau} \frac{\partial L}{\partial z} \mathrm{d}\theta\right) \left(\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s + \frac{\partial L}{\partial\, {}_A^C D_t^{\alpha} q_s} \delta\, {}_A^C D_t^{\alpha} q_s\right) \mathrm{d}t \tag{17}$$

where $\vartheta(t) = \exp\left(-\int_A^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)$. According to Formula (12), $\delta z(A) = 0$, Equation (17) is written as

$$\delta z(t)\vartheta(t) = \int_A^t \vartheta(t)\left(\frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s + \frac{\partial L}{\partial\, {}_A^C D_t^\alpha q_s}\delta\, {}_A^C D_t^\alpha q_s\right)\mathrm{d}t \tag{18}$$

Considering that $z(B) \to \text{extr.}$, we have

$$\delta z(B) = 0 \tag{19}$$

Equation (18) holds for any $t \in [A, B]$. To be specific, take $t = B$, and we obtain

$$\int_A^B \left(\frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s + \frac{\partial L}{\partial\, {}_A^C D_t^\alpha q_s}\delta\, {}_A^C D_t^\alpha q_s\right)\vartheta(t)\mathrm{d}t = 0 \tag{20}$$

Let

$$\widetilde{L} = \widetilde{L}\left(t, q_s, \dot{q}_\sigma, {}_A^C D_t^\alpha q_s, z\right) = L\left(t, q_s, \dot{q}_\sigma, \varphi_\beta(t, q_s, \dot{q}_\sigma), {}_A^C D_t^\alpha q_s, z\right) \tag{21}$$

be the expression obtained by eliminating $\dot{q}_{\varepsilon+\beta}$ with the help of the constraint (13), and we have

$$\frac{\partial \widetilde{L}}{\partial t} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial t} \tag{22}$$

$$\frac{\partial \widetilde{L}}{\partial q_s} = \frac{\partial L}{\partial q_s} + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial q_s} \tag{23}$$

$$\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma} = \frac{\partial L}{\partial \dot{q}_\sigma} + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} \tag{24}$$

According to the Appell–Chetaev condition (14) and the commutative relation (15), we can easily obtain

$$\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\delta \dot{q}_\sigma = \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s - \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\delta q_\sigma \tag{25}$$

Substituting Equations (23)–(25) into (20) yields

$$\begin{aligned}&\int_A^B \vartheta(t)\left(\frac{\partial \widetilde{L}}{\partial q_s}\delta q_s - \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial q_s}\delta q_s\right.\\ &\left.+\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\delta q_\sigma + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\delta \dot{q}_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_s}\delta\, {}_A^C D_t^\alpha q_s\right)\mathrm{d}t = 0\end{aligned} \tag{26}$$

Using Formula (6), and taking into account the boundary conditions (11) and commutative relation (15), we have

$$\begin{aligned}&\int_A^B \vartheta(t)\left(\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_s}\delta\, {}_A^C D_t^\alpha q_s\right)\mathrm{d}t\\ &= \int_A^B \delta q_s\, {}_t D_B^\alpha\left[\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_s}\right]\mathrm{d}t + {}_t D_B^{\alpha-1}\left[\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_s}\right]\delta q_s\Big|_A^B\\ &= \int_A^B \delta q_\sigma\left\{{}_t D_B^\alpha\left[\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma}\right] + \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\, {}_t D_B^\alpha\left[\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}}\right]\right\}\mathrm{d}t\end{aligned} \tag{27}$$

and

$$\int_A^B \vartheta(t)\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\delta \dot{q}_\sigma \mathrm{d}t = -\int_A^B \vartheta(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma} - \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\frac{\partial L}{\partial z}\right)\delta q_\sigma \mathrm{d}t \tag{28}$$

Substituting Equations (27) and (28) into Equation (26), due to the lemma of variational calculus [44] and the independence of δq_σ, we obtain

$$\vartheta(t)\left[\frac{\partial\widetilde{L}}{\partial q_\sigma}+\frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}\frac{\partial L}{\partial z}+\frac{\partial\widetilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}-\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}+\frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\chi}}\frac{\partial\varphi_\chi}{\partial\dot{q}_\sigma}\right)\right.$$
$$\left.-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}\right]+{}_tD_B^\alpha\left[\vartheta(t)\frac{\partial\widetilde{L}}{\partial\,{}_A^CD_t^\alpha q_\sigma}\right]+\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}\,{}_tD_B^\alpha\left[\vartheta(t)\frac{\partial\widetilde{L}}{\partial\,{}_A^CD_t^\alpha q_{\varepsilon+\beta}}\right]=0$$
$$(\sigma=1,2,\cdots,\varepsilon) \tag{29}$$

Equation (29) is the fractional-order equations of motion of the Herglotz type for the fractional Lagrange system with nonholonomic constraints.

4. Herglotz-Type Noether Symmetry

The infinitesimal transformations are provided as

$$t^*=t+\Delta t,q_s^*(t^*)=q_s(t)+\Delta q_s \tag{30}$$

and their expanding formulas are

$$t^*=t+v\xi_0\left(t,q_l,\dot{q}_l,{}_A^CD_t^\alpha q_l\right)$$
$$q_s^*(t^*)=q_s(t)+v\xi_s\left(t,q_l,\dot{q}_l,{}_A^CD_t^\alpha q_l\right),(s,l=1,2,\cdots,n) \tag{31}$$

where ξ_0 and ξ_s are time and space infinitesimal generating functions, and v is a small parameter.

The functions $\xi_0(t,q_l,\dot{q}_l,{}_A^CD_t^\alpha q_l)$ and $\xi_s(t,q_l,\dot{q}_l,{}_A^CD_t^\alpha q_l)$ are called the generating functions. The key to seeking the conserved quantities of mechanical systems by using the Noether theorem is to find these generating functions. Under the integer-order calculus, the generating function is generally dependent on time, generalized coordinates, and generalized velocity functions, that is, $\xi_0(t,q_l,\dot{q}_l)$ and $\xi_s(t,q_l,\dot{q}_l)$; such a transformation constitutes a Lie group, which is geometrically preserved [45]. Sarlet and Cantrijin [46] discuss in detail the problem of functional dependence for generating functions. Since we study fractional-order nonconservative systems with nonholonomic constraints and their invariance, we extend the range of generating functions by introducing fractional-order derivative terms.

After the transformation, the Hamilton–Herglotz action changes accordingly

$$\Delta z(t)=\bar{z}(\bar{t})-z(t) \tag{32}$$

There is the relation between nonsimultaneous variation Δ and simultaneous variation δ [45]

$$\Delta\varpi=\delta\varpi+\dot{\varpi}\Delta t \tag{33}$$

and

$$\Delta\dot{\varpi}=\frac{\mathrm{d}}{\mathrm{d}t}\Delta\varpi-\dot{\varpi}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{34}$$

where $\varpi(t)$ is an arbitrary function. According to Formula (33), Equation (10) is written as

$$\Delta\dot{z}=\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial\dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial\,{}_A^CD_t^\alpha q_s}\Delta\,{}_A^CD_t^\alpha q_s+\frac{\partial L}{\partial z}\Delta z \tag{35}$$

From Formulas (34) and (35), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta z=M+\frac{\partial L}{\partial z}\Delta z \tag{36}$$

where

$$M = L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t + \frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial q_s}\Delta q_s + \frac{\partial L}{\partial \dot{q}_s}\Delta \dot{q}_s + \frac{\partial L}{\partial\, {}^C_A D^\alpha_t q_s}\Delta\, {}^C_A D^\alpha_t q_s \tag{37}$$

Equation (36) can be solved as

$$\Delta z(t)\vartheta(t) - \Delta z(A) = \int_A^t \vartheta(t) M \mathrm{d}t \tag{38}$$

Using Formulas (14) and (33), we can obtain

$$\Delta q_{\varepsilon+\beta} = \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}(\Delta q_\sigma - \dot{q}_\sigma \Delta t) + \dot{q}_{\varepsilon+\beta}\Delta t \tag{39}$$

Taking the derivative of Equation (39), and using Formula (34), we obtain

$$\begin{aligned}\Delta \dot{q}_{\varepsilon+\beta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\delta q_\sigma + \frac{\partial \varphi_\beta}{\partial t}\Delta t + \frac{\partial \varphi_\beta}{\partial q_s}\dot{q}_s \Delta t \\ + \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta q_\sigma - \dot{q}_\sigma \frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\end{aligned} \tag{40}$$

From Formula (33), we have

$$\Delta\, {}^C_A D^\alpha_t q_s(t) = {}^C_A D^\alpha_t \delta q_s(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left({}^C_A D^\alpha_t q_s(t)\right)\Delta t \tag{41}$$

By substituting Formulas (22)–(24) into (37), and combining Equations (40) and (41), Equation (37) can be expressed as

$$\begin{aligned} M =& \widetilde{L}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t + \frac{\partial \widetilde{L}}{\partial t}\Delta t + \frac{\partial \widetilde{L}}{\partial q_s}\Delta q_s + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta q_\sigma - \dot{q}_\sigma \frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right) \\ &+ \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} - \frac{\partial \varphi_\beta}{\partial q_\sigma} - \frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\chi}}\frac{\partial \varphi_\chi}{\partial \dot{q}_\sigma}\right)\delta q_\sigma \\ &+ \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_\sigma}\, {}^C_A D^\alpha_t \delta q_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_s}\frac{\mathrm{d}}{\mathrm{d}t}\left({}^C_A D^\alpha_t q_s\right)\Delta t \\ &+ \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_{\varepsilon+\beta}}\, {}^C_A D^\alpha_t\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\delta q_\sigma\right) \end{aligned} \tag{42}$$

Substituting Equation (42) into Equation (38) yields

$$\begin{aligned} \Delta z(t)\vartheta(t) =& \int_A^t \Bigg\{ \Bigg[\widetilde{L}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t + \frac{\partial \widetilde{L}}{\partial t}\Delta t + \frac{\partial \widetilde{L}}{\partial q_s}\Delta q_s + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta q_\sigma - \dot{q}_\sigma \frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right) \\ &+ \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} - \frac{\partial \varphi_\beta}{\partial q_\sigma} - \frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\chi}}\frac{\partial \varphi_\chi}{\partial \dot{q}_\sigma}\right)\delta q_\sigma \\ &+ \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_\sigma}\, {}^C_A D^\alpha_t \delta q_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_s}\frac{\mathrm{d}}{\mathrm{d}t}\left({}^C_A D^\alpha_t q_s\right)\Delta t \\ &+ \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_{\varepsilon+\beta}}\, {}^C_A D^\alpha_t\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\delta q_\sigma\right)\Bigg]\vartheta(t)\Bigg\}\mathrm{d}t \end{aligned} \tag{43}$$

Due to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L} = \frac{\partial \widetilde{L}}{\partial t} + \frac{\partial \widetilde{L}}{\partial q_s}\dot{q}_s + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\ddot{q}_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_s}\frac{\mathrm{d}}{\mathrm{d}t}\, {}^C_A D^\alpha_t q_s + \frac{\partial \widetilde{L}}{\partial z}L \tag{44}$$

Formula (43) can be written as

$$
\begin{aligned}
\Delta z(t)\vartheta(t)= \int_A^t \Bigg\{ \Bigg\{ \vartheta(t) \Bigg[& \frac{\partial \widetilde{L}}{\partial q_\sigma} + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\frac{\partial L}{\partial z} + \frac{\partial \widetilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma} \\
& - \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\partial \varphi_\beta}{\partial q_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} + \frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\chi}}\frac{\partial \varphi_\chi}{\partial \dot{q}_\sigma} \right) \Bigg] + {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma} \right) \\
& + \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\, {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}} \right) \Bigg\}\delta q_\sigma \\
& + \frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{ \vartheta(t)\left(\widetilde{L}\Delta t + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\delta q_\sigma \right) + \int_A^t \Bigg[\vartheta(t)\Bigg(\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma}\, {}_A^C D_t^\alpha \delta q_\sigma \\
& + \frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}}\, {}_A^C D_t^\alpha\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\delta q_\sigma \right) \Bigg)\mathrm{d}t - \delta q_\sigma\, {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma} \right) \\
& - \delta q_\sigma \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\, {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}} \right) \Bigg]\mathrm{d}t \Bigg\} \Bigg\}\mathrm{d}t
\end{aligned}
\tag{45}
$$

Representing Formulas (43) and (45) with generating functions ξ_0 and ξ_σ, we can obtain

$$
\begin{aligned}
\Delta z(t)\vartheta(t)= \int_A^t \Bigg\{ \Bigg[& \frac{\partial \widetilde{L}}{\partial t}\xi_0 + \frac{\partial \widetilde{L}}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} - \frac{\partial \varphi_\beta}{\partial q_\sigma} - \frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\chi}}\frac{\partial \varphi_\chi}{\partial \dot{q}_\sigma} \right)\overline{\xi}_\sigma \\
& + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\dot{\xi}_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma}\left({}_A^C D_t^\alpha \overline{\xi}_\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\left({}_A^C D_t^\alpha q_\sigma \right)\xi_0 \right) + \left(\widetilde{L} - \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\dot{q}_\sigma \right)\dot{\xi}_0 \\
& + \frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}}\left({}_A^C D_t^\alpha\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma \right) + \frac{\mathrm{d}}{\mathrm{d}t}\left({}_A^C D_t^\alpha q_{\varepsilon+\beta} \right)\xi_0 \right) \Bigg]\vartheta(t) \Bigg\} \upsilon\mathrm{d}t
\end{aligned}
\tag{46}
$$

and

$$
\begin{aligned}
\Delta z(t)\vartheta(t)= \int_A^t \upsilon \Bigg\{ \Bigg\{ \vartheta(t) \Bigg[& \frac{\partial \widetilde{L}}{\partial q_\sigma} + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\frac{\partial L}{\partial z} + \frac{\partial \widetilde{L}}{\partial q_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma} \\
& - \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\partial \varphi_\beta}{\partial q_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} + \frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\chi}}\frac{\partial \varphi_\chi}{\partial \dot{q}_\sigma} \right) \Bigg] \\
& + {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma} \right) + \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\, {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}} \right) \Bigg\}\overline{\xi}_\sigma \\
& + \frac{\mathrm{d}}{\mathrm{d}t}\left[\vartheta(t)\left(\widetilde{L}\xi_0 + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma \right) \right] - \overline{\xi}_\sigma\, {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma} \right) \\
& + \vartheta(t)\left(\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_\sigma}\, {}_A^C D_t^\alpha \overline{\xi}_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}}\, {}_A^C D_t^\alpha\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma \right) \right) \\
& - \overline{\xi}_\sigma \frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\, {}_tD_B^\alpha\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}_A^C D_t^\alpha q_{\varepsilon+\beta}} \right) \Bigg\}\mathrm{d}t
\end{aligned}
\tag{47}
$$

where $\overline{\xi}_\sigma = \xi_\sigma - \dot{q}_\sigma\xi_0$. The variational Formulas (46) and (47) are important foundations for establishing Noether symmetry criterion equations for fractional Lagrange systems with nonholonomic constraints.

We now establish the definition of Herglotz-type Noether symmetry and derive the criterion equation.

Definition 1. *The infinitesimal transformations of time and space for the fractional Lagrange system with nonholonomic constraints are called Herglotz-type Noether symmetric, if and only if*

$$\Delta z(B) = 0 \tag{48}$$

where $z(B)$ is the fractional Hamilton–Herglotz action.

Criterion 1. *If the generating functions ξ_0 and ξ_σ of time and space satisfy the generalized Noether identity*

$$\begin{aligned} &\frac{\partial \widetilde{L}}{\partial t}\xi_0 + \frac{\partial \widetilde{L}}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} - \frac{\partial \varphi_\beta}{\partial q_\sigma} - \frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\chi}}\frac{\partial \varphi_\chi}{\partial \dot{q}_\sigma}\right)\overline{\xi}_\sigma \\ &+\frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\dot{\xi}_\sigma + \left(\widetilde{L} - \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\dot{q}_\sigma\right)\dot{\xi}_0 + \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_\sigma}\left({}^C_A D^\alpha_t \overline{\xi}_\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\left({}^C_A D^\alpha_t q_\sigma\right)\xi_0\right) \\ &+\frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_{\varepsilon+\beta}}\left({}^C_A D^\alpha_t\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left({}^C_A D^\alpha_t q_{\varepsilon+\beta}\right)\xi_0\right) = 0 \end{aligned} \tag{49}$$

then the infinitesimal transformations of time and space are Herglotz-type Noether symmetric.

5. Noether Theorem

Theorem 1. *If the infinitesimal transformations of time and space for a fractional Lagrange system with nonholonomic constraints are Herglotz-type Noether symmetric, then*

$$\begin{aligned} I = \vartheta(t)\left(\widetilde{L}\xi_0 + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma\right) + \int_A^t\Bigg[\vartheta(t)\left(\frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_\sigma}\,{}^C_A D^\alpha_t \overline{\xi}_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_{\varepsilon+\beta}}\,{}^C_A D^\alpha_t\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma,\right)\right) \\ -\overline{\xi}_\sigma\, {}_t D^\alpha_B\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_\sigma}\right) - \overline{\xi}_\sigma\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\, {}_t D^\alpha_B\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_{\varepsilon+\beta}}\right)\Bigg]\mathrm{d}t = \text{const.} \end{aligned} \tag{50}$$

is a conserved quantity.

Proof. According to Definition 1 and using Equations (47) and (29), we acquire

$$\begin{aligned} &\int_A^B v\Bigg\{\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\vartheta(t)\left(\widetilde{L}\xi_0 + \frac{\partial \widetilde{L}}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma\right) \\ &+\int_A^t\Bigg[\vartheta(t)\left(\frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_\sigma}\,{}^C_A D^\alpha_t \overline{\xi}_\sigma + \frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_{\varepsilon+\beta}}\,{}^C_A D^\alpha_t\left(\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\overline{\xi}_\sigma,\right)\right) \\ &-\overline{\xi}_\sigma\, {}_t D^\alpha_B\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_\sigma}\right) - \overline{\xi}_\sigma\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}\, {}_t D^\alpha_B\left(\vartheta(t)\frac{\partial \widetilde{L}}{\partial\, {}^C_A D^\alpha_t q_{\varepsilon+\beta}}\right)\Bigg]\mathrm{d}t\Bigg\}\Bigg\}\mathrm{d}t = 0 \end{aligned} \tag{51}$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}I = 0 \tag{52}$$

Consequently, integrating (52), we obtain the conserved quantity (50). The proof of the theorem is provided. □

Theorem 1 can be called a Noether theorem of the Herglotz type, which generalizes the classical Noether theorem to a fractional Lagrange system with nonholonomic constraints.

6. Examples

Example 1. *Let the Herglotz-type Lagrangian be*

$$L = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_2^2 - \frac{\gamma}{2}\,{}^C_A D^\alpha_t q_1 - z \tag{53}$$

The system is constrained by

$$f = \dot{q}_2 - t\dot{q}_1 = 0 \tag{54}$$

where the damping coefficient γ and the mass m are fixed constants.

From Equation (10), the Hamilton–Herglotz action z satisfies

$$\dot{z} = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_2^2 - \frac{\gamma}{2}\,{}_A^C D_t^\alpha q_1 - z \tag{55}$$

Comparing Equation (13) with Equation (54), we obtain $\dot{q}_2 = t\dot{q}_1 = \varphi_1$*, and substituting it into (53), we have*

$$\widetilde{L} = \tfrac{1}{2}m\left(1+t^2\right)\dot{q}_1^2 - \tfrac{\gamma}{2}\,{}_A^C D_t^\alpha q_1 - z \tag{56}$$

According to Equation (29), we obtain

$$e^{A-t}\left[-m\left(1+t^2\right)\ddot{q}_1 - m\left(t^2+t+1\right)\dot{q}_1\right] - \tfrac{\gamma}{2}\,{}_t D_B^\alpha e^{A-t} = 0 \tag{57}$$

or write as

$$\ddot{q}_1 = \frac{t^2+t+1}{1+t^2}\dot{q}_1 + \frac{\gamma}{2m\left(1+t^2\right)}e^{t-A}\,{}_t D_B^\alpha e^{A-t} \tag{58}$$

Equations (57) or (58) is the fractional-order Herglotz-type differential equation of motion for the system under study.

According to Formula (49), the generalized Noether identity is

$$\begin{aligned} &mt\dot{q}_1^2\xi_0 + mt\dot{q}_1\left(\xi_1 - \dot{q}_1\xi_0\right) + m\left(1+t^2\right)\dot{q}_1\dot{\xi}_1 \\ &+\left(\tfrac{1}{2}m\left(1+t^2\right)\dot{q}_1^2 - \tfrac{\gamma}{2}\,{}_A^C D_t^\alpha q_1 - z - m\left(1+t^2\right)\dot{q}_1^2\right)\dot{\xi}_0 \\ &-\tfrac{\gamma}{2}\left({}_A^C D_t^\alpha\left(\xi_1 - \dot{q}_1\xi_0\right) + \tfrac{\mathrm{d}}{\mathrm{d}t}\left({}_A^C D_t^\alpha q_1\right)\xi_0\right) = 0 \end{aligned} \tag{59}$$

Equation (59) has a solution

$$\xi_0 = 1, \xi_1 = 0 \tag{60}$$

According to Theorem 1, we obtain

$$\begin{aligned} I = e^{A-t}&\left[-\tfrac{\gamma}{2}\,{}_A^C D_t^\alpha q_1 - z - \tfrac{1}{2}m\left(1+t^2\right)\dot{q}_1^2\right] \\ -\tfrac{\gamma}{2}\int_a^t &\left[e^{A-t}\,{}_A^C D_t^\alpha \dot{q}_1 - \dot{q}_1\,{}_t D_B^\alpha e^{A-t}\right]\mathrm{d}t = \text{const.} \end{aligned} \tag{61}$$

Equation (61) is a fractional Hegroz-type Noether conserved quantity.

When the fractional derivative term disappears, Equation (61) degenerates to the Noether conserved quantity of the Herglotz type:

$$I = -e^{A-t}\left[\frac{1}{2}m\left(1+t^2\right)\dot{q}_1^2 + z\right] = \text{const.} \tag{62}$$

Example 2. *We study the nonsliding rolling of a homogeneous sphere on a completely rough horizontal plane. Suppose that the sphere is subject to nonconservative forces, and take the spherical center coordinates* q_4 *and* q_5*, and the three Euler angles* q_1*,* q_2 *and* q_3 *as generalized coordinates; thus, the Herglotz-type Lagrangian is*

$$L = \frac{1}{2}m\left(\dot{q}_4^2 + \dot{q}_5^2\right) + \frac{1}{5}ma^2\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2\dot{q}_1\dot{q}_3\cos q_2\right) + \frac{\gamma}{2}\,{}_A^C D_t^\alpha q_4 - \gamma z \tag{63}$$

and the nonholonomic constraints are

$$\dot{q}_4 = -a\left(\dot{q}_3\cos q_1 \sin q_2 - \dot{q}_2 \sin q_1\right) \tag{64}$$

$$\dot{q}_5 = -a\left(\dot{q}_3\sin q_1 \sin q_2 + \dot{q}_2 \cos q_1\right) \tag{65}$$

where a, m and γ *are constants, and* $0 < \alpha < 1$*.*

Substituting Equations (64) and (65) into Equation (63), we obtain

$$\widetilde{L}=\frac{1}{2}ma^2\left[\frac{7}{5}\dot{q}_2^2+\dot{q}_3^2\sin^2 q_2+\frac{2}{5}\left(\dot{q}_1^2+\dot{q}_3^2+2\dot{q}_1\dot{q}_3\cos q_2\right)\right]+\frac{\gamma}{2}\,{}_A^C D_t^\alpha q_4-\gamma z \tag{66}$$

From Equation (29), the fractional Herglotz-type equations of the system are

$$\begin{gathered}
-e^{\gamma(t-A)}\Big\{\tfrac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{2}{5}ma^2(\dot{q}_1+\dot{q}_3\cos q_2)\right]+\tfrac{2}{5}\gamma ma^2(\dot{q}_1+\dot{q}_3\cos q_2)\\
+ma\dot{q}_4(\dot{q}_2\cos q_1+\dot{q}_3\sin q_1\sin q_2)+ma\dot{q}_5(\dot{q}_2\sin q_1-\dot{q}_3\cos q_1\sin q_2)\Big\}=0,\\
e^{\gamma(t-A)}\Big\{ma^2\left(\dot{q}_3^2\sin q_2\cos q_2-\tfrac{2}{5}\dot{q}_1\dot{q}_3\sin q_2\right)-\tfrac{7}{5}\gamma ma^2\dot{q}_2-\tfrac{7}{5}ma^2\ddot{q}_2\\
+ma\dot{q}_4(\dot{q}_1\cos q_1+\dot{q}_3\cos q_1\cos q_2)+ma\dot{q}_5(\dot{q}_1\sin q_1+\dot{q}_3\sin q_1\cos q_2)\Big\}\\
+\tfrac{\gamma}{2}a\sin q_1\,{}_tD_B^\alpha e^{\gamma(t-A)}=0,\\
-e^{\gamma(t-A)}\Big\{\tfrac{\mathrm{d}}{\mathrm{d}t}\left[ma^2\dot{q}_3\sin^2 q_2+\tfrac{2}{5}ma^2(\dot{q}_3+\dot{q}_1\cos q_2)\right]\\
+\gamma ma^2\left[\dot{q}_3\sin^2 q_2+\tfrac{2}{5}(\dot{q}_3+\dot{q}_1\cos q_2)\right]+ma\dot{q}_4\tfrac{\mathrm{d}}{\mathrm{d}t}(\cos q_1\sin q_2)\\
+ma\dot{q}_5\tfrac{\mathrm{d}}{\mathrm{d}t}(\sin q_1\sin q_2)\Big\}-\tfrac{\gamma}{2}a\cos q_1\sin q_2\,{}_tD_B^\alpha e^{\gamma(t-A)}=0\\
e^{\gamma(t-A)}\Big\{ma^2\left(\dot{q}_3^2\sin q_2\cos q_2-\tfrac{2}{5}\dot{q}_1\dot{q}_3\sin q_2\right)-\tfrac{7}{5}\gamma ma^2\dot{q}_2-\tfrac{7}{5}ma^2\ddot{q}_2\\
+ma\dot{q}_4(\dot{q}_1\cos q_1+\dot{q}_3\cos q_1\cos q_2)+ma\dot{q}_5(\dot{q}_1\sin q_1+\dot{q}_3\sin q_1\cos q_2)\Big\}\\
+\tfrac{\gamma}{2}a\sin q_1\,{}_tD_B^\alpha e^{\gamma(t-A)}=0,
\end{gathered} \tag{67}$$

Substituting Equations (64) and (65) into Equation (67), we obtain

$$\begin{gathered}
\ddot{q}_1+\ddot{q}_3\cos q_2-\dot{q}_2\dot{q}_3\sin q_2+\gamma(\dot{q}_1+\dot{q}_3\cos q_2)=0,\\
-\tfrac{7}{5}mae^{\gamma(t-A)}(\dot{q}_1\dot{q}_3\sin q_2+\gamma\dot{q}_2+\ddot{q}_2)+\tfrac{\gamma}{2}\sin q_1\,{}_tD_B^\alpha e^{\gamma(t-A)}=0,\\
e^{\gamma(t-A)}ma\Big\{\tfrac{2}{5}\ddot{q}_1\cos q_2+\ddot{q}_3\sin^2 q_2+\tfrac{2}{5}\ddot{q}_3-\tfrac{7}{5}\dot{q}_1\dot{q}_2\sin q_2+\dot{q}_2\dot{q}_3\sin q_2\cos q_2\\
+\gamma\left[\dot{q}_3\sin^2 q_2+\tfrac{2}{5}ma^2(\dot{q}_3+\dot{q}_1\cos q_2)\right]\Big\}+\tfrac{\gamma}{2}\cos q_1\sin q_2\,{}_tD_B^\alpha e^{\gamma(t-A)}=0
\end{gathered} \tag{68}$$

By Formula (49), the generalized Noether identity is

$$\begin{gathered}
\widetilde{L}\dot{\xi}_0+ma^2\dot{q}_3\sin q_2\left(\dot{q}_3\cos q_2-\tfrac{2}{5}\dot{q}_1\right)\xi_2-ma^2\dot{q}_3\sin q_2(\dot{q}_1+\dot{q}_3\cos q_2)(\xi_2-\dot{q}_2\xi_0)\\
+ma^2\dot{q}_2\sin q_2(\dot{q}_1+\dot{q}_3\cos q_2)(\xi_3-\dot{q}_3\xi_0)+\tfrac{2}{5}ma^2(\dot{q}_1+\dot{q}_3\cos q_2)\left(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0\right)\\
\tfrac{7}{5}ma^2\dot{q}_2\left(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0\right)+ma^2\left[\dot{q}_3\sin^2 q_2+\tfrac{2}{5}(\dot{q}_3+\dot{q}_1\cos q_2)\right]\left(\dot{\xi}_3-\dot{q}_3\dot{\xi}_0\right)\\
+\tfrac{\gamma}{2}\,{}_A^C D_t^\alpha\left[a\sin q_1(\xi_2-\dot{q}_2\xi_0)\right]+\tfrac{\gamma}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left({}_A^C D_t^\alpha q_4\right)\xi_0\\
-\tfrac{\gamma}{2}\,{}_A^C D_t^\alpha\left[a\cos q_1\sin q_2(\xi_3-\dot{q}_3\xi_0)\right]=0
\end{gathered} \tag{69}$$

Equation (69) has a solution

$$\xi_0=0,\xi_1=1,\xi_2=0,\xi_3=0 \tag{70}$$

According to Theorem 1, we obtain

$$I=\tfrac{2}{5}ma^2e^{\gamma(t-A)}(\dot{q}_1+\dot{q}_3\cos q_2)=\text{const.} \tag{71}$$

This is the conserved quantity of a purely rolling sphere with nonconservative forces applied.

7. Conclusions

Fractional calculus is a more precise tool for studying and describing complex systems' physical processes and dynamic behavior. In this study, the Herglotz-type Noether theorem is presented for fractional Lagrange systems with nonholonomic constraints. Two variational formulas are found for the fractional Hamilton–Herglotz action, as well as the criterion and definition of fractional Noether symmetry. Symmetry is closely related to conserved quantities. The fractional Herglotz-type Noether theorem is an extension of classical Noether's theorem, which shows how the conserved quantity and the system's symmetry relate to one another. The issue degenerates to the variational problem of fractional Lagrange systems with classical nonholonomic constraints when the Lagrangian function does not explicitly contain z, and Equation (50) degenerates to Noether's theorem of fractional Lagrange systems with nonholonomic constraints. As a result, this paper's results are more broadly applicable to holonomic and nonholonomic systems as well as conservative and nonconservative processes. The approach and findings in this paper can also be used to investigate Herglotz-type Noether theorems for other kinds of constrained mechanical systems under fractional models.

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