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Multivalued Variational Inequalities with Generalized Fractional Φ -Laplacians

Vy Khoi Le 

Department of Mathematics and Statistics, Missouri University of Science and Technology,
Rolla, MO 65409, USA; vy@mst.edu

Abstract: In this article, we examine variational inequalities of the form

$$\begin{cases} \langle \mathcal{A}(u), v - u \rangle + \langle \mathcal{F}(u), v - u \rangle \geq 0, \forall v \in K \\ u \in K, \end{cases} \text{ , where } \mathcal{A} \text{ is a generalized fractional } \Phi\text{-Laplace operator, } K \text{ is a closed convex set in a fractional Musielak–Orlicz–Sobolev space, and } \mathcal{F} \text{ is a multivalued integral operator. We consider a functional analytic framework for the above problem, including conditions on the multivalued lower order term } \mathcal{F} \text{ such that the problem can be properly formulated in a fractional Musielak–Orlicz–Sobolev space, and the involved mappings have certain useful monotonicity–continuity properties. Furthermore, we investigate the existence of solutions contingent upon certain coercivity conditions.}$$

ator, K is a closed convex set in a fractional Musielak–Orlicz–Sobolev space, and $\mathcal{F}$ is a multivalued integral operator. We consider a functional analytic framework for the above problem, including conditions on the multivalued lower order term $\mathcal{F}$ such that the problem can be properly formulated in a fractional Musielak–Orlicz–Sobolev space, and the involved mappings have certain useful monotonicity–continuity properties. Furthermore, we investigate the existence of solutions contingent upon certain coercivity conditions.

Keywords: variational inequality; pseudomonotone mapping; multivalued mapping; fractional Laplacian; fractional Musielak–Orlicz space; fractional Musielak–Orlicz–Sobolev space

1. Introduction

We are concerned in this article with variational inequalities driven by a generalized fractional Φ -Laplacian, perturbed by a multivalued lower order term, over a closed convex set in a fractional Musielak–Orlicz–Sobolev space.

The study of variational inequalities traces its origins back to the calculus of variations, yet its systematic development only began in the 1960s, initiated by the works of Fichera [1] and Stampacchia [2,3], which was motivated by problems in mechanics, like obstacle problems in elasticity—the Signorini problem, and Potential theory (the study of set capacities). Following the groundbreaking contributions of Lions and Stampacchia in [4], the exploration of variational inequalities gained momentum, evolving into a significant domain within nonlinear analysis, calculus of variations, optimization theory, optimal control and various branches of mechanics, mathematical physics and engineering.

The principal operators involved in variational inequalities span a wide range, depending on the modeling necessities; they range from Laplacians and p -Laplacians to more complex, nonlinear operators of Leray–Lions types, etc. Our focus in this paper is on multivalued variational inequalities, which encompass, as special instances, variational inequalities containing Clarke’s generalized gradients of locally Lipschitz functionals, usually referred to as variational hemi-variational inequalities. Such inequalities typically emerge in the context of mechanical problems characterized by nonconvex and potentially nonsmooth energy functionals. This situation arises particularly when nonmonotone, multivalued constitutive laws are considered, as illustrated in studies like [5,6]. Variational inequalities with general multivalued terms, which are not Clarke’s generalized gradients, do not generally have variational structures. For further details on variational inequalities related to those discussed here, readers are referred, e.g., to [5,7,8], and the references cited therein, which provide extensive discussions on this subject matter.

On the other hand, since the seminal articles by Caffarelli, Salsa and Silvestre [9,10], there has been a surge in research interest surrounding nonlocal problems, particularly



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those involving fractional operators, including fractional Laplace operators of different types, and the corresponding fractional Sobolev type spaces. These problems have garnered attention due to both their intriguing theoretical abstract structures and their practical applications in diverse fields, including fluid mechanics, mathematical finance, phase transitions, optimization, anomalous diffusion, materials science and image processing (see, e.g., [11–13] and the references therein).

Recently, Azroul, Benkirane, Shimi and Strati [14,15] and de Albuquerque, de Assis, Carvalho and Salort [16] studied equations driven by the s -fractional $\Phi_{x,y}$ -Laplacian operator $(-\Delta_{\Phi_{x,y}})^s$ and properties of its associated generalized fractional Musielak–Sobolev spaces $W^{s,\Phi_{x,y}}(\Omega)$. The operator $(-\Delta_{\Phi_{x,y}})^s$ is defined by

$$(-\Delta_{\Phi_{x,y}})^s u(x) := 2 \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a_{x,y} \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{dy}{|x - y|^{N+s}},$$

where $s \in (0, 1)$, $\Phi_{x,y}(t) = \int_0^{|t|} \tau a_{x,y}(\tau) d\tau$ is a generalized N -function, and $a : \Omega \times \Omega \times \mathbb{R}$ is a Carathéodory function, which is symmetric with respect to x and y . The appropriate function space for such problems is the generalized fractional Musielak–Sobolev space $W^{s,\Phi_{x,y}}(\Omega)$ associated with the fraction s and the generalized N -function $\Phi_{x,y}$. The fractional $\Phi_{x,y}$ -Laplacian operator $(-\Delta_{\Phi_{x,y}})^s$ and its associated fractional Musielak–Sobolev space $W^{s,\Phi_{x,y}}(\Omega)$ extend several concepts and functional frameworks in the literature.

In this paper, we are concerned with variational inequalities of the form

$$\begin{cases} \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x,y)|) D_s u(x,y) (D_s v(x,y) - D_s u(x,y)) d\mu \\ + \int_{\Omega} f(x,u)(v - u) dx \geq 0, \quad \forall v \in K \\ u \in K, \end{cases} \quad (1)$$

where K is a closed convex set in the fractional Musielak–Orlicz–Sobolev space $W^{s,\Phi_{x,y}}(\Omega)$, $\mathcal{F}$ is a multivalued integral operator, and

$$\langle \mathcal{A}(u), v \rangle = \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x,y)|) D_s u(x,y) D_s v(x,y) d\mu$$

represents a generalized fractional Φ -Laplacian in variational form, with $D_s u(x,y) = \frac{u(x) - u(y)}{|x - y|^s}$ and $d\mu = \frac{dx dy}{|x - y|^N}$.

Note that in the case $K = W_0^{s,\Phi_{x,y}}(\Omega)$ (where $W_0^{s,\Phi_{x,y}}(\Omega) = \{u \in W^{s,\Phi_{x,y}}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$) and f is single-valued, this variational inequality reduces to the variational equation

$$\int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x,y)|) D_s u(x,y) D_s v(x,y) d\mu + \int_{\Omega} f(x,u) v dx = 0,$$

for all $v \in W_0^{s,\Phi_{x,y}}(\Omega)$, which is the weak form of the Dirichlet boundary value problem

$$\begin{cases} (-\Delta_{\Phi_{x,y}})^s u + f(x,u) = 0, & \text{in } \Omega \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

This problem, together with its particular cases, has been extensively investigated in recent times. Some other choices of K formulate obstacle and unilateral problems. For example,

$$K = \{u \in W_0^{s,\Phi_{x,y}}(\Omega) : u \geq g \text{ a.e. on } \Omega\},$$

where g is a measurable function defined on Ω , corresponds to obstacle problems.

We focus here in the more general case where f is a multivalued function; that is, f is a function from $\Omega \times \mathbb{R}$ to $2^{\mathbb{R}}$. In this case, the variational inequality (1) is basically interpreted as follows. A function $u \in K$ is a solution of (1) if there exists a function η defined on Ω such that

$$\eta(x) \in f(x, u(x)) \text{ for } x \in \Omega, \quad (2)$$

and

$$\begin{aligned} & \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x,y)|) D_s u(x,y) (D_s v(x,y) - D_s u(x,y)) d\mu \\ & + \int_{\Omega} \eta(v - u) dx \geq 0, \quad \forall v \in K. \end{aligned} \quad (3)$$

In the particular case where $K = W_0^{s, \Phi_{x,y}}(\Omega)$, the multivalued variational inequality (2) and (3) is the weak form of the inclusion

$$\begin{cases} (-\Delta_{\Phi_{x,y}})^s u + f(x, u) \ni 0, & \text{in } \Omega \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The main goal of this article is to establish a functional analytic framework and derive existence results for the variational inequality (1) which involves a general convex set K in X and a multivalued lower term f . To the best of our knowledge, previous works related to fractional Laplacians have primarily focused on equations involving single-valued functions, and equations or inequalities that incorporate fractional Laplacians or generalized fractional Laplacians with multivalued lower order terms have not been systematically investigated. We would like to point out that classical variational methods are not directly applicable to problems (2) and (3) due to its general multivalued nature. In this article, we utilize a combination of topological and monotonicity methods to investigate the existence and related properties of solutions to problems (2) and (3). The analytical framework and results presented here seem useful for studying related problems involving fractional Laplace operators and fractional Musielak–Orlicz–Sobolev spaces as well.

The paper is organized as follows. Section 2 provides a preliminary discussion on fractional Musielak–Orlicz–Sobolev spaces and related topics. In Section 2.1, we present basic definitions and properties of N -functions, generalized N -functions, Musielak–Orlicz spaces and fractional Musielak–Orlicz–Sobolev spaces. In Section 2.2, we prove properties of generalized N -functions that satisfy Δ_2 conditions. The main topic of multivalued variational inequalities in fractional Musielak–Orlicz–Sobolev spaces is discussed in Section 3. The problem's setting, along with some basic assumptions, is provided in Section 3.1. In Sections 3.2 and 3.3, we focus on the multivalued lower order term and prove some crucial continuity and monotonicity properties of this term. This allows us to establish a functional analytic framework for our problem in an appropriate fractional Musielak–Orlicz–Sobolev space, suitable for investigation using topological and monotonicity approaches. Section 3.4 is devoted to the main existence results. We demonstrate the solvability of our multivalued variational inequality under certain conditions concerning the nonlocal fractional main operator, the multivalued term and the closed and convex set of constraints. Some properties presented in Sections 2.2 and 3.3 are necessary for subsequent discussions and also appear to be useful for the investigation of related problems in Musielak–Orlicz spaces and Musielak–Orlicz–Sobolev spaces.

2. Preliminaries on N -Functions and Fractional Musielak–Orlicz–Sobolev Spaces

In this section, we will present some basic facts about N -functions and fractional Musielak–Orlicz–Sobolev spaces, as well as related topics that are essential for our discussion in the following sections, such as properties of Musielak–Orlicz spaces and fractional Musielak–Orlicz–Sobolev spaces corresponding to N -functions that satisfy Δ_2 conditions.

2.1. Musielak–Orlicz–Sobolev Spaces and Fractional Musielak–Orlicz–Sobolev Spaces

We will discuss in this section basic concepts and results about N -functions and their associated Musielak–Orlicz spaces, Musielak–Orlicz–Sobolev spaces and fractional Musielak–Orlicz–Sobolev spaces.

2.1.1. N -Functions

In this section, we summarize some facts about the classes of N -functions and generalized N -functions; refer, e.g., to [17–21] for more details.

Definition 1. Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a right continuous, increasing function such that $\psi(0) = 0$, $\lim_{t \rightarrow \infty} \psi(t) = \infty$ and $\psi(t) > 0$, $\forall t > 0$. Then, the function Ψ defined by

$$\Psi(t) = \int_0^{|t|} \psi(s) ds, \quad t \in \mathbb{R},$$

is called an N -function.

Equivalently, a function $\Psi : \mathbb{R} \rightarrow [0, \infty)$ is an N -function if and only if Ψ is continuous, even, and convex with $\lim_{t \rightarrow 0} \frac{\Psi(t)}{t} = 0$, $\lim_{t \rightarrow \infty} \frac{\Psi(t)}{t} = \infty$ and $\Psi(t) > 0$, $\forall t > 0$.

In this case $\psi = \Psi'_+$, the right derivative of Ψ , satisfies the conditions in Definition 1 and $\Psi(t) = \int_0^{|t|} \psi(s) ds$, $t \in \mathbb{R}$. In some equivalent variants of the above definition, one can extend ψ to an odd function defined on $\mathbb{R}$, or restrict Ψ to $[0, \infty)$. We denote by $\mathcal{N}$ the set of all N -functions.

Definition 2. Let $\mathcal{U}$ be a measurable subset of $\mathbb{R}^d$ ($d \geq 1$). A function $\Psi : \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be a generalized N -function on $\mathcal{U}$ (Ψ is an $N(\mathcal{U})$ -function for short) if

- (i) For all $t \in \mathbb{R}$, $x \mapsto \Psi(x, t)$ is measurable on $\mathcal{U}$,
- (ii) For almost all $x \in \mathcal{U}$, $t \mapsto \Psi(x, t)$ is an N -function.

We denote by $\mathcal{N}(\mathcal{U})$ the set of all generalized N -functions on $\mathcal{U}$. Let $L^0(\mathcal{U})$ the set of all (equivalent classes of) real valued measurable functions on $\mathcal{U}$. Definition 2 means that

$$\mathcal{N}(\mathcal{U}) = \{\Psi : \mathcal{U} \times \mathbb{R} \rightarrow \mathbb{R} : \Psi(\cdot, t) \in L^0(\mathcal{U}), \forall t \in \mathbb{R}, \text{ and } \Psi(x, \cdot) \in \mathcal{N}, \text{ for a.e. } x \in \mathcal{U}\}.$$

Remark 1. Let $\Psi : (x, t) \mapsto \Psi(x, t)$ be a function defined on $\mathcal{U} \times \mathbb{R}$. In the sequel, we will use the notation Ψ_x for $\Psi(x, \cdot)$ or to simply emphasize the fact that Ψ also depends on $x \in \mathcal{U}$, in addition to its dependence on $t \in \mathbb{R}$.

Definition 3. (a) A function $\Psi \in \mathcal{N}$ is said to satisfy a (global) Δ_2 condition if there exists a constant $\kappa > 0$ such that

$$\Psi(2t) \leq \kappa \Psi(t), \quad \text{for all } t \geq 0. \quad (4)$$

(b) Let $\mathcal{U}$ be a measurable subset of $\mathbb{R}^d$. A function $\Psi = \Psi_x$ in $\mathcal{N}(\mathcal{U})$ is said to satisfy a (global) Δ_2 condition if there exists a constant $\kappa > 0$ (independent of x and t) such that

$$\Psi_x(2t) \leq \kappa \Psi_x(t), \quad \text{for a.e. } x \in \mathcal{U} \text{ and all } t \geq 0. \quad (5)$$

Definition 4. For $\Psi \in \mathcal{N}$ or $\mathcal{N}(\mathcal{U})$, let $\bar{\Psi}$ denote the Hölder conjugate (the Young complementary) function of Ψ , defined by

$$\bar{\Psi}(t) = \sup\{ts - \Psi(s) : s \in \mathbb{R}\} \quad (t \in \mathbb{R}),$$

or

$$\bar{\Psi}(x, t) = \sup\{ts - \Psi(x, s) : s \in \mathbb{R}\} \quad ((x, t) \in \mathcal{U} \times \mathbb{R}).$$

It follows from this definition that $\bar{\Psi}$ is also an N -function (or $N(\mathcal{U})$ -function respectively), and $\bar{\bar{\Psi}} = \Psi$.

Let Ω be an open domain in $\mathbb{R}^N$. In what follows, we concentrate on the cases where $\mathcal{U} = \Omega \subset \mathbb{R}^N$ and $\mathcal{U} = \Omega \times \Omega \subset \mathbb{R}^{2N}$. If $\Phi = \Phi_{x,y}$ is a function in $\mathcal{N}(\Omega \times \Omega)$, then the function $\hat{\Phi} = \hat{\Phi}_x$ defined by $\hat{\Phi}_x = \Phi_{x,x}$ is clearly a function in $\mathcal{N}(\Omega)$. Moreover, if both $\Phi_{x,y}$ and its conjugate $\bar{\Phi}_{x,y}$ satisfy Δ_2 conditions on $\Omega \times \Omega$, then $\hat{\Phi}_x$ and its conjugate $\bar{\hat{\Phi}}_x$ also satisfy Δ_2 conditions on Ω , with the same constants κ in formula (5) for $\Phi_{x,y}$ and $\hat{\Phi}_x$, and for $\bar{\Phi}_{x,y}$ and $\bar{\hat{\Phi}}_x$, respectively.

The following definition gives some comparison for growth rates of N -functions.

Definition 5. (a) Let $\Psi_1, \Psi_2 \in \mathcal{N}$; one says that Ψ_2 dominates Ψ_1 (near ∞) and writes

$$\Psi_1 \preceq \Psi_2 \text{ (or } \Psi_2 \succeq \Psi_1),$$

provided there exist constants $t_0, k \geq 0$ such that

$$\Psi_1(t) \leq \Psi_2(kt), \quad t \geq t_0.$$

(b) Let $\Psi_1, \Psi_2 \in \mathcal{N}(\mathcal{U})$; one says that Ψ_2 dominates Ψ_1 (near ∞) and writes

$$\Psi_1 \preceq \Psi_2 \text{ (or } \Psi_2 \succeq \Psi_1),$$

provided there exist constants $t_0, k \geq 0$ such that

$$\Psi_1(x, t) \leq \Psi_2(x, kt),$$

for all $t \geq t_0$ and a.e. $x \in \mathcal{U}$.

We also mention a notion of relative growth of N -functions which will play a role in our later considerations (cf. [17,19,20]).

Definition 6. (a) An N -function Ψ_1 is said to grow essentially more slowly than another N -function Ψ_2 , abbreviated by $\Psi_1 \ll \Psi_2$, if

$$\lim_{t \rightarrow \infty} \frac{\Psi_1(t)}{\Psi_2(kt)} = 0, \quad \forall k > 0.$$

(b) For $\Psi_1, \Psi_2 \in \mathcal{N}(\mathcal{U})$, we say that Ψ_1 grows essentially more slowly than Ψ_2 and denote by $\Psi_1 \ll \Psi_2$, if for all $k > 0$,

$$\lim_{t \rightarrow \infty} \frac{\Psi_1(x, t)}{\Psi_2(x, kt)} = 0,$$

uniformly for a.e. $x \in \mathcal{U}$.

2.1.2. Orlicz Spaces and Musielak–Orlicz Spaces

Let Ω be a domain in $\mathbb{R}^N$.

Definition 7. Let Ψ be an N -function. The Orlicz space $L^\Psi = L^\Psi(\Omega)$ is the set of all (equivalent classes of) measurable functions u on Ω such that

$$\int_{\Omega} \Psi\left(\frac{|u(x)|}{k}\right) dx < \infty, \quad \text{for some } k > 0.$$

With this definition, we see $L^\Psi(\Omega)$ is a Banach space when equipped with the (Luxemburg) norm

$$\|u\|_\Psi = \inf \left\{ k > 0 : \int_\Omega \Psi \left(\frac{|u|}{k} \right) dx \leq 1 \right\}.$$

The Hölder inequality is generalized to Orlicz spaces as follows.

Theorem 1. If $u \in L^\Psi(\Omega)$ and $v \in L^{\bar{\Psi}}(\Omega)$ then $uv \in L^1(\Omega)$ and

$$\left| \int_\Omega u(x)v(x)dx \right| \leq 2\|u\|_\Psi\|v\|_{\bar{\Psi}}.$$

Definition 8. Let $\Psi = \Psi_x$ be an $N(\Omega)$ -function. The Musielak–Orlicz space $L^\Psi(\Omega) = L^{\Psi_x}(\Omega)$ is the set of all functions $u \in L^0(\Omega)$ such that

$$\int_\Omega \Psi_x \left(\frac{u(x)}{k} \right) dx < \infty, \text{ for some } k > 0.$$

$L^{\Psi_x}(\Omega)$ is a Banach space when equipped with the Luxemburg norm

$$\|u\|_{\Psi_x} = \inf \left\{ k > 0 : \int_\Omega \Psi_x \left(\frac{u(x)}{k} \right) dx \leq 1 \right\}.$$

Let $\Phi = \Phi_{x,y}$ be a function in $\mathcal{N}(\Omega \times \Omega)$. Corresponding to the $N(\Omega)$ -function $\hat{\Phi}_x = \Phi_{x,x}$, we have Musielak–Orlicz space

$$L^{\hat{\Phi}_x}(\Omega) = \left\{ u \in L^0(\Omega) : \int_\Omega \hat{\Phi}_x \left(\frac{u(x)}{k} \right) dx < \infty, \text{ for some } k > 0 \right\}.$$

Under the assumption that both $\Phi_{x,y}$ and its conjugate satisfy Δ_2 conditions and thus both $\hat{\Phi}_x$ and its conjugate also satisfy Δ_2 conditions, $L^{\hat{\Phi}_x}(\Omega)$ is a separable and reflexive Banach space when endowed with the Luxemburg norm

$$\|u\|_{\hat{\Phi}_x} := \inf \left\{ k > 0 : \int_\Omega \hat{\Phi}_x \left(\frac{u(x)}{k} \right) dx \leq 1 \right\}.$$

We also have the following Hölder's type inequality

$$\left| \int_\Omega uv dx \right| \leq 2\|u\|_{\hat{\Phi}_x}\|v\|_{\bar{\hat{\Phi}}_x}$$

for all $u \in L^{\hat{\Phi}_x}(\Omega)$ and $v \in L^{\bar{\hat{\Phi}}_x}(\Omega)$.

2.1.3. Fractional Musielak–Orlicz–Sobolev Spaces

Let Ω be an open domain in $\mathbb{R}^N$, $\Phi = \Phi_{x,y} \in \mathcal{N}(\Omega \times \Omega)$, and $s \in (0, 1)$. The fractional Musielak–Orlicz–Sobolev space $W^{s,\Phi_{x,y}}(\Omega)$ is the set of all functions $u \in L^{\hat{\Phi}_x}(\Omega)$ such that

$$\int_\Omega \int_\Omega \Phi_{x,y} \left(D_s \left(\frac{u(x,y)}{k} \right) \right) d\mu < \infty,$$

for some $k > 0$, where $D_s u(x,y) := \frac{u(x) - u(y)}{|x - y|^s}$ and $d\mu := \frac{dx dy}{|x - y|^N}$. Note that $d\mu$ is a regular Borel measure on $\Omega \times \Omega$.

For $u \in W^{s,\Phi_{x,y}}(\Omega)$, we define

$$[u]_{s,\Phi_{x,y}} := \inf \left\{ k > 0 : \int_\Omega \int_\Omega \Phi_{x,y} \left(D_s \left(\frac{u(x,y)}{k} \right) \right) d\mu \leq 1 \right\}.$$

Then, $[\cdot]_{s,\Phi_{x,y}}$ is a seminorm on $W^{s,\Phi_{x,y}}(\Omega)$ (called the Gagliardo seminorm associated with s and $\Phi_{x,y}$), and the space $W^{s,\Phi_{x,y}}(\Omega)$ equipped with the norm

$$\|u\|_{W^{s,\Phi_{x,y}}(\Omega)} := \|u\|_{\widehat{\Phi}_x} + [u]_{s,\Phi_{x,y}},$$

is a Banach space. Moreover, under the assumption supposed here that $\Phi_{x,y}$ and $\overline{\Phi}_{x,y}$ both satisfy Δ_2 conditions, $W^{s,\Phi_{x,y}}(\Omega)$ is a reflexive and separable Banach space (cf. [14]).

Let $\Phi \in \mathcal{N}(\Omega \times \Omega)$. We suppose in what follows the following fractional boundedness condition: There are constants $C_1, C_2 > 0$, for which

$$C_1 \leq \Phi_{x,y}(1) \leq C_2, \quad \text{for a.e. } (x,y) \in \Omega \times \Omega, \quad (6)$$

Let $\widehat{\Phi}_x = \Phi_{x,x} \in \mathcal{N}(\Omega)$. Suppose that

$$\int_0^1 \frac{\widehat{\Phi}_x^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau < \infty \quad \text{and} \quad \int_1^\infty \frac{\widehat{\Phi}_x^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau = \infty, \quad \text{for a.e. } x \in \Omega, \quad (7)$$

where $\widehat{\Phi}_x^{-1}$ is the inverse function of $\widehat{\Phi}_x$. As in [17], the inverse Musielak–Orlicz–Sobolev conjugate function of $\widehat{\Phi}$, defined by $\widehat{\Phi}_{s,x}^*$, as follows:

$$\left(\widehat{\Phi}_{s,x}^*\right)^{-1}(t) := \left(\widehat{\Phi}_s^*\right)^{-1}(x,t) = \int_0^t \frac{\widehat{\Phi}_x^{-1}(\tau)}{\tau^{\frac{N+s}{N}}} d\tau, \quad \text{for } x \in \Omega \text{ and } t \geq 0,$$

is also a function in $\mathcal{N}(\Omega)$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. We have the following embedding results.

Proposition 1 (Theorems 2.1 and 2.2, [15]). *Let $s \in (0,1)$ and $\Phi \in \mathcal{N}(\Omega \times \Omega)$.*

Under conditions (6) and (7), the embedding $W^{s,\Phi_{x,y}}(\Omega) \hookrightarrow L^{\widehat{\Phi}_{s,x}^}(\Omega)$ is continuous. Moreover, for any $\Psi \in \mathcal{N}(\Omega)$ such that $\Psi \ll \widehat{\Phi}_s^*$, the embedding $W^{s,\Phi_{x,y}}(\Omega) \hookrightarrow L^{\Psi_x}(\Omega)$ is compact.*

Let us consider the following closed subspace of $W^{s,\Phi_{x,y}}(\Omega)$ defined by

$$W_0^{s,\Phi_{x,y}}(\Omega) = \left\{ u \in W^{s,\Phi_{x,y}}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

We have the following generalized Poincaré type inequality.

Proposition 2 (Theorem 2.3, [15]). *Let $s \in (0,1)$ and $\Phi \in \mathcal{N}(\Omega \times \Omega)$ such that both Φ and $\overline{\Phi}$ satisfy Δ_2 conditions. Assume that (6) holds. Then, there exists a positive constant C such that*

$$\|u\|_{\widehat{\Phi}_x} \leq C[u]_{s,\Phi_{x,y}}$$

for all $u \in W_0^{s,\Phi_{x,y}}(\Omega)$.

It follows from this result that $[\cdot]_{s,\Phi_{x,y}}$ is a norm on $W_0^{s,\Phi_{x,y}}(\Omega)$ which is equivalent to the usual norm $\|\cdot\|_{s,\Phi_{x,y}}$.

Note that the results in Propositions 1 and 2 were established in [15] under the assumption that $\phi_{x,y}$ is a strictly increasing and odd homeomorphism from $\mathbb{R}$ onto itself. However, the arguments in their proofs can be directly extended to the more general case where $\phi_{x,y}$ satisfies the conditions in Definition 1.

Remark 2. (a) Since generalized N -functions are natural extensions and unifications of N -functions and power functions with variable exponents, Musielak–Orlicz spaces generalize Orlicz

spaces and Lebesgue spaces with variable exponents and fractional Musielak–Orlicz–Sobolev spaces generalize fractional Orlicz–Sobolev spaces and fractional Sobolev spaces with variable exponents.

More specifically, for the case $\Phi_{x,y}(t) = \Phi(t)$ is independent of the variables x and y , we see that $L^{\hat{\Phi}}(\Omega)$ and $W^{s,\Phi}(\Omega)$ are the regular Orlicz spaces and fractional Orlicz–Sobolev spaces. For example, we have in this case

$$W^{s,\Phi}(\Omega) = \left\{ u \in L^{\hat{\Phi}}(\Omega) : \int_{\Omega} \int_{\Omega} \Phi \left(D_s \left(\frac{u(x,y)}{k} \right) \right) d\mu < \infty, \text{ for some } k > 0 \right\}.$$

On the other hand, in the case $\Phi_{x,y}(t) = |t|^{p(x,y)}$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, where $p : \overline{\Omega} \times \overline{\Omega} \rightarrow (1, \infty)$ is a continuous function that is symmetric in x and y , and

$$1 < p^- = \min_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) \leq p^+ = \max_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x,y) < \infty,$$

then the Musielak–Orlicz space $L^{\hat{\Phi}_x}(\Omega)$ becomes the Lebesgue space with variable exponents $L^{\hat{p}(x)}(\Omega)$ (with $\hat{p}(x) = p(x,x)$, $x \in \overline{\Omega}$), and the fractional Musielak–Orlicz–Sobolev space $W^{s,\Phi_{x,y}}(\Omega)$ becomes the fractional Sobolev space with variable exponents $W^{s,p(x,y)}(\Omega)$,

$$L^{\hat{\Phi}_x}(\Omega) = L^{\hat{p}(x)}(\Omega) = \left\{ u \in L^0(\Omega) : \int_{\Omega} \left(\frac{|u(x)|}{k} \right)^{\hat{p}(x)} dx < \infty, \text{ for some } k > 0 \right\},$$

and

$$\begin{aligned} W^{s,\Phi_{x,y}}(\Omega) &= W^{s,p(x,y)}(\Omega) \\ &= \left\{ u \in L^{\hat{p}(x)}(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{k^{p(x,y)} |x - y|^{sp(x,y)+N}} dx dy < \infty, \text{ for some } k > 0 \right\}. \end{aligned}$$

We refer to [11,14–16,22] and the references therein for more details on fractional Orlicz–Sobolev spaces, fractional Sobolev spaces with variable exponents, and fractional Musielak–Orlicz–Sobolev spaces, along with more detailed discussions and additional examples of particular cases of these spaces.

(b) We also refer to the recent works in [23–25] and the references therein for discussions on the limiting cases of fractional Orlicz–Sobolev spaces where $s \rightarrow 0^+$ or $s \rightarrow 1^-$, and for certain properties of fractional Orlicz–Sobolev spaces in one-dimensional domains.

2.2. Auxiliary Results Related to Δ_2 Conditions

We consider in this section some useful properties of $N(\mathcal{U})$ -functions that satisfy Δ_2 conditions. For these functions, we demonstrate the existence of certain gauge functions, under which some direct relations between integrals containing $N(\mathcal{U})$ -functions and norms or Gagliardo seminorms in the corresponding Musielak–Orlicz spaces or fractional Musielak–Orlicz–Sobolev spaces are established. These relations are necessary for the estimates in Section 3.

Theorem 2. Let $\mathcal{U} \subset \mathbb{R}^d$ be measurable and let $\Psi = \Psi_x$ be an $N(\mathcal{U})$ -function. Then, the following conditions are equivalent.

- (i) Ψ satisfies a Δ_2 condition.
- (ii) There exists a continuous, strictly increasing function $\mu_{\Psi}^+ : [0, \infty) \rightarrow [0, \infty)$ such that $\mu_{\Psi}^+(0) = 0$, $\mu_{\Psi}^+(t) \rightarrow \infty$ as $t \rightarrow \infty$ and

$$\Psi_x(tu) \leq \mu_{\Psi}^+(t) \Psi_x(u), \text{ for a.e. } x \in \mathcal{U}, \text{ for all } t \geq 0, \text{ all } u \in \mathbb{R}. \quad (8)$$

(iii) There exists a continuous, strictly increasing function $\mu_{\Psi}^{-} : [0, \infty) \rightarrow [0, \infty)$ such that $\mu_{\Psi}^{-}(0) = 0$, $\mu_{\Psi}^{-}(t) \rightarrow \infty$ as $t \rightarrow \infty$ and

$$\Psi_x(tu) \geq \mu_{\Psi}^{-}(t)\Psi_x(u), \text{ for a.e. } x \in \mathcal{U}, \text{ for all } t \geq 0, \text{ all } u \in \mathbb{R}. \quad (9)$$

Proof. [(i) $\Rightarrow$ (ii)] Suppose that Ψ satisfies (5). Since $\Psi(x, \cdot)$ is even, we only need to prove (8) for all $u \geq 0$. Also, without loss of generality, we can assume that $\Psi = \Psi_x$ satisfies (5) with $\kappa > 1$. If $t \in [0, 1]$, then from the convexity of $\Psi(x, \cdot)$, we have

$$\Psi(x, tu) \leq t\Psi(x, u) \leq \kappa t\Psi(x, u). \quad (10)$$

Let $t \geq 1$. Then, there is a unique nonnegative integer n such that $2^n \leq t < 2^{n+1}$. Hence $n \leq \ln t / \ln 2$. It follows from (5) that

$$\begin{aligned} \Psi(x, tu) &\leq \Psi(x, 2^{n+1}u) \\ &\leq \kappa^{n+1}\Psi(x, u) \\ &\leq \kappa \kappa^{\frac{\ln t}{\ln 2}} \Psi(x, u) \\ &= \kappa t^{\frac{\ln \kappa}{\ln 2}} \Psi(x, u). \end{aligned} \quad (11)$$

Define $\mu_{\Psi}^{+} : [0, \infty) \rightarrow [0, \infty)$ by

$$\mu_{\Psi}^{+}(t) = \begin{cases} \kappa t, & 0 \leq t \leq 1 \\ \kappa t^{\frac{\ln \kappa}{\ln 2}}, & t > 1. \end{cases} \quad (12)$$

Since $\frac{\ln \kappa}{\ln 2} > 0$, μ_{Ψ}^{+} is continuous, strictly increasing on $[0, \infty)$ with $\mu_{\Psi}^{+}(0) = 0$, $\mu_{\Psi}^{+}(t) \rightarrow \infty$ as $t \rightarrow \infty$. Moreover, (10) and (11) imply (8).

[(ii) $\Rightarrow$ (iii)] Suppose Ψ and μ_{Ψ}^{+} be as in (ii). Define $\mu_{\Psi}^{-} : [0, \infty) \rightarrow [0, \infty)$ by

$$\mu_{\Psi}^{-}(t) = \begin{cases} \frac{1}{\mu_{\Psi}^{+}(1/t)}, & t > 0 \\ 0, & t = 0. \end{cases} \quad (13)$$

Straightforward arguments show that μ_{Ψ}^{-} is strictly increasing and continuous on $(0, \infty)$. Moreover, as $t \rightarrow \infty$, $\mu_{\Psi}^{+}(1/t) \rightarrow 0^{+}$ and thus $\mu_{\Psi}^{-}(t) \rightarrow \infty$. Similarly, $\mu_{\Psi}^{-}(t) \rightarrow 0^{+}$ as $t \rightarrow 0^{+}$. Hence, μ_{Ψ}^{-} is a strictly increasing homeomorphism of $[0, \infty)$ onto itself. When $t = 0$, (9) trivially holds. For $t > 0$ and $u \in \mathbb{R}$, we have from (8) that $\Psi_x(u) = \Psi_x((1/t)tu) \leq \mu_{\Psi}^{+}(1/t)\Psi_x(tu)$ and thus $\Psi_x(tu) \geq \frac{1}{\mu_{\Psi}^{+}(1/t)}\Psi_x(u) = \mu_{\Psi}^{-}(t)\Psi_x(u)$.

[(iii) $\Rightarrow$ (i)] It follows from (9) that $\Psi_x(u) \geq \mu_{\Psi}^{-}(1/2)\Psi_x(2u)$, that is, Ψ_x satisfies (5) with $\kappa = \frac{1}{\mu_{\Psi}^{-}(1/2)}$. $\square$

From Theorem 2 and its proof, we have following immediate corollary.

Corollary 1. (a) Suppose $\Psi = \Psi_x$ is an $N(\mathcal{U})$ -function that satisfies a Δ_2 condition. Then, there exist functions $\mu_{\Psi}^{+}, \mu_{\Psi}^{-} : [0, \infty) \rightarrow [0, \infty)$ such that

$$\mu_{\Psi}^{+}, \mu_{\Psi}^{-} \text{ are continuous and strictly increasing,} \quad (14)$$

$$\mu_{\Psi}^{+}(0) = \mu_{\Psi}^{-}(0) = 0, \mu_{\Psi}^{+}(t), \mu_{\Psi}^{-}(t) \rightarrow \infty \text{ as } t \rightarrow \infty, \quad (15)$$

$$\mu_{\Psi}^{-}(t)\mu_{\Psi}^{+}(1/t) = 1, \forall t > 0, \quad (16)$$

and for a.e. $x \in \mathcal{U}$, for all $t \geq 0$, all $u \in \mathbb{R}$,

$$\mu_{\Psi}^{-}(t)\Psi_x(u) \leq \Psi_x(tu) \leq \mu_{\Psi}^{+}(t)\Psi_x(u), \quad (17)$$

(b) In particular, if $\Phi = \Phi_{x,y}$ is an $N(\Omega \times \Omega)$ -function that satisfies a Δ_2 condition, then there are functions $\mu_{\Phi}^{+}, \mu_{\Phi}^{-} : [0, \infty) \rightarrow [0, \infty)$ satisfying (14)–(16) such that for a.e. $(x, y) \in \Omega \times \Omega$, for all $t \geq 0$, all $u \in \mathbb{R}$,

$$\mu_{\Phi}^{-}(t)\Phi_{x,y}(u) \leq \Phi_{x,y}(tu) \leq \mu_{\Phi}^{+}(t)\Phi_{x,y}(u). \quad (18)$$

Moreover, if $\widehat{\Phi}_x = \Phi_{x,x}$ with $\Phi = \Phi_{x,y} \in \mathcal{N}(\Omega \times \Omega)$ as in (a), then $\mu_{\widehat{\Phi}}^{\pm}$ can be chosen the same as $\mu_{\Phi}^{\pm}$, that is,

$$\mu_{\widehat{\Phi}}^{-}(t)\widehat{\Phi}_x(u) \leq \widehat{\Phi}_x(tu) \leq \mu_{\widehat{\Phi}}^{+}(t)\widehat{\Phi}_x(u), \quad (19)$$

for a.e. $x \in \Omega$, for all $t \geq 0$, all $u \in \mathbb{R}$.

It follows from this corollary the following estimates for the integrals $\int_{\mathcal{U}} \Psi_x(|u(x)|)dx$ and $\int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x, y))d\mu$.

Corollary 2. (a) Suppose $\Psi = \Psi_x$ is an $N(\mathcal{U})$ -function that satisfies a Δ_2 condition and let $\mu_{\Psi}^{+}, \mu_{\Psi}^{-}$ be given in Corollary 1. Then

$$\mu_{\Psi}^{-}(\|u\|_{\Psi_x}) \leq \int_{\mathcal{U}} \Psi_x(|u(x)|)dx \leq \mu_{\Psi}^{+}(\|u\|_{\Psi_x}), \quad \forall u \in L^{\Psi_x}(\Omega). \quad (20)$$

(b) Suppose $\Phi = \Phi_{x,y}$ is an $N(\Omega \times \Omega)$ -function that satisfies a Δ_2 condition. Let $\widehat{\Phi} = \widehat{\Phi}_x = \Phi_{x,x}$ and let $\mu_{\Phi}^{+}, \mu_{\Phi}^{-}$ be given in Corollary 1. Then

$$\mu_{\Phi}^{-}([u]_{s, \Phi_{x,y}}) \leq \int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x, y))d\mu \leq \mu_{\Phi}^{+}([u]_{s, \Phi_{x,y}}), \quad \forall u \in W^{s, \Phi_{x,y}}(\Omega), \quad (21)$$

and

$$\mu_{\widehat{\Phi}}^{-}(\|u\|_{\widehat{\Phi}_x}) \leq \int_{\Omega} \widehat{\Phi}_x(|u(x)|)dx \leq \mu_{\widehat{\Phi}}^{+}(\|u\|_{\widehat{\Phi}_x}), \quad \forall u \in L^{\widehat{\Phi}_x}(\Omega). \quad (22)$$

Proof. The estimates in (a) and (b) follow directly from Corollary 1 and the definitions of $\|u\|_{\Psi_x}$, $[u]_{s, \Phi_{x,y}}$ and $\|u\|_{\widehat{\Phi}_x}$. The proof for (21) is briefly presented here for completeness; those for (20) and (22) follow the same lines.

Let $\Phi = \Phi_{x,y}$ satisfy a Δ_2 condition. From the definition of $[u]_{s, \Phi_{x,y}}$, there is a sequence $\{\lambda_n\}$ in $(0, \infty)$ such that $\lambda_n \rightarrow [u]_{s, \Phi_{x,y}}$ and $\int_{\Omega} \int_{\Omega} \Phi_{x,y}\left(\frac{D_s u(x, y)}{\lambda_n}\right)d\mu \leq 1$ for all n . It follows from (17) that for a.e. $x, y \in \Omega$,

$$\mu_{\Phi}^{-}(1/\lambda_n)\Phi_{x,y}(D_s u(x, y)) \leq \Phi_{x,y}\left(\frac{D_s u(x, y)}{\lambda_n}\right).$$

Hence, for all $n \in \mathbb{N}$,

$$\mu_{\Phi}^{-}(1/\lambda_n) \int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x, y))d\mu \leq \int_{\Omega} \int_{\Omega} \Phi_{x,y}\left(\frac{D_s u(x, y)}{\lambda_n}\right)d\mu \leq 1,$$

and thus $\int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x, y))d\mu \leq \frac{1}{\mu_{\Phi}^{-}(1/\lambda_n)} = \mu_{\Phi}^{+}(\lambda_n)$. Letting $n \rightarrow \infty$, noting the continuity of μ_{Φ}^{+} , we see that $\int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x, y))d\mu \leq \mu_{\Phi}^{+}([u]_{s, \Phi_{x,y}})$. Let us prove the other inequality of (21). Since $\mu_{\Phi}^{-}(0)$, this inequality is obvious when $[u]_{s, \Phi_{x,y}} =$

0. Suppose $[u]_{s, \Phi_{x,y}} > 0$. For $n \in \mathbb{N}$, since $\lambda'_n := \frac{n}{n+1} [u]_{s, \Phi_{x,y}} < [u]_{s, \Phi_{x,y}}$, we have

$$\int_{\Omega} \int_{\Omega} \Phi_{x,y} \left(\frac{D_s u(x,y)}{\lambda'_n} \right) d\mu > 1. \text{ As}$$

$$\mu_{\Phi}^+(1/\lambda'_n) \Phi_{x,y}(D_s u(x,y)) \geq \Phi_{x,y} \left(\frac{D_s u(x,y)}{\lambda'_n} \right).$$

a.e. on $\Omega \times \Omega$, we have

$$\mu_{\Phi}^+(1/\lambda'_n) \int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x,y)) d\mu \geq \int_{\Omega} \int_{\Omega} \Phi_{x,y} \left(\frac{D_s u(x,y)}{\lambda'_n} \right) d\mu > 1, \forall n \in \mathbb{N}.$$

Therefore, $\int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x,y)) d\mu > \frac{1}{\mu_{\Phi}^+(1/\lambda'_n)} = \mu_{\Phi}^-(\lambda'_n)$. Noting that $\lambda'_n \rightarrow [u]_{s, \Phi_{x,y}}$

and using the continuity of μ_{Φ}^- , we obtain $\int_{\Omega} \int_{\Omega} \Phi_{x,y}(D_s u(x,y)) d\mu \geq \mu_{\Phi}^-([u]_{s, \Phi_{x,y}})$.

The proofs of (22) and (20) are similar. $\square$

As another consequence of Theorem 2, we have the following form of Young's inequality, that will be useful in the sequel.

Corollary 3 (Young's inequality with ε). Suppose $\Psi \in \mathcal{N}(\mathcal{U})$ and Ψ and $\bar{\Psi}$ satisfy Δ_2 conditions. Then, there exists a continuous, strictly increasing function $\sigma : [0, \infty) \rightarrow [0, \infty)$ such that $\sigma(0) = 0$, $\sigma(t) \rightarrow \infty$ as $t \rightarrow \infty$ and

$$uv \leq \varepsilon \Psi(x, u) + \sigma(\varepsilon^{-1}) \bar{\Psi}(x, v), \text{ for a.e. } x \in \mathcal{U}, \text{ for all } \varepsilon > 0, \text{ all } u, v \in \mathbb{R}. \quad (23)$$

Proof. Let μ_{Ψ}^+ and $\mu_{\bar{\Psi}}^+$ be the gauge functions associated with Ψ and $\bar{\Psi}$, respectively, as given in Theorem 2. For any $\lambda > 0$ and $u, v \in \mathbb{R}$, we have from Young's inequality, together with (8) and its counterpart for $\bar{\Psi}$ that for a.e. $x \in \Omega$,

$$\begin{aligned} uv = (\lambda u)(\lambda^{-1}v) &\leq \Psi(x, \lambda u) + \bar{\Psi}(x, \lambda^{-1}v) \\ &\leq \mu_{\Psi}^+(\lambda) \Psi(x, u) + \mu_{\bar{\Psi}}^+(\lambda^{-1}) \bar{\Psi}(x, v). \end{aligned} \quad (24)$$

Let us define $\sigma(t) = \mu_{\bar{\Psi}}^+ \left(\frac{1}{(\mu_{\Psi}^+)^{-1}(1/t)} \right)$ for $t > 0$ and $\sigma(0) = 0$, where $(\mu_{\Psi}^+)^{-1}$ is the inverse function of μ_{Ψ}^+ . It is clear that σ is a function from $[0, \infty)$ into itself. Moreover, it follows directly from the strictly increasing property of μ_{Ψ}^+ and $\mu_{\bar{\Psi}}^+$ and thus of $(\mu_{\Psi}^+)^{-1}$ and $(\mu_{\bar{\Psi}}^+)^{-1}$, that σ is strictly increasing on $(0, \infty)$. From the continuity of $(\mu_{\Psi}^+)^{-1}$ and $\mu_{\bar{\Psi}}^+$ that σ is continuous on $(0, \infty)$. Furthermore, as $t \rightarrow 0^+$, we have $1/t \rightarrow \infty$ and thus $(\mu_{\Psi}^+)^{-1}(1/t) \rightarrow \infty$, which implies that $\frac{1}{(\mu_{\Psi}^+)^{-1}(1/t)} \rightarrow 0^+$ and thus $\sigma(t) = \mu_{\bar{\Psi}}^+ \left(\frac{1}{(\mu_{\Psi}^+)^{-1}(1/t)} \right) \rightarrow 0^+$. Similarly, $\sigma(t) \rightarrow \infty$ as $t \rightarrow \infty$. In particular, σ is continuous and strictly increasing on $[0, \infty)$ as well.

For $\varepsilon > 0$, letting $\lambda = (\mu_{\Psi}^+)^{-1}(\varepsilon)$, we have from (24) that

$$\begin{aligned} uv &\leq \varepsilon \Psi(x, u) + \mu_{\bar{\Psi}}^+ \left(\frac{1}{(\mu_{\Psi}^+)^{-1}(\varepsilon)} \right) \bar{\Psi}(x, v) \\ &= \varepsilon \Psi(x, u) + \sigma(\varepsilon^{-1}) \bar{\Psi}(x, v). \end{aligned} \quad (25)$$

$\square$

The following characterizations for Δ_2 conditions of $N(\mathcal{U})$ -functions are direct generalizations for Δ_2 conditions of N -functions, given, e.g., in [19,20].

Theorem 3. Let $\Psi = \Psi_x$ be an $N(\mathcal{U})$ -function, $\bar{\Psi} = \bar{\Psi}_x$ be its conjugate, and $\psi = \psi_x$, $\bar{\psi} = \bar{\psi}_x$, be their corresponding right partial derivatives with respect to t . Then, the following conditions are equivalent.

- (i) Ψ satisfies a Δ_2 condition.
- (ii) There exists a constant $\alpha \in (1, \infty)$ such that

$$\frac{t\psi_x(t)}{\Psi_x(t)} \leq \alpha, \text{ for a.e. } x \in \mathcal{U}, \text{ for all } t > 0. \quad (26)$$

- (iii) There exists a constant $\bar{\alpha} \in (1, \infty)$ such that

$$\frac{t\bar{\psi}_x(t)}{\bar{\Psi}_x(t)} \geq \bar{\alpha}, \text{ for a.e. } x \in \mathcal{U}, \text{ for all } t > 0. \quad (27)$$

Proof. [(i) $\Rightarrow$ (ii)] Suppose Ψ_x satisfies (5). We have, for $x \in \mathcal{U}$ and $t > 0$,

$$\kappa\Psi_x(t) \geq \Psi_x(2t) = \int_0^{2t} \psi_x(s)ds \geq \int_t^{2t} \psi_x(s)ds \geq t\psi_x(t),$$

since $\psi_x(\cdot)$ is increasing. Since $\kappa > 1$, (ii) follows with any $\alpha \geq \kappa$.

[(ii) $\Rightarrow$ (iii)] Let $x \in \mathcal{U}$ and $t > 0$. We have $\psi_x(\bar{\psi}_x(t)) \geq t$ by properties of right inverses. Since the function $u \mapsto \frac{\bar{\Psi}_x(u)}{u}$ is increasing and positive for $u > 0$, we have $\frac{\bar{\Psi}_x(\psi_x(\bar{\psi}_x(t)))}{\psi_x(\bar{\psi}_x(t))} \geq \frac{\bar{\Psi}_x(t)}{t}$, and thus $0 < \frac{\psi_x(\bar{\psi}_x(t))}{\bar{\Psi}_x(\psi_x(\bar{\psi}_x(t)))} \leq \frac{t}{\bar{\Psi}_x(t)}$. Choosing $s = \bar{\psi}_x(t) > 0$, we have from (26) that $s\psi_x(s)/\Psi_x(s) \leq \alpha$. Moreover, using Young's equality $s\psi_x(s) = \Psi_x(s) + \bar{\Psi}_x(\psi_x(s))$, we obtain

$$\begin{aligned} \frac{t\bar{\psi}_x(t)}{\bar{\Psi}_x(t)} &\geq \frac{\psi_x(\bar{\psi}_x(t)) \cdot \bar{\psi}_x(t)}{\bar{\Psi}_x(\psi_x(\bar{\psi}_x(t)))} = \frac{s\psi_x(s)}{\bar{\Psi}_x(\psi_x(s))} = \frac{s\psi_x(s)}{s\psi_x(s) - \Psi_x(s)} \\ &= \frac{s\psi_x(s)/\Psi_x(s)}{[s\psi_x(s)/\Psi_x(s)] - 1} \geq \frac{\alpha}{\alpha - 1} := \bar{\alpha} (> 1). \end{aligned}$$

by the decreasing property of the function $u \mapsto u/(u-1)$ on $(1, \infty)$. This proves (iii).

[(iii) $\Rightarrow$ (i)] Assume (iii) and let $\lambda = 2^{\frac{1}{\bar{\alpha}-1}}$, where $\bar{\alpha}$ is given in (27). Then, $\lambda > 1$ and $\lambda^{\bar{\alpha}} = 2\lambda$. For $t > 0$ and $x \in \mathcal{U}$, we have from (27) that

$$\ln \frac{\bar{\Psi}_x(\lambda t)}{\bar{\Psi}_x(t)} = \int_t^{\lambda t} \frac{\bar{\psi}_x(s)}{\bar{\Psi}_x(s)} ds \geq \int_t^{\lambda t} \frac{\bar{\alpha}}{s} ds = \bar{\alpha} \ln \lambda = \ln(2\lambda).$$

It means that

$$\bar{\Psi}_x(\lambda t) \geq 2\lambda \bar{\Psi}_x(t), \quad (28)$$

for all $t \geq 0$, a.e. $x \in \mathcal{U}$. Define the function $M_x \in N(\mathcal{U})$ by $M_x(t) = \frac{1}{2\lambda} \bar{\Psi}_x(\lambda t)$ ($x \in \mathcal{U}, t \in \mathbb{R}$). We have (cf. (2.5) in Chapter I, [19]), the complement $\bar{M}_x$ of M_x is given by $\bar{M}_x(t) = \frac{1}{2\lambda} \bar{\bar{\Psi}}_x(2t) = \frac{1}{2\lambda} \Psi_x(2t)$. It follows from (28) that $M_x(t) \geq \bar{\Psi}_x(t)$ for all $t \geq 0$ and thus $\frac{1}{2\lambda} \Psi_x(2t) = \bar{M}_x(t) \leq \bar{\bar{\Psi}}_x(t) = \Psi_x(t)$ for all $t \geq 0$. This shows that (5) holds with $\kappa = 2\lambda = 2^{\frac{\bar{\alpha}}{\bar{\alpha}-1}}$. $\square$

The following result is a direct corollary of Theorem 3.

Corollary 4. Let $\Psi = \Psi_x$ be an $N(\mathcal{U})$ -function. Then, both Ψ_x and its conjugate $\bar{\Psi}_x$ satisfy Δ_2 conditions if and only if there are numbers $\alpha_{\bar{\Psi}}^-, \alpha_{\Psi}^+ \in (1, \infty)$ such that

$$\alpha_{\bar{\Psi}}^- \leq \frac{t\psi_x(t)}{\Psi_x(t)} \leq \alpha_{\Psi}^+, \text{ for a.e. } x \in \mathcal{U}, \text{ for all } t > 0. \quad (29)$$

In particular, if $\Phi = \Phi_{x,y}$ be an $N(\Omega \times \Omega)$ -function, then both $\Phi_{x,y}$ and its conjugate $\bar{\Phi}_{x,y}$ satisfy Δ_2 conditions if and only if there are numbers $\alpha_{\bar{\Phi}}^-, \alpha_{\Phi}^+ \in (1, \infty)$ such that

$$\alpha_{\bar{\Phi}}^- \leq \frac{t\phi_{x,y}(t)}{\Phi_{x,y}(t)} \leq \alpha_{\Phi}^+, \text{ for a.e. } (x, y) \in \Omega \times \Omega, \text{ for all } t > 0. \quad (30)$$

Remark 3. (a) In Definition 3, if we replace the condition “ $t \geq 0$ ” in (4) and (5) with the condition “ $t \geq t_0$ ” for some $t_0 \geq 0$, then we obtain the concept of Δ_2 conditions at infinity.

The arguments in Theorem 3 and Corollary 4 are extended straightforwardly in this case, with the replacement of “ $t > 0$ ” by “ $t > t_0$ ” for some $t_0 \geq 0$. In this instance, the equivalences in Theorem 3 show that the continuity of the derivatives of the corresponding N -functions can be removed in Theorem 4.3, Chapter I, [19].

(b) Conditions like (29) and (30) have been used in, e.g., [14–16,26,27] and it was proved there that they are sufficient for the Δ_2 conditions of the corresponding N -function and its conjugate (cf., e.g., Lemma 2.7, [26] and Proposition 2.3, [27]). Corollary 4 shows that conditions such as (29) and (30) are, in fact, both necessary and sufficient for the Δ_2 conditions of the corresponding N -function and its conjugate. Hence, assuming that an $N(\mathcal{U})$ -function Ψ satisfies a Δ_2 condition, together with its conjugate, implies the existence of numbers $\alpha_{\bar{\Psi}}^-$ and α_{Ψ}^+ such that (29) holds.

The following result is a direct consequence of the above theorems, whose proof is straightforward and is therefore omitted.

Corollary 5. (a) Let $\Psi = \Psi_x$ be an $N(\mathcal{U})$ -function that satisfies (29). Then, we have the estimates (17) and thus (20) with

$$\mu_{\bar{\Psi}}^-(t) = \min\{t^{\alpha_{\bar{\Psi}}^-}, t^{\alpha_{\Psi}^+}\} \text{ and } \mu_{\Psi}^+(t) = \max\{t^{\alpha_{\bar{\Psi}}^-}, t^{\alpha_{\Psi}^+}\}, \quad t \geq 0. \quad (31)$$

(b) Similarly, let $\Phi = \Phi_{x,y}$ be an $N(\Omega \times \Omega)$ -function that satisfies (30). Then, we have the estimates (18) and (19) and thus (21) and (22), with

$$\mu_{\bar{\Phi}}^-(t) = \min\{t^{\alpha_{\bar{\Phi}}^-}, t^{\alpha_{\Phi}^+}\} \text{ and } \mu_{\Phi}^+(t) = \max\{t^{\alpha_{\bar{\Phi}}^-}, t^{\alpha_{\Phi}^+}\}, \quad t \geq 0. \quad (32)$$

3. Nonlocal Fractional Type Multivalued Variational Inequalities

In this section, we will be studying the solvability of multivalued variational inequalities of the form (1) in fractional Musielak–Orlicz–Sobolev spaces. In Sections 3.1–3.3, we establish a functional analytic framework for our problem in an appropriate fractional Musielak–Orlicz–Sobolev space that is well-suited for investigation using topological and monotonicity methods. The solvability of our multivalued variational inequality will be discussed in Section 3.4.

3.1. Assumptions—Setting of the Problem

In this subsection, we discuss the problem’s setting, along with some basic assumptions about the problem. Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary. Let $a : \Omega \times \Omega \times (0, \infty) \rightarrow [0, \infty)$ and define $\phi : \Omega \times \Omega \times [0, \infty) \rightarrow [0, \infty)$ by

$$\phi(x, y, t) = \begin{cases} ta(x, y, t), & t > 0 \\ 0, & t = 0. \end{cases}$$

Suppose that $\phi(\cdot, \cdot, t)$ is measurable on $\Omega \times \Omega$ for each $t \geq 0$ and $\phi_{x,y}$ satisfies the conditions in Definition 1, that is, for a.e. $(x, y) \in \Omega \times \Omega$, $\phi_{x,y}$ is right continuous and increasing on $[0, +\infty)$, $\phi_{x,y}(t) > 0$ for $t > 0$ and $\lim_{t \rightarrow \infty} \phi_{x,y}(t) = \infty$.

As a consequence, the function $\Phi : \Omega \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\Phi_{x,y}(t) := \Phi(x, y, t) = \int_0^{|t|} \phi(x, y, s) ds = \int_0^{|t|} s a(x, y, s) ds,$$

belongs to $\mathcal{N}(\Omega \times \Omega)$. As in the previous sections, we also consider the function $\widehat{\Phi}_x = \Phi_{x,x}$, that is, $\widehat{\Phi} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\widehat{\Phi}_x(t) := \widehat{\Phi}(x, t) = \int_0^{|t|} \widehat{\phi}_x(s) ds = \int_0^{|t|} s \widehat{a}_x(s) ds$$

where $\widehat{a}_x = a_{x,x}$ and $\widehat{\phi}_x = \phi_{x,x}$ ($x \in \Omega$). Suppose that a is symmetric with respect to x and y , that is, $a(x, y, t) = a(y, x, t)$, for a.e. $(x, y) \in \Omega \times \Omega$ and $t > 0$. Consequently, ϕ and Φ are also symmetric with respect to x and y , i.e., $\phi_{x,y} = \phi_{y,x}$ and $\Phi_{x,y} = \Phi_{y,x}$ for a.e. $x, y \in \Omega$.

In the following presentation, we suppose that $\Phi_{x,y}$ satisfies conditions (6) and (7) so that Propositions 1 and 2 hold. We also assume that both $\Phi_{x,y}$ and its Hölder conjugate $\overline{\Phi}_{x,y}$ satisfy Δ_2 conditions. Consequently, we have, among others, the estimates in Corollaries 4 and 5.

Let $X := W^{s, \Phi_{x,y}}(\Omega)$ and $X_0 := W_0^{s, \Phi_{x,y}}(\Omega)$ be the fractional Musielak–Orlicz–Sobolev spaces defined in Section 2. For $s \in (0, 1)$, let $(-\Delta_{\Phi_{x,y}})^s$ be the s fractional $\Phi_{x,y}$ -Laplacian defined by

$$(-\Delta_{\Phi_{x,y}})^s u(x) := 2 \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a_{x,y} \left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{u(x) - u(y)}{|x - y|^s} \frac{dy}{|x - y|^{N+s}}.$$

In the case where $a_{x,y} = a$ is independent of x and y , it follows from Theorem 6.12 of [22] that for $u, v \in W_0^{s, \Phi}(\Omega)$, we have the variational representation formula for $(-\Delta_\Phi)^s$:

$$\langle (-\Delta_\Phi)^s(u), v \rangle := \int_\Omega \int_\Omega a(|D_s u(x, y)|) D_s u(x, y) D_s v(x, y) d\mu, \quad (33)$$

where

$$D_s u(x, y) := \frac{u(x) - u(y)}{|x - y|^s} \text{ and } d\mu := \frac{dx dy}{|x - y|^N}.$$

Motivated by this representation, we define the mapping $\mathcal{A} = (-\Delta_{\Phi_{x,y}})^s$ from X to X^* by

$$\langle \mathcal{A}u, v \rangle = \langle (-\Delta_{\Phi_{x,y}})^s(u), v \rangle := \int_\Omega \int_\Omega a_{x,y}(|D_s u(x, y)|) D_s u(x, y) D_s v(x, y) d\mu, \quad (34)$$

for all $u, v \in X$.

For a closed and convex subset K of X and a function f defined on $\Omega \times \mathbb{R}$, which may be multivalued, let us consider the following variational inequality on K :

$$\begin{cases} \int_\Omega \int_\Omega a_{x,y}(|D_s u(x, y)|) D_s u(x, y) (D_s v(x, y) - D_s u(x, y)) d\mu \\ \quad + \int_\Omega f(x, u)(v - u) dx \geq 0, \quad \forall v \in K, \\ u \in K. \end{cases} \quad (35)$$

Note that in the case $K = X_0$ and f is single-valued, this variational inequality reduces to the variational equation

$$\int_\Omega \int_\Omega a_{x,y}(|D_s u(x, y)|) D_s u(x, y) D_s v(x, y) d\mu + \int_\Omega f(x, u) v dx = 0, \quad \forall v \in X_0, \quad (36)$$

which is the variational form of the Dirichlet boundary value problem for the s fractional $\Phi_{x,y}$ Laplacian:

$$\begin{cases} (-\Delta_{\Phi_{x,y}})^s u + f(x, u) = 0, & \text{in } \Omega \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (37)$$

Some other choices of K formulate obstacle and unilateral problems. For example,

$$K = \{u \in X : u \geq g \text{ a.e. on } \Omega\},$$

or

$$K = \{u \in X_0 : u \geq g \text{ a.e. on } \Omega\},$$

where g is a measurable function defined on Ω , correspond to obstacle problems.

Let $f = f(x, u)$ be a multivalued function defined on $\Omega \times \mathbb{R}$. We are interested here with following multivalued variational inequality, which is a natural extension of (35) when the lower order terms have sets as values: Find $u \in K$ and $\eta \in L^0(\Omega)$ such that

$$\eta(x) \in f(x, u(x)) \text{ for a.e. } x \in \Omega, \quad (38)$$

$$\eta v \in L^1(\Omega) \text{ for all } v \in K, \quad (39)$$

and

$$\begin{cases} \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x, y)|) D_s u(x, y) (D_s v(x, y) - D_s u(x, y)) d\mu \\ + \int_{\Omega} \eta(v - u) dx \geq 0, \quad \forall v \in K, \\ u \in K. \end{cases} \quad (40)$$

In the particular case where $K = X_0$, problem (38)–(40) is the weak form of the inclusion

$$\begin{cases} (-\Delta_{\Phi_{x,y}})^s u + f(x, u) \ni 0, & \text{in } \Omega \\ u = 0, & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (41)$$

3.2. Assumptions on the Multivalued Term—Inclusion Formulation

To obtain a precise functional analytic formulation of (38)–(40), we will begin by discussing certain specific conditions on the multivalued term f . Together with the standard notation for Musielak–Orlicz and fractional Musielak–Orlicz–Sobolev spaces introduced in Section 2, we also use in the sequel the notation

$$\mathcal{K}(S) = \{E \subset S : E \neq \emptyset, E \text{ is closed and convex}\},$$

for a Banach space S and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{S^*, S}$ the duality pairing between S and its topological dual S^* .

Let f be a function from $\Omega \times \mathbb{R}$ to $\mathcal{K}(\mathbb{R})$ that has the following properties.

(F1) f is superpositionally measurable, that is, if u is a measurable function on Ω then the (multivalued) function $f(\cdot, u(\cdot)), x \mapsto f(x, u(x))$ is measurable on Ω .

Note that if f is graph measurable on $\Omega \times \mathbb{R}$, that is, $\text{Gr}(f) = \{(x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R} : \xi \in f(x, u)\}$ belongs to $[\mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R})] \times \mathcal{B}(\mathbb{R})$ ($\mathcal{L}(\Omega)$ is the family of Lebesgue measurable subsets of Ω and $\mathcal{B}(\mathbb{R})$ is the σ -algebra of Borel sets in $\mathbb{R}$), then f is superpositionally measurable.

Furthermore, if f is measurable from $\Omega \times \mathbb{R}$ to $\mathcal{K}(\mathbb{R})$ in the regular sense, that is $f^-(U) := \{(x, u) \in \Omega \times \mathbb{R} : f(x, u) \cap U \neq \emptyset\} \in \mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R})$ for all $U \subset \mathbb{R}$ open, then f is graph measurable on $\Omega \times \mathbb{R}$ and thus superpositionally measurable.

(F2) For a.e. $x \in \Omega$, the function $f(x, \cdot) : \mathbb{R} \rightarrow \mathcal{K}(\mathbb{R})$ is upper semicontinuous. This means that, for every $u \in \mathbb{R}$ and every open $U \subset \mathbb{R}$ such that $f(x, u) \subset U$, there is $\delta > 0$ such that $|v - u| < \delta$ implies $f(x, v) \subset U$.

Note that since $f(x, u)$ is a compact interval in $\mathbb{R}$, condition (F2) is equivalent to the Hausdorff upper semicontinuity (h-u.s.c.) of $f(x, \cdot)$ for a.e. $x \in \Omega$ (cf. Theorem 2.68, Chap. 1, [28]). In many places in the sequel, we also need the following subcritical growth condition on f :

(F3) There exists a generalized N -function $P = P_x \in N(\Omega)$ satisfying a Δ_2 condition, together with its Hölder conjugate, such that

$$P_x \ll \widehat{\Phi}_x^*, \quad (42)$$

and for some $a_1 \in L^{\bar{P}_x}(\Omega)$, $b_1 \in (0, \infty)$,

$$|\eta| \leq a_1(x) + b_1 \bar{P}_x^{-1}(P_x(|u|)), \quad (43)$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, all $\eta \in f(x, u)$.

Let u be any measurable function on Ω . From (F1), the function $f(\cdot, u(\cdot))$, $x \mapsto f(x, u(x))$, is also a measurable function from Ω to $\mathcal{K}(\mathbb{R})$. Let $\tilde{f}(u)$ be the set of all measurable selections of $f(\cdot, u(\cdot))$, that is,

$$\tilde{f}(u) = \{\eta : \Omega \rightarrow \mathbb{R} : \eta \text{ is measurable on } \Omega \text{ and } \eta(x) \in f(x, u(x)) \text{ for a.e. } x \in \Omega\}. \quad (44)$$

We know that $\tilde{f}(u) \neq \emptyset$ whenever u is measurable on Ω since $f(\cdot, u(\cdot))$ is measurable.

For $P = P_x$ satisfying (42), we have from Proposition 1 that the embedding $i_{P_x} : X \hookrightarrow L^{P_x}(\Omega)$ is compact. Therefore its adjoint $i_{P_x}^*$, which is the projection from $L^{\bar{P}_x}(\Omega) \equiv [L^{P_x}(\Omega)]^*$ to X^* , is also compact. Note that $i_{P_x}(u) = u$ for $u \in X$, that is, $i_{P_x}(u)(x) = u(x)$ for a.e. $x \in \Omega$. Thus, to simplify the notation in the sequel, we shall use in several places u instead of $i_{P_x}(u)$. Similarly, $i_{P_x}^*$ is the restriction of elements in $L^{\bar{P}_x}(\Omega) \equiv [L^{P_x}(\Omega)]^*$ on the functions in X , i.e., for $\eta \in L^{\bar{P}_x}(\Omega)$, $i_{P_x}^*(\eta) = \eta|_X$,

$$\langle i_{P_x}^*(\eta), v \rangle_{X^*, X} = \langle \eta, i_{P_x}(v) \rangle_{L^{\bar{P}_x}(\Omega), L^{P_x}(\Omega)} = \int_{\Omega} \eta v dx, \forall v \in X. \quad (45)$$

Therefore, if f satisfies the growth condition (43) in (F3) with P_x satisfying (42), then for any η satisfying (38), we have $\eta \in L^{\bar{P}_x}(\Omega)$, which implies (39).

Moreover, if the growth condition (43) is fulfilled then $\tilde{f}(u) \subset L^{\bar{P}_x}(\Omega)$ whenever $u \in L^{P_x}(\Omega)$.

Letting $\eta^* = i_{P_x}^*(\eta)$, we can reformulate problems (38)–(40) in an equivalent way as follows:

(P) Find $u \in K$ and $\eta^* \in (i_{P_x}^* \tilde{f} i_{P_x})(u)$ such that

$$\langle \mathcal{A}u, v - u \rangle_{X^*, X} + \langle \eta^*, v - u \rangle_{X^*, X} \geq 0, \forall v \in K, \quad (46)$$

or equivalently,

$$I_K(v) - I_K(u) + \langle \mathcal{A} + \eta^*, v - u \rangle_{X^*, X} \geq 0, \forall v \in X, \quad (47)$$

where I_K is the indicator functional of K , $I_K(w) = 0$ if $w \in K$ and $I_K(w) = \infty$ if $w \in X \setminus K$. Since K is closed and convex and since X is reflexive, $I_K : X \rightarrow [0, \infty]$ is convex and lower semicontinuous in both the norm and the weak topologies of X . Let ∂I_K be the subdifferential of I_K in the sense of Convex Analysis. The variational inequality (47) is, in its turn, equivalent to the following inclusion:

$$\partial I_K(u) + \mathcal{A}(u) + \eta^* \ni 0, \quad (48)$$

that is,

$$(\partial I_K + \mathcal{A} + i_{P_x}^* \tilde{f} i_{P_x})(u) \ni 0. \quad (49)$$

3.3. Topological and Monotonicity Properties of the Multivalued Term

In order to study the solvability of the inclusion (49), that is, of Problem (P), we will prove in this subsection certain crucial continuity and monotonicity properties of its lower order term. Let us begin with some essential properties of the mappings $u \mapsto \tilde{f}(u)$ and $u \mapsto (i_{P_x}^* \tilde{f} i_{P_x})(u)$.

Lemma 1. *Let conditions (F1)–(F3) be satisfied.*

- (a) *If $u \in L^{P_x}(\Omega)$ then, $\tilde{f}(u)$ is a bounded, closed and convex subset of $L^{\bar{P}_x}(\Omega)$; in particular, $\tilde{f}(u) \in \mathcal{K}(L^{\bar{P}_x}(\Omega))$. Moreover, the mapping $\tilde{f} : u \mapsto \tilde{f}(u)$ is a bounded mapping from $L^{P_x}(\Omega)$ to $\mathcal{K}(L^{\bar{P}_x}(\Omega))$.*
- (b) *If $u \in X$ then $\mathcal{F}(u) := i_{P_x}^* \tilde{f} i_{P_x}(u)$ is a convex and weakly-compact subset of X^* . Moreover, the mapping $\mathcal{F} : u \mapsto \mathcal{F}(u)$ is a bounded mapping from X to 2^{X^*} .*

Proof. (a) The convexity of $\tilde{f}(u)$ follows directly from the fact that $f(x, u)$ is a closed interval. Let $u \in L^{P_x}(\Omega)$ and $\eta \in \tilde{f}(u)$. From (43),

$$|\eta(x)| \leq a_1(x) + b_1 \bar{P}_x^{-1} P_x(|u(x)|), \quad \text{a.e. } x \in \Omega. \quad (50)$$

Since $\int_{\Omega} P_x(|u(x)|) dx < \infty$, we see from (50) that $\bar{P}_x^{-1} P_x(|u|) \in L^{\bar{P}_x}(\Omega)$ and also that $\tilde{f}(u)$ is a bounded subset of $L^{\bar{P}_x}(\Omega)$. Inequality (50) also proves that if W is a bounded set in $L^{P_x}(\Omega)$ then $\tilde{f}(W) = \bigcup_{u \in W} \tilde{f}(u)$ is a bounded set in $L^{\bar{P}_x}(\Omega)$. This means that $\tilde{f}$ is a bounded mapping from $L^{P_x}(\Omega)$ to $2^{L^{\bar{P}_x}(\Omega)}$.

To verify that $\tilde{f}(u)$ is closed in $L^{\bar{P}_x}(\Omega)$, let $\{\eta_n\}$ be a sequence in $\tilde{f}(u)$ such that $\eta_n \rightarrow \eta$ in $L^{\bar{P}_x}(\Omega)$. By passing to a subsequence if necessary, we can assume that $\eta_n(x) \rightarrow \eta(x)$ for a.e. $x \in \Omega$. Since $\eta_n(x) \in f(x, u(x))$ for a.e. $x \in \Omega$, all $n \in \mathbb{N}$, and $f(x, u(x))$ is closed in $\mathbb{R}$, we have $\eta(x) \in f(x, u(x))$. Thus $\eta \in \tilde{f}(u)$, which proves the closedness of $\tilde{f}(u)$ in $L^{\bar{P}_x}(\Omega)$.

(b) As $L^{\bar{P}_x}(\Omega)$ is a reflexive Banach space, we obtain from (a) that for every $u \in L^{P_x}(\Omega)$, $\tilde{f}(u)$ is a convex, closed and bounded subset of $L^{\bar{P}_x}(\Omega)$. Therefore, $\tilde{f}(u)$ is a weakly compact subset of $L^{\bar{P}_x}(\Omega)$. We also note that the mapping $i_{P_x}^*$ is continuous from $L^{\bar{P}_x}(\Omega) = [L^{P_x}(\Omega)]^*$ to X^* both equipped with the norm topologies. Therefore, $i_{P_x}^*$ is also continuous with both $L^{\bar{P}_x}(\Omega)$ and X^* both equipped with the weak topologies. Let $u \in X$. Since the set $\tilde{f}(u) = \tilde{f}(i_{P_x} u)$ is convex and weakly compact in $L^{\bar{P}_x}(\Omega)$, it follows that the set $\mathcal{F}(u) = (i_{P_x}^* \tilde{f} i_{P_x})(u) = i_{P_x}^*[\tilde{f}(u)]$ is convex and weakly compact in X^* . Moreover, the boundedness of $\tilde{f}$ implies that of $\mathcal{F}$. $\square$

Next, let us prove an essential lemma about the upper semicontinuity of multivalued mappings between Musielak–Orlicz spaces.

Lemma 2. *Let $\Psi_j = \Psi_{jx}$, $j = 0, 1$, be functions in $\mathcal{N}(\Omega)$ that satisfy Δ_2 conditions. Assume F satisfies the following conditions:*

- (i) *For a.e. $x \in \Omega$, all $u \in \mathbb{R}$, $F(x, u)$ is a nonempty closed and bounded interval in $\mathbb{R}$.*
- (ii) *F is super positionally measurable.*
- (iii) *For a.e. $x \in \Omega$, the function $u \mapsto F(x, u)$ is Hausdorff-upper semicontinuous (h-u.s.c.).*
- (iv) *There exist $a \in L^{\Psi_{0x}}(\Omega)$ and $b > 0$ such that*

$$|v| \leq a(x) + b \Psi_{0x}^{-1} \Psi_{1x}(|u|), \quad (51)$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, all $v \in F(x, u)$.

Thus, for each $u \in L^{\Psi_{1x}}(\Omega)$, $\tilde{F}(u)$ is a (nonempty) closed subset of $L^{\Psi_{0x}}$ and the mapping $\tilde{F} : u \mapsto \tilde{F}(u)$ is h -u.s.c. from $L^{\Psi_{1x}}(\Omega)$ to $2^{L^{\Psi_{0x}}(\Omega)}$, that is, for each $u_0 \in L^{\Psi_{1x}}(\Omega)$, the function

$$u \mapsto h_{L^{\Psi_{0x}}(\Omega)}^*(\tilde{F}(u), \tilde{F}(u_0))$$

is continuous at u_0 , where

$$h_{L^{\Psi_{0x}}(\Omega)}^*(A, B) = \sup_{u \in A} \left(\inf_{v \in B} \|u - v\|_{L^{\Psi_{0x}}(\Omega)} \right),$$

for $A, B \subset L^{\Psi_{0x}}(\Omega)$.

Proof. First, as in the proof of Lemma 1 (a), we note from (i)–(iii) that for each $u \in L^{\Psi_{1x}}(\Omega)$, $\tilde{F}(u)$ is a nonempty closed, convex and bounded subset of $L^{\Psi_{0x}}(\Omega)$. Assume $\{u_n\}$ is a sequence in $L^{\Psi_{1x}}(\Omega)$ such that

$$u_n \rightarrow u \text{ in } L^{\Psi_{1x}}(\Omega). \quad (52)$$

Let $\varepsilon > 0$. We prove that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sup_{v' \in \tilde{F}(u_n)} \left(\inf_{v \in \tilde{F}(u)} \|v' - v\|_{L^{\Psi_{0x}}(\Omega)} \right) \leq \varepsilon, \quad (53)$$

i.e., $\inf_{v \in \tilde{F}(u)} \|v' - v\|_{L^{\Psi_{0x}}(\Omega)} \leq \varepsilon, \forall v' \in \tilde{F}(u_n)$. In fact, we have from (52) that

$$\int_{\Omega} \Psi_{1x}(|u_n(x) - u(x)|) dx \rightarrow 0.$$

Hence, by passing to a subsequence if necessary, we can assume that

$$u_n(x) \rightarrow u(x) \text{ for a.e. } x \in \Omega, \quad (54)$$

and there exists $h^1 \in L^1(\Omega)$ such that $\Psi_{1x}(|u_n(x) - u(x)|) \leq h^1(x)$ for a.e. $x \in \Omega$, all $n \in \mathbb{N}$. Thus,

$$|u_n(x)| \leq h^0(x) := \Psi_{1x}^{-1}[h^1(x)] + |u(x)|, \quad (55)$$

for a.e. $x \in \Omega$, all $n \in \mathbb{N}$, where $h^0 \in L^{\Psi_{1x}}(\Omega)$. Let $n \in \mathbb{N}$. For $v' \in \tilde{F}(u_n)$, let us consider the following functions $w_1 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $w_1(x, \eta) = \Psi_{0x}(|\eta - v'(x)|)$ ($x \in \Omega, \eta \in \mathbb{R}$) and $F_1 : \Omega \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$, $F_1(x) = F(x, u(x))$. From Theorem 3.24 in [28] (with “inf” instead of “sup”) applied to w_1 and F_1 (note the growth condition (51)), we obtain

$$\inf_{v \in \tilde{F}(u)} \int_{\Omega} w_1(x, v(x)) dx = \int_{\Omega} \inf_{\eta \in F(x, u(x))} w_1(x, \eta) dx, \text{ that is,}$$

$$\inf_{v \in \tilde{F}(u)} \int_{\Omega} \Psi_{0x}(|v(x) - v'(x)|) dx = \int_{\Omega} \inf_{\eta \in F(x, u(x))} \Psi_{0x}(|\eta - v'(x)|) dx. \quad (56)$$

Let $w_2(x, \eta') = \inf_{\eta \in F(x, u(x))} \Psi_{0x}(|\eta - \eta'|)$ ($x \in \Omega, \eta' \in \mathbb{R}$). Then, w_2 is a measurable mapping from $\Omega \times \mathbb{R}$ into $\mathbb{R} = [-\infty, +\infty]$. Using w_2 and $F_2 : \Omega \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$, $F_2(x) = \tilde{F}(x, u_n(x))$, in Theorem 3.24 of [28], and again taking into account condition (51), we have $\sup_{v' \in \tilde{F}(u_n)} \int_{\Omega} w_2(x, v'(x)) dx = \int_{\Omega} \sup_{\eta' \in F(x, u_n(x))} w_2(x, \eta') dx$, i.e.,

$$\begin{aligned} & \sup_{v' \in \tilde{F}(u_n)} \left[\int_{\Omega} \inf_{\eta \in F(x, u(x))} \Psi_{0x}(|\eta - v'(x)|) dx \right] \\ &= \int_{\Omega} \sup_{\eta' \in F(x, u_n(x))} \left[\inf_{\eta \in F(x, u(x))} \Psi_{0x}(|\eta - \eta'|) \right] dx. \end{aligned} \quad (57)$$

Combining (56) with (57) yields

$$\begin{aligned} & \int_{\Omega} \sup_{\eta' \in F(x, u_n(x))} \left[\inf_{\eta \in F(x, u(x))} \Psi_{0x}(|\eta - \eta'|) \right] dx \\ &= \sup_{v' \in \tilde{F}(u_n)} \left[\inf_{v \in \tilde{F}(u)} \int_{\Omega} \Psi_{0x}(|v(x) - v'(x)|) dx \right]. \end{aligned} \quad (58)$$

For a.e. $x \in \Omega$, from (54) and the Hausdorff-upper semicontinuity of $F(x, \cdot)$, we see that

$$\begin{aligned} & \sup_{\eta' \in F(x, u_n(x))} \left[\inf_{\eta \in F(x, u(x))} |\eta - \eta'| \right] \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and thus} \\ & \sup_{\eta' \in F(x, u_n(x))} \left[\inf_{\eta \in F(x, u(x))} \Psi_{0x}(|\eta - \eta'|) \right] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (59)$$

Since Ψ_{0x}, Ψ_{1x} satisfies a Δ_2 condition, it follows from Corollary 1 that for each $\lambda \in (0, \infty)$, with $C_{\lambda}^* = \max\{\mu_{\Psi_{0x}}^+(\lambda), \mu_{\Psi_{1x}}^+(\lambda)\} > 0$, we have

$$\Psi_{jx}(\lambda t) \leq C_{\lambda}^* \Psi_{jx}(t), \text{ for a.e. } x \in \Omega, \text{ for all } t \in [0, \infty), j = 0, 1. \quad (60)$$

As a consequence, for $x \in \Omega, l \in \mathbb{N}$ and $t_1, \dots, t_l \in [0, \infty)$ and $j = 0, 1$, we have

$$\Psi_{jx}\left(\sum_{s=1}^l t_s\right) \leq \frac{1}{l} \sum_{s=1}^l \Psi_{jx}(lt_s) \leq \frac{C_l^*}{l} \sum_{s=1}^l \Psi_{jx}(x_s). \quad (61)$$

From (51) and (55), we have for all $n \in \mathbb{N}$, almost all $x \in \Omega$ and all $\eta \in F(x, u(x))$, all $\eta' \in F(x, u_n(x))$,

$$\begin{aligned} \Psi_{0x}(|\eta(x) - \eta'(x)|) &\leq \Psi_{0x}(|\eta(x)| + |\eta'(x)|) \\ &\leq \frac{C_2^*}{2} [\Psi_{0x}(|\eta(x)|) + \Psi_{0x}(|\eta'(x)|)] \\ &\leq \frac{C_2^*}{2} \Psi_{0x} \left[a(x) + b \Psi_{0x}^{-1} \Psi_{1x}(|u(x)|) \right] \\ &\quad + \frac{C_2^*}{2} \Psi_{0x} \left[a(x) + b \Psi_{0x}^{-1} \Psi_{1x}(|u_n(x)|) \right] \\ &\leq \frac{(C_2^*)^2}{2} \Psi_{0x}[a(x)] + \frac{(C_2^*)^2}{4} \Psi_{0x} \left[b \Psi_{0x}^{-1} \Psi_{1x}(|u(x)|) \right] \\ &\quad + \frac{(C_2^*)^2}{4} \Psi_{0x} \left[b \Psi_{0x}^{-1} \Psi_{1x}(|u_n(x)|) \right] \\ &\leq \frac{(C_2^*)^2}{2} \Psi_{0x}[a(x)] + \frac{(C_2^*)^2}{4} [C_b^* \Psi_{1x}(|u(x)|)] \\ &\quad + \frac{(C_2^*)^2}{4} [C_b^* \Psi_{1x}(|u_n(x)|)] \\ &\leq \frac{(C_2^*)^2}{2} \Psi_{0x}[a(x)] + \frac{(C_2^*)^2}{4} [C_b^* \Psi_{1x}(|u(x)|)] \\ &\quad + \frac{(C_2^*)^2}{4} [C_b^* \Psi_{1x}(h^0(x))] \\ &:= H(x). \end{aligned}$$

Since $a \in L^{\Psi_{0x}}(\Omega)$ and $u, h^0 \in L^{\Psi_{1x}}(\Omega)$, we see that $H \in L^1(\Omega)$. Therefore,

$$\sup_{\eta' \in F(x, u_n(x))} \left[\inf_{\eta \in F(x, u(x))} \Psi_{0x}(|\eta - \eta'|) \right] \leq H(x), \quad (62)$$

for a.e. $x \in \Omega$, all $n \in \mathbb{N}$. It follows from (58), (59) and (62) that

$$\lim_{n \rightarrow \infty} \sup_{v' \in \tilde{F}(u_n)} \left[\inf_{v \in \tilde{F}(u)} \int_{\Omega} \Psi_{0x}(|v(x) - v'(x)|) dx \right] = 0. \quad (63)$$

Since Ψ_{0x} satisfies a Δ_2 condition, the modular convergence and norm convergence are equivalent in $L^{\Psi_{0x}}(\Omega)$. Therefore, given any $\varepsilon > 0$, there exists $\varepsilon_1 > 0$ such that for any $w \in L^{\Psi_{0x}}(\Omega)$,

$$\int_{\Omega} \Psi_{0x}(|w|)dx < \varepsilon_1 \text{ implies } \|w\|_{L^{\Psi_{0x}}(\Omega)} < \varepsilon. \quad (64)$$

From (63), for that chosen $\varepsilon_1 > 0$, there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sup_{v' \in \tilde{F}(u_n)} \left[\inf_{v \in \tilde{F}(u)} \int_{\Omega} \Psi_{0x}(|v(x) - v'(x)|)dx \right] < \varepsilon_1.$$

Thus, for all $n \geq n_0$, all $v' \in \tilde{F}(u_n)$, there exists $v \in \tilde{F}(u)$ such that $\int_{\Omega} \Psi_{0x}(|v(x) - v'(x)|)dx < \varepsilon_1$. According to (64), $\|v - v'\|_{L^{\Psi_{0x}}(\Omega)} < \varepsilon$. Hence, for all $n \geq n_0$, all $v' \in \tilde{F}(u_n)$, we have $\inf_{v \in \tilde{F}(u)} \|v - v'\|_{L^{\Psi_{0x}}(\Omega)} < \varepsilon$. This implies that (53) holds true for all $n \geq n_0$, which completes the proof. $\square$

An immediate consequence of Lemma 2 is the following continuity property of $\tilde{f}$.

Corollary 6. Under assumptions (F1)-(F2)-(F3), $\tilde{f}$ is Hausdorff upper semicontinuous (h-u.s.c.) from $L^{P_x}(\Omega)$ to $\mathcal{K}(L^{\bar{P}_x}(\Omega))$.

This means that, for each $u_0 \in L^{P_x}(\Omega)$, the function

$$u \mapsto h_{L^{\bar{P}_x}(\Omega)}^*(\tilde{f}(u), \tilde{f}(u_0)) \quad (65)$$

is continuous at u_0 , where

$$h_{L^{\bar{P}_x}(\Omega)}^*(A, B) = \sup_{u \in A} \left(\inf_{v \in B} \|u - v\|_{L^{\bar{P}_x}(\Omega)} \right), \quad (66)$$

for $A, B \subset L^{\bar{P}_x}(\Omega)$. The following properties of $\mathcal{F}$ are crucial for later developments.

Theorem 4. Suppose f satisfies conditions (F1)-(F2)-(F3).

(a) The mapping $\mathcal{F} = i_{P_x}^* \tilde{f} i_{P_x}$ is weak-weak closed in $X \times X^*$ in the following sense. If $\{(u_k, u_k^*)\}_{k \in \mathbb{N}}$ is a sequence in $X \times X^*$ satisfying the following conditions:

$$u_k^* \in \mathcal{F}(u_k), \quad \forall k \in \mathbb{N}, \quad (67)$$

$$u_k \rightharpoonup u_0 \text{ weakly in } X, \quad (68)$$

and

$$u_k^* \rightharpoonup u_0^* \text{ weakly in } X^*, \quad (69)$$

then

$$u_0^* \in \mathcal{F}(u_0), \quad (70)$$

and

$$\langle u_k^*, u_k \rangle_{X^*, X} \rightarrow \langle u_0^*, u_0 \rangle_{X^*, X} \text{ as } k \rightarrow \infty. \quad (71)$$

(b) The mapping $\mathcal{F}$ is generalized pseudomonotone with domain $D(\mathcal{F}) = X$.

Proof. (a) Assume (67) and (68) and note that $i_{P_x}(z) = z$ for $z \in X$ and $i_{P_x}^*(\eta) = \eta|_{X^*}$ for $\eta \in L^{\bar{P}_x}(\Omega)$. From (67), for each $k \in \mathbb{N}$, there exists $\eta_k \in \tilde{f} i_{P_x}(u_k) = \tilde{f}(u_k) \subset L^{\bar{P}_x}(\Omega)$ such that $u_k^* = i_{P_x}^*(\eta_k) = \eta_k|_{X^*}$. From (68) and the compactness of i_{P_x} , we have

$$u_k = i_{P_x}(u_k) \rightarrow i_{P_x}(u_0) = u_0 \text{ (strongly) in } L^{P_x}(\Omega). \quad (72)$$

Hence, from the h-upper semicontinuity of $\tilde{f}$ from $L^{P_x}(\Omega)$ to $\mathcal{K}(L^{\bar{P}_x}(\Omega))$ in Lemma 2 and Corollary 6, we have

$$h^*(\tilde{f}(u_k), \tilde{f}(u_0)) \rightarrow 0, \quad (73)$$

where h^* is given in (66). Since $\eta_k \in \tilde{f}(u_k)$,

$$\inf_{v \in \tilde{f}(u_0)} \|\eta_k - v\|_{L^{\bar{P}_x}(\Omega)} \leq h^*(\tilde{f}(u_k), \tilde{f}(u_0)).$$

Hence, $\inf_{v \in \tilde{f}(u_0)} \|\eta_k - v\|_{L^{\bar{P}_x}(\Omega)} \rightarrow 0$ and there exists a sequence $\{\tilde{\eta}_k\} \subset \tilde{f}(u_0)$ such that

$$\lim \|\eta_k - \tilde{\eta}_k\|_{L^{\bar{P}_x}(\Omega)} = 0. \quad (74)$$

This implies that

$$\lim \|u_k^* - i_{P_x}^*(\tilde{\eta}_k)\|_{X^*} = \lim \|i_{P_x}^*(\eta_k) - i_{P_x}^*(\tilde{\eta}_k)\|_{X^*} = 0. \quad (75)$$

On the other hand, we have $\{\tilde{\eta}_k\}$ is a sequence in $\tilde{f}(u_0)$ and thus $\{i_{P_x}^*(\tilde{\eta}_k)\}$ is a sequence in $i_{P_x}^*\tilde{f}(u_0) = \mathcal{F}(u_0)$. From the complete continuity of $i_{P_x}^*$ from $L^{\bar{P}_x}(\Omega)$ to X^* , the weak-compactness of $\tilde{f}(u_0)$ in $L^{\bar{P}_x}(\Omega)$ (cf. Lemma 1) by passing to a subsequence if necessary, we can assume that

$$\tilde{\eta}_k \rightharpoonup \tilde{\eta}_0 \text{ weakly in } L^{\bar{P}_x}(\Omega), \quad (76)$$

for some $\tilde{\eta}_0 \in \tilde{f}(u_0)$ and thus

$$i_{P_x}^*(\tilde{\eta}_k) \rightarrow i_{P_x}^*(\tilde{\eta}_0) \text{ (strongly) in } X^*, \quad (77)$$

with $i_{P_x}^*(\tilde{\eta}_0) \in i_{P_x}^*(\tilde{f}(u_0)) = \mathcal{F}(u_0)$. Combining (75) and (77) yields

$$u_k^* \rightarrow i_{P_x}^*(\tilde{\eta}_0) \text{ in } X^*. \quad (78)$$

In view of (69), we see from (78) that $u_0^* = i_{P_x}^*(\tilde{\eta}_0) \in \mathcal{F}(u_0)$.

To prove (71), we note that

$$\begin{aligned} \langle u_k^*, u_k \rangle_{X^*, X} &= \langle i_{P_x}^*(\eta_k), u_k \rangle_{X^*, X} \\ &= \langle \eta_k, i_{P_x}(u_k) \rangle_{L^{\bar{P}_x}(\Omega), L^{P_x}(\Omega)} = \langle \eta_k, u_k \rangle_{L^{\bar{P}_x}(\Omega), L^{P_x}(\Omega)}. \end{aligned} \quad (79)$$

On the other hand, it follows from (74) and (76) that

$$\eta_k \rightharpoonup \tilde{\eta}_0 \text{ weakly in } L^{\bar{P}_x}(\Omega). \quad (80)$$

Hence, from (79), (72) and (80),

$$\begin{aligned} \langle u_k^*, u_k \rangle_{X^*, X} &= \langle \eta_k, u_k \rangle_{L^{\bar{P}_x}(\Omega), L^{P_x}(\Omega)} \\ &\rightarrow \langle \tilde{\eta}_0, u_0 \rangle_{L^{\bar{P}_x}(\Omega), L^{P_x}(\Omega)} = \langle \tilde{\eta}_0, i_{P_x}(u_0) \rangle_{L^{\bar{P}_x}(\Omega), L^{P_x}(\Omega)} \\ &= \langle i_{P_x}^*(\tilde{\eta}_0), u_0 \rangle_{X^*, X} \\ &= \langle u_0^*, u_0 \rangle_{X^*, X}. \end{aligned}$$

This proves (71).

(b) It follows from (F1) that $\mathcal{F}(u) \neq \emptyset$ for all $u \in X$, i.e., $D(\mathcal{F}) = X$. The generalized pseudomonotonicity of $\mathcal{F}$ follows directly from (a). $\square$

Combining Lemma 1 with Theorem 4, we arrive at the following result, whose proof is a direct application of Proposition 4 of [29]. We refer to [29,30] or [8] for more com-

plete definitions and properties of pseudomonotone and generalized pseudomonotone mappings.

Corollary 7. *The mapping $\mathcal{F} = i_{P_x}^* \tilde{f} i_{P_x}$ is pseudomonotone from X to 2^{X^*} .*

3.4. Existence Theorems

As a consequence of the preceding discussions, we can now establish the following basic existence result for Problem (P) under an appropriate coercivity condition.

Theorem 5. *Under the conditions (F1)-(F2)-(F3), let following coercivity condition be satisfied:*

There exist $w_0 \in K$ and a positive constant $R > \|w_0\|$ such that for all $w \in K$ such that $\|w\| = R$,

$$\inf_{w^* \in \mathcal{F}(w)} \langle \mathcal{A}(w) + w^*, w - w_0 \rangle > 0, \quad (81)$$

(alternatively, $\inf_{w^ \in \mathcal{F}(w)} \langle \mathcal{A}(w) + w^*, w - w_0 \rangle \geq 0$).*

Then, there exists a solution u to Problem (P) satisfying $\|u\| < R$ (alternatively, $\|u\| \leq R$).

Proof. Straightforward calculations (cf., e.g., Lemma 3.1, [14]) show that $\mathcal{A}$ is a monotone continuous mapping from X to X^* with domain $D(\mathcal{A}) = X$. Hence, $\mathcal{A}$ is maximal monotone, according to Theorem 1.33, [28]. On the other hand, from Rockafellar's theorem, since I_K is convex and lower semicontinuous, ∂I_K is maximal monotone. Observe that $D(\partial I_K) = K$ and $D(\partial I_K) \cap [D(\mathcal{A})]^\circ = K \cap X^\circ = K \cap X = K \neq \emptyset$. According to Rockafellar's theorem on sums of maximal monotone mappings (cf., e.g., Theorem 32.1, [30]), the mapping $\mathcal{A} + \partial I_K$ is maximal monotone.

Concerning the multivalued lower order term $\mathcal{F}$, we obtain from Lemma 1 and Corollary 7 that $\mathcal{F}$ is a bounded pseudomonotone mapping from X to 2^{X^*} , in particular, $\mathcal{F}$ satisfies conditions (B1'), (B2) and (B4) in [31]. Thus the mappings $\mathcal{A} + \partial I_K$ and $\mathcal{F}$ satisfy all the conditions needed in Corollary 4.1 of [31] with $f_0 = 0$. According to that result, under the coercivity condition (81) (or its weaker version), the inclusion (49) has a solution with the given norm condition. $\square$

As corollaries of this result, let us consider some sublinear growth and/or boundedness conditions on $f(x, u)$ that imply the above general coercivity condition. In fact, we have the following existence result.

Theorem 6. *Suppose $f = g + h$ where g and h satisfy (F1)-(F2) and the following conditions:*

(F_g) The function g satisfies (F3) and there exist $u_0 \in K$ and $a_g \in L^1(\Omega)$, $a_g \geq 0$ a.e. on Ω , $b_g \in (0, \infty)$, such that for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, all $\eta \in g(x, u)$,

$$\eta(u - u_0(x)) \geq b_g \hat{\Phi}_x(u) - a_g(x), \quad (82)$$

and

(F_h) There exists a generalized N-function $Q = Q_x \in \mathcal{N}(\Omega)$ satisfying a Δ_2 condition, together with its Hölder conjugate, such that

$$Q_x \ll \hat{\Phi}_x, \quad (83)$$

and for some $a_h \in L^{\bar{Q}_x}(\Omega)$, $b_h \in (0, \infty)$,

$$|\zeta| \leq a_h(x) + b_h \bar{Q}_x^{-1}(Q_x(|u|)), \quad (84)$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, all $\zeta \in h(x, u)$.

Then, Problem (P) has a solution.

Proof. Since $\widehat{\Phi}_x \ll \widehat{\Phi}_x^*$, (83) and (84) imply that h satisfies (F3) and thus $f = g + h$ also satisfies (F3). Let us check that the coercivity condition (81) is satisfied for $R > 0$ sufficiently large with $w_0 = u_0$ given in (F_g) .

In fact, let $u, v \in X$ and $u^* = u_g^* + u_h^* \in \mathcal{F}(u)$, where $u_g^* \in \mathcal{G}(u) := (i_{P_x}^* \widetilde{g} i_{P_x})(u)$ and $u_h^* \in \mathcal{H}(u) := (i_{Q_x}^* \widetilde{h} i_{Q_x})(u)$. We have

$$u_g^* = i_{P_x}^* \eta i_{P_x} \text{ and } u_h^* = i_{Q_x}^* \zeta i_{Q_x}, \quad (85)$$

where $\eta \in \widetilde{g}(u)$ and $\zeta \in \widetilde{h}(u)$, that is, $\eta(x) \in g(x, u(x))$ and $\zeta(x) \in h(x, u(x))$ for a.e. $x \in \Omega$. It follows from (82) that

$$\langle u_g^*, u - u_0 \rangle = \int_{\Omega} \eta(u - u_0(x)) \geq b_g \int_{\Omega} \widehat{\Phi}_x(u) dx - \int_{\Omega} a_g dx. \quad (86)$$

On the other hand, since $\zeta(x) \in h(x, u(x))$ for a.e. $x \in \Omega$, we have from (84) that

$$|\zeta(x)| \leq a_h(x) + b_h \overline{Q}_x^{-1}(Q_x(|u(x)|)), \quad (87)$$

for a.e. $x \in \Omega$. Therefore, by Young's inequality,

$$\begin{aligned} \left| \int_{\Omega} \zeta v dx \right| &\leq \int_{\Omega} a_h |v| dx + b_h \int_{\Omega} \overline{Q}_x^{-1}(Q_x(|u|)) |v| dx \\ &\leq \int_{\Omega} \overline{Q}_x(a_h) dx + \int_{\Omega} Q_x(|v|) dx + b_h \int_{\Omega} Q_x(|u|) dx + b_h \int_{\Omega} Q_x(|v|) dx \\ &= \int_{\Omega} \overline{Q}_x(a_h) dx + b_h \int_{\Omega} Q_x(|u|) dx + (1 + b_h) \int_{\Omega} Q_x(|v|) dx. \end{aligned} \quad (88)$$

In particular,

$$\begin{aligned} |\langle u_h^*, u - u_0 \rangle| &= \left| \int_{\Omega} \zeta(u - u_0) dx \right| \\ &\leq \left| \int_{\Omega} \zeta u dx \right| + \left| \int_{\Omega} \zeta u_0 dx \right| \\ &\leq \int_{\Omega} \overline{Q}_x(a_h) dx + (1 + 2b_h) \int_{\Omega} Q_x(|u|) dx \\ &\quad + \int_{\Omega} \overline{Q}_x(a_h) dx + b_h \int_{\Omega} Q_x(|u|) dx + (1 + b_h) \int_{\Omega} Q_x(|u_0|) dx \\ &= 2 \int_{\Omega} \overline{Q}_x(a_h) dx + (1 + 3b_h) \int_{\Omega} Q_x(|u|) dx + (1 + b_h) \int_{\Omega} Q_x(|u_0|) dx. \end{aligned} \quad (89)$$

Let ε be a positive number. From (83), there exists $C_{\varepsilon} > 0$ such that

$$(1 + 3b_h)Q_x(t) \leq \varepsilon \widehat{\Phi}_x(t) + C_{\varepsilon}, \quad (90)$$

for a.e. $x \in \Omega$, all $t \in \mathbb{R}$. For convenience of notation, without confusion, we shall use in the sequel C_{ε} for a generic positive constant, that generally depends on ε , but does not depend on u and x and may change its value from line to line. We obtain as a consequence of (89) and (90),

$$|\langle u_h^*, u - u_0 \rangle| = \left| \int_{\Omega} \zeta(u - u_0) dx \right| \leq \varepsilon \int_{\Omega} \widehat{\Phi}_x(|u|) dx + C_{\varepsilon}. \quad (91)$$

Combining (86) with (91) yields

$$\langle u^*, u - u_0 \rangle \geq \langle u_g^*, u - u_0 \rangle - |\langle u_h^*, u - u_0 \rangle| \geq (b_g - \varepsilon) \int_{\Omega} \widehat{\Phi}_x(|u|) dx - C_{\varepsilon}. \quad (92)$$

Regarding the generalized N -functions in $\mathcal{N}(\Omega \times \Omega)$, we have the following pointwise estimates, which are available for N -functions (cf., e.g., [19,20]) and are thus extended directly to functions in $\mathcal{N}(\Omega \times \Omega)$. For a.e. $x, y \in \Omega$, we have

$$\Phi_{x,y}(t) \leq t\phi_{x,y}(t), \forall t \geq 0, \quad (93)$$

and

$$\overline{\Phi}_{x,y}[\phi_{x,y}(t)] \leq \Phi_{x,y}(2t), \forall t \geq 0. \quad (94)$$

For any $u \in X$ and $\varepsilon > 0$, we have

$$\begin{aligned} \langle \mathcal{A}u, u - u_0 \rangle &= \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x, y)|) D_s u(x, y) [D_s u(x, y) - D_s u_0(x, y)] d\mu \\ &\geq \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x, y)|) |D_s u(x, y)|^2 d\mu \\ &\quad - \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x, y)|) |D_s u(x, y)| |D_s u_0(x, y)| d\mu. \end{aligned} \quad (95)$$

It follows from (93) that

$$\begin{aligned} \int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x, y)|) |D_s u(x, y)|^2 d\mu &= \int_{\Omega} \int_{\Omega} \phi_{x,y}(|D_s u(x, y)|) |D_s u(x, y)| d\mu \\ &\geq \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x, y)|) d\mu. \end{aligned} \quad (96)$$

Since $\Phi_{x,y}$ and $\overline{\Phi}_{x,y}$ satisfy Δ_2 conditions, we obtain from (94) and Young's inequality with ε (Corollary 3) that for a.e. $x, y \in \Omega$, $x \neq y$,

$$\begin{aligned} &a_{x,y}(|D_s u(x, y)|) |D_s u(x, y)| |D_s u_0(x, y)| \\ &= \phi_{x,y}(|D_s u(x, y)|) |D_s u_0(x, y)| \\ &\leq \varepsilon \overline{\Phi}_{x,y}(\phi_{x,y}(|D_s u(x, y)|)) + C_{\varepsilon} \Phi_{x,y}(|D_s u_0(x, y)|) \\ &\leq \varepsilon \Phi_{x,y}(2|D_s u(x, y)|) + C_{\varepsilon} \Phi_{x,y}(|D_s u_0(x, y)|) \\ &\leq \varepsilon \kappa \Phi_{x,y}(|D_s u(x, y)|) + C_{\varepsilon} \Phi_{x,y}(|D_s u_0(x, y)|). \end{aligned} \quad (97)$$

Consequently,

$$\begin{aligned} &\int_{\Omega} \int_{\Omega} a_{x,y}(|D_s u(x, y)|) |D_s u(x, y)| |D_s u_0(x, y)| d\mu \\ &\leq \varepsilon \kappa \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x, y)|) d\mu + C_{\varepsilon} \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u_0(x, y)|) d\mu. \end{aligned} \quad (98)$$

Combining (95), (96) and (98) yields

$$\begin{aligned} \langle \mathcal{A}u, u - u_0 \rangle &\geq (1 - \varepsilon \kappa) \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x, y)|) d\mu - C_{\varepsilon} \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u_0(x, y)|) d\mu \\ &= (1 - \varepsilon \kappa) \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x, y)|) d\mu - C_{\varepsilon}. \end{aligned} \quad (99)$$

For $u \in X$ and $u^* \in \mathcal{F}(u)$, by choosing $\varepsilon > 0$ sufficiently small, we see from (92) and (99) that there are positive constants D_1 and D_2 independent of u and u^* such that

$$\begin{aligned} \langle \mathcal{A}u + u^*, u - u_0 \rangle &\geq (1 - \varepsilon \kappa) \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x, y)|) d\mu \\ &\quad + (b_g - \varepsilon) \int_{\Omega} \widehat{\Phi}_x(|u|) dx - C_{\varepsilon} \\ &\geq D_1 \left(\int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x, y)|) d\mu + \int_{\Omega} \widehat{\Phi}_x(|u|) dx \right) - D_2. \end{aligned} \quad (100)$$

Since

$$\int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x,y)|) d\mu + \int_{\Omega} \widehat{\Phi}_x(|u|) dx \rightarrow \infty \text{ as } \|u\|_{s,\Phi_{x,y}} \rightarrow \infty,$$

we immediately obtain (81) for R sufficiently large. The existence of solutions of Problem (P) now follows from Theorem 5. $\square$

Next, let us consider another corollary of Theorem 5 in the case where K is a subset of $X_0 = W_0^{s,\Phi_{x,y}}(\Omega)$ related to the gauge functions $\mu^{\pm}$ of N -functions introduced in Theorem 2. Suppose f has the following growth condition.

(F_f) There exists a generalized N -function $Q = Q_x \in \mathcal{N}(\Omega)$ satisfying a Δ_2 condition, together with its Hölder conjugate, such that $Q_x \ll \widehat{\Phi}_x^*$ and

$$|\zeta| \leq a_h(x) + b_h \overline{Q}_x^{-1}(Q_x(|u|)), \quad (101)$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, all $\zeta \in h(x, u)$, with $a_f \in L^{\overline{Q}_x}(\Omega)$, $b_f \in (0, \infty)$.

Moreover, $\mu_Q^+ \ll \mu_{\Phi}^-$ in the following sense:

$$\lim_{t \rightarrow \infty} \frac{\mu_Q^+(kt)}{\mu_{\Phi}^-(t)} = 0, \quad \forall k > 0. \quad (102)$$

Theorem 7. Suppose $K \subset X_0$ and f satisfies (F1), (F2) and (F_f). Then, Problem (P) has a solution.

Before proving this theorem, we note that since Q_x , $\overline{Q}_x$ and $\Phi_{x,y}$, $\overline{\Phi}_{x,y}$ satisfy Δ_2 conditions, according to Corollary 4, there are $\alpha_Q^-, \alpha_Q^+, \alpha_{\Phi}^-, \alpha_{\Phi}^+ \in (1, \infty)$ such that

$$\alpha_Q^- \leq \frac{tq_x(t)}{Q_x(t)} \leq \alpha_Q^+ \text{ and } \alpha_{\Phi}^- \leq \frac{t\phi_{x,y}(t)}{\Phi_{x,y}(t)} \leq \alpha_{\Phi}^+,$$

for a.e. $x, y \in \Omega$, for all $t > 0$, where q_x is the right derivative of Q_x .

As a consequence of Corollary 5, the functions μ_Q^+ and μ_{Φ}^- can be chosen as:

$$\mu_Q^+(t) = \max\{t^{\alpha_Q^-}, t^{\alpha_Q^+}\} \text{ and } \mu_{\Phi}^-(t) = \min\{t^{\alpha_{\Phi}^-}, t^{\alpha_{\Phi}^+}\}, \quad t \geq 0.$$

In particular, $\mu_Q^+(t) = t^{\alpha_Q^+}$ and $\mu_{\Phi}^-(t) = t^{\alpha_{\Phi}^-}$ for $t \geq 1$. Consequently, in this case, condition (102) is equivalent to

$$\alpha_Q^+ < \alpha_{\Phi}^-. \quad (103)$$

Proof of Theorem 7. Let $\mu_{\Phi}^{\pm}$ and $\mu_{Q_x}^{\pm}$ be the gauge functions associated with $\Phi_{x,y}$ and Q_x given in Corollaries 1 and 2. For $\varepsilon > 0$, sufficiently small, the estimate in (99), together with (17), gives

$$\begin{aligned} \langle \mathcal{A}u, u - u_0 \rangle &\geq (1 - \varepsilon\kappa) \int_{\Omega} \int_{\Omega} \Phi_{x,y}(|D_s u(x,y)|) d\mu - C_{\varepsilon} \\ &\geq \frac{1}{2} \mu_{\Phi}^-([u]_{s,\Phi_{x,y}}) - C. \end{aligned} \quad (104)$$

On the other hand, due to the embedding $W_0^{s,\Phi_{x,y}}(\Omega) \hookrightarrow L^{Q_x}(\Omega)$ and the equivalence of the norms $[\cdot]_{s,\Phi_{x,y}}$ and $\|\cdot\|_{s,\Phi_{x,y}}$ on $X_0 = W_0^{s,\Phi_{x,y}}(\Omega)$ (cf. Propositions 1 and 2), there is a constant $k_0 > 0$ such that

$$\|u\|_{L^{Q_x}(\Omega)} \leq k_0 [u]_{s,\Phi_{x,y}}, \quad \forall u \in X_0. \quad (105)$$

Hence, for any $u \in X_0$, $u^* \in \mathcal{F}(u)$, (101), (105), Corollaries 1 and 2 applied to Q_x and the calculations in (88) and (89) yield

$$\begin{aligned}
|\langle u_h^*, u - u_0 \rangle| &\leq 2 \int_{\Omega} \overline{Q}_x(a_f) dx + (1 + 3b_f) \int_{\Omega} Q_x(|u|) dx + (1 + b_f) \int_{\Omega} Q_x(|u_0|) dx \\
&= C \left(1 + \int_{\Omega} Q_x(|u|) dx \right) \\
&\leq C \left(1 + \mu_Q^+(\|u\|_{L^{Q_x}(\Omega)}) \right) \\
&\leq C \left(1 + \mu_Q^+(k_0[u]_{s, \Phi_{x,y}}) \right).
\end{aligned} \tag{106}$$

It follows from (104) and (106) that

$$\begin{aligned}
\langle \mathcal{A}u + u^*, u - u_0 \rangle &\geq \frac{1}{2} \mu_{\Phi}^-([u]_{s, \Phi_{x,y}}) - C \left(1 + \mu_Q^+(k_0[u]_{s, \Phi_{x,y}}) \right) \\
&\geq \mu_{\Phi}^-([u]_{s, \Phi_{x,y}}) \left[\frac{1}{2} - C \left(\frac{1}{\mu_{\Phi}^-([u]_{s, \Phi_{x,y}})} + \frac{\mu_Q^+(k_0[u]_{s, \Phi_{x,y}})}{\mu_{\Phi}^-([u]_{s, \Phi_{x,y}})} \right) \right].
\end{aligned} \tag{107}$$

As $[u]_{s, \Phi_{x,y}} \rightarrow \infty$, we obtain from (102) and properties of $\mu_{\Phi}^{\pm}$ and $\mu_Q^{\pm}$ stated in Theorem 2 and Corollaries 1 and 2, that

$$\mu_{\Phi}^-([u]_{s, \Phi_{x,y}}) \rightarrow \infty \text{ and } \frac{\mu_Q^+(k_0[u]_{s, \Phi_{x,y}})}{\mu_{\Phi}^-([u]_{s, \Phi_{x,y}})} \rightarrow 0.$$

Thus,

$$\lim_{[u]_{s, \Phi_{x,y}} \rightarrow \infty, u \in X_0} \left(\inf_{u^* \in \mathcal{F}(u)} \langle \mathcal{A}(u) + u^*, u - u_0 \rangle \right) = \infty,$$

that is, (81) is satisfied for $R > 0$ sufficiently large. The existence of solutions of (P) now follows from Theorem 5. $\square$

Remark 4. In cases where coercivity conditions such as (81), (83) and (84) or (101) and (102), are not satisfied, by integrating the above arguments and results with an adaptation of the sub-supersolution approach for multivalued variational inequalities in Sobolev spaces, as presented in, e.g., [32], to our current context of fractional Musielak–Orlicz–Sobolev spaces, we can define appropriate concepts of sub and supersolutions for Problem (P) and demonstrate the existence and some qualitative properties of solutions to Problem (P) between such sub and supersolutions. The details of this approach will be provided in a forthcoming work.

In conclusion, this paper focuses on studying a general class of variational inequalities driven by generalized fractional Φ -Laplacian type operators, perturbed by multivalued lower order terms, over closed convex sets of fractional Musielak–Orlicz–Sobolev spaces. We establish a suitable functional analytic framework for analyzing such variational inequalities, and examine their solvability under appropriate conditions pertaining to the nonlocal fractional main operators, the multivalued terms and the convex sets of constraints.

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