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Investigation of Well-Posedness for a Direct Problem for a Nonlinear Fractional Diffusion Equation and an Inverse Problem

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Abstract: In this paper, we consider a direct problem and an inverse problem involving a nonlinear fractional diffusion equation, which can be applied to many physical situations. The equation contains a Caputo fractional derivative, a symmetric uniformly elliptic operator and a source term consisting of the sum of two terms, one of which is linear and the other is nonlinear. The well-posedness of the direct problem is examined and the results are used to investigate the stability of an inverse problem of determining a function in the linear part of the source. The main tools in our study are the generalized eigenfunction expansions theory for nonlinear fractional diffusion equations, contraction mapping, Young's convolution and generalized Grönwall's inequalities. We present a stability estimate for the solution of the inverse source problem by means of observation data at a given point in the domain.

Keywords: nonlinear fractional diffusion equation; fixed point theory; direct problem; inverse problem; stability

MSC: 35R11

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1. Introduction

The diffusion of microscopic particles, which can be seen in a variety of natural and man-made processes, is often characterized by chaotic movements with unpredictable behavior. Anomalous diffusion can occur, which describes slower or faster diffusion than normal [1]. Fractional diffusion equations depict this type of diffusion. They have been used for a number of physical situations, including thin saturated areas within porous media, protein-folding models for non-Markovian dynamical phenomena and anomalous transport in disordered systems [2]. Due to their widespread applications in nature, such as physics, geology, complex viscoelastic materials, hydrology, and ecology, fractional diffusion equations have attracted much interest over the last two decades [3].

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$,

$$\Omega = \begin{cases} (0,b_1), & \text{for } d=1, \\ (0,b_1)\times(0,b_2), & \text{for } d=2, \\ (0,b_1)\times(0,b_2)\times(0,b_3), & \text{for } d=3, \end{cases}$$

for positive $b_1, b_2, b_3 \in \mathbb{R}$ and $T > 0$ be a fixed value. In this work, we deal with the nonlinear fractional diffusion equation

$$\partial_t^\alpha u(x,t) = (Lu)(x,t) + p(x)r(t) + g(u(x,t)), \ \ (x,t) \in \Omega \times (0,T), \tag{1}$$

where $\alpha \in (0,1)$ and $\partial_t^\alpha u(x,t)$ denote the α-th order Caputo fractional derivative of $u(x,t)$ with respect to t. If $\alpha = 1$ and L, p, r, g are chosen in a particular way, as in [4], there exist some biochemical models of enzyme systems for Equation (1), such as artificial membranes coupled with electrodes and a glucose oxidase membrane.

The definition of the fractional derivative is given as follows:

$$\partial_t^\alpha v(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int\limits_0^t \frac{\partial_\tau^m v(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & 0 \le m-1 < \alpha < m, \\ \partial_t^m v(t), & \alpha = m \in \mathbb{N} \end{cases} \tag{2}$$

(see [5]). Here, Γ indicates the Gamma function. The operator L is defined as

$$Lu = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right), x \in \overline{\Omega}, \tag{3}$$

where we suppose that

$$a_{ij} \in C^\infty(\overline{\Omega}), \quad a_{ij} = a_{ji} \tag{4}$$

for every integer $0 \le i, j \le d$ and the inequality

$$\delta \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \tag{5}$$

holds for a constant $\delta > 0$. Moreover, we consider the boundary and initial conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{6}$$

$$u(x,0) = 0, \quad x \in \Omega. \tag{7}$$

In this paper, we discuss the following problems:

Problem 1. *Find $u(x,t)$, satisfying Equation (1) with boundary condition (6) and initial condition (7). Here, $\alpha \in (0,1)$; p, r and g are given functions.*

Problem 2. *Determine the pair of functions (u,r) satisfying the initial-boundary value problem (1), (6), (7) by the additional data $u(x_0,t) = h(t)$ for $0 < t < T$. Here, $\alpha \in (0,1)$; p and g are given functions; $x_0 \in \Omega$ is a given point.*

Problem 1 is a direct problem, while Problem 2 is an inverse problem. We note that when $g(u(x,t)) = 0$ in (1), we obtain the same direct and inverse problems considered in [6].

The method we use to solve the initial-boundary value problem involving a fractional-order diffusion equation relies on the Laplace transform and the theory of boundary-value problems for elliptic equations. It was introduced by [7,8] for a homogeneous equation, further investigated by [9] and was applied to an inhomogeneous case by [6]. Then, the technique for examining the existence, uniqueness and regularity of the solution for a nonlinear model was developed by [10] and generalized by [11–15]. We note that the last five works employ a contraction mapping method by assuming the solution as the fixed point.

With regard to the theory of inverse problems for partial differential equations, there are many studies on the stability investigation for different equations, for instance, [16–19]. Recently, inverse problems for nonlinear fractional partial differential equations have become a widely studied subject. In [10,20–23], numerical techniques are employed to obtain the solution of some inverse problems for nonlinear time-fractional diffusion equations. On the other hand, the theoretical aspects have rarely been investigated. In [3], the inverse problems of determining the fractional order and determining the function that defines the nonlinear term in a nonlinear fractional diffusion equation are considered.

Our main objectives are to investigate the existence, uniqueness and regularity of the solution of the direct problem and to prove that the solution of the inverse problem is stable. For this purpose, with the help of tools used in [3,10] for nonlinear fractional differential equations, we use the approach of [6] and generalize to a nonlinear equation in our study. First, we estimate the solution of the direct problem, and then we use it to investigate the stability of the corresponding inverse problem. However, since [6] has no nonlinear terms, the assumptions in [6] are insufficient to solve our problem.

The outline of the paper is as follows: The next Section 2 provides the fundamental theoretical tools which are necessary in our proofs. Section 3 is devoted to our first main result Theorem 1 for the initial-boundary value problem. In Section 4, using the results of the previous section, we show the stability for the solution of the inverse problem, which is our second main result: Theorem 2. Finally, Section 5 concludes the paper with our final remarks on the Problems 1 and 2.

2. Materials and Methods

In this work, $L^2(\Omega)$ denotes the Lebesgue space for $p=2$, $H_0^1(\Omega)$ and $H^s(\Omega)$, $s\in\mathbb{Z}^+$ denote the usual Sobolev spaces. When $s\in\mathbb{R}^+$, $H^s(\Omega)$ denotes the fractional Sobolev spaces, for which we refer to [14,24,25]. Definition of the spaces $C\big([0,T];L^2(\Omega)\big)$ can be seen from Section 5.9 of [26].

The spaces $\widetilde{H}^s(\Omega)$, $s\geq 0$ are defined by the association of an elliptic operator. Due to $\Omega\subset\mathbb{R}^d$, $d\in\{1,2,3\}$ and the Sobolev imbedding theorem, we can write

$$-L : H^2(\Omega)\cap H_0^1(\Omega)\to L^2(\Omega). \tag{8}$$

The intersection written on (8) is given in Section 8.3 of [27]. Considering (3)–(5), the theory in Section 6.5 of [26] can be used. There exists an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2(\Omega)$, $\varphi_n\in H^2(\Omega)\cap H_0^1(\Omega)$ denotes an eigenfunction corresponding to λ_n and

$$\begin{cases} -L\varphi_n=\lambda_n\varphi_n, & \text{in } \Omega,\\ \varphi_n=0, & \text{on } \partial\Omega \end{cases} \tag{9}$$

for $n=1,2,\ldots$. Here, we have $0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\ldots$ and $\lambda_n\to\infty$ when $n\to\infty$ for the eigenvalues. The space of $\widetilde{H}^s(\Omega)$ is a Hilbert space and it is defined by

$$\widetilde{H}^s(\Omega)=\{w\in L^2(\Omega):\sum_{n=1}^{\infty}\lambda_n^s|(w,\varphi_n)|^2<\infty\},$$

with the norm

$$\|w\|^2_{\widetilde{H}^s(\Omega)}=\sum_{n=1}^{\infty}\lambda_n^s|(w,\varphi_n)|^2 \tag{10}$$

for real $s\geq 0$, which corresponds to the space $D\Big((-L)^{s/2}\Big)$ given by [6]. By Section 5.4 of [27], it is known that

$$w = \sum_{n=1}^{\infty}(w,\varphi_n)\varphi_n, \tag{11}$$

$$\|w\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty}|(w,\varphi_n)|^2, \tag{12}$$

for every $w\in L^2(\Omega)$. We can write

$$\|w\|^2_{\widetilde{H}^s(\Omega)}=\sum_{n=1}^{\infty}\Big|\Big((-L)^{s/2}w,\varphi_n\Big)\Big|^2=\Big\|(-L)^{s/2}w\Big\|^2_{L^2(\Omega)}. \tag{13}$$

We have $\widetilde{H}^s(\Omega)\subset H^s(\Omega)$ for any $s>0$,

$$\widetilde{H}^0(\Omega) = L^2(\Omega),\ \widetilde{H}^1(\Omega) = H_0^1(\Omega),\ \widetilde{H}^2(\Omega) = H^2(\Omega) \cap H_0^1(\Omega),$$

$$\widetilde{H}^s(\Omega) \subset L^2(\Omega) \subset \left(\widetilde{H}^s(\Omega)\right)' \tag{14}$$

and

$$\widetilde{H}^{-s}(\Omega) = \left\{ w \in L^2(\Omega) : \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} |(w, \varphi_n)|^2 < \infty \right\} = \left(\widetilde{H}^s(\Omega)\right)',$$

(see [14]).

We have the following lemmata which are necessary in the proceeding sections:

Lemma 1 ([26])**.** *Suppose that* $u_1 \in H^s\left(\mathbb{R}^d\right)$*,*

$$u_1 = \begin{cases} u_2, & in\ \Omega, \\ 0, & in\ \mathbb{R}^d \backslash \Omega, \end{cases}$$

and $s > d/2$*. Then, we have* $u_1 \in L^\infty\left(\mathbb{R}^d\right)$ *and*

$$\|u_1\|_{L^\infty\left(\mathbb{R}^d\right)} \le C_1 \|u_1\|_{H^s\left(\mathbb{R}^d\right)}, \tag{15}$$

where the constant C_1 *depends on only* d *and* s*.*

Lemma 2 ([6])**.** *If all eigenvalues of operator* $-L$ *are represented by the set* $\{\lambda_n\}_{n \in \mathbb{N}}$*, then we have* $\lambda_n \ge C_2 n^{2/d}$ *for every* $n \in \mathbb{N}$*.*

We consider the two-parameter Mittag–Leffler functions

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},\ \alpha, \beta > 0,\ z \in \mathbb{C},$$

which are important tools in fractional analysis and the generalized form of the significant function $f(z) = e^z$ for the theory of classical differential equations, [5,14].

Lemma 3 ([5,6,14,28])**.**
(i) *We have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\mu t^\alpha) = -\mu t^{\alpha - m} E_{\alpha, \alpha - m + 1}(-\mu t^\alpha)$$

for $t, \alpha, \mu > 0$ *and positive* $m \in \mathbb{Z}$*.*
(ii) $E_{\alpha,1}(-\mu)$ *is completely monotonic for* $\mu \in \mathbb{R}^+$*and* $\alpha \in (0,1)$*.*
(iii) *Suppose that* $0 < \alpha < 2$*,* β *is a real number and* $\eta \in (\pi\alpha, \min\{\pi, \pi\alpha\})$*. Then, the inequality*

$$\left|E_{\alpha,\beta}(\mu)\right| \le \frac{C_3}{1 + |\mu|},\ \eta \le |\arg(\mu)| \le \pi \tag{16}$$

is satisfied for a constant $C_3 = C_3(\alpha, \beta, \eta)$ *and for* $|\mu| \ge 0$*.*
(iv) *When* $\alpha, \beta, \mu \in \mathbb{C}$*,* $Re(\alpha) > 0$*,* $n \in \mathbb{N}$*, we have the Laplace transform*

$$\mathcal{L}\left[t^{\beta-1} E_{\alpha,\beta}(\mu t^\alpha)\right](s) = \frac{s^{\alpha-\beta}}{s^\alpha - \mu},\ Re(s) > 0,\ \left|\mu s^{-\alpha}\right| < 1.$$

We also use the well-known Young's convolution inequality and the generalized Grönwall inequality, which can be found in Appendix A of [29]. For Lebesgue's theorem, we refer to Section 2.1.7 of [30]. From the first chapter of [29], we know the formula

$$\int_{\theta}^{t} (\tau-\theta)^{q-1}(t-\tau)^{z-1} d\tau = \frac{\Gamma(q)\Gamma(z)}{\Gamma(q+z)}(t-\theta)^{q+z-1} \tag{17}$$

for $q, z > 0$ and $\theta < t$.

Using the eigenfunction expansions, the weak solution of Problem 1 is sought in the following form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x), \tag{18}$$

where $\varphi_n(x)$ are the solutions of (9) and

$$u_n(t) = (u(.,t), \varphi_n)_{L^2(\Omega)}.$$

Multiplying (1) with $\varphi_n(x)$, integrating the result on Ω and analyzing the terms, yields the equation

$$\partial_t^{\alpha} u_n(t) = -\lambda_n u_n(t) + F_n(u(t)),\ 0 < t < T,$$

where

$$F_n(u(t)) = r(t)(p, \varphi_n)_{L^2(\Omega)} + (g(u(\cdot,\tau)), \varphi_n)_{L^2(\Omega)}.$$

Therefore, considering (7), the Problem 1 is converted into solving the system of Problems (9) and

$$\begin{cases} \partial_t^{\alpha} u_n(t) = -\lambda_n u_n(t) + F_n(u(t)),\ 0 < t < T, \\ u_n(0) = 0. \end{cases} \tag{19}$$

Then, using the Laplace transform with Lemma 3, the solution of the initial value Problem (19) can be written as

$$u_n(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^{\alpha}\right) F_n(u(t)) d\tau. \tag{20}$$

Writing (20) in (18), the solution of Problem 1 is obtained as

$$\begin{aligned} u(x,t) \ &= \ \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) r(t)(p,\varphi_n) d\tau \right) \varphi_n(x) \\ &\quad + \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-\tau)^{\alpha}) (g(u(\cdot,\tau)), \varphi_n) d\tau \right) \varphi_n(x). \end{aligned} \tag{21}$$

For further details, see [13,15].

3. Solvability of Problem 1

In this section, we present some estimates for the solution u of Problem 1 and the nonlinear term $g(u)$, which are useful for investigating the stability of the solution of Problem 2.

Since representation (21) is an integral equation, to prove Theorem 1, we define a map which enables us to apply fixed point theory. The existence and uniqueness of the solution are investigated by employing the method of [14], which can be considered as a generalization of Bielecki's method (see Section 2.4 in [31]). In that investigation, we apply Banach's fixed-point theorem, which can be seen from Section 9.2 in [26]. Finally, we derive useful inequalities for both the solution u and the nonlinear term $g(u)$.

Now, we present our main result for Problem 1.

Theorem 1. *Let* $p \in L^2(\Omega)$, $r \in C[0,T]$, $g(0) = 0$ *and*

$$\left|\sup_{\phi\in\mathbb{R}} \frac{d^i g}{d\phi^i}(\phi)\right| < \infty, \ i = 0, 1. \tag{22}$$

Then, there exists a unique solution of Problem 1 satisfying (21) and $u \in C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big)$ *with* $\partial_t^\alpha u \in C\big([0,T]; L^2(\Omega)\big)$*. Additionally, we have the inequality*

$$\|u\|_{C([0,T];L^2(\Omega))} \leq C_4 \|p\|_{L^2(\Omega)} \|r\|_{C[0,T]}, \tag{23}$$

for a positive constant C_4.

It should be noted that for $g \in C^1(\mathbb{R})$,

$$g(x) = \frac{x}{1+x^2} \tag{24}$$

is an example for satisfying the conditions $g(0) = 0$ and (22).

The proof of Theorem 1 is lengthy and is separated with different steps by Lemmata 4–7. In Lemma 4, we show the existence and uniqueness of the solution. By Lemma 5, we obtain the inequality (23). With Lemma 6, we obtain an estimate for the nonlinear term. As a result of Lemma 7, we have $u \in C\big([0,T]; H^2(\Omega) \cap H_0^1(\Omega)\big)$. Finally, we complete the proof by showing that $\partial_t^\alpha u \in C\big([0,T]; L^2(\Omega)\big)$.

Lemma 4. *Under the hypotheses of Theorem 1, there exists a unique solution of Problem 1 satisfying (21).*

Proof of Lemma 4. Since the representation of the solution to Problem 1 is in the form of (21) and it is an integral equation, we can employ the technique of [14]. We denote $C\big([0,T]; L^2(\Omega)\big)_k$ by the space $C\big([0,T]; L^2(\Omega)\big)$ equipped with the norm

$$\|u\|_k = \max_{t\in[0,T]} \left\{ \left\| e^{-kt} u(t) \right\|_{L^2(\Omega)} \right\}, \tag{25}$$

and from [14,31], we know that (25) is equivalent to the standard norm of $C\big([0,T]; L^2(\Omega)\big)$ for any fixed $k > 0$. We define a map

$$M : C\Big([0,T]; L^2(\Omega)\Big)_k \to C\Big([0,T]; L^2(\Omega)\Big)_k$$

by

$$\begin{aligned} M(u(x,t)) \quad := \quad & \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) r(t) (p, \varphi_n) d\tau \right) \varphi_n(x) \\ & + \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) (g(u(\cdot,\tau)), \varphi_n) d\tau \right) \varphi_n(x). \end{aligned} \tag{26}$$

For any $k > 0$, $u \in C\big([0,T]; L^2(\Omega)\big)$ is a solution of Problem 1 if, and only if, u is a fixed point of the map M. Therefore, we need to prove that for some $k > 0$, the map M has a unique fixed point. For this purpose, we set the notations

$$I_1(\tau) := (g(u(\cdot,\tau)) - g(v(\cdot,\tau)), \varphi_n)_{L^2(\Omega)},$$

$$I_2(t) := e^{-2kt} \|M(u(\cdot,t)) - M(v(\cdot,t))\|_{L^2(\Omega)}^2$$

for any $u, v \in C\big([0,T]; L^2(\Omega)\big)$ and

$$I_3(t) := \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^{\alpha}\big) I_1(\tau) d\tau.$$

Now, we start to evaluate the term I_2. By implementing Lebesgue's theorem to the series, using the Cauchy–Schwarz inequality, (12), the properties of $\{\varphi_n\}_{n=1}^{\infty}$, we have

$$\begin{aligned} I_2(t) &= e^{-2kt} \left\| \sum_{n=1}^{\infty} \left(\int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^{\alpha}\big) I_1(\tau) d\tau \right) \varphi_n(x) \right\|^2_{L^2(\Omega)} \\ &= \sum_{n=1}^{\infty} \left| e^{-kt} I_3(t) \right|^2. \end{aligned} \tag{27}$$

For examining the right-hand side of (27), we use (16) with the notation

$$I_4 := \max_{0 \le \tau \le t} \left\{ e^{-k\tau} I_1(\tau) \right\}$$

and we write

$$\begin{aligned} I_3(t) &= \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^{\alpha}\big) I_1(\tau) e^{k\tau - k\tau} d\tau \\ &\le I_4 \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^{\alpha}\big) e^{-k(t-\tau)} d\tau \\ &\le C_5 I_4 \int\limits_0^t (t-\tau)^{\alpha-1} e^{-k(t-\tau)} d\tau = C_5 I_4 \int\limits_0^t \tau^{\alpha-1} e^{-k\tau} d\tau. \end{aligned} \tag{28}$$

By the change of variable $\varepsilon = \tau/t$, we obtain

$$\begin{aligned} \int\limits_0^t \tau^{\alpha-1} e^{-k\tau} d\tau &= t^{\alpha} \int\limits_0^1 \varepsilon^{\alpha-1} e^{-kt\varepsilon} dw \\ &= k^{1-\alpha} t \int\limits_0^1 (kt\varepsilon)^{\alpha-1} e^{-kt\varepsilon} dw \\ &\le k^{1-\alpha} t \int\limits_0^1 \left[\max_{\varepsilon \in [0,1]} \frac{(kt\varepsilon)^{\alpha/2}}{e^{kt\varepsilon}} \right] (kt\varepsilon)^{(\alpha/2)-1} d\varepsilon \\ &= \frac{C_6 t^{\alpha/2}}{k^{\alpha/2}} \int\limits_0^1 \varepsilon^{(\alpha/2)-1} d\varepsilon = \frac{2C_6}{\alpha} \left(\frac{t}{k} \right)^{\alpha/2} \end{aligned} \tag{29}$$

and considering (27)–(29) with the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} I_2(t) &\le \left[\frac{C_{17} t^{\alpha/2}}{k^{\alpha/2}} \right]^2 \max_{0 \le \tau \le T} \left| \frac{\sum\limits_{n=1}^{\infty} \big(g(u(\cdot,\tau)) - g(v(\cdot,\tau)), \varphi_n\big)^2}{e^{2k\tau}} \right| \\ &= \frac{C_7^2 t^{\alpha}}{k^{\alpha}} \max_{0 \le \tau \le T} \left| \frac{\| g(u(\cdot,\tau)) - g(v(\cdot,\tau)) \|^2_{L^2(\Omega)}}{e^{2k\tau}} \right|. \end{aligned} \tag{30}$$

By using the mean value theorem, there exists a $\theta(x,t) \in (\min\{v,w\}, \max\{v,w\})$ for any $v, w \in C\big([0,T]; L^2(\Omega)\big)$ satisfying

$$\|g(v) - g(w)\|_{L^2(\Omega)} = \big\|g'(\theta)[v-w]\big\|_{L^2(\Omega)}.$$

With (22), we obtain

$$\|g(v) - g(w)\|_{L^2(\Omega)} \le C_0 \|v - w\|_{L^2(\Omega)} \tag{31}$$

for any $v, w \in C\big([0,T]; L^2(\Omega)\big)$ and for a constant $C_0 > 0$ independent from x, t, v, w. By (25) and (31), inequality (30) becomes

$$I_2(t) \le \frac{C_0^2 C_7^2 t^\alpha}{k^\alpha} \max_{0 \le \tau \le T} \left| \frac{\|u(\cdot,\tau) - v(\cdot,\tau)\|^2_{L^2(\Omega)}}{e^{2k\tau}} \right| = \frac{C_0^2 C_7^2 t^\alpha}{k^\alpha} \|u - v\|_k^2. \tag{32}$$

By taking the maximum of the inequality (32) with respect to t and writing (25), we have

$$\|M(u) - M(v)\|_k \le \frac{C_0 C_7 T^{\alpha/2}}{k^{\alpha/2}} \|u - v\|_k. \tag{33}$$

With the choice of $k^{\alpha/2} > C_0 C_7 T^{\alpha/2}$, inequality (33) shows that M is a contraction map. By Banach's fixed-point theorem, the map M has a unique fixed point in $C\big([0,T]; L^2(\Omega)\big)_k$. Therefore, the solution u exists.

As for the uniqueness of the solution u, it is obvious by the selection of k, definition (26), inequality (33) and the fact that every norm is nonnegative. □

Lemma 5. *Under the hypotheses of Theorem 1, we have (23) and* $u \in C\big([0,T]; L^2(\Omega)\big)$.

Proof of Lemma 5. By using the notations

$$\begin{aligned}
I_5(x,t) &:= \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) r(\tau) (p, \varphi_n) d\tau \right) \varphi_n(x) \\
I_6(x,t) &:= \sum_{n=1}^{\infty} \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) (g(u(\cdot,\tau)), \varphi_n) d\tau \right) \varphi_n(x), \\
I_7(x,t) &:= \int_0^t \left[\sum_{n=1}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) (g(u(\cdot,t-\tau)), \varphi_n) \varphi_n(x) \right] d\tau,
\end{aligned}$$

we write

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \le 2\|I_5(\cdot,t)\|^2_{L^2(\Omega)} + 2\|I_6(\cdot,t)\|^2_{L^2(\Omega)}. \tag{34}$$

With the help of (12), (16), (22) for $i = 0$ and Lebesgue's theorem, it can be shown that the order of integrations and summations of the terms on the right-hand side of (34) can change. Indeed, for the terms I_5 and I_6, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) r(\tau) (p, \varphi_n) \varphi_n(x) &\le C_8 (t-\tau)^{\alpha-1} r(\tau) \sum_{n=1}^{\infty} p(x) \varphi_n(x) \\
&\le C_8 (t-\tau)^{\alpha-1} r(\tau) p(x),
\end{aligned} \tag{35}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) (g(u(\cdot,t-\tau)), \varphi_n) \varphi_n(x) &\le \left| C_9 \tau^{\alpha-1} \sum_{n=1}^{\infty} (g(u(\cdot,t-\tau)), \varphi_n) \varphi_n(x) \right| \\
&= \left| C_9 \tau^{\alpha-1} g(u(x,t-\tau)) \right|
\end{aligned} \tag{36}$$

and the right-hand sides of (35)–(36) are integrable on the domain of t by

$$\begin{aligned} C_8 p(x) \int_0^t (t-\tau)^{\alpha-1} r(\tau) d\tau &\le C_8 p(x) \|r\|_{C[0,T]} \int_0^t (t-\tau)^{\alpha-1} d\tau \\ &= C_8 p(x) \|r\|_{C[0,T]} \frac{T^\alpha}{\alpha} < \infty \end{aligned}$$

$$\begin{aligned} C_9 \int_0^t \left| \tau^{\alpha-1} g(u(x,t-\tau)) \right| d\tau &\le C_9 \|g\|_{L^\infty(\mathbb{R})} \int_0^t (t-\tau)^{\alpha-1} d\tau \\ &= C_9 \|g\|_{L^\infty(\mathbb{R})} \frac{T^\alpha}{\alpha} < \infty \end{aligned}$$

Now, we can estimate the term I_5. By Lemma 3 and (12), we have

$$\begin{aligned} \|I_5(\cdot,t)\|_{L^2(\Omega)} &= \left\| \int_0^t \sum_{n=1}^{\infty} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^\alpha\right) r(\tau) (p,\varphi_n) \varphi_n(x) d\tau \right\|_{L^2(\Omega)} \\ &\le \int_0^t \left\| \sum_{n=1}^{\infty} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^\alpha\right) r(\tau) (p,\varphi_n) \varphi_n(x) \right\|_{L^2(\Omega)} d\tau \\ &= \int_0^t \left[\left\| \sum_{n=1}^{\infty} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^\alpha\right) r(\tau) (p,\varphi_n) \varphi_n(x) \right\|_{L^2(\Omega)}^2 \right]^{1/2} d\tau \\ &= \int_0^t \left[\sum_{n=1}^{\infty} \left| (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n (t-\tau)^\alpha\right) r(\tau) (p,\varphi_n) \right|^2 \right]^{1/2} d\tau \end{aligned}$$

and

$$\begin{aligned} \|I_5(\cdot,t)\|_{L^2(\Omega)} &\le C_{10} \int_0^t (t-\tau)^{\alpha-1} r(\tau) \left[\sum_{n=1}^{\infty} |(p,\varphi_n)|^2 \right]^{1/2} d\tau \\ &= C_{10} I_8(t) \end{aligned} \tag{37}$$

for

$$I_8(t) := \int_0^t (t-\tau)^{\alpha-1} r(\tau) \|p\|_{L^2(\Omega)} d\tau.$$

In order to estimate the term I_6, we obtain

$$\begin{aligned} \|I_6(\cdot,t)\|^2_{L^2(\Omega)} &= \|I_7(\cdot,t)\|^2_{L^2(\Omega)} \\ &= \left(\int_0^t \sum_{n=1}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) (g(u(\cdot,t-\tau)),\varphi_n) \varphi_n(x) d\tau, I_7(\cdot,t) \right)_{L^2(\Omega)} \\ &= \int_0^t \left(\sum_{n=1}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) (g(u(\cdot,t-\tau)),\varphi_n) \varphi_n(x), I_7(\cdot,t) \right)_{L^2(\Omega)} d\tau \\ &\le \|I_7(\cdot,t)\|_{L^2(\Omega)} \int_0^t \left\| \sum_{n=1}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) (g(u(\cdot,t-\tau)),\varphi_n) \varphi_n(x) \right\|_{L^2(\Omega)} d\tau \end{aligned}$$

and

$$\begin{aligned}
\|I_7(\cdot,t)\|_{L^2(\Omega)} &\le \int_0^t \left\| \sum_{n=1}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)(g(u(\cdot,t-\tau)),\varphi_n)\varphi_n(x) \right\|_{L^2(\Omega)} d\tau \\
&= I_9(t),
\end{aligned} \tag{38}$$

by using the Fubini theorem, the Cauchy–Schwarz inequality and the notation

$$I_9(t) := \int_0^t \left[\left\| \sum_{n=1}^{\infty} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)(g(u(\cdot,t-\tau)),\varphi_n)\varphi_n(x) \right\|^2_{L^2(\Omega)} \right]^{1/2} d\tau.$$

Considering (12), (16) and the properties of $\{\varphi_n\}_{n=1}^{\infty}$, the term $I_9(t)$ can be written as

$$\begin{aligned}
I_9(t) &= \int_0^t \left[\sum_{n=1}^{\infty} \left| \left(\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)(g(u(\cdot,t-\tau)),\varphi_n)_{L^2(\Omega)}\varphi_n, \varphi_n \right)_{L^2(\Omega)} \right|^2 \right]^{1/2} d\tau \\
&= \int_0^t \left[\sum_{n=1}^{\infty} \left| \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)(g(u(\cdot,t-\tau)),\varphi_n) \right|^2 \right]^{1/2} d\tau \\
&\le C_{11} \int_0^t \tau^{\alpha-1} \left[\sum_{n=1}^{\infty} |(g(u(\cdot,t-\tau)),\varphi_n)|^2 \right]^{1/2} d\tau \\
&= C_{11} \int_0^t \tau^{\alpha-1} \|g(u(\cdot,t-\tau))\|_{L^2(\Omega)} d\tau.
\end{aligned} \tag{39}$$

Writing inequalities (37), (38) and (39) in (34), we obtain

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C_{12} I_8(t) + C_{13} \int_0^t (t-\tau)^{\alpha-1} \|g(u(\cdot,\tau))\|_{L^2(\Omega)} d\tau. \tag{40}$$

On the other hand, by hypothesis $g(0) = 0$ and (31), we obtain

$$\|g(u(\cdot,t))\|_{L^2(\Omega)} \le C_0 \|u(\cdot,t)\|_{L^2(\Omega)} \tag{41}$$

for $0 < t < T$ and by (40), (41), we write

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le C_{12} I_8(t) + C_0 C_{13} \int_0^t (t-\tau)^{\alpha-1} \|u(\cdot,\tau)\|_{L^2(\Omega)} d\tau. \tag{42}$$

With Young's convolution inequality, it can be shown that

$$\begin{aligned}
\|I_8(t)\|_{L^1(0,T)} &= \|p\|_{L^2(\Omega)} \|I_{10} * r\|_{L^1(0,T)} \\
&\le \|p\|_{L^2(\Omega)} \|I_{10}\|_{L^1(0,T)} \|r\|_{L^1(0,T)} \\
&= \|p\|_{L^2(\Omega)} \|I_{10}\|_{L^1(0,T)} \int_0^T |r(\tau)| d\tau \\
&\le \frac{T^{\alpha+1}}{\alpha} \|p\|_{L^2(\Omega)} \|r\|_{C[0,T]} < \infty
\end{aligned} \tag{43}$$

for

$$I_{10}(\tau) := \tau^{\alpha-1}.$$

By (43), we have $C_{12}I_8(t) \in L^1(0,T)$ and using the generalized Grönwall's inequality, we obtain

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq C_{12}I_8(t) + C_{12}C_{14}e^{C_{15}t}\int\limits_0^t (t-\tau)^{\alpha-1}I_8(\tau)d\tau \tag{44}$$

from (42). Here, for the second term in the right-hand side of (44), by using formula (17), we write

$$\begin{aligned}\int\limits_0^t (t-\tau)^{\alpha-1}I_8(\tau)d\tau &= \|p\|_{L^2(\Omega)}\int\limits_0^t (t-\tau)^{\alpha-1}\int\limits_0^\tau (\tau-\theta)^{\alpha-1}r(\theta)d\theta d\tau \\ &= \int\limits_0^t \left[\int\limits_\theta^t (\tau-\theta)^{\alpha-1}(t-\tau)^{\alpha-1}d\tau\right] r(\theta)d\theta \\ &= \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int\limits_0^t (t-\theta)^{2\alpha-1}r(\theta)d\theta. \end{aligned} \tag{45}$$

With (45), inequality (44) becomes

$$\begin{aligned}\|u(\cdot,t)\|_{L^2(\Omega)} &\leq C_{12}\|p\|_{L^2(\Omega)}\left[\int\limits_0^t (t-\theta)^{\alpha-1}r(\theta)d\theta + C_{14}e^{C_{15}t}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int\limits_0^t (t-\theta)^{2\alpha-1}r(\theta)d\theta\right] \\ &\leq C_{12}\|p\|_{L^2(\Omega)}\|r\|_{C[0,T]}\left[\int\limits_0^t (t-\theta)^{\alpha-1}d\theta + C_{14}e^{C_{15}t}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int\limits_0^t (t-\theta)^{2\alpha-1}d\theta\right] \\ &= \frac{C_{12}}{\alpha}\|p\|_{L^2(\Omega)}\|r\|_{C[0,T]}\left[t^\alpha + C_{14}e^{C_{15}t}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\frac{t^{2\alpha}}{2}\right]. \end{aligned} \tag{46}$$

Finally, taking the maximum of inequality (46) with respect to t on $[0,T]$, we obtain (23) and $u \in C\big([0,T];L^2(\Omega)\big)$. □

Lemma 6. *Under the hypotheses of Theorem 1, we have*

$$\|g(u(\cdot,t))\|_{L^2(\Omega)} \leq \frac{C_0C_{16}}{\alpha}\left(t^\alpha + \frac{t^{2\alpha}}{2}\right)\|p\|_{L^2(\Omega)}\|r\|_{C[0,T]},\ 0 < t < T. \tag{47}$$

Proof of Lemma 6. By (41), (46) and the notation

$$I_{11}(t) := \|g(u(\cdot,t))\|_{L^2(\Omega)},$$

we obtain

$$\begin{aligned}I_{11}(t) &\leq C_0C_{12}\|p\|_{L^2(\Omega)}\left[\int\limits_0^t (t-\theta)^{\alpha-1}r(\theta)d\theta + C_{14}e^{C_{15}t}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int\limits_0^t (t-\theta)^{2\alpha-1}r(\theta)d\theta\right] \\ &\leq C_0C_{12}\|p\|_{L^2(\Omega)}\|r\|_{C[0,T]} \\ &\times\left[\int\limits_0^t (t-\theta)^{\alpha-1}d\theta + C_{14}e^{C_{15}t}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int\limits_0^t (t-\theta)^{2\alpha-1}d\theta\right] \end{aligned} \tag{48}$$

for $0 < t < T$, which leads to (47). □

Lemma 7. *Under the hypotheses of Theorem 1, we have* $u \in C\big([0,T];H^2(\Omega)\cap H_0^1(\Omega)\big)$.

Proof of Lemma 7. By considering the term Lu with (12), (47), Lemma 3, Lemma 6 and the notations

$$\begin{aligned} I_{12}(x,t) &:= L\left(\sum_{n=1}^{\infty}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^{\alpha}\right)r(\tau)(p,\varphi_n)d\tau\right)\varphi_n(x)\right), \\ I_{13}(x,t) &:= L\left(\sum_{n=1}^{\infty}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^{\alpha}\right)(g(u(\cdot,\tau)),\varphi_n)d\tau\right)\varphi_n(x)\right), \end{aligned}$$

we obtain

$$\begin{aligned} \|Lu(\cdot,t)\|^2_{L^2(\Omega)} &\leq 2\|I_{12}(\cdot,t)\|^2_{L^2(\Omega)} + 2\|I_{13}(\cdot,t)\|^2_{L^2(\Omega)} \\ &= 2\left\|\sum_{n=1}^{\infty}\left(\int_0^t -\partial_\tau\left[E_{\alpha,1}\left(-\lambda_n(t-\tau)^{\alpha}\right)\right]r(\tau)(p,\varphi_n)d\tau\right)\varphi_n(x)\right\|^2_{L^2(\Omega)} \\ &\quad +2\left\|\sum_{n=1}^{\infty}\left(\int_0^t -\partial_\tau\left[E_{\alpha,1}\left(-\lambda_n(t-\tau)^{\alpha}\right)\right](g(u(\cdot,\tau)),\varphi_n)d\tau\right)\varphi_n(x)\right\|^2_{L^2(\Omega)} \\ &= 2\sum_{n=1}^{\infty}\left|\int_0^t -\partial_\tau\left[E_{\alpha,1}\left(-\lambda_n(t-\tau)^{\alpha}\right)\right]r(\tau)(p,\varphi_n)d\tau\right|^2 \\ &\quad +2\sum_{n=1}^{\infty}\left|\int_0^t -\partial_\tau\left[E_{\alpha,1}\left(-\lambda_n(t-\tau)^{\alpha}\right)\right](g(u(\cdot,\tau)),\varphi_n)d\tau\right|^2 \\ &\leq 2\|r\|^2_{C[0,T]}\sum_{n=1}^{\infty}|(p,\varphi_n)|^2\left|\int_0^t \partial_\tau\left[E_{\alpha,1}\left(-\lambda_n(t-\tau)^{\alpha}\right)\right]d\tau\right|^2 \\ &\quad +2\sum_{n=1}^{\infty}\max_{0\leq\tau\leq T}|(g(u(\cdot,\tau)),\varphi_n)|^2\left|\int_0^t \partial_\tau\left[E_{\alpha,1}\left(-\lambda_n(t-\tau)^{\alpha}\right)\right]d\tau\right|^2 \\ &\leq 2C_{17}^2\left[\|r\|^2_{C[0,T]}\|p\|^2_{L^2(\Omega)} + \max_{0\leq\tau\leq T}\|g(u(\cdot,\tau))\|^2_{L^2(\Omega)}\right] \\ &\leq 2C_{17}^2\|p\|^2_{L^2(\Omega)}\|r\|^2_{C[0,T]} \\ &\quad +2C_{17}^2\max_{0\leq\tau\leq T}\left[\frac{C_0C_{16}}{\alpha}\|p\|_{L^2(\Omega)}\|r\|_{C[0,T]}\left(\tau^{\alpha}+\frac{\tau^{2\alpha}}{2}\right)\right]^2 \\ &\leq C_0^2C_{18}^2\|p\|^2_{L^2(\Omega)}\|r\|^2_{C[0,T]}. \end{aligned} \tag{49}$$

By (12) and (13), inequality (49) becomes

$$\|u(\cdot,t)\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq C_0C_{18}\|p\|_{L^2(\Omega)}\|r\|_{C[0,T]}. \tag{50}$$

On $[0,T]$, taking the maximum of (50) with respect to the time variable, we write $u \in C\left([0,T];H^2(\Omega)\cap H_0^1(\Omega)\right)$. □

Now, we can prove Theorem 1.

Proof of Theorem 1. The proof of the existence and uniqueness of solution (21) can be seen from Lemma 4. Inequality (23) is shown in Lemma 5. Considering Lemmata 5 and 7, $u \in C\left([0,T];H^2(\Omega)\cap H_0^1(\Omega)\right)$ is obtained. Now, we only need to show that $\partial_t^{\alpha}u \in C\left([0,T];L^2(\Omega)\right)$. By considering Equation (1) with inequalities (47) and (49), we obtain

$$\begin{aligned}\|\partial_t^\alpha u(\cdot,t)\|^2_{L^2(\Omega)} &\le 4\|Lu(\cdot,t)\|^2_{L^2(\Omega)} + 4[r(t)]^2\|p\|^2_{L^2(\Omega)} + 2\|g(u(\cdot,t))\|^2_{L^2(\Omega)} \\ &\le C_0^2 C_{19}^2 \|p\|^2_{L^2(\Omega)}\|r\|^2_{C[0,T]} \end{aligned} \tag{51}$$

and by taking the maximum of (51) with respect to the variable t on $[0,T]$, we have $\partial_t^\alpha u \in C\big([0,T]; L^2(\Omega)\big)$, which concludes the proof. □

4. Stability of the Inverse Problem

In this section, we consider Problem 2 and analyze the stability of the inverse problem's solution. From the previous Section 3, we use the solution representation (21) of Problem 1 and Lemma 6, where the estimate (23) for Problem 1 plays a key role. With additional data and conditions, we obtain a stability estimate for the solution of Problem 2.

Let us remark that the inverse problem by [6] is a special case of Problem 2 and that the problem has a Lipschitz stable solution. For the solution of Problem 2, we also obtain a stability estimate of Lipschitz type.

Now, we present our main result for Problem 2.

Theorem 2. *Suppose that $g(0) = 0$ and we have (22) for the nonlinear term. Let*

$$p \in \widetilde{H}^{2\beta}(\Omega),\ \beta > 1 + \frac{3d}{4} \tag{52}$$

and let u satisfy (1), (6), (7) for $r \in C[0,T]$. We also assume that $p(x_0) \neq 0$. Then, there exists a positive constant C_{20} satisfying

$$\|r\|_{C[0,T]} \le C_{20}\Big(\|\partial_t^\alpha h\|_{C[0,T]} + \|g(h)\|_{C[0,T]}\Big). \tag{53}$$

Proof of Theorem 2. From Theorem 1, the solution u is in the form of (21). By writing the solution (21) in Equation (1), we obtain

$$\begin{aligned}&\partial_t^\alpha u(x,t)\\ =\ &\sum_{n=1}^{\infty}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right)r(\tau)(p,\varphi_n)d\tau\right)(-\lambda_n\varphi_n(x))\\ &+\sum_{n=1}^{\infty}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right)(g(u(\cdot,\tau)),\varphi_n)d\tau\right)(-\lambda_n\varphi_n(x))\\ &+p(x)r(t)+g(u(x,t))\end{aligned} \tag{54}$$

and by taking the maximum norm of (54) from both sides with respect to the variable t on $[0,T]$, we obtain

$$\|\partial_t^\alpha u(x,\cdot)\|^2_{C[0,T]} \le 4\Big\{[J_1(x)]^2 + [J_2(x)]^2 + [p(x)]^2\|r\|^2_{C[0,T]} + \|g(u(x,\cdot))\|^2_{C[0,T]}\Big\}, \tag{55}$$

where

$$J_1(x) := \left\|\sum_{n=1}^{\infty}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right)r(\tau)(p,\varphi_n)d\tau\right)\lambda_n\varphi_n(x)\right\|_{C[0,T]},$$

$$J_2(x) := \left\|\sum_{n=1}^{\infty}\left(\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^\alpha\right)(g(u(\cdot,\tau)),\varphi_n)d\tau\right)\lambda_n\varphi_n(x)\right\|_{C[0,T]}.$$

In order to write $x = x_0$ in (55), we need to investigate the convergence of the inequality. Since $p(x_0), \partial_t^\alpha u(x_0,t) = \partial_t^\alpha h(t)$ and $g(u(x_0,t)) = g(h(t)), 0 < t < T$ are known, we consider the terms J_1 and J_2. For the notations

$$J_3 := |esssup_{x\in\Omega}\lambda_n(p,\varphi_n)\varphi_n(x)|,$$

$$J_4(t) := \left|\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)r(t-\tau)d\tau\right|,$$

we have

$$\begin{aligned} J_1(x) &= \max_{0\le t\le T}\sum_{n=1}^\infty\left(\lambda_n\varphi_n(x)\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)r(\tau)(p,\varphi_n)d\tau\right) \\ &\le \max_{0\le t\le T}\sum_{n=1}^\infty|esssup_{x\in\Omega}\lambda_n(p,\varphi_n)\varphi_n(x)|\left|\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)r(t-\tau)d\tau\right| \\ &= \max_{0\le t\le T}\sum_{n=1}^\infty J_3\cdot J_4(t) \end{aligned} \tag{56}$$

and

$$\|\varphi_n\|_{L^\infty(\Omega)} \le C_{21}\|\varphi_n\|_{H^{2\beta-2-d}(\Omega)} \le C_{22}\|\varphi_n\|_{\widetilde{H}^{2\beta-2-d}(\Omega)} = C_{22}\left\|(-L)^{\beta-1-\frac{d}{2}}\varphi_n\right\|_{L^2(\Omega)} \tag{57}$$

by using the hypothesis (52), inequality (15), Lemma 1 with the properties of $\widetilde{H}^s(\Omega)$ spaces. From (57) and the properties of $\{\varphi_n\}_{n=1}^\infty$, we write

$$\begin{aligned} J_3 &= \lambda_n(p,\varphi_n)\|\varphi_n\|_{L^\infty(\Omega)} \\ &\le C_{22}\lambda_n(p,\varphi_n)\left\|(-L)^{\beta-1-d/2}\varphi_n\right\|_{L^2(\Omega)} \\ &= C_{22}\lambda_n(p,\varphi_n)\left\|\lambda_n^{\beta-1-d/2}\varphi_n\right\|_{L^2(\Omega)} \\ &= C_{22}\lambda_n^{-d/2}\left\|\left(\lambda_n^\beta p,\varphi_n\right)\varphi_n\right\|_{L^2(\Omega)} \\ &= C_{22}\lambda_n^{-d/2}\left((-L)^\beta p,\varphi_n\right)_{L^2(\Omega)}. \end{aligned} \tag{58}$$

Using Lemma 3, we obtain

$$\begin{aligned} J_4(t) &\le \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^\alpha)|r(t-\tau)|d\tau \\ &\le \frac{\|r\|_{C[0,T]}}{\lambda_n}\int_0^t -\frac{d}{d\tau}E_{\alpha,1}(-\lambda_n\tau^\alpha)d\tau \\ &\le \frac{\|r\|_{C[0,T]}}{\lambda_n}\left[2[E_{\alpha,1}(-\lambda_n t^\alpha)]^2 + 2[E_{\alpha,1}(0)]^2\right]^{1/2} \\ &\le \frac{C_{23}\|r\|_{C[0,T]}}{\lambda_n}. \end{aligned} \tag{59}$$

Writing inequalities (58)–(59) in (56) and by using (13), Lemma 2, the Cauchy–Schwarz inequality with the assumption $d \in \{1,2,3\}$, we obtain

$$\begin{aligned} J_1(x) &\leq \sum_{n=1}^{\infty} C_{22}\left((-L)^{\beta}p,\varphi_n\right)\frac{C_{23}\|r\|_{C[0,T]}}{\lambda_n^{1+d/2}} \\ &\leq \frac{C_{22}C_{23}}{C_{24}}\|r\|_{C[0,T]}\left[\sum_{n=1}^{\infty}\left|\left((-L)^{\beta}p,\varphi_n\right)\right|^2\right]^{1/2}\left[\sum_{n=1}^{\infty}\frac{1}{\left(n^{(2/d)+1}\right)^2}\right]^{1/2} \end{aligned}$$

and

$$\begin{aligned} J_1(x) &\leq \frac{C_{22}C_{23}}{C_{24}}\|r\|_{C[0,T]}\|p\|_{\widetilde{H}^{2\beta}(\Omega)}\left[\sum_{n=1}^{\infty}\frac{1}{n^{(4/d)+2}}\right]^{1/2} \\ &\leq C_{25}\|r\|_{C[0,T]}. \end{aligned}$$

A similar investigation was performed for the term J_2. Using the inequalities of Cauchy–Schwarz and (48), we have

$$\begin{aligned} J_2(x) &\leq \max_{0\leq t\leq T}\sum_{n=1}^{\infty}(\lambda_n\varphi_n(x))\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^{\alpha}\right)\|g(u(\cdot,\tau))\|_{L^2(\Omega)}d\tau \\ &\leq \max_{0\leq t\leq T}\sum_{n=1}^{\infty}\Bigg\{\frac{C_0C_{16}}{\alpha}\|p\|_{L^2(\Omega)}\|r\|_{C[0,T]}(\lambda_n\varphi_n(x)) \\ &\quad\times\left[\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(t-\tau)^{\alpha}\right)\left(\tau^{\alpha}+\frac{\tau^{2\alpha}}{2}\right)d\tau\right]\Bigg\} \\ &\leq \frac{C_0C_{16}\|r\|_{C[0,T]}}{\alpha}\max_{0\leq t\leq T}\left(t^{\alpha}+\frac{t^{2\alpha}}{2}\right)\sum_{n=1}^{\infty}\{J_5\cdot J_6(t)\} \end{aligned} \tag{60}$$

for

$$J_5 := \left|\lambda_n\|p\|_{L^2(\Omega)}\varphi_n(x)\right|$$

and

$$J_6(t) := \left|\int_0^t\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n\tau^{\alpha})d\tau\right|.$$

Using (14) and the hypothesis $p\in\widetilde{H}^{2\beta}(\Omega)$, we write $p\in L^2(\Omega)$. Thus, we obtain

$$\begin{aligned} J_5 &\leq C_{26}\|p\|_{L^2(\Omega)}\lambda_n\left\|(-L)^{\beta-1-d/2}\varphi_n\right\|_{L^2(\Omega)} \\ &= C_{26}\|p\|_{L^2(\Omega)}\left\|\lambda_n^{\beta-d/2}\varphi_n\right\|_{L^2(\Omega)} \\ &= C_{26}\lambda_n^{-d/2}\left\|\left[\sum_{n=1}^{\infty}\lambda_n^{2\beta}|(p,\varphi_n)|^2\right]^{1/2}\varphi_n\right\|_{L^2(\Omega)} \\ &= C_{26}\lambda_n^{-d/2}\|p\|_{\widetilde{H}^{2\beta}(\Omega)}, \end{aligned} \tag{61}$$

by (10), (13) and (57). Considering (59), it can be easily seen that

$$J_6(t)\leq\frac{C_{27}}{\lambda_n}. \tag{62}$$

Writing (61), (62) in (60) and taking into account Lemma 2, we obtain the inequality

$$\begin{aligned} J_2(x) &\leq C_0 C_{28} \|p\|_{\widetilde{H}^{2\beta}(\Omega)} \|r\|_{C[0,T]} \max_{0\leq t\leq T} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{(2/d)+1}} \\ &\leq C_0 \frac{C_{28}}{C_{29}} \|p\|_{\widetilde{H}^{2\beta}(\Omega)} \|r\|_{C[0,T]} \max_{0\leq t\leq T} \sum_{n=1}^{\infty} \frac{1}{n^{(4/d)+2}} \\ &\leq C_0 C_{30} \|r\|_{C[0,T]}. \end{aligned}$$

Therefore, in equality (54), the series are convergent in $C\big(\overline{\Omega} \times [0,T]\big)$ and we can write

$$\begin{aligned} \partial_t^\alpha u(x_0,t) &= \sum_{n=1}^{\infty} -\lambda_n \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) r(\tau)(p,\varphi_n) d\tau \right) \varphi_n(x_0) \\ &\quad + \sum_{n=1}^{\infty} -\lambda_n \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big) (g(u(\cdot,\tau)),\varphi_n) d\tau \right) \varphi_n(x_0) \\ &\quad + p(x_0) r(t) + g(u(x_0,t)) \end{aligned} \tag{63}$$

for every t in the domain. In order to estimate (63), we set

$$\begin{aligned} J_7(x,\tau) &:= \sum_{n=1}^{\infty} -\lambda_n E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)(p,\varphi_n)\varphi_n(x), \\ J_8(x;t,\tau) &:= \sum_{n=1}^{\infty} -\lambda_n E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big)\big(g(u(\cdot,\tau)),\varphi_n\big)\varphi_n(x), \\ J_9(x,\tau) &:= \sum_{n=1}^{\infty} |\lambda_n (g(u(\cdot,\tau)),\varphi_n)\varphi_n(x)|, \\ J_{10}(\tau) &:= \left| \int_0^\tau \Big[(\tau-\theta)^{\alpha-1} + (\tau-\theta)^{2\alpha-1} \Big] r(\theta) d\theta \right|. \end{aligned}$$

By equalities (10), (12), (13) and the inequalities of Cauchy–Schwarz, (48), (58), (61) with Lemmata 1, 2, 3, we obtain

$$\begin{aligned} \|J_7(x,\cdot)\|_{C[0,T]} &\leq \max_{0\leq \tau\leq T} \left| \sum_{n=1}^{\infty} -\lambda_n (p,\varphi_n) E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)\varphi_n(x) \right| \\ &\leq \max_{0\leq \tau\leq T} \left| \sum_{n=1}^{\infty} \|\lambda_n (p,\varphi_n)\varphi_n\|_{L^\infty(\Omega)} |E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)| \right| \\ &\leq \left| C_{31} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{d/2}} \Big((-L)^\beta p, \varphi_n \Big)_{L^2(\Omega)} \right| \\ &\leq \left| \frac{C_{31}}{C_{32}} \left[\sum_{n=1}^{\infty} \frac{1}{n^2} \right]^{1/2} \left[\sum_{n=1}^{\infty} \Big((-L)^\beta p, \varphi_n \Big)^2 \right]^{1/2} \right| \\ &\leq C_{33} \|p\|_{\widetilde{H}^{2\beta}(\Omega)} \\ &\leq C_{34}, \end{aligned} \tag{64}$$

$$\begin{aligned} |J_8(x;t,\tau)| &\leq \sum_{n=1}^{\infty} \left| -\lambda_n E_{\alpha,\alpha}\big(-\lambda_n (t-\tau)^\alpha\big)\big(g(u(\cdot,\tau)),\varphi_n\big)\varphi_n(x) \right| \\ &\leq C_{35} J_9(x,\tau) \end{aligned} \tag{65}$$

and

$$\begin{aligned} J_9(x,\tau) &\leq \left| \left[\sum_{n=1}^{\infty} [\lambda_n \varphi_n(x)]^2 \right]^{1/2} \left[\sum_{n=1}^{\infty} |(g(u(\cdot,\tau)),\varphi_n)|^2 \right]^{1/2} \right| \\ &= \left| \left[\sum_{n=1}^{\infty} [\lambda_n \varphi_n(x)]^2 \right]^{1/2} \|g(u(\cdot,\tau))\|_{L^2(\Omega)} \right| \\ &\leq \Bigg| C_0 C_{12} \left[\sum_{n=1}^{\infty} \left\| \lambda_n \|p\|_{L^2(\Omega)} \varphi_n(x) \right\|^2_{L^\infty(\Omega)} \right]^{1/2} \\ &\quad \times \left[\int_0^\tau (\tau-\theta)^{\alpha-1} r(\theta)d\theta + C_{14} e^{C_{15}\tau} \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)} \int_0^\tau (\tau-\theta)^{2\alpha-1} r(\theta)d\theta \right] \Bigg| \\ &\leq C_0 C_{36} \|p\|_{\widetilde{H}^{2\beta}(\Omega)} \left[\sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n^{d/2}} \right)^2 \right]^{1/2} \\ &\quad \times \left| \int_0^\tau \left[(\tau-\theta)^{\alpha-1} + (\tau-\theta)^{2\alpha-1} \right] r(\theta)d\theta \right| \\ &\leq C_{37} J_{10}(\tau) \end{aligned} \tag{66}$$

for any $0 < \tau < T$. Using the notation

$$J_{11}(\tau) := \tau^{\alpha-1} + \tau^{2\alpha-1},$$

and Young's convolution inequality, we also obtain

$$\begin{aligned} \|J_{10}\|_{L^1(0,t)} &= \|J_{11} * r\|_{L^1(0,t)} \\ &\leq \|J_{11}\|_{L^1(0,t)} \|r\|_{L^1(0,t)} \\ &\leq \left(\frac{T^\alpha}{\alpha} + \frac{T^{2\alpha}}{2\alpha} \right) \int_0^T |r(\tau)| d\tau \\ &\leq \left(\frac{T^{\alpha+1}}{\alpha} + \frac{T^{2\alpha+1}}{2\alpha} \right) \|r\|_{C[0,T]}. \end{aligned} \tag{67}$$

By (64)–(67), we have $J_7, J_9 \in C[0,T]$ for any $x \in \Omega$ and in particular $J_7(x_0,t), J_9(x_0,t) \in C[0,T]$. We can use Lebesgue's theorem and the order of integrations and summations in (63) can be changed. Now, by (54) and the additional data $h(t) = u(x_0,t)$, we can write

$$\begin{aligned} \partial_t^\alpha h(t) &= \int_0^t (t-\tau)^{\alpha-1} J_7(x_0, t-\tau) r(\tau) d\tau + \int_0^t (t-\tau)^{\alpha-1} J_8(x_0; t, \tau) d\tau \\ &\quad + p(x_0) r(t) + g(h(t)) \end{aligned} \tag{68}$$

for $0 < t < T$. Using the hypothesis $p(x_0) \neq 0$ in (68), we have

$$\begin{aligned} r(t) &= \frac{1}{p(x_0)} \Bigg[\partial_t^\alpha h(t) - \int_0^t (t-\tau)^{\alpha-1} J_7(x_0, t-\tau) r(\tau) d\tau \\ &\quad - \int_0^t (t-\tau)^{\alpha-1} J_8(x_0; t, \tau) d\tau - g(h(t)) \Bigg] \end{aligned}$$

and

$$\begin{aligned} |r(t)|^2 \quad \le \quad & \frac{2}{[p(x_0)]^2}\Big[2|\partial_t^\alpha h(t)|^2 + 2|g(h(t))|^2 \\ & + \left|\int_0^t (t-\tau)^{\alpha-1} J_7(x_0, t-\tau) r(\tau) d\tau + \int_0^t (t-\tau)^{\alpha-1} J_8(x_0; t, \tau) d\tau\right|^2\Big]. \end{aligned}$$

By the change of variable $\varepsilon = t - \tau$, we obtain

$$\begin{aligned} |r(t)|^2 \quad \le \quad & \frac{2}{[p(x_0)]^2}\Big[2\max_{0\le t\le T}|\partial_t^\alpha h(t)|^2 + 2\max_{0\le t\le T}|g(h(t))|^2 \\ & + \left|\int_0^t \varepsilon^{\alpha-1}\left[\max_{0\le\varepsilon\le T} J_7(x_0,\varepsilon)\right] r(t-\varepsilon) d\varepsilon + C_{35}\int_0^t (t-\tau)^{\alpha-1} J_9(x_0,\tau) d\tau\right|^2\Big] \\ \le \quad & \frac{2}{[p(x_0)]^2}\Big[2\max_{0\le t\le T}|\partial_t^\alpha h(t)|^2 + 2\max_{0\le t\le T}|g(h(t))|^2 \\ & + \Big| \|J_7(x_0,\cdot)\|_{C[0,T]}\int_0^t (t-\tau)^{\alpha-1}|r(\tau)| d\tau \\ & + C_{35}C_{36}\int_0^t (t-\tau)^{\alpha-1}\int_0^\tau \left[(\tau-\theta)^{\alpha-1} + (\tau-\theta)^{2\alpha-1}\right]|r(\theta)| d\theta d\tau\Big|^2\Big]. \end{aligned} \tag{69}$$

Using the notation

$$J_{12}(t) := \int_0^t (t-\tau)^{\alpha-1}\int_0^\tau \left[(\tau-\theta)^{\alpha-1} + (\tau-\theta)^{2\alpha-1}\right] r(\theta) d\theta d\tau,$$

the last term on the right-hand side of (69) can be written as

$$\begin{aligned} J_{12}(t) \quad = \quad & \int_0^t r(\theta)\int_\theta^t (\tau-\theta)^{\alpha-1}(t-\tau)^{\alpha-1} d\tau d\theta \\ & + \int_0^t r(\theta)\int_\theta^t (\tau-\theta)^{\alpha-1}(t-\tau)^{2\alpha-1} d\tau d\theta, \end{aligned} \tag{70}$$

and the terms on the right-hand side of equality (70) yield two special cases of formula (17). By writing $q = z = \alpha$ and $q = \alpha, z = 2\alpha$ in (17), equality (70) becomes

$$\begin{aligned} J_{12}(t) \quad = \quad & \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int_0^t (t-\theta)^{2\alpha-1} r(\theta) d\theta + \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\Gamma(3\alpha)}\int_0^t (t-\theta)^{3\alpha-1} r(\theta) d\theta \\ \le \quad & \frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\int_0^t \left[\max_{0\le\theta\le t}(t-\theta)^\alpha\right](t-\theta)^{\alpha-1} r(\theta) d\theta \\ & + \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\Gamma(3\alpha)}\int_0^t \left[\max_{0\le\theta\le t}(t-\theta)^{2\alpha}\right](t-\theta)^{\alpha-1} r(\theta) d\theta \end{aligned}$$

and

$$J_{12}(t) \leq \left[\frac{(\Gamma(\alpha))^2 t^\alpha}{\Gamma(2\alpha)} + \frac{\Gamma(\alpha)\Gamma(2\alpha)t^{2\alpha}}{\Gamma(3\alpha)} \right] \int_0^t (t-\theta)^{\alpha-1} r(\theta) d\theta. \quad (71)$$

Considering (69) with (71), we obtain

$$\begin{aligned} |r(t)| \leq{}& C_{38}\left(\|\partial_t^\alpha h\|_{C[0,T]} + \|g(h)\|_{C[0,T]} \right) \\ &+C_{38}\left(\|J_7(x_0,\cdot)\|_{C[0,T]} + \frac{(\Gamma(\alpha))^2 t^\alpha}{\Gamma(2\alpha)} + \frac{\Gamma(\alpha)\Gamma(2\alpha)t^{2\alpha}}{\Gamma(3\alpha)} \right) J_{13}(t) \end{aligned} \quad (72)$$

for $t \in (0,T)$, where

$$J_{13}(t) := \int_0^t (t-\theta)^{\alpha-1} |r(\theta)| d\theta.$$

Since, we have

$$C_{38}\left(\|\partial_t^\alpha h\|_{C[0,T]} + \|g(h)\|_{C[0,T]} \right) \in L^1(0,T),$$

the generalized Grönwall inequality can be used for (72). Therefore, we write

$$\begin{aligned} |r(t)| \leq{}& C_{38}\left(\|\partial_t^\alpha h\|_{C[0,T]} + \|g(h)\|_{C[0,T]} \right) \\ &+C_{38}C_{39}e^{C_{40}t} \int_0^t (t-\tau)^{\alpha-1} \left(\|\partial_t^\alpha h\|_{C[0,T]} + \|g(h)\|_{C[0,T]} \right) d\tau \\ ={}& C_{38}\left(\|\partial_t^\alpha h\|_{C[0,T]} + \|g(h)\|_{C[0,T]} \right) \left(1 + C_{39}e^{C_{40}t} \int_0^t (t-\theta)^{\alpha-1} d\theta \right) \end{aligned} \quad (73)$$

and

$$\int_0^t (t-\theta)^{\alpha-1} d\theta = \frac{t^\alpha}{\alpha} \leq \frac{T^\alpha}{\alpha} \quad (74)$$

for any t in the domain. With (74), inequality (73) becomes

$$|r(t)| \leq C_{38}\left(1 + C_{39}e^{C_{40}T} \frac{T^\alpha}{\alpha} \right) \left(\|\partial_t^\alpha h\|_{C[0,T]} + \|g(h)\|_{C[0,T]} \right). \quad (75)$$

Finally, taking the maximum with respect to t on both sides of (75), we obtain the stability inequality (53) for the solution (u,r) of Problem 2.

This completes the proof. □

5. Concluding Remarks

In this study, we first consider a direct problem for a nonlinear time-fractional partial differential equation with initial and boundary conditions. We study the well-posedness of the problem by the methodology of [15]. Then, we apply the results to an inverse problem with additional data and we obtain the stability of the solution (u,r). These two problems are generalizations of the problems that were previously discussed by [6].

In this paper, we consider the case of $p(x_0) \neq 0$. On the other hand, as far as we know, the case of $p(x_0) = 0$ is still an open problem, which is important in applications. In the linear case, it was studied in [32]. For further research, this paper can be used to explore other direct and inverse problems for nonlinear fractional partial differential equations.

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References

1. Alves, S.B.; de Oliveira, G.F.; de Oliveira, L.C.; de Silans, T.P.; Chevrollier, M.; Oriá, M.; de S. Cavalcante, H.L.D. Characterization of Diffusion Processes: Normal and Anomalous Regimes. *Phys. A Stat. Mech. Appl.* **2016**, *447*, 392–401. [CrossRef]
2. Lenzi, E.K.; Mendes, R.S.; Kwok Sau, F.A.; Da Silva, L.R.; Lucena, L.S. Solutions for a Fractional Nonlinear Diffusion Equation: Spatial Time Dependent Diffusion Coefficient and External Forces. *J. Math. Phys.* **2004**, *45*, 3444–3452. [CrossRef]
3. Jing, X.; Jia, J.; Peng, J. Well-posedness for a Nonlocal Nonlinear Diffusion Equation and Applications to Inverse Problems. *Appl. Anal.* **2020**, *99*, 2609–2623. [CrossRef]
4. Kernevez, J.P. *Enzyme Mathematics*; North-Holland: Amsterdam, The Netherlands, 1980.
5. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
6. Sakamoto, K.; Yamamoto, M. Initial Value/boundary Value Problems for Fractional Diffusion-wave Equations and Applications to Some Inverse Problems. *J. Math. Anal. Appl.* **2011**, *382*, 426–447. [CrossRef]
7. Cheng, J.; Nakagawa, J.; Yamamoto, M.; Yamazaki, T. Uniqueness in an Inverse Problem for a One-dimensional Fractional Diffusion Equation. *Inverse Probl.* **2009**, *25*, 115002. [CrossRef]
8. Luchko, Y. Some Uniqueness and Existence Results for the Initial-boundary-value Problems for the Generalized Time-fractional Diffusion Equation. *Comput. Math. Appl.* **2010**, *59*, 1766–1772. [CrossRef]
9. Beckers, S.; Yamamoto, M. Regularity and Unique Existence of Solution to Linear Diffusion Equation with Multiple Time-fractional Derivatives. In *Control and Optimization with PDE Constraints*; Bredies, K., Clason, C., Kunisch, K., Winckel, G., Eds.; Springer: Basel, Switzerland; Heidelberg, Germany, 2013; pp. 45–55.
10. Luchko, Y.; Rundell, W.; Yamamoto, M.; Zuo, L. Uniqueness and Reconstruction of an Unknown Semilinear Term in a Time-fractional Reaction-diffusion Equation. *Inverse Probl.* **2013**, *29*, 065019. [CrossRef]
11. Kian, Y.; Yamamoto, M. On Existence and Uniqueness of Solutions for Semilinear Fractional Wave Equations. *Fract. Calc. Appl. Anal.* **2017**, *20*, 117–138. [CrossRef]
12. Jin, B.; Li, B.; Zhou, Z.H.I. Numerical Analysis of Nonlinear Subdiffusion Equations. *SIAM J. Numer. Anal.* **2018**, *56*, 1–23. [CrossRef]
13. Ngoc, T.B.; Tuan, N.H.; O'Regan, D. Existence and Uniqueness of Mild Solutions for a Final Value Problem for Nonlinear Fractional Diffusion Systems. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *78*, 104882. [CrossRef]
14. Jin, B. *Fractional Differential Equations: An Approach Via Fractional Derivatives*, 1st ed.; Springer: Berlin/Heidelberg, Germany, 2021.
15. Arıbaş, Ö.; Gölgeleyen, İ.; Yıldız, M. On the Solvability of an Initial-boundary Value Problem for a Non-linear Fractional Diffusion Equation. *AIMS Math.* **2023**, *8*, 5432–5444. [CrossRef]
16. Yamamoto, M. Carleman Estimates for Parabolic Equations and Applications. *Inverse Probl.* **2009**, *25*, 123013. [CrossRef]
17. Yamamoto, M.; Zhang, Y. Conditional Stability in Determining a Zeroth-order Coefficient in a Half-order Fractional Diffusion Equation by a Carleman Estimate. *Inverse Probl.* **2012**, *28*, 105010. [CrossRef]
18. Huang, X.; Imanuvilov, O.Y.; Yamamoto, M. Stability for Inverse Source Problems by Carleman Estimates. *Inverse Probl.* **2020**, *242*, 125006. [CrossRef]
19. Gölgeleyen, İ.; Kaytmaz, Ö. Conditional Stability for a Cauchy Problem for the Ultrahyperbolic Schrödinger Equation. *Appl. Anal.* **2022**, *101*, 1505–1516. [CrossRef]
20. Au, V.V.; Jafari, H.; Hammouch, Z.; Tuan, N.H. On a Final Value Problem for a Nonlinear Fractional Pseudo-parabolic Equation. *Electron. Res. Arch.* **2021**, *29*, 1709–1734. [CrossRef]
21. Guo, Z.; Yao, W.; Sun, J.; Wu, B. Nonlinear Fractional Diffusion Model for Deblurring Images With Textures. *Inverse Probl. Imaging Vol.* **2019**, *13*, 1161–1188. [CrossRef]
22. Tuan, N.H.; Huynh, L.N.; Ngoc, T.B.; Zhou, Y. On a Backward Problem for Nonlinear Fractional Diffusion Equations. *Appl. Math. Lett.* **2019**, *92*, 76–84. [CrossRef]
23. Oulmelk, A.; Afraites, L.; Hadri, A. An Inverse Problem of Identifying the Coefficient in a Nonlinear Time-fractional Diffusion Equation. *Comput. Appl. Math.* **2023**, *42*, 65. [CrossRef]
24. Triebel, H. *Theory of Function Spaces II*; Birkhauser Verlag: Basel, Switzerland, 1992.
25. Di Nezza, E.; Palatucci, G.; Valdinoci, E. Hitchhiker's Guide to the Fractional Sobolev Spaces. *Bull. Des Sci. Math.* **2012**, *136*, 521–573. [CrossRef]

26. Evans, L.C. *Partial Differential Equations*; American Mathematical Society: Providence, RI, USA, 1998.
27. Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, 1st ed.; Springer: New York, NY, USA, 2011.
28. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
29. Kubica, A.; Ryszewska, K.; Yamamoto, M. *Time-Fractional Differential Equations: A Theoretical Introduction*; Springer: Tokyo, Japan, 2020.
30. Mikhailov, V.P. *Partial Differential Equations*; Mir Publishers: Moscow, Russia, 1978.
31. Tarafdar, E.U.; Chowdhury, M.S.R. *Topological Methods for Set-Valued Nonlinear Analysis*; World Scientific: Singapore, 2008.
32. Liu, Y.; Rundell, W.; Masahiro, Y. Strong Maximum Principle for Fractional Diffusion Equations and an Application to an Inverse Source Problem. *Fract. Calc. Appl. Anal.* **2016**, *19*, 888–906. [CrossRef]