Fixed Point Results with Applications to Fractional Differential Equations of Anomalous Diffusion

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Abstract: The main objective of this manuscript is to define the concepts of $F(\lambda,h)$-contraction and $(\alpha,\eta)$-Reich type interpolative contraction in the framework of orthogonal $F$-metric space and prove some fixed point results. Our primary result serves as a cornerstone, from which established findings in the literature emerge as natural consequences. To enhance the clarity of our novel contributions, we furnish a significant example that not only strengthens the innovative findings but also facilitates a deeper understanding of the established theory. The concluding section of our work is dedicated to the application of these results in establishing the existence and uniqueness of a solution for a fractional differential equation of anomalous diffusion.

Keywords: fixed point; orthogonal $F$-metric space; $F(\lambda,h)$-contraction; fractional differential equations; anomalous diffusion

1. Introduction

Fixed point theory is a thriving field of functional analysis which provides a powerful tool with diverse applications in different areas. This field has demonstrably shaped the development of essential concepts and methodologies and remains an active area of investigation and advancement. At its core is the investigation of metric spaces (MSs), which provide a framework for defining distances between elements of a set. This idea proves surprisingly useful in many areas of science, leading to many successful applications (see [1,2]). In pursuit of broader applicability, researchers have expanded the concept of MSs, paving the way for fruitful advancements in this theory. In contrast to the standard triangle inequality of a MS, Branciari [3] proposed a generalized metric based on a more relaxed inequality with four terms, termed the rectangular inequality. This kind of MS is famous in literature as the Branciari MS. While classical MSs rely on a continuous metric to define their topology, Bakhtin [4] developed the idea of $b$-metric spaces ($b$-MSs), which relaxes this requirement. Later on, Czerwik [5] provided a more comprehensive description of $b$-MSs. Gordji et al. [6] took things a step further by incorporating orthogonality into MSs, leading to the concept of orthogonal metric spaces ($O$-MSs) and associated some results for contraction mappings. Later, Mani et al. [7] and Gungor et al. [8] utilized the concept of $O$-MSs and investigated fixed points for various generalized contractions. In 2018, Jleli et al. [9] began the motion of $\tilde{F}$-metric space ($\tilde{F}$-MS) as a broadened framework encompassing both MSs and $b$-MSs. Afterwards, Hussain et al. [10] studied the solution for a differential equation by obtaining some new results in this direction. Subsequently, Asif et al. [11] and Faraji et al. [12] presented fixed point theorems in $\tilde{F}$-MSs and investigated the solutions of integral equations. Kanwal et al. [13] made a significant contribution by merging the concepts of $\tilde{F}$-MSs and orthogonal sets. This novel concept, termed orthogonal...
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\(\mathfrak{F}\)-MSs (\(O\mathfrak{F}\)-MSs), has opened doors for new fixed point theorems. By generalizing MSs, we achieve enhanced adaptability in constructing models that accurately represent real-world phenomena. This expanded framework allows us to address problems that may not be effectively captured by conventional metric structures. Ahmad et al. [14] introduced a new idea called an \((a, \perp, \mathcal{F})\)-contraction in the background of \(O\mathfrak{F}\)-MSs and presented some results.

Stefan Banach [15] is considered a trailblazer in the field of this theory who introduced the idea of contraction in the framework of MSs and established a fundamental theorem. This theorem has found widespread application across various disciplines including physics, chemistry, computer science, machine learning, engineering, economics, and numerous other areas. Subsequent research efforts have focused on refining and extending this theorem in diverse ways, contributing to its enduring relevance and impact. Building upon the concept of traditional contractions, Wardowski [16] introduced the idea of \(F\)-contractions in 2011, offering a more general framework. This notion involves the incorporation of an additional function, represented by \(F\), which encodes the relationship between the distances of corresponding points within a MS. This generalization facilitates a more refined investigation of contractive properties, thereby furnishing an effective tool for analyzing the convergence patterns of iterative processes. Samet et al. [17] put forward the idea of \(\alpha\)-admissibility which extends the reach of admissibility by introducing a function \(\alpha(e, \kappa)\) that influences the significance of contractiveness based on the specific pairs of points involved. Ramezani et al. [18] bridged the gap between the concepts of orthogonality and \(\alpha\)-admissibility by introducing the concept of an orthogonal \(\alpha\)-admissible mapping. This innovative concept offers a new perspective for studying properties of mappings in certain mathematical settings. Ansari et al. [19,20] introduced some C-class functions to establish a result as an expansion of the Banach contraction principle. For a more comprehensive exploration of this topic, we direct readers to references [21–26].

In spite of that, fixed point theory plays a pivotal part in the study of solutions to differential equations, spanning a broad spectrum from ordinary differential equations to more intricate fractional differential equations. In the realm of ordinary differential equations, fixed point techniques are employed to investigate the uniqueness, existence, and stability of solutions. The theory’s adaptability extends seamlessly to fractional differential equations, where it provides valuable insights into the behavior of solutions involving fractional derivatives. These days, fractional differential equations are increasingly utilized across diverse sectors like biology, economics, physics, and engineering (see [27,28]). Driven by these practical applications, numerous scientists specializing in the application of fixed point theory have oriented their research towards resolving a variety of tangible problems through the application of integral and differential equations.

In the current research work, we define the notions of \(F_1\)-(\(\lambda, h\))-contraction and \((\alpha, \eta)\)-Reich type interpolative contraction in the framework of an \(O\mathfrak{F}\)-MS and prove some fixed point results. Our primary findings yield well-established results from existing literature as natural consequences. As a practical application, we investigate the solution for a fractional differential equation.

2. Preliminaries

We start this section with the famous and pioneering fixed point theorem the “Banach Contraction Principle” in this way.

**Theorem 1.** Let \((\mathcal{P}, d)\) be a complete MS and \(B : \mathcal{P} \rightarrow \mathcal{P}\). If there exists some constant \(\lambda \in [0,1)\) such that

\[
d(Be, B\kappa) \leq \lambda d(e, \kappa)
\]

for all \(e, \kappa \in \mathcal{P}\), then \(B\) has a unique fixed point.
In a recent development, Wardowski [16] introduced the concept of F-contractions, a novel type of contractive mapping. This innovation led to the establishment of new fixed point theorems applicable within the framework of complete MSs.

**Definition 1.** Let $\Omega$ be the family of the continuous functions $F : (0, +\infty) \to \mathbb{R}$ satisfying the following:

1. For all $t_1, t_2 \in (0, +\infty)$ such that $t_1 < t_2 \implies F(t_1) < F(t_2)$;
2. For every sequence $\{t_n\} \subseteq (0, +\infty)$, $\lim_{n \to \infty} t_n = 0$ and $\lim_{n \to \infty} F(t_n) = -\infty$ are equivalent;
3. There exists $r \in (0, 1)$ such that $\lim_{t \to 0^+} t^r F(t) = 0$.

A self-mapping $B : \mathcal{P} \to \mathcal{P}$ is called an F-contraction if there exist a function $F \in \Omega$ satisfying (F1)-(F3) and a constant $\tau > 0$ such that

$$d(B\epsilon, B\kappa) > 0 \implies \tau + F(d(B\epsilon, B\kappa)) \leq F(d(\epsilon, \kappa))$$

for all $\epsilon, \kappa \in \mathcal{P}$.

**Theorem 2** ([16]). If $(\mathcal{P}, d)$ is a complete MS and $B : \mathcal{P} \to \mathcal{P}$ satisfies the properties of an F-contraction, then $B$ possesses a unique fixed point in $\mathcal{P}$.

Czerwik [5] presented the concept of the b-metric by altering the triangular inequality of MS in this manner:

- for all $\epsilon, \omega, \kappa \in \mathcal{P}$ and for some $b \geq 1$,
  $$d(\epsilon, \kappa) \leq b[d(\epsilon, \omega) + d(\omega, \kappa)].$$

Gordji et al. [6] put forward the idea of the orthogonal set ($O$-set) and reinforced the concept of the MS by introducing the orthogonal MS ($O$-MS) in 2017.

**Definition 2.** Let $\mathcal{P} \neq \emptyset$. $\mathcal{P}$ is called an $O$-set if there exists some binary relation $\perp \subseteq \mathcal{P} \times \mathcal{P}$ fulfilling the following axiom

there exists $\epsilon_0$ such that $\kappa \perp \epsilon_0$ or $\epsilon_0 \perp \kappa$ for all $\kappa \in \mathcal{P}$.

Furthermore, $\epsilon_0$ is termed an orthogonal point, and an $O$-set is denoted as $(\mathcal{P}, \perp)$.

**Definition 3** ([6]). A sequence $\{\epsilon_n\}$ in $O$-set $(\mathcal{P}, \perp)$ is considered as an orthogonal sequence ($O$-sequence) if $\epsilon_n \perp \epsilon_{n+1}$ or $\epsilon_{n+1} \perp \epsilon_n$ for all $n \in \mathbb{N}$.

**Definition 4** ([6]). Let $(\mathcal{P}, \perp)$ be an $O$-set. A mapping $B : \mathcal{P} \to \mathcal{P}$ is known as $\perp$-preserving if $\epsilon \perp \kappa$ implies $B\epsilon \perp B\kappa$.

Jleli et al. [9] began the notion of $\mathfrak{F}$-metric space ($\mathfrak{F}$-MS) in this fashion.

Let $\mathfrak{F}$ be the class of functions $\xi : (0, +\infty) \to \mathbb{R}$ satisfying

1. $0 < \epsilon_1 < \epsilon_2 \Rightarrow \xi(\epsilon_1) \leq \xi(\epsilon_2)$;
2. For all $\{\epsilon_n\} \subseteq \mathbb{R}^+$, $\lim_{n \to \infty} \epsilon_n = 0 \iff \lim_{n \to \infty} \xi(\epsilon_n) = -\infty$.

**Definition 5** ([9]). Let $\mathcal{P} \neq \emptyset$ and $d : \mathcal{P} \times \mathcal{P} \to [0, +\infty)$. Assume that there exists $(\xi, \nu) \in \mathfrak{F} \times [0, +\infty)$ such that

1. $(\epsilon, \kappa) \in \mathcal{P} \times \mathcal{P}, d(\epsilon, \kappa) = 0 \iff \epsilon = \kappa$;
2. $d(\epsilon, \kappa) = d(\kappa, \epsilon)$, for all $\epsilon, \kappa \in \mathcal{P}$;
3. For all $(\epsilon, \kappa) \in \mathcal{P} \times \mathcal{P}$, and $(\epsilon_i)_{i=1}^N \subset \mathcal{P}$, with $(\epsilon_1, \epsilon_N) = (\epsilon, \kappa)$ and $N \geq 2$, we have
\[ d(\epsilon, \kappa) > 0 \Rightarrow \xi(d(\epsilon, \kappa)) \leq \xi\left(\sum_{i=1}^{N-1} d(\epsilon_i, \epsilon_{i+1})\right) + b. \]

Then, \((\mathcal{P}, d)\) is recognized as an \(\mathcal{F}\)-MS.

**Example 1** ([9]). Let \(\mathcal{P} = \mathbb{N}\) and \(\xi(t) = \ln(t)\). Then, \(d : \mathcal{P} \times \mathcal{P} \to [0, +\infty)\) is an \(\mathcal{F}\)-metric defined by
\[
d(\epsilon, \kappa) = \begin{cases} 
(\epsilon - \kappa)^2 & \text{if } (\epsilon, \kappa) \in [0, 3] \times [0, 3] \\
|\epsilon - \kappa| & \text{if } (\epsilon, \kappa) \notin [0, 3] \times [0, 3]
\end{cases}
\]
with \(b = \ln(3)\).

**Definition 6** ([9]). Let \((\mathcal{P}, d)\) be an \(\mathcal{F}\)-MS.

(i) Within set \(\mathcal{P}\), a sequence \(\{\epsilon_n\}\) is considered as an \(\mathcal{F}\)-convergent if
\[
\lim_{n \to \infty} d(\epsilon_n, \epsilon) = 0.
\]

(ii) A sequence \(\{\epsilon_n\}\) is said to be an \(\mathcal{F}\)-Cauchy in \(\mathcal{P}\), if
\[
\lim_{n,m \to \infty} d(\epsilon_n, \epsilon_m) = 0.
\]

In due course, Kanwal et al. [13] united the concepts of the \(\mathcal{F}\)-MS and the \(\mathcal{O}\)-set and put forward the concept of an orthogonal \(\mathcal{F}\)-metric space in this manner.

**Definition 7** ([13]). Let \((\mathcal{P}, \perp)\) be an \(\mathcal{O}\)-set and \(d : \mathcal{P} \times \mathcal{P} \to [0, +\infty)\) be an \(\mathcal{F}\)-metric, then \((\mathcal{P}, \perp, d)\) is said to be an \(\mathcal{O}\mathcal{F}\)-MS.

**Example 2** ([13]). Let \(\mathcal{P} = [0, 1]\). Define an \(\mathcal{F}\)-metric \(d\) given as
\[
d(\epsilon, \kappa) = \begin{cases} 
\epsilon^{|\kappa - \epsilon|} & \text{if } \epsilon \neq \kappa \\
0 & \text{if } \epsilon = \kappa,
\end{cases}
\]
for all \(\epsilon, \kappa \in \mathcal{P}\), \(\xi(t) = -\frac{1}{t}, t > 0\) and \(b = 1\). Define \(\epsilon \perp \kappa \iff e\kappa \leq e\epsilon \leq \kappa\). Then, for all \(\epsilon \in \mathcal{P}\), \(0 \perp \epsilon\), so \((\mathcal{P}, \perp)\) is an \(\mathcal{O}\)-set. Then, \((\mathcal{P}, \perp, d)\) is an \(\mathcal{O}\mathcal{F}\)-MS.

**Definition 8** ([13]). Let \((\mathcal{P}, d, \perp)\) be an orthogonal \(\mathcal{F}\)-metric space and \(B : \mathcal{P} \to \mathcal{P}\). Then, \(B\) is called an \(\mathcal{O}\)-continuous or \(\perp\)-continuous at \(\epsilon \in \mathcal{P}\) if, for each \(\mathcal{O}\)-sequence \(\{\epsilon_n\}\) in \(\mathcal{P}\) if \(\epsilon_n \to \epsilon\), then \(B\epsilon_n \to B\epsilon\). Also, the mapping \(B\) is \(\perp\)-continuous on \(\mathcal{P}\) if the mapping \(B\) is \(\perp\)-continuous in each \(\epsilon \in \mathcal{P}\).

**Definition 9** ([13]). Consider \((\mathcal{P}, d, \perp)\) as an orthogonal \(\mathcal{F}\)-metric space. It is designated as a complete \(\mathcal{O}\mathcal{F}\)-MS if each Cauchy \(\mathcal{O}\)-sequence converges in \(\mathcal{P}\).

Samet et al. [17] initiated the concept of \(\alpha\)-admissibility in such a way.

**Definition 10.** A mapping \(B : \mathcal{P} \to \mathcal{P}\) is characterized as an \(\alpha\)-admissible if
\[
\alpha(\epsilon, \kappa) \geq 1 \implies \alpha(B\epsilon, B\kappa) \geq 1.
\]

Ramezani [18] introduced the idea of orthogonal \(\alpha\)-admissible mapping in such wise.

**Definition 11.** A mapping \(B : \mathcal{P} \to \mathcal{P}\) is called an orthogonal \(\alpha\)-admissible mapping if
\[
\epsilon \perp \kappa \text{ and } \alpha(\epsilon, \kappa) \geq 1 \implies \alpha(B\epsilon, B\kappa) \geq 1.
\]
In a recent study, Ansari et al. [20] employed the function pair \((\lambda, h)\) within contractive inequalities, leading to the establishment of fixed point theorems.

**Definition 12.** A pair of functions \((\lambda, h)\) defined on \(\mathbb{R}^+ \times \mathbb{R}\) are called C-class functions if the pair satisfy the following conditions:

- (i) for \(\sigma \geq 1 \implies h(1, \varsigma) \leq h(\sigma, \varsigma)\);
- (ii) \(0 \leq \ell \leq 1 \implies \lambda(\ell, \varsigma) \leq \lambda(1, \varsigma)\);
- (iii) \(h(1, \varsigma) \leq \lambda(\ell, \varsigma) \implies \varsigma \leq \ell \varsigma\)

\(\forall \varsigma, \ell \in \mathbb{R}\).

**Example 3.** Considering the functions \(h : \mathbb{R}^+ \times \mathbb{R} \to (-\infty, +\infty)\) and \(\lambda : \mathbb{R}^+ \times \mathbb{R} \to (-\infty, +\infty)\), where \(h(\sigma, \varsigma) = \varsigma\) and \(\lambda(\ell, \varsigma) = \ell \varsigma\), then \(\lambda\) and \(h\) are the C-class functions.

**3. Main Results**

**3.1. Fixed Point Theorems for \(F-(\lambda, h)\)-Contractions**

To equip our exploration of this subsection, we formally define \(F-(\lambda, h)\)-contractions, paving the way for their powerful applications within the framework of orthogonal \(\mathcal{G}\)-metric space \((\mathcal{P}, d, \perp)\).

**Definition 13.** A mapping \(\mathcal{B} : (\mathcal{P}, d, \perp) \to (\mathcal{P}, d, \perp)\) is considered as an \(F-(\lambda, h)\)-contraction if the following conditions hold:

- There exists a positive constant \(\tau > 0\);
- \(\lambda\) belongs to a specific class of functions \(\Omega\);
- \(\alpha : \mathcal{P} \times \mathcal{P} \to \mathbb{R}^+\) is a function defined on the product of set \(\mathcal{P}\) with itself, taking values in the positive reals;
- \(\lambda\) and \(h\) are the C-class functions;
- the mapping \(\mathcal{B}\) satisfies a contractive condition

\[
d(\mathcal{B}e, \mathcal{B}x) > 0 \implies h(\alpha(e, \kappa), \tau + F(d(\mathcal{B}e, \mathcal{B}x))) \leq \lambda(1, F(d(e, \kappa)))
\]

for all \(e, \kappa \in \mathcal{P}\) with \(e \perp \kappa\) for \(\kappa \perp e\).

**Theorem 3.** Let \((\mathcal{P}, d, \perp)\) be a complete orthogonal \(\mathcal{G}\)-metric space; \(\mathcal{B} : \mathcal{P} \to \mathcal{P}\) is \(\perp\)-preserving and an \(F-(\lambda, h)\)-contraction. Assume that the following conditions hold:

- (i) \(\mathcal{B}\) is an orthogonal \(\alpha\)-admissible mapping;
- (ii) \(e_0 \in \mathcal{P}\) such that \(e_0 \perp \mathcal{B}e_0\) or \(\mathcal{B}e_0 \perp e_0\) and \(\alpha(e_0, B_0) \geq 1\);
- (iii) Either \(\mathcal{B} : \mathcal{P} \to \mathcal{P}\) is \(\perp\)-continuous or if \(\{e_n\}\) is an \(O\)-sequence in \(\mathcal{P}\) such that \(e_n \to e^*\) and \(\alpha(e_n, e_{n+1}) \geq 1\) for all \(n \in \mathbb{N}\), then \(\alpha(e_n, e^*) \geq 1\) for all \(n \in \mathbb{N}\).

Then, \(\mathcal{B}\) has a fixed point. Moreover, if \(\alpha(e, \kappa) \geq 1\) for all \(e, \kappa \in \text{Fix}(\mathcal{B})\), then the fixed point of the mapping \(\mathcal{B}\) is unique.

**Proof.** We consider an arbitrary element \(e_0\) belonging to set \(\mathcal{P}\) such that

\[
e_0 \perp \mathcal{B}e_0\text{ or } \mathcal{B}e_0 \perp e_0
\]

and

\[
\alpha(e_0, e_1) = \alpha(e_0, B_0) \geq 1.
\]

Now, we define a sequence \(\{e_n\}\) in this way

\[
e_1 = \mathcal{B}e_0, \ldots, e_{n+1} = \mathcal{B}e_n = B^{n+1}e_0
\]

for all \(n \geq 0\). By (2) and (4), we have

\[
e_0 \perp e_1 \text{ or } e_1 \perp e_0.
\]
Since \( B : \mathcal{P} \rightarrow \mathcal{P} \) is \( \perp \)-preserving, by (5) we have
\[
e_1 = B\epsilon_0 \perp B\epsilon_1 = e_2 \quad \text{or} \quad e_2 = B\epsilon_1 \perp B\epsilon_0 = e_1.
\]
Continuing in this way, we obtain
\[
e_{n-1} \perp e_n \quad \text{or} \quad e_n \perp e_{n-1}
\]
for all \( n \in \mathbb{N} \cup \{0\} \). Hence, \( \{e_n\} \) is an \( O \)-sequence. Also, since \( B \) is an orthogonal \( \alpha \)-admissible mapping, by (3), we have
\[
a(e_1, e_2) = a(B\epsilon_0, B\epsilon_1) \geq 1.
\]
By continuing this process, we obtain
\[
a(e_{n-1}, e_n) = a(B\epsilon_{n-2}, B\epsilon_{n-1}) \geq 1.
\]
for every natural number \( n \). It holds that, if \( e_{n_0} \) equals \( e_{n_0+1} \), for a certain number \( n_0 \in \mathbb{N} \cup \{0\} \), then it is clear that \( e_{n_0} \) serves as a fixed point of \( B \). To explore the implications further, let us investigate the case where \( e_n \neq e_{n+1} \), \( \forall n \in \mathbb{N} \cup \{0\} \). In this instance, we assume the following:
\[
d(B\epsilon_{n-1}, B\epsilon_n) = d(e_n, e_{n+1}) > 0,
\]
for all \( n \in \mathbb{N} \cup \{0\} \). Building upon the results established in (1), we can now conclude
\[
h(1, \tau + F(d(e_n, e_{n+1})))
\]
\[
= h(1, \tau + F(d(B\epsilon_{n-1}, B\epsilon_n)))
\]
\[
\leq h(a(e_{n-1}, e_n)), \tau + F(d(B\epsilon_{n-1}, B\epsilon_n))
\]
\[
\leq h(1, F(d(e_{n-1}, e_n))),
\]
which implies that
\[
\tau + F(d(e_n, e_{n+1})) \leq F(d(e_{n-1}, e_n)).
\]
It further implies that
\[
F(d(e_n, e_{n+1})) \leq F(d(e_{n-1}, e_n)) - \tau,
\]
for all \( \mathbb{N} \cup \{0\} \). Repeatedly applying inequality (7) yields
\[
F(d(e_n, e_{n+1})) \leq F(d(e_{n-1}, e_n)) - \tau \leq \ldots \leq F(d(e_0, e_1)) - n\tau.
\]
Letting \( n \to \infty \) in (8), we obtain
\[
\lim_{n \to \infty} F(d(e_n, e_{n+1})) = -\infty.
\]
By \((F_2)\), we have
\[
\lim_{n \to \infty} d(e_n, e_{n+1}) = 0.
\]
To show that \( \{e_n\} \) is a Cauchy sequence, from \((F_3)\), there exists \( r \in (0,1) \) such that
\[
\lim_{n \to \infty} (d(e_n, e_{n+1}))'F(d(e_n, e_{n+1})) = 0.
\]
By (8), the following holds for all \( n \in \mathbb{N} \), we have

\[
(d(\varepsilon_n, \varepsilon_{n+1}))' F(d(\varepsilon_n, \varepsilon_{n+1})) - (d(\varepsilon_n, \varepsilon_{n+1}))' F(d(\varepsilon_0, \varepsilon_1)) \\
\leq (d(\varepsilon_n, \varepsilon_{n+1}))' (F(d(\varepsilon_0, \varepsilon_1)) - n \tau) - (d(\varepsilon_n, \varepsilon_{n+1}))' F(d(\varepsilon_0, \varepsilon_1)) \\
\leq - (d(\varepsilon_n, \varepsilon_{n+1}))' n \tau \\
\leq 0.
\]  

(11)

Letting \( n \to \infty \) in (11) and using (9) and (10), we obtain

\[
\lim_{n \to \infty} n(d(\varepsilon_n, \varepsilon_{n+1}))' = 0.
\]  

(12)

From (12), there exists \( n_1 \in \mathbb{N} \) such that \( n(d(\varepsilon_n, \varepsilon_{n+1}))' \leq 1 \) for all \( n \geq n_1 \). Consequently, we have

\[
d(\varepsilon_n, \varepsilon_{n+1}) \leq \frac{1}{n \tau},
\]  

(13)

for all \( n \geq n_1 \). This yields

\[
\sum_{i=n}^{m-1} d(\varepsilon_i, \varepsilon_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i \tau}
\]

for \( m > n \). Since \( \sum_{i=n}^{\infty} \frac{1}{i \tau} \) is convergent,

\[
0 < \sum_{i=n}^{m-1} \frac{1}{i \tau} < \sum_{i=n}^{\infty} \frac{1}{i \tau} < \delta,
\]

for \( n > n_1 \). Let \( \epsilon > 0 \) be fixed and \( (\xi, b) \in \mathcal{F} \times [0, +\infty) \) such that \((D_3)\) holds. By (\( \mathfrak{F}_2)\), there exists \( \delta > 0 \) such that

\[
0 < t < \delta \implies \xi(t) < \xi(t) - b.
\]  

(14)

Hence, by (13), (14), and (\( \mathfrak{F}_1),\) we have

\[
\xi \left( \sum_{i=n}^{m-1} d(\varepsilon_i, \varepsilon_{i+1}) \right) \leq \xi \left( \sum_{i=n}^{m-1} \frac{1}{i \tau} \right) \leq \xi \left( \sum_{i=n}^{\infty} \frac{1}{i \tau} \right) < \xi(\epsilon) - b,
\]  

(15)

where \( m > n \geq n_1 \). Thus, by \((D_3)\) and (15) for \( d(\varepsilon_n, \varepsilon_m) > 0, m > n \geq n_1 \), we have

\[
\xi(d(\varepsilon_n, \varepsilon_m)) \leq \xi \left( \sum_{i=n}^{m-1} d(\varepsilon_i, \varepsilon_{i+1}) \right) + b \\
\leq \xi \left( \sum_{i=n}^{\infty} \frac{1}{i \tau} \right) + b \\
< \xi(\epsilon),
\]

which, from \((\mathfrak{F}_1),\) gives

\[
d(\varepsilon_n, \varepsilon_m) < \epsilon,
\]

for all \( m > n \geq n_1 \). Therefore, \( \{\varepsilon_n\} \) is a Cauchy \( \mathcal{O} \)-sequence in \( (\mathcal{P}, \perp, d) \). As \((\mathcal{P}, \perp, d)\) is \( \mathcal{O} \)-complete, there exists \( \varepsilon^* \in \mathcal{P} \) such that \( \lim_{n \to \infty} \varepsilon_n \to \varepsilon^* \). Now, we show that \( \varepsilon^* = B \varepsilon^* \). Now, if \( B : \mathcal{P} \to \mathcal{P} \) is \( \perp \)-continuous, then we have \( B\varepsilon_n \to B\varepsilon^* \) as \( n \to \infty \). Thus,

\[
B\varepsilon^* = \lim_{n \to \infty} B\varepsilon_n = \lim_{n \to \infty} \varepsilon_{n+1} = \varepsilon^*.
\]
Now, if \( \{ \epsilon_n \} \) is an \( O \)-sequence in \( P \) such that \( \epsilon_n \to e^* \) and \( a(\epsilon_n, \epsilon_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then by assumption (iii) we have \( a(\epsilon_n, e^*) \geq 1 \) for all \( n \in \mathbb{N} \). We suppose on the contrary that \( e^* \) is not the fixed point of \( B \). Then, by the assumption, we obtain

\[
\tau = \alpha(\epsilon_n, e^*) \leq \lambda(1, d(\epsilon_n, e^*)) ,
\]

which implies that

\[
\tau + F(d(\epsilon_{n+1}, Be^*)) \leq F(d(\epsilon_n, e^*)) ,
\]

that

\[
F(d(\epsilon_{n+1}, Be^*)) \leq F(d(\epsilon_n, e^*)) - \tau .
\]

Letting \( n \to \infty \) in the above inequality and since both \( F \) and \( d \) are continuous functions, we obtain

\[
F(d(\epsilon^*, Be^*)) = -\infty .
\]

By utilizing the condition \( (F_3) \), it follows that \( d(\epsilon^*, Be^*) = 0 \) leading to a contradiction. Consequently, \( e^* = Be^* \), establishing \( e^* \) as a fixed point of \( B \). Furthermore, suppose \( e^* \) represents another fixed point of \( B \), such that \( Be^* = e^* \neq e^\prime = Be^\prime \) and \( e^* \perp e^\prime \) or \( e^\prime \perp e^* \).

Then, by the assumption, we obtain \( a(e^*, e^\prime) \geq 1 \). Now, by (1), we have

\[
\tau + F(d(e^*, e^\prime)) \leq F(d(e^*, e^\prime)).
\]

This leads to a contradiction since \( \tau > 0 \), affirming that \( e^* = e^\prime \) thereby establishing the uniqueness of the fixed point. \( \square \)

The following result is the main result of Ahmad et al. [14] which is an immediate consequence of our leading Theorem 3.

**Corollary 1.** Let \( (P,d,\perp) \) be a complete \( O \)-MS and \( B : P \to P \) be \( \perp \)-preserving. Assume that there exist a positive constant \( \tau > 0 \) and the functions \( F \in \Omega, \alpha: P \times P \to \mathbb{R}^+ \) such that \( (\epsilon, \kappa) \geq 1 \) and

\[
d(Be^*, Bx) > 0 \implies \tau + F(d(Be^*, Bx)) \leq F(d(\epsilon, \kappa))
\]

for all \( \epsilon, \kappa \in P \) with \( \epsilon \perp \kappa \). Assume that these assertions hold:

(i) \( B \) is an orthogonal \( \alpha \)-admissible mapping;

(ii) \( \exists \epsilon_0 \in P \) such that \( \epsilon_0 \perp B\epsilon_0 \) or \( B\epsilon_0 \perp \epsilon_0 \) and \( a(\epsilon_0, B\epsilon_0) \geq 1 \);

(iii) Either \( B : P \to P \) is \( \perp \)-continuous or if \( \{ \epsilon_n \} \) is an \( O \)-sequence in \( P \) such that \( \epsilon_n \to e^* \) and \( a(\epsilon_n, e^*) \geq 1 \) for all \( n \in \mathbb{N} \), then \( a(\epsilon_n, e^*) \geq 1 \) for all \( n \in \mathbb{N} \).

Then, \( B \) has a fixed point. Moreover, if \( a(\epsilon, \leq) \geq 1 \) for all \( \epsilon, \kappa \in Fix(B) \), then the fixed point of the mapping \( B \) is unique.

**Proof.** Introduce the functions \( h \) and \( \lambda \) on \( \mathbb{R}^+ \times \mathbb{R} \) as follows: \( h(\sigma, \zeta) = \gamma \) and \( \lambda(\ell, \varphi) = \ell \varphi \) in Theorem 3. \( \square \)
Corollary 2. Let \((\mathcal{P}, d, \perp)\) be a complete \(O\mathfrak{S}\)-MS; \(B : \mathcal{P} \to \mathcal{P}\) is \(\perp\)-preserving and \(B : \mathcal{P} \to \mathcal{P}\) is \(\perp\)-continuous. Assume that there is \(\tau > 0\) and a function \(F \in \Omega\) such that

\[d(B\epsilon, B\kappa) > 0 \implies \tau + F(d(B\epsilon, B\kappa)) \leq F(d(\epsilon, \kappa)),\]

for all \(\epsilon, \kappa \in \mathcal{P}\) with \(\epsilon \perp \kappa\). Then, there exists a unique point \(\epsilon\) in \(\mathcal{P}\) such that \(B\epsilon = \epsilon\).

Proof. Define the functions \(\alpha : \mathcal{P} \times \mathcal{P} \to \mathbb{R}^+, h\), and \(\land\) by \(\alpha(\epsilon, \kappa) = 1\), \(h(\sigma, \zeta) = \zeta\) and \(\land(\ell, \phi) = \ell \phi\) in Theorem 3. \(\square\)

Example 4. Define the sequence \(\{\epsilon_n\}\) as follows:

\[
\begin{align*}
\epsilon_1 &= \ln(1) \\
\epsilon_2 &= \ln(3) \\
&\quad \vdots \\
\epsilon_n &= \ln(1 + 3 + 5 + \ldots + (2n - 1)) = 2\ln(n)
\end{align*}
\]

for all \(n \in \mathbb{N}\). Let \(\mathcal{P} = \{\epsilon_n : n \in \mathbb{N}\}\) furnished with the \(\mathfrak{S}\) metric \(d\) defined by

\[
d(\epsilon, \kappa) = \begin{cases} 
\exp|\epsilon - \kappa|, & \text{if } \epsilon \neq \kappa \\
0, & \text{if } \epsilon = \kappa
\end{cases}
\]

with \(\xi(t) = \frac{1}{t}\) and \(b = 1\). For all \(\epsilon_n, \epsilon_m \in \mathcal{P}\), define \(\epsilon_n \perp \epsilon_m\) if and only if \((m \geq 2 \land n = 1)\). Hence, \((\mathcal{P}, \perp, d)\) is a complete \(O\mathfrak{S}\)-MS. Define \(B : \mathcal{P} \to \mathcal{P}\) by

\[
B(\epsilon_n) = \begin{cases} 
\epsilon_1, & \text{if } n = 1, \\
\epsilon_{n-1}, & \text{if } n > 1
\end{cases}
\]

and \(\alpha : \mathcal{P} \times \mathcal{P} \to [1, +\infty)\) by

\[
\alpha(\epsilon_n, \epsilon_m) = \begin{cases} 
1, & \text{if } \epsilon_n \neq \epsilon_m \\
0, & \text{if } \epsilon_n = \epsilon_m
\end{cases}
\]

Define the function \(h\) and \(\land\) on \(\mathbb{R}^+ \times \mathbb{R}\) by \(h(\sigma, \zeta) = \zeta\) and \(\land(\ell, \phi) = \ell \phi\). Evidently,

\[
\lim_{n \to \infty} \frac{d(B(\epsilon_n), B(\epsilon_1))}{d(\epsilon_n, \epsilon_1)} = 1.
\]

Therefore, \(B\) fails to satisfy the contractive condition of the Banach contraction principle.

Verifying the \(\perp\)-continuity and \(\perp\)-preserving properties of \(B\) is a simple process. Consider the mapping \(F : (0, \infty) \to (-\infty, +\infty)\) given by

\[
F(t) = \ln t + t, \quad t > 0.
\]

Since establishing \(F \in \Omega\) is a straightforward process, we can now focus on proving \(B\) is an \(F(\alpha, h)\)-contraction. This requires us to show that \(d(B(\epsilon_n), B(\epsilon_m)) \neq 0\) implies

\[
\tau + \ln d(B(\epsilon_n), B(\epsilon_m)) + d(B(\epsilon_n), B(\epsilon_m)) \leq \ln d(\epsilon_n, \epsilon_m) + d(\epsilon_n, \epsilon_m)
\]

for \(\tau > 0\). The aforementioned condition aligns precisely with

\[
d(B(\epsilon_n), B(\epsilon_m)) \neq 0 \implies e^{\tau + \ln d(B(\epsilon_n), B(\epsilon_m)) + d(B(\epsilon_n), B(\epsilon_m))} \leq e^{\ln d(\epsilon_n, \epsilon_m) + d(\epsilon_n, \epsilon_m)}.
\]

To proceed, we need to ensure that

\[
d(B(\epsilon_n), B(\epsilon_m)) \neq 0 \implies \frac{d(B(\epsilon_n), B(\epsilon_m))}{d(\epsilon_n, \epsilon_m)} e^{d(B(\epsilon_n), B(\epsilon_m)) - d(\epsilon_n, \epsilon_m)} \leq e^{-\tau}.
\]
For every \( m \in \mathbb{N}, m \geq 2 \), we have
\[
\frac{d(B(e_m), B(e_1))}{d(e_m, e_1)} \neq 0 \implies \frac{d(B(e_m), B(e_1))}{d(e_m, e_1)} \cdot e^{d(e_m, B(e_1)) - d(e_m, e_1)} \leq e^{-\tau}
\]
Based on inequality (1), it holds true when \( \tau = 1 > 0 \). This confirms that \( B \) satisfies the properties of an \( F(\lambda, h) \)-contraction. By applying Theorem 4, we can deduce that \( \epsilon = \ln(1) \) is the unique fixed point of \( B \).

3.2. Fixed Point Theorems for Interpolative Contractions

\textbf{Definition 14.} A mapping \( B : \mathcal{P} \to \mathcal{P} \) is called an \((\alpha, \eta)\)-Reich type interpolative contraction if there exist the functions \( \alpha, \eta : \mathcal{P} \times \mathcal{P} \to [0, +\infty) \) and some constant \( \lambda \in [0, 1) \) and positive reals \( p, q \) with \( p + q < 1 \) such that
\[
\epsilon \perp \kappa, \quad \alpha(\epsilon, \kappa) \geq \eta(\epsilon, \kappa) \tag{16}
\]
implies
\[
d(B\epsilon, B\kappa) \leq \lambda [M(\epsilon, \kappa)],
\]
where
\[
M(\epsilon, \kappa) = \left\{ d(\epsilon, \kappa)^{\alpha}, [d(\epsilon, B\epsilon)]^{p}, [d(\kappa, B\kappa)]^{1-p-q} \right\}.
\]
for all \( \epsilon, \kappa \in \mathcal{P}\setminus \text{Fix}(B) \).

\textbf{Theorem 4.} Let \( B : \mathcal{P} \to \mathcal{P} \) be a \( \perp \)-preserving and \((\alpha, \eta)\)-Reich type interpolative contraction. Let us examine the following assertions:
(i) \( B \) is an orthogonal \( \alpha \)-admissible mapping with respect to \( \eta \);
(ii) \( \exists e_0 \in \mathcal{P} \) such that \( e_0 \perp B e_0 \) or \( B e_0 \perp e_0 \) and \( \alpha(e_0, B e_0) \geq \eta(e_0, B e_0) \);
(iii) \( B : \mathcal{P} \to \mathcal{P} \) is \( \perp \)-continuous.

Then, \( B \) has a fixed point.

\textbf{Proof.} Let \( e_0 \in \mathcal{P} \) be an arbitrary point such that
\[
e_0 \perp B e_0 \quad \text{or} \quad B e_0 \perp e_0, \tag{17}
\]
and
\[
\alpha(e_0, e_1) = \alpha(e_0, B e_0) \geq \eta(e_0, B e_0) = \eta(e_0, e_1). \tag{18}
\]
Now, we define a sequence \( \{e_n\} \) in this way
\[
e_1 = B e_0, \ldots, e_{n+1} = B e_n = B^{n+1} e_0, \tag{19}
\]
for all \( n \geq 0 \). By (17) and (19), we have
\[
e_0 \perp e_1 \quad \text{or} \quad e_1 \perp e_0. \tag{20}
\]
Since \( B : \mathcal{P} \to \mathcal{P} \) is \( \perp \)-preserving, by (20) we have
\[
e_1 = B e_0 \perp B e_1 = e_2 \quad \text{or} \quad e_2 = B e_1 \perp B e_0 = e_1.
\]
Continuing in this way, we obtain

\[ \varepsilon_{n-1} \perp \varepsilon_n \text{ or } \varepsilon_n \perp \varepsilon_{n-1}, \]

for all \( n \in \mathbb{N} \). Hence, \( \{\varepsilon_n\} \) is an \( O \)-sequence. Also, since \( B \) is an orthogonal \( \alpha \)-admissible mapping with respect to \( \eta \), by (18), we have

\[ a(\varepsilon_1, \varepsilon_2) = a(B\varepsilon_0, B\varepsilon_1) \geq \eta(B\varepsilon_0, B\varepsilon_1) = \eta(\varepsilon_1, \varepsilon_2). \]

By continuing this process, we obtain

\[ a(\varepsilon_{n-1}, \varepsilon_n) = a(B\varepsilon_{n-2}, B\varepsilon_{n-1}) \geq \eta(B\varepsilon_{n-2}, B\varepsilon_{n-1}) = \eta(\varepsilon_{n-1}, \varepsilon_n), \]

for all \( n \in \mathbb{N} \). Now, if \( \varepsilon_{n_0} = \varepsilon_{n_0+1} \), for some \( n_0 \in \mathbb{N} \), then clearly \( \varepsilon_{n_0} \) is a fixed point of \( B \). Thus, we posit the equality \( \varepsilon_n \neq \varepsilon_{n+1} \), for every \( n \) in the set of natural numbers. Consequently, we make the assumption that

\[ d(\varepsilon_n, \varepsilon_{n+1}) = d(B\varepsilon_{n-1}, B\varepsilon_n) > 0, \tag{21} \]

for all \( n \in \mathbb{N} \). From (16) and (21), we derive the following information:

\[ d(\varepsilon_n, \varepsilon_{n+1}) = d(B\varepsilon_{n-1}, B\varepsilon_n) \leq \lambda[M_1(\varepsilon_{n-1}, \varepsilon_n)], \]

where

\[
\begin{align*}
    d(\varepsilon_n, \varepsilon_{n+1}) &= d(B\varepsilon_{n-1}, B\varepsilon_n) \\
    &\leq \lambda \left[ d(\varepsilon_{n-1}, \varepsilon_n)^p \cdot [d(\varepsilon_{n-1}, B\varepsilon_{n-1})]^1 \cdot [d(\varepsilon_n, B\varepsilon_n)]^{1-p-q} \right] \\
    &= \lambda \left[ d(\varepsilon_{n-1}, \varepsilon_n)^p \cdot [d(\varepsilon_{n-1}, B\varepsilon_{n-1})]^1 \cdot [d(\varepsilon_n, B\varepsilon_n)]^{1-p-q} \right] \\
    &= \lambda \left[ d(\varepsilon_{n-1}, \varepsilon_n)^{p+q} \cdot [d(\varepsilon_n, B\varepsilon_n)]^{1-p-q} \right].
\end{align*}
\]

We derive

\[ [d(\varepsilon_n, \varepsilon_{n+1})]^{p+q} \leq \lambda [d(\varepsilon_{n-1}, \varepsilon_n)]^{p+q}. \]

So, we conclude that

\[ d(\varepsilon_n, \varepsilon_{n+1}) \leq d(\varepsilon_n, \varepsilon_n), \tag{22} \]

for all \( n \in \mathbb{N} \) and the sequence \( \{d(\varepsilon_{n-1}, \varepsilon_n)\} \) exhibits non-increasing behavior with all terms being non-negative. Consequently, a non-negative constant, denoted by \( \varkappa \), exists such that

\[ \lim_{n \to \infty} d(\varepsilon_{n-1}, \varepsilon_n) = \varkappa. \]

Note that \( \varkappa \geq 0 \). Indeed, from (22), we have

\[ d(\varepsilon_n, \varepsilon_{n+1}) \leq \lambda d(\varepsilon_{n-1}, \varepsilon_n) \leq \lambda d(\varepsilon_{n-2}, \varepsilon_n) \leq \ldots \leq \lambda^n d(\varepsilon_0, \varepsilon_1). \tag{23} \]

Since \( \lambda < 1 \), and by taking \( n \to \infty \) in the inequality (23), we deduce that \( \varkappa = 0 \). Now, by (23) for \( m > n \), we have

\[ \sum_{i=n}^{m-1} d(\varepsilon_i, \varepsilon_{i+1}) \leq \frac{\lambda^n}{1-\lambda} d(\varepsilon_0, \varepsilon_1). \tag{24} \]

Since \( \lambda < 1 \), we have

\[ \lim_{n \to \infty} \frac{\lambda^n}{1-\lambda} d(\varepsilon_0, \varepsilon_1) = 0, \]
that is, there exists some \( n_0 \in \mathbb{N} \) such that

\[
0 < \frac{\lambda^n}{1 - \lambda} d(\epsilon_0, \epsilon_1) < \delta,
\]

for \( n \geq n_0 \). Let \( \epsilon > 0 \) be fixed and \((\xi, \eta) \in \mathfrak{F} \times [0, +\infty)\) such that \((D_3)\) holds. By \((\mathfrak{H}_2)\), there exists \( \delta > 0 \) such that

\[
0 < t < \delta \implies \xi(t) < \xi(t) - \eta.
\]

Hence, by \((24), (25), \) and \((\mathfrak{H}_1)\), we have

\[
\xi \left( \sum_{i=n}^{m-1} d(\epsilon_i, \epsilon_{i+1}) \right) \leq \frac{\lambda^n}{1 - \lambda} d(\epsilon_0, \epsilon_1) < \xi(\epsilon) - \eta,
\]

where \( m > n \geq n_0 \). Thus, by \((D_3)\) and \((26)\) for \( d(\epsilon_n, \epsilon_m) > 0, m > n \geq n_0 \), we have

\[
\xi(d(\epsilon_n, \epsilon_m)) \leq \xi \left( \sum_{i=n}^{m-1} d(\epsilon_i, \epsilon_{i+1}) \right) + \eta
\]

\[
\leq \xi \left( \frac{\lambda^n}{1 - \lambda} d(\epsilon_0, \epsilon_1) \right) + \eta
\]

\[
< \xi(\epsilon),
\]

which, from \((\mathfrak{H}_1)\), gives

\[
d(\epsilon_n, \epsilon_m) < \epsilon,
\]

for all \( m > n \geq n_0 \). Therefore, \( \{\epsilon_n\} \) is a Cauchy \( \mathcal{O} \)-sequence in \((\mathcal{P}, \perp, d)\). As \((\mathcal{P}, \perp, d)\) is \( \mathcal{O} \)-complete, there exists \( \epsilon^* \in \mathcal{P} \) such that \( \lim_{n \to \infty} \epsilon_n = \epsilon^* \). Next, we demonstrate that \( \epsilon^* = B\epsilon^* \). Given that \( B : \mathcal{P} \to \mathcal{P} \) is \( \perp \)-continuous, it follows that \( B\epsilon_n \to B\epsilon^* \) as \( n \to \infty \). Thus,

\[
B\epsilon^* = \lim_{n \to \infty} B\epsilon_n = \lim_{n \to \infty} \epsilon_{n+1} = \epsilon^*.
\]

\(\square\)

**Theorem 5.** Let \( B : \mathcal{P} \to \mathcal{P} \) be a \( \perp \)-preserving and \((\alpha, \eta)\)-Reich type interpolative contraction. Assume that these assertions hold:

(i) \( B \) is an orthogonal \( \alpha \)-admissible mapping with respect to \( \eta \);

(ii) \( \exists \, \epsilon_0 \in \mathcal{P} \) such that \( \alpha(\epsilon_0, B\epsilon_0) \geq \eta(\epsilon_0, B\epsilon_0) \);

(iii) If \( \{\epsilon_n\} \) is an \( \mathcal{O} \)-sequence in \( \mathcal{P} \) such that \( \epsilon_n \to \epsilon^* \) and \( \alpha(\epsilon_n, \epsilon_{n+1}) \geq \eta(\epsilon_n, \epsilon_{n+1}) \) for all \( n \in \mathbb{N} \), then \( \alpha(\epsilon_n, \epsilon^*) \geq \eta(\epsilon_n, \epsilon^*) \) for all \( n \in \mathbb{N} \).

Then, \( B \) has a fixed point.

**Proof.** Along the lines of the proof of Theorem 4, we obtain that \( \{\epsilon_n\} \) is an \( \mathcal{O} \)-sequence in \( \mathcal{P} \) such that \( \epsilon_n \to \epsilon^* \) and \( \alpha(\epsilon_n, \epsilon_{n+1}) \geq \eta(\epsilon_n, \epsilon_{n+1}) \) for all \( n \in \mathbb{N} \); then, by assumption (iii) we have \( \alpha(\epsilon_n, \epsilon^*) \geq \eta(\epsilon_n, \epsilon^*) \) for all \( n \in \mathbb{N} \). We suppose on the contrary that \( \epsilon^* \) is not the fixed point of \( B \). Then, \( d(B\epsilon_n, \epsilon^*) \neq 0 \). Now, by \((D_3)\) and \((16)\), we have

\[
\xi(d(\epsilon^*, \epsilon_n)) \leq \xi(d(\epsilon^*, B\epsilon_n) + d(B\epsilon_n, \epsilon^*)) + \eta
\]

\[
\leq \xi \left( \frac{\lambda d(\epsilon^*, \epsilon_n)^q \cdot [d(\epsilon^*, B\epsilon_n)]^p \cdot d(\epsilon_n, B\epsilon_n)]^{1-p-q}}{d(B\epsilon_n, \epsilon^*)} + \frac{\lambda d(\epsilon^*, \epsilon_n)^q \cdot [d(\epsilon^*, B\epsilon_n)]^p \cdot d(\epsilon_n, \epsilon_{n+1})]^{1-p-q}}{d(B\epsilon_n, \epsilon^*)} \right) + \eta.
\]

(27)
Letting \( n \to \infty \), in the aforementioned inequality and recalling that \( \lim_{n \to \infty} d(\xi_n, \xi^*) = 0 \) together with \( \lim_{n \to \infty} d(\xi_n, \xi^*) = 0 \), we obtain

\[
\lim_{n \to \infty} \xi_n = \begin{cases} \lambda d(\xi^*, \xi_n) + \eta d(\xi_n, \xi_n) & \text{if } \eta = 0, \\ \lambda d(\xi^*, \xi_n) + \eta d(\xi_n, \xi_n) & \text{if } \eta \neq 0. \end{cases}
\]

Thus, by (27), we have \( \xi(\lim_{n \to \infty} d(\xi, \xi_n)) = -\infty \). Therefore, according to (32), we derive \( d(\xi, \xi_n) = 0 \), which presents a contradiction, leading to the conclusion that \( \xi^* = \varepsilon^* \). □

**Corollary 3.** Let \( B : \mathcal{P} \to \mathcal{P} \) be \( \bot \)-preserving. Suppose that there exist a function \( \alpha : \mathcal{P} \times \mathcal{P} \to [0, \infty) \) and the constants \( \lambda \in [0, 1) \) and positive reals \( p, q \) with \( p + q < 1 \) such that

\[ \varepsilon \bot \kappa, \quad \alpha(\varepsilon, \kappa) \geq 1 \]

implies

\[ d(\mathcal{B}_\varepsilon B_\kappa) \leq \lambda \left[ \mathcal{M}(\varepsilon, \kappa) \right], \]

where

\[ \mathcal{M}(\varepsilon, \kappa) = \left\{ d(\varepsilon, \kappa) | d(\varepsilon, \mathcal{B}_\varepsilon \mathcal{B}_\kappa) | d(\kappa, \mathcal{B}_\kappa) \right\} \]

for all \( \varepsilon, \kappa \in \mathcal{P} \setminus \text{Fix}(\mathcal{B}) \). Assume that these assertions hold:

(i) \( \mathcal{B} \) is orthogonal \( \alpha \)-admissible;
(ii) \( \exists \varepsilon_0 \in \mathcal{P} \) such that \( \alpha(\varepsilon_0, \mathcal{B}_\varepsilon_0) \geq 1 \);
(iii) \( \mathcal{B} : \mathcal{P} \to \mathcal{P} \) is \( \bot \)-continuous.

Then, \( \mathcal{B} \) has a fixed point.

**Proof.** Take \( \eta : \mathcal{P} \times \mathcal{P} \to [0, \infty) \) by \( \eta(\varepsilon, \kappa) = 1 \), for all \( \varepsilon, \kappa \in \mathcal{P} \) in Theorem 4. □

**Corollary 4.** Let \( B : \mathcal{P} \to \mathcal{P} \) be \( \bot \)-preserving. Suppose that there exist a function \( \alpha : \mathcal{P} \times \mathcal{P} \to [0, \infty) \) and some constant \( \lambda \in [0, 1) \) and positive reals \( p, q \) with \( p + q < 1 \) such that

\[ \varepsilon \bot \kappa, \quad \alpha(\varepsilon, \kappa) \geq 1 \]

implies

\[ d(\mathcal{B}_\varepsilon B_\kappa) \leq \lambda \left[ \mathcal{M}(\varepsilon, \kappa) \right], \]

where

\[ \mathcal{M}(\varepsilon, \kappa) = \left\{ d(\varepsilon, \kappa) | d(\varepsilon, \mathcal{B}_\varepsilon \mathcal{B}_\kappa) | d(\kappa, \mathcal{B}_\kappa) \right\} \]

for all \( \varepsilon, \kappa \in \mathcal{P} \setminus \text{Fix}(\mathcal{B}) \). Assume that these assertions hold:

(i) \( \mathcal{B} \) is an orthogonal \( \alpha \)-admissible mapping;
(ii) \( \exists \varepsilon_0 \in \mathcal{P} \) such that \( \alpha(\varepsilon_0, \mathcal{B}_\varepsilon_0) \geq 1 \);
(iii) If \( \{ \varepsilon_n \} \) is an \( \mathcal{O} \)-sequence in \( \mathcal{P} \) such that \( \varepsilon_n \to \varepsilon^* \) and \( \alpha(\varepsilon_n, \varepsilon_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then

\[ \alpha(\varepsilon_n, \varepsilon^*) \geq 1 \]

for all \( n \in \mathbb{N} \).

Then, \( \mathcal{B} \) has a fixed point.

**Proof.** Take \( \eta : \mathcal{P} \times \mathcal{P} \to [0, \infty) \) by \( \eta(\varepsilon, \kappa) = 1 \), for all \( \varepsilon, \kappa \in \mathcal{P} \) in Theorem 5. □

**Corollary 5.** Let \( B : \mathcal{P} \to \mathcal{P} \) be \( \bot \)-preserving and \( \bot \)-continuous. Suppose that there exists some constant \( \lambda \in [0, 1) \) and positive reals \( p, q \) with \( p + q < 1 \) such that

\[ d(\kappa, \mathcal{B}_\kappa) \leq \lambda d(\varepsilon, \kappa), \]

implies

\[ d(\mathcal{B}_\varepsilon B_\kappa) \leq \lambda \left[ \mathcal{M}(\varepsilon, \kappa) \right], \]
where
\[ M(\epsilon, \kappa) = \left\{ d(\epsilon, \kappa)^q, [d(\epsilon, B\kappa)]^p, [d(\kappa, B\kappa)]^{1-p-q} \right\}, \]
for all \( \epsilon, \kappa \in \mathcal{P} \setminus \text{Fix}(B) \) and \( \epsilon \perp \kappa \). Then, there exists \( \epsilon^* \in \mathcal{P} \) such that \( B\epsilon^* = \epsilon^* \).

**Proof.** Define \( a, \eta : \mathcal{P} \times \mathcal{P} \rightarrow [0, +\infty) \) by
\[ a(\epsilon, \kappa) = d(\epsilon, \kappa) \text{ and } \eta(\epsilon, \kappa) = d(\epsilon, B\kappa), \]
for all \( \epsilon, \kappa \in \mathcal{P} \) with \( \epsilon \perp \kappa \). Then, from \( d(\kappa, B\kappa) \leq d(\epsilon, \kappa) \), we have
\[ \eta(\epsilon, \kappa) \leq a(\epsilon, \kappa). \] (28)
Then, all the conditions (i)-(iii) of Theorem 4 are satisfied. Thus, inequality (28) implies that
\[ d(B\epsilon, B\kappa) \leq \lambda [M(\epsilon, \kappa)], \]
where
\[ M(\epsilon, \kappa) = \left\{ d(\epsilon, \kappa)^q, [d(\epsilon, B\kappa)]^p, [d(\kappa, B\kappa)]^{1-p-q} \right\}. \]
Therefore, all the prerequisites outlined in Theorem 4 are met, affirming the existence of a fixed point for \( B \). \( \square \)

3.3. Consequences
3.3.1. Fixed Point Theorems in Complete \( \mathfrak{g} \)-Metric Spaces

The subsequent exploration unfolds within the realm of a complete \( \mathfrak{g} \)-MS, \( (\mathcal{P}, d) \), endowed with the distinctive properties of an \( \mathfrak{g} \)-metric.

**Corollary 6.** Consider a self-mapping \( B : \mathcal{P} \rightarrow \mathcal{P} \). Suppose there exists a positive constant \( \tau > 0 \), a function \( F \in \Omega \), another function \( a : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}_+ \) and the \( \mathcal{C} \)-class functions \( \lambda \) and \( h \) that satisfy the following condition
\[ d(B\epsilon, B\kappa) > 0 \implies h(a(\epsilon, \kappa), \tau + F(d(B\epsilon, B\kappa))) \leq \lambda(1, F(d(\epsilon, \kappa))), \] (29)
for all \( \epsilon, \kappa \in \mathcal{P} \). Assume that these conditions hold:
(i) \( B \) is an \( \alpha \)-admissible mapping;
(ii) \( \exists \epsilon_0 \in \mathcal{P} \) such that \( \epsilon_0 \perp B\epsilon_0 \) or \( B\epsilon_0 \perp \epsilon_0 \) and \( a(\epsilon_0, B\epsilon_0) \geq 1 \);
(iii) either \( B : \mathcal{P} \rightarrow \mathcal{P} \) is continuous or if \( \{\epsilon_n\} \) is a sequence in \( \mathcal{P} \) such that \( \epsilon_n \rightarrow \epsilon^* \) and \( a(\epsilon_n, \epsilon^*) \geq 1 \) for all \( n \in \mathbb{N} \), then \( a(\epsilon_n, \epsilon^*) \geq 1 \) for all \( n \in \mathbb{N} \). Then, \( B \) has a fixed point. Furthermore, if \( a(\epsilon, \kappa) \geq 1 \) for all \( \epsilon, \kappa \in \text{Fix}(B) \), then the fixed point of the mapping \( B \) is unique.

**Proof.** Assume that \( \epsilon \perp \kappa \) if and only if \( d(B\epsilon, B\kappa) > 0 \).
Fix \( \epsilon_0 \in \mathcal{P} \). Since \( B \) satisfies inequality (29), for all \( \kappa \in \mathcal{P}, \epsilon_0, \kappa \). Consequently, it follows that \( (\mathcal{P}, \perp) \) constitutes an \( O \)-set. Additionally, it is evident that \( P \) is \( O \)-complete, and \( B \) is \( \perp \)-continuous and \( \perp \)-preserving. Thus, by utilizing Theorem 3, we deduce that \( B \) possesses a unique fixed point within the set \( \mathcal{P} \). \( \square \)

**Corollary 7.** Let \( B : \mathcal{P} \rightarrow \mathcal{P} \) be a self-mapping and suppose that there is some constant \( \tau > 0 \) and the functions \( F \in \Omega, a : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}_+ \) such that \( a(\epsilon, \kappa) \geq 1 \) and
\[ \tau + F(d(B\epsilon, B\kappa)) \leq F(d(\epsilon, \kappa)), \]
for all \( \epsilon, \kappa \in \mathcal{P} \). Assume that these conditions hold:
(i) \( B \) is an \( \alpha \)-admissible mapping;
(ii) \( \exists \epsilon_0 \in \mathcal{P} \) such that \( \epsilon_0 \perp B\epsilon_0 \) or \( B\epsilon_0 \perp \epsilon_0 \) and \( a(\epsilon_0, B\epsilon_0) \geq 1 \);
(iii) Either \( B : \mathcal{P} \to \mathcal{P} \) is continuous or if \( \{ \varepsilon_n \} \) is a sequence in \( \mathcal{P} \) such that \( \varepsilon_n \to \varepsilon^* \) and \( \alpha(\varepsilon_n, \varepsilon_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then \( \alpha(\varepsilon_n, \varepsilon^*) \geq 1 \) for all \( n \in \mathbb{N} \). Then, \( B \) has a fixed point. Moreover, if \( \alpha(\varepsilon, \varepsilon) \geq 1 \) for all \( \varepsilon, \kappa \in \text{Fix}(B) \), then the fixed point of the mapping \( B \) is unique.

**Proof.** Introduce \( h \) and \( \lambda \) on \( \mathbb{R}^+ \times \mathbb{R} \) by \( h(\varepsilon, \zeta) = \zeta \) and \( \lambda(\ell, \varphi) = \ell \varphi \) in Corollary 6. \( \square \)

**Corollary 8** ([11]). Let \( B : \mathcal{P} \to \mathcal{P} \) be a self-mapping which is continuous. Assume that there exist some constant \( \tau > 0 \) and \( F \in \Omega \) such that

\[
\tau + F(d(B\varepsilon, B\kappa)) \leq F(d(\varepsilon, \kappa)),
\]

for all \( \varepsilon, \kappa \in \mathcal{P} \). Then, \( B \) has a unique fixed point.

3.3.2. Fixed Point Results in Complete Orthogonal Metric Space

Letting the function \( \xi(t) \) be the natural logarithm of \( t \), for \( t > 0 \) and \( b = 1 \) in Definition 7, then \( O\delta-\text{MS} \) reduces to \( O-M \text{S} \) which leads to the following result.

**Corollary 9.** Consider a mapping \( B \) that maps elements of an \( O-M \text{S} (\mathcal{P}, d, \perp) \) to itself, preserving the orthogonality relation \( \perp \). Assume that there exists a positive constant \( \tau \) and \( F \) belonging to a set \( \Omega, \alpha : \mathcal{P} \times \mathcal{P} \to \mathbb{R}^+ \) and \( C \)-class functions \( \lambda \) and \( h \) that satisfy the following condition

\[
d(B\varepsilon, B\kappa) > 0 \implies h(\alpha(\varepsilon, \kappa)), \tau + F(d(B\varepsilon, B\kappa))) \leq \lambda(1, F(d(\varepsilon, \kappa))),
\]

for all \( \varepsilon, \kappa \in \mathcal{P} \) with \( \varepsilon \perp \kappa \). Assume that these assertions hold:

(i) \( B \) is orthogonal \( \alpha \)-admissible;
(ii) \( \exists \varepsilon_0 \in \mathcal{P} \) such that \( \varepsilon_0 \perp B\varepsilon_0 \) or \( B\varepsilon_0 \perp \varepsilon_0 \) and \( \alpha(\varepsilon_0, B\varepsilon_0) \geq 1 \);
(iii) Either \( B : \mathcal{P} \to \mathcal{P} \) is \( \perp \)-continuous or if \( \{ \varepsilon_n \} \) is an \( O \)-sequence in \( \mathcal{P} \) such that \( \varepsilon_n \to \varepsilon^* \) and \( \alpha(\varepsilon_n, \varepsilon_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then \( \alpha(\varepsilon_n, \varepsilon^*) \geq 1 \) for all \( n \in \mathbb{N} \).

Then, \( B \) has a fixed point. In addition, if \( \alpha(\varepsilon, \kappa) \geq 1, \forall \varepsilon, \kappa \in \text{Fix}(B) \), then the fixed point of the mapping \( B \) is unique.

**Corollary 10.** Consider a mapping \( B \) that maps elements of an \( O-M \text{S} (\mathcal{P}, d, \perp) \) to itself, preserving the orthogonality relation \( \perp \). Assume that there exist a positive constant \( \tau \) and \( F \) belonging to a set \( \Omega, \alpha : \mathcal{P} \times \mathcal{P} \to \mathbb{R}^+ \) such that \( \alpha(\varepsilon, \kappa) \geq 1 \) and

\[
d(B\varepsilon, B\kappa) > 0 \implies \tau + F(d(B\varepsilon, B\kappa)) \leq F(d(\varepsilon, \kappa)),
\]

for all \( \varepsilon, \kappa \in \mathcal{P} \) with \( \varepsilon \perp \kappa \). Assume that these assertions hold:

(i) \( B \) is orthogonal \( \alpha \)-admissible;
(ii) \( \exists \varepsilon_0 \in \mathcal{P} \) such that \( \varepsilon_0 \perp B\varepsilon_0 \) or \( B\varepsilon_0 \perp \varepsilon_0 \) and \( \alpha(\varepsilon_0, B\varepsilon_0) \geq 1 \);
(iii) Either \( B : \mathcal{P} \to \mathcal{P} \) is \( \perp \)-continuous or if \( \{ \varepsilon_n \} \) is an \( O \)-sequence in \( \mathcal{P} \) such that \( \varepsilon_n \to \varepsilon^* \) and \( \alpha(\varepsilon_n, \varepsilon_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then \( \alpha(\varepsilon_n, \varepsilon^*) \geq 1 \) for all \( n \in \mathbb{N} \).

Then, \( B \) has a fixed point. In addition, if \( \alpha(\varepsilon, \kappa) \geq 1, \forall \varepsilon, \kappa \in \text{Fix}(B) \), then the fixed point of the mapping \( B \) is unique.

**Proof.** Define the functions \( h \) and \( \lambda \) on \( \mathbb{R}^+ \times \mathbb{R} \) as follows: \( h(\varepsilon, \zeta) = \zeta \) and \( \lambda(\ell, \varphi) = \ell \varphi \) in Corollary 9. \( \square \)

**Remark 1.** If we take \( \alpha(\varepsilon, \kappa) = 1 \) for all \( \varepsilon, \kappa \in \mathcal{P} \) in Corollary 10 then we can derive the main result of Mani et al. [7].

4. Applications

In recent years, fractional differential equations (FDEs) have gained significant traction as versatile mathematical tools, finding applications across diverse scientific and engineering disciplines. Initially introduced as mathematical curiosities, FDEs have found profound utility in describing complex phenomena characterized by non-local, memory-dependent,
and anomalous behaviors. In the realm of physics, FDEs provide a sophisticated framework for modeling subdiffusion and superdiffusion phenomena, where particles exhibit slower or faster-than-classical random motion, respectively. The application of FDEs has become integral to understanding anomalous diffusion in biological systems, such as the movement of molecules in cellular environments, with implications for drug delivery and biomolecular interactions.

Let us consider a specific application in anomalous diffusion involving a fractional differential equation:

$$C^D_\eta (e(t)) = g(t, e(t)), \ (0 < t < 1, \ 1 < \eta \leq 2)$$  \hspace{1cm} (31)

along with the integral boundary conditions

$$e(0) = 0, \ e(1) = \int_0^\pi e(s)ds, \ (0 < \pi < 1), \ (32)$$

where $C^D_\eta$ represents the Caputo fractional derivative of order $\eta$ defined by

$$C^D_\eta g(t) = \frac{1}{\Gamma(j-\eta)} \int_0^t (t-s)^{j-\eta-1} g^j(s)ds,$$

$\ (j-1 < \eta < j, \ j = [\eta] + 1)$ and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}^+$ is continuous function.

In anomalous diffusion, the relationship between the diffusion coefficient $D$ and the fractional derivative order $\eta$ can be established through fractional differential equations. One common model for describing anomalous diffusion is the fractional diffusion equation, which is a generalization of the classical diffusion equation to incorporate fractional derivatives.

The fractional diffusion equation in one dimension can be written as follows:

$$\frac{\partial^\eta g(e,t)}{\partial t^\eta} = D \frac{\partial^2 g(e,t)}{\partial e^2},$$

where

- $g(e,t)$ is the concentration of particles;
- $t$ is time;
- $e$ is position;
- $\partial^\eta / \partial t^\eta$ is the Caputo fractional derivative with respect to time of order $\eta$;
- $\partial^2 / \partial e^2$ is the classical second derivative with respect to position.

In this equation, the diffusion coefficient $D$ appears as a proportionality constant relating the concentration gradient to the rate of change of concentration with respect to time. The fractional derivative order $\eta$ characterizes the degree of memory or long-range dependence in the diffusion process. A smaller $\eta$ implies more memory or longer-range dependence, leading to slower spreading of particles and, hence, a smaller effective diffusion coefficient. Conversely, a larger $\eta$ indicates less memory or shorter-range dependence, resembling classical diffusion with a larger effective diffusion coefficient.

We now present a key lemma for the subsequent proof.

**Lemma 1 ([13]).** Consider $\mathcal{P} = \{ e : e \in C([0,1], \mathbb{R}) \}$ with supremum norm $\| e \|_\infty = \sup_{t \in [0,1]} |e(t)|$; then, the Banach space $(\mathcal{P}, \| \cdot \|_\infty)$ furnished with $d$ defined by

$$d(e, \kappa) = \| e - \kappa \|_\infty = \sup_{t \in [0,1]} |e(t) - \kappa(t)|$$

for $e, \kappa \in \mathcal{P}$ and orthogonal relation $e \perp \kappa \iff e \kappa \geq 0$ is an orthogonal $\mathfrak{F}$-MS.

**Theorem 6.** Suppose that
(i) For some positive constant \( \tau \), let a function \( g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}^+ \), satisfying the following condition

\[
|g(t, \varepsilon) - g(t, \kappa)| \leq \frac{\Gamma(\eta + 1)}{5} e^{-\tau |t - \kappa|},
\]

for \( t \in [0, 1] \) and for all \( \varepsilon, \kappa \in \mathcal{P} \) such that \( e(t) \kappa(t) \geq 0 \).

(ii) There exists \( e_0 \in \mathcal{P} \) such that \( e_0(t)B_0(t) \geq 0 \), \( \forall t \in [0, 1] \), where \( B : \mathcal{P} \rightarrow \mathcal{P} \) is defined as

\[
B \varepsilon(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s, \varepsilon(s)) ds - \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^1 (1-s)^{\eta-1} g(s, \varepsilon(s)) ds
\]

\[+ \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^\pi \left( \int_0^s (s-m)^{\eta-1} g(m, \varepsilon(m)) dm \right) ds.
\]

(iii) For all \( t \in [0, 1] \) and \( \varepsilon, \kappa \in \mathcal{P} \), \( e(t) \kappa(t) \geq 0 \implies B \varepsilon(t)B \kappa(t) \geq 0 \).

(iv) If \( e_n \) is an \( O \)-sequence with \( e_n(t) \rightarrow e(t) \) as \( n \rightarrow \infty \) and \( e_n(t)e_{n+1}(t) \geq 0 \), for all \( t \in [0, 1] \), then \( e_n(t)e(t) \geq 0 \).

Then, the differential Equation (31) with the boundary conditions in Equation (32) has a unique solution.

**Proof.** Suppose that \( \perp \subseteq \mathcal{P} \times \mathcal{P} \) is an orthogonality relation on \( \mathcal{P} \) defined by

\[
\varepsilon \perp \kappa \iff e(t) \kappa(t) \geq 0,
\]

for all \( t \in [0, 1] \). Under this relation \( \perp \), the set \( \mathcal{P} \) is orthogonal because, for every \( \varepsilon \in \mathcal{P} \), there exists \( \kappa(t) = 0 \), for all \( t \in [0, 1] \) such that \( e(t) \kappa(t) = 0 \). We define \( d : \mathcal{P} \times \mathcal{P} \rightarrow [0, +\infty) \) by

\[
d(\varepsilon, \kappa) = ||\varepsilon - \kappa||_\infty = \sup_{t \in [0,1]} ||e(t) - \kappa(t)||
\]

for all \( \varepsilon, \kappa \in \mathcal{P} \); then, \( (\mathcal{P}, d, \perp) \) is a complete orthogonal \( \mathfrak{F} \)-MS. Evidently, the mapping \( B \) is \( \perp \)-continuous. It is well known from [13] that a function \( e \in \mathcal{P} \) is a solution of (31) if \( e \in \mathcal{P} \) is a fixed point of the mapping \( B \). In order to prove the existence of a fixed point of \( B \), we suppose that \( B \) is \( \perp \)-preserving. Suppose that \( e(t) \perp \kappa(t) \), for all \( t \in [0, 1] \). Now, we have

\[
B \varepsilon(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s, \varepsilon(s)) ds - \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^1 (1-s)^{\eta-1} g(s, \varepsilon(s)) ds
\]

\[+ \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^\pi \left( \int_0^s (s-m)^{\eta-1} g(m, \varepsilon(m)) dm \right) ds > 0,
\]

which implies by orthogonality relation \( \perp \) that \( B \varepsilon \perp B \kappa \), i.e., \( B \) is \( \perp \)-preserving. Now, for \( t \in [0, 1] \) accompanied by \( e(t) \perp \kappa(t) \), we obtain

\[
|B \varepsilon(t) - B \kappa(t)| = \left| \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s, \varepsilon(s)) ds - \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^1 (1-s)^{\eta-1} g(s, \varepsilon(s)) ds
\]

\[+ \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^\pi \left( \int_0^s (s-m)^{\eta-1} g(m, \varepsilon(m)) dm \right) ds
\]

\[\quad - \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s, \kappa(s)) ds + \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^1 (1-s)^{\eta-1} g(s, \kappa(s)) ds
\]

\[- \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^\pi \left( \int_0^s (s-m)^{\eta-1} g(m, \kappa(m)) dm \right) ds,\]

which implies
\[ |B\epsilon(t) - B\kappa(t)| \leq \frac{1}{\Gamma(\eta)} \int_0^t |t - s|^{\eta-1} |g(s, \epsilon(s)) - g(s, \kappa(s))| ds \\
+ \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^1 |1 - s|^{\eta-1} |g(s, \epsilon(s)) - g(s, \kappa(s))| ds \\
+ \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^\pi \int_0^s |s - m|^{\eta-1} |g(m, \epsilon(m)) - g(m, \kappa(m))| dm ds \]

Then, from (34) and the above concepts, we have

\[ e^{|\epsilon - \kappa|} \leq \sup_{t \in [0,1]} e^{\tau|t - s|^{\eta-1}} \left( \frac{1}{\Gamma(\eta)} \int_0^t |t - s|^{\eta-1} ds \\
+ \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^1 |1 - s|^{\eta-1} ds \\
+ \frac{2t}{(2 - \pi^2)\Gamma(\eta)} \int_0^\pi \int_0^s |s - m|^{\eta-1} dm ds \right) \]

that is,

\[ |B\epsilon(t) - B\kappa(t)| \leq \frac{\Gamma(\eta + 1)}{5} e^{-\tau} \|\epsilon - \kappa\|_\infty \]

Hence, for each \( \epsilon, \kappa \in \mathcal{P} \) with \( \epsilon(t) \perp \kappa(t) \) for all \( t \in [0,1] \), we have

\[ \|B\epsilon - B\kappa\|_\infty \leq e^{-\tau} \|\epsilon - \kappa\|_\infty \]

or

\[ d(B\epsilon, B\kappa) \leq e^{-\tau} d(\epsilon, \kappa). \]

By passing to logarithm, we write

\[ \ln(d(B\epsilon, B\kappa)) \leq \ln(e^{-\tau} d(\epsilon, \kappa)), \]

and, hence,

\[ \tau + \ln(d(B\epsilon, B\kappa)) \leq \ln(d(\epsilon, \kappa)). \] (34)

Now, define the function \( F : (0, +\infty) \rightarrow (-\infty, +\infty) \) by \( F(t) = \ln t \); we have

\[ \tau + F(d(B\epsilon, B\kappa)) \leq F(d(\epsilon, \kappa)), \]

and define the functions \( h \) and \( \lambda \) on \( \mathbb{R}^+ \times \mathbb{R} \) as follows: \( h(\sigma, \zeta) = \zeta \) and \( \lambda(\ell, \phi) = \ell \phi \). Then, \( \lambda \) and \( h \) are C-class functions. Also, define \( a : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}^+ \) by

\[ a(\epsilon, \kappa) = \begin{cases} 1, & \text{if } \epsilon(t) \perp \kappa(t) \text{ for } t \in [0,1] \\ 0, & \text{otherwise.} \end{cases} \]

Then, from (34) and the above concepts, we have

\[ h(a(\epsilon, \kappa)), \tau + F(d(B\epsilon, B\kappa)) \leq \lambda(1, F(d(\epsilon, \kappa))). \]

Now, from the definition of \( a : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}^+ \) and assumption (iii), we obtain that \( B \) is orthogonal \( \alpha \)-admissible. Now, let \( (\epsilon_n) \) be an \( \mathcal{O} \)-sequence converging in \( \mathcal{P} \) and \( \epsilon_n(t) \epsilon_{n+1}(t) \geq 0 \), for all \( t \in [0,1] \).
We will explore two distinct cases for discussion:

**Case 1.** If $\epsilon_n(t) \geq 0$ holds for all natural numbers $n$ and every $t$ in the interval $[0, 1]$, then, for each $t$ in $[0, 1]$, there exists a convergent sequence of positive values approaching $\epsilon(t)$. Consequently, we establish $\epsilon(t) \geq 0$, for every $t$ in $[0, 1]$, indicating that $\epsilon_n(t) \perp \epsilon(t)$.

**Case 2.** The case where $\epsilon_n(t) \leq 0$, for all $n \in \mathbb{N}$, for all natural numbers $n$ must be disregarded.

As a result, by Theorem 3, the mapping $B$ has a unique fixed point. Thus, the differential Equation (31) with the boundary conditions in Equation (32) has a unique solution. 

5. Conclusions

This manuscript has ventured into the fertile ground of fixed point theory within orthogonal $\mathfrak{g}$-metric spaces. We not only introduced novel concepts like $F-(\lambda, \mu)$-contraction and $(\alpha, \eta)$-Reich type interpolative contraction but also forged them into powerful tools, culminating in the establishment of several fixed point theorems. Our prime result stands as a robust pillar, from which established results in the literature seamlessly flow as corollaries, solidifying its centrality. Beyond theoretical elegance, we provided an illuminating example that not only serves as a testament to our innovative findings but also sheds light on the existing theory, fostering a deeper understanding. An application of our work is to solve a FDE that models anomalous diffusion phenomena.

Expanding on our current work, future research will explore the fascinating behavior of fixed points for multi-valued and fuzzy mappings within the rich structure of orthogonal $\mathfrak{g}$-metric space. Moreover, the proposed contractions can be generalized to even more comprehensive metric spaces beyond orthogonal $\mathfrak{g}$-metric spaces. The applicability of these results can be explored to other types of fractional differential equations or even integral equations. Also, some efficient computational algorithms can be developed to solve fixed point problems arising from the proposed contractions.

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