fractal and fractional

Article

Darbo's Fixed-Point Theorem: Establishing Existence and Uniqueness Results for Hybrid Caputo–Hadamard Fractional Sequential Differential Equations

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Abstract: This work explores the existence and uniqueness criteria for the solution of hybrid Caputo–Hadamard fractional sequential differential equations (HCHFSDEs) by employing Darbo's fixed-point theorem. Fractional differential equations play a pivotal role in modeling complex phenomena in various areas of science and engineering. The hybrid approach considered in this work combines the advantages of both the Caputo and Hadamard fractional derivatives, leading to a more comprehensive and versatile model for describing sequential processes. To address the problem of the existence and uniqueness of solutions for such hybrid fractional sequential differential equations, we turn to Darbo's fixed-point theorem, a powerful mathematical tool that establishes the existence of fixed points for certain types of mappings. By appropriately transforming the differential equation into an equivalent fixed-point formulation, we can exploit the properties of Darbo's theorem to analyze the solutions' existence and uniqueness. The outcomes of this research expand the understanding of HCHFSDEs and contribute to the growing body of knowledge in fractional calculus and fixed-point theory. These findings are expected to have significant implications in various scientific and engineering applications, where sequential processes are prevalent, such as in physics, biology, finance, and control theory.

Keywords: Darbo's fixed-point theorem; hybrid Caputo–Hadamard fractional sequential differential equations; Caputo derivative; Hadamard fractional derivative

Citation: Yaseen, M.; Mumtaz, S.; George, R.; Hussain, A.; Nabwey, H.A. Darbo's Fixed-Point Theorem: Establishing Existence and Uniqueness Results for Hybrid Caputo–Hadamard Fractional Sequential Differential Equations. *Fractal Fract.* **2024**, *8*, 326. https://doi.org/10.3390/fractalfract8060326

Academic Editors: Wei-Shih Du and Sameerah Jamal

Received: 25 April 2024
Revised: 23 May 2024
Accepted: 24 May 2024
Published: 29 May 2024

1. Introduction

Fractional calculus is a significant branch of mathematics that is an extension of classical calculus, which involves the differentiation and the integration of a non-integer order. Fractional differential models have several applications in science and engineering. Meral et al. [1] delved into the application of fractional calculus in viscoelastic materials. Oldham [2] explored the utility of fractional differential equations in the domain of electrochemistry. The study investigated the dynamic behavior of electrochemical systems using fractional calculus, providing insights into complex processes, such as charge transfer and electrode kinetics. Balachandran et al. [3] focused on the controllability aspects of linear fractional dynamical systems. The work contributed to understand the manipulability of systems governed by fractional differential equations, which is crucial in control theory and engineering applications. Hadamard [4] presented an essay on functions represented by their Taylor series expansion. The fundamental work provided insights into analytical techniques that may be applicable in the analysis of functions involved in fractional differential equations. Ahmad and Nieto [5] explored analytical techniques to solve sequential fractional differential equations with various boundary conditions extending the applicability of fractional calculus to sequential systems. Ahmad and Nieto [6] explored boundary value problems arising in a particular class of sequential integro-differential equations of a fractional order. Ahmad and Ntouyas [7] considered higher-order nonlocal boundary value problems for sequential fractional differential equations. The research expanded the scope of boundary value problems in fractional calculus, providing new insights into the behavior of higher-order fractional systems. Aqlan et al. [8] developed existence theory for sequential fractional differential equations with anti-periodic-type boundary conditions, which enhanced the theoretical foundation for solving such equations, addressing a specific class of boundary value problems relevant in mathematical modeling. Klimek [9] delved into the realm of sequential fractional differential equations employing the Hadamard derivative. Mahmudov et al. [10] investigated nonlinear sequential fractional differential equations with nonlocal boundary conditions contributing to understanding the behavior of nonlinear systems with fractional-order dynamics. Ye and Huang [11] delved into the realm of nonlinear fractional differential equations utilizing the Caputo sequential fractional derivative. Aljoudi et al. explored a coupled system of Hadamard-type sequential differential equations with coupled strip conditions. Kilbas et al. [12] wrote a fundamental text on the theory and applications of fractional differential equations. The seminal work served as a comprehensive reference for researchers and practitioners interested in fractional calculus. Mohammadi et al. [13] tackled a hybrid fractional boundary value problem incorporating both Caputo and Hadamard derivatives. Jarad et al. [14] introduced a Caputo-type modification of the Hadamard fractional derivatives providing a bridge between two commonly used fractional derivative operators. Baitiche et al. [15] investigated boundary value problems for hybrid Caputo sequential fractional differential equations. Benchohra et al. [16] explored the measure of noncompactness and its application to fractional differential equation in Banach spaces. Darbo [17] investigated fixed points in noncompact transformation, laying the groundwork for the theory of fixed-point theorems in noncompact spaces. Banas and Olszowy [18] explored a class of noncompactness in Banach algebras and their applications to nonlinear integral equations. Aghajani et al. [19] presented some generalizations of Darbo's fixed-point theorem and their applications. The study provided new tools for the analysis of nonlinear phenomena, with potential applications in various fields. Shunan and Bingyang [20] proposed a novel approach to understanding phonon heat transport beyond traditional hydrodynamics. Marawan [21] explored the application of fractional quantum mechanics to systems with electrical screening effects in their study. Dubey and Chakraverty [22] developed hybrid techniques for the approximate analytical solution of space- and time-fractional telegraph equations. Inspired by the research studies, we discuss a new idea that is based on the sequential definition of a Caputo–Hadamard fractional operator. For the hybrid Caputo–Hadamard fractional sequential differential equations (CHFSDEs), we analyze the following initial value problem:

$$[{}^{CH}\mathfrak{D}_a^{w+1} + \tau {}^{CH}\mathfrak{D}_a^{w}][\frac{s(\ell)}{f(\ell, s(\ell))}] = y(\ell), \qquad \ell \in \mathcal{E} = [1, e], \tag{1}$$

$$\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)\bigg|_{\ell=a} = 0, \quad \left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)}\bigg|_{\ell=a} = 0, \quad \left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(2)}\bigg|_{\ell=a} = 0, \tag{2}$$

where $w \in (1, 2]$, $\ell \in \mathcal{E} = [a, e]$, $1 \leq a < e$, ${}^{CH}\mathfrak{D}_a^w$ is a Caputo–Hadamard fractional derivative of order w, $f : \mathcal{E} \times \Re \to \Re \setminus \{0\}$ and $y : \mathcal{E} \to \Re$ are continuous functions, and τ is a real number. In this direction, we will use techniques related to measures of noncompactness in Banach algebras and Darbo's fixed-point theorem.

This manuscript is structured as follows: Section 2 offers a fundamental review for the readers' reference. Section 3 is devoted to proving our main results utilizing Darbo's fixed-point theorem. Section 4 includes an explanatory example that authenticates our theoretical findings. Finally, this study concludes with a summary of the key insights and implications.

2. Preliminaries

Now, we will provide some useful concepts and basic definitions used in proving our main results.

The Banach space of all continuous functions, $s : \mathcal{E} \to \Re$, will be denoted by $C(\mathcal{E}, \Re)$. A norm on this space is given by

$$\|s\|_\infty := \sup\{|s(\ell)| : \ell \in \mathcal{E}\}.$$

Definition 1 ([12,13,23]). *The Caputo–Hadamard integral of fractional order w for a continuous function $s\colon (a,b) \to \Re$ is defined as follows*

$${}^{CH}\Im^{w}_{a^+}(s(\ell)) = \frac{1}{\Gamma(w)} \int_a^\ell \left(\ln\frac{\ell}{x}\right)^{w-1} s(x)\, \frac{dx}{x}$$

whenever the RHS integral exists.

Lemma 1 ([14]). *Let $\Re(w) \geq 0$, $n = [\Re(w)] + 1$ and $\Re(H) > 0$. Then*

$${}^{CH}\mathfrak{D}^{w}_{a}(\ln\frac{\ell}{a})^{H-1} = \frac{\Gamma(H)}{\Gamma(H-w)}(\ln\frac{\ell}{a})^{H-w-1}, \quad \Re(H) > n. \tag{3}$$

$${}^{CH}\mathfrak{D}^{w}_{b}(\ln\frac{b}{\ell})^{H-1} = \frac{\Gamma(H)}{\Gamma(H-w)}(\ln\frac{b}{\ell})^{H-w-1}, \quad \Re(H) > n. \tag{4}$$

$${}^{CH}\mathfrak{D}^{w}_{a}(\ln\frac{\ell}{a})^{k} = 0, \ {}^{CH}\mathfrak{D}^{w}_{b}(\ln\frac{b}{\ell})^{k} = 0 \qquad k = 0, 1, 2, \ldots n-1. \tag{5}$$

In particular cases, ${}^{CH}\mathfrak{D}^{w}_{a}1 = 0$ and ${}^{CH}\mathfrak{D}^{w}_{b}1 = 0$.

Lemma 2 ([12]). *For each $w_1, w_2 \in \Re^+$, the following equality*

$${}^{CH}\Im^{w_1}_{a^+}{}^{CH}\Im^{w_2}_{a^+}(s(\ell)) = {}^{CH}\Im^{w_1+w_2}_{a^+}(s(\ell))$$

holds true for almost all $\ell \in [1, e]$.

Lemma 3 ([12,23]). *Assume that $s \in AC^m_{\Re}([a,b])$ so that $\beta - 1 < w \leq \beta$, a general solution for the Caputo–Hadamard differential equation ${}^{CH}\mathfrak{D}^{w}_{a^+}(s(\ell)) = 0$ is of the form $s(\ell) = \sum_{k=0}^{\beta-1} d_k \left(\ln\frac{\ell}{a}\right)^k$, and we obtain*

$${}^{CH}\Im^{w}_{a^+}{}^{CH}\mathfrak{D}^{w}_{a^+}(s(\ell)) = s(\ell) + d_0 + d_1\left(\ln\frac{\ell}{a}\right) + d_2\left(\ln\frac{\ell}{a}\right)^2 + \cdots + d_{\beta-1}\left(\ln\frac{\ell}{a}\right)^{\beta-1},$$

where $d_0, d_1, \ldots, d_{\beta-1}$ are real constants and $\beta = [w] + 1$.

Lemma 4 ([15]). *Let $w > 0$, $p \in \mathcal{L}^1[1,e]$. Then, for almost all $\ell \in [1,e]$, we have*

$$\Im^{w+1}_{a^+}p(\ell) \leq \|\Im^{w}_{a^+}p\|_{\mathcal{L}^1}.$$

Proof. Let $p \in \mathcal{L}^1[1,e]$ from Lemma 2, and we have

$$\Im^{w+1}_{a^+}p(\ell) = \Im^{1}_{a^+}p(\ell)\Im^{w}_{a^+}p(\ell) = \int_1^\ell \Im^{w}_{a^+}p(x)\,\frac{dx}{x} \leq \int_1^e |\Im^{w}_{a^+}p(x)|\,\frac{dx}{x} = \|\Im^{w}_{a^+}p\|_{\mathcal{L}^1}. \quad \square$$

Lemma 5 ([12]). *The integral $\Im^{w}_{a^+}, w > 0$ is bounded in $\mathcal{L}^1[1,e]$ with*

$$\|\Im^{w}_{a^+}p\|_{\mathcal{L}^1} \leq \frac{\|p\|_{\mathcal{L}^1}}{\Gamma(w+1)}.$$

Darbo's fixed-point theorem (DFPT) is very important in our discussion, as given below.

Theorem 1 ([17]). *Let $\omega \subset \mathcal{N}$ be nonempty, closed, bounded, and convex. Also, assume $\mathcal{Q}\colon \omega \to \omega$ is a continuous function. Consider there is $\ell \in [0,1)$ such that*

$$\gamma(\mathcal{QC}) \leq \ell\gamma(\mathcal{C}) \tag{6}$$

for any $\mathcal{C} \subset \omega$, where γ is a measure of noncompactness in $\mathcal{N}$. Then, $\mathcal{Q}$ has a fixed point in ω.

The following is an extension of Theorem 1 that will be very beneficial in our research.

Theorem 2 ([18]). *Assume $\omega \subset \mathcal{N}$ is nonempty, closed, bounded, and convex. Also, consider $\mathcal{Q}\colon \omega \to \omega$ to be a continuous mapping satisfying*

$$\gamma(\mathcal{QC}) \leq \varphi(\gamma \cdot \mathcal{C}), \tag{7}$$

for any $\mathcal{C} \subset \omega$, where the measure of noncompactness is γ and $\varphi\colon \Re_+ \to \Re_+$ is an increasing function such that $\lim_{n\to 0} \varphi^n(\ell) = 0$ for each $\ell \in \Re_+$, when the n-iteration of φ is denoted by φ^n. Then, $\mathcal{Q}$ has at least one fixed point in ω.

Lemma 6 ([19]). *Assume $\varphi\colon \Re_+ \to \Re_+$ is an increasing and upper semicontinuous map. Then, the following conditions are equivalent:*

1. $\lim_{n\to 0} \varphi^n(\ell) = 0$, *for any $\ell \geq 0$;*
2. $\varphi(\ell) < \ell$ *for any $\ell > 0$.*

Now, we consider that the space $\mathcal{N}$ has a Banach algebra structure. We will present the product of two elements $c, b \in \mathcal{C}$ by cb, and by using $\mathcal{CB}$, we express the set defined by

$$\mathcal{CB} = \{cb\colon c \in \mathcal{C}, b \in \mathcal{B}\}.$$

Definition 2 ([18]). *Assume that $\mathcal{N}$ is a Banach algebra. A measure of noncompactness γ in $\mathcal{N}$ is said to satisfy condition (m) if it meets the criteria given below:*

$$\gamma(\mathcal{CB}) \leq \|\mathcal{C}\|\gamma(\mathcal{B}) + \|\mathcal{B}\|\gamma(\mathcal{C})$$

for any $\mathcal{C}, \mathcal{B} \in \mathcal{U}_{\mathcal{N}}$. A Banach space with a standard norm is the family of all continuous and real-valued functions defined on an interval $\mathcal{E}$ with the norm

$$\|\mathcal{C}\| = \sup\{|c(\ell)|, \ell \in \mathcal{E}\}.$$

Definition 3 ([16]). *Assume that $(C(\mathcal{E}), \|.\|)$ is a Banach algebra, with the standard product of real functions as the multiplication. To define the measure, consider a set $\mathcal{C}$ in $C(\mathcal{E})$. For $c \in \mathcal{C}$ and for any given $\varepsilon > 0$, denote $\lambda(c, \varepsilon)$ as the modulus of continuity of c by*

$$\lambda(c, \varepsilon) = \sup\{|c(\ell) - c(s)|\colon \ell, s \in \mathcal{E}, |\ell - s| \leq \varepsilon\}.$$

Put

$$\lambda(\mathcal{C}, \varepsilon) = \sup\{\lambda(c, \varepsilon)\colon c \in \mathcal{C}\},$$

and

$$\lambda_0(\mathcal{C}) = \lim_{\varepsilon\to 0} \lambda(\mathcal{C}, \varepsilon). \tag{8}$$

The function λ_0 is a measure of noncompactness in space $C(\mathcal{E})$.

3. Main Results

Lemma 7. *Consider $f \in C(\mathcal{E} \times \Re, \Re \setminus \{0\})$. For every $y \in C(\mathcal{E}, \Re)$, the unique solution for a hybrid CHFSDE*

$$[{}^{CH}\mathfrak{D}_a^{w+1} + \tau {}^{CH}\mathfrak{D}_a^{w}][\frac{s(\ell)}{f(\ell, s(\ell))}] = y(\ell), \quad \ell \in \mathcal{E} = [1, e], \tag{9}$$

$$\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)\bigg|_{\ell=a} = 0, \quad \left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)}\bigg|_{\ell=a} = 0, \quad \left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(2)}\bigg|_{\ell=a} = 0, \tag{10}$$

is given by

$$s(\ell) = f(\ell, s(\ell))\{-\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{x^{-\tau}} - 1\right)\Im_a^{w-1} y(x)\,\frac{dx}{x}\}. \tag{11}$$

Proof. By applying ${}^{CH}\Im_a^{w+1}$ on both sides of (9),

$$\left(\frac{s(\ell)}{f(\ell, s(\ell))} - b_1(\ln \ell)^w - b_2(\ln \ell)^{w-1} - b_3(\ln \ell)^{w-2}\right) + \tau \Im_a^1\Big(\frac{s(\ell)}{f(\ell, s(\ell))} - c_1(\ln \ell)^{w-1}$$
$$-c_2(\ln \ell)^{w-2}\Big) = {}^{CH}\Im_a^{w+1} y(\ell).$$

By using the first condition given in (10), we obtain $b_3 = 0$ so that

$$\left(\frac{s(\ell)}{f(\ell, s(\ell))} - b_1(\ln \ell)^w - b_2(\ln \ell)^{w-1}\right) + \tau \Im_a^1\Big(\frac{s(\ell)}{f(\ell, s(\ell))} - c_1(\ln \ell)^{w-1}$$
$$-c_2(\ln \ell)^{w-2}\Big) = {}^{CH}\Im_a^{w+1} y(\ell).$$

Now, taking the first ordinary derivative of the above equation,

$$\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)} - \frac{b_1 w}{\ell}(\ln \ell)^{w-1} - \frac{b_2(w-1)}{\ell}(\ln \ell)^{w-2} + \frac{\tau}{\ell}\Big(\frac{s(\ell)}{f(\ell, s(\ell))} - c_1(\ln \ell)^{w-1}$$
$$-c_2(\ln \ell)^{w-2}\Big) = \frac{1}{\ell}{}^{CH}\Im_a^{w} y(\ell).$$

Multiplying both sides by ℓ,

$$\left[\ell\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)} + \tau\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)\right] - (b_1 w + \tau c_1)(\ln \ell)^{w-1}$$
$$- (b_2(w-1) + \tau c_2)(\ln \ell)^{w-2} = {}^{CH}\Im_a^{w} y(\ell).$$

According to the second condition, we obtain $b_2(w-1) + \tau c_2 = 0$, and we have

$$\left[\ell\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)} + \tau\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)\right] - (b_1 w + \tau c_1)(\ln \ell)^{w-1} = {}^{CH}\Im_a^{w} y(\ell).$$

Now, by taking the second ordinary derivative of the above equation, we obtain:

$$\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)} + \ell\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(2)} + \tau\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)} - (b_1 w + \tau c_1)\frac{(w-1)}{\ell}(\ln \ell)^{w-2}$$
$$= \frac{1}{\ell}{}^{CH}\Im_a^{w-1} y(\ell).$$

By utilizing the third condition given in (10), we obtain $(b_1 w + \tau c_1) = 0$. Multiplying both sides of the previous equation by ℓ, we obtain

$$\ell^2\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(2)} + (1+\tau)\ell\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)} = {}^{CH}\Im_a^{w-1} y(\ell). \tag{12}$$

It is a Cauchy–Euler differential equation of second order that has a general solution:

$$\frac{s(\ell)}{f(\ell, s(\ell)} = \frac{s_c(\ell)}{f(\ell, s_c(\ell)} + \frac{s_p(\ell)}{f(\ell, s_p(\ell)},$$

where $\frac{s_c(\ell)}{f(\ell, s_c(\ell)}$ and $\frac{s_p(\ell)}{f(\ell, s_p(\ell)}$ are complementary and a particular solution of (12). Consider the solutions $\frac{s_1(\ell)}{f(\ell, s_1(\ell)} = \ell^{m_1}$ and $\frac{s_2(\ell)}{f(\ell, s_2(\ell)} = \ell^{m_2}$ for the homogeneous equation,

$$\ell^2\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(2)} + (1+\tau)\ell\left(\frac{s(\ell)}{f(\ell, s(\ell))}\right)^{(1)} = 0 \tag{13}$$

where the distinct real roots of the characteristic equation $m^2 + \tau m = 0$ are $m_1 = 0$ and $m_2 = -\tau$. The complementary solution of the homogeneous Equation (13) is

$$\frac{s_c(l)}{f(\ell, s_c(\ell)} = a_1 + a_2 \ell^{-\tau}$$

for some constants a_1 and a_2. These two constants can be evaluated by the initial conditions:

$$\frac{s(a)}{f(a, s(a))} = \left(\frac{s(a)}{f(a, s(a))}\right)^{(1)} = 0$$

given in (10).

$$\begin{cases} a_1 + a_2 a^{-\tau} = 0 \\ -\tau a_2 a^{-\tau-1} = 0. \end{cases}$$

The only solution for these algebraic equations is $a_1 = a_2 = 0$. Because $m_1 \neq m_2$. The Wronskian W for the solutions $\frac{s_1(\ell)}{f(\ell, s_1(\ell))} = x_1$ and $\frac{s_2(\ell)}{f(\ell, s_2(\ell))} = x_2$ is

$$\begin{aligned} W(x_1, x_2) &= \begin{vmatrix} 1 & \ell^{-\tau} \\ 0 & -\tau \ell^{-\tau-1} \end{vmatrix} \\ &= -\tau \ell^{-\tau-1} \neq 0. \end{aligned}$$

Applying the variation of parameter technique, we can obtain the particular solution

$$\frac{s_p(\ell)}{f(\ell, s_p(\ell))} = -\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{x^{-\tau}} - 1\right) \Im_a^{w-1} y(x) \frac{dx}{x}.$$

Therefore, the general solution is $\frac{s(\ell)}{f(\ell, s(\ell))} = \frac{s_p(\ell)}{f(\ell, s_p(\ell))}$.

Thanks to Lemma 7, the following integral equation is equivalent to the presented problem.

$$s(\ell) = f(\ell, s(\ell))\{-\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{x^{-\tau}} - 1\right) \Im_a^{w-1} y(x, s(x)) \frac{dx}{x}\}.$$

□

We consider the following assumptions to be satisfied in order to obtain our major findings:

$(\mathcal{S}_1)$ $f \in C(\mathcal{E} \times \Re, \Re \setminus \{0\})$ and $y \in C(\mathcal{E} \times \Re, \Re)$.
$(\mathcal{S}_2)$ There exists an upper semicontinuous function $\varrho\colon \Re_+ \to \Re_+$ such that $\varrho(\ell) < \ell$ for any $\ell > 0$, ϱ is non-decreasing, and

$$|f(\ell, s_1(\ell)) - f(\ell, s_2(\ell))| \leq \varrho(|s_1(\ell) - s_2(\ell)|),\ \ell \in \mathcal{E},\ s_1(\ell), s_2(\ell) \in \Re.$$

$(\mathcal{S}_3)$ There exists functions $\theta \in \mathcal{L}^1(\mathcal{E}, \Re_+)$ and $\sigma\colon \Re_+ \to \Re_+$ continuous and non-decreasing such that $|y(\ell, s(\ell))| \leq \theta(\ell)\sigma(|s(\ell)|)$, $\ell \in \mathcal{E}$, $s(\ell) \in \Re$.
$(\mathcal{S}_4)$ There is $r^* > 0$ such that $[\varrho(r^*) + H]\|\theta\|_{\mathcal{L}^1}\|A\|\sigma(r^*) \leq \tau\Gamma(w)r^*$, and $\|\theta\|_{\mathcal{L}^1}\|A\|\sigma(r^*) < \tau\Gamma(w)$, where $H = \sup_{\ell \in \mathcal{E}} |f(\ell, 0, 0)|$.

Theorem 3. *Consider that $(\mathcal{S}_1)$–$(\mathcal{S}_4)$ are satisfied. Then, the problem (1)–(2) has a unique solution.*

Proof. According to Theorem 2, we assume an operator $\mathcal{Q}$ defined on the Banach algebra $C(\mathcal{E})$ as given below:

$$\mathcal{Q}s(\ell) = f(\ell, s(\ell))\{-\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{s^{-\tau}} - 1\right) \Im_a^{w-1} y(x, s(x))\, \frac{dx}{x}\}$$

for $\ell \in \mathcal{E}$. A fixed point of $\mathcal{Q}$ provides us with the required result according to Lemma 7. The operators $\mathcal{F}$ and $\mathcal{G}$ on the Banach algebra $C(\mathcal{E})$ are described as $\mathcal{F}s(\ell) = f(\ell, s(\ell))$ and $\mathcal{G}s(\ell) = -\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{s^{-}\tau} - 1\right) \Im_a^{w-1} y(x, s(x))\, \frac{dx}{x}$ for $\ell \in \mathcal{E}$. Consequently, $\mathcal{Q}s = (\mathcal{F}s) \cdot (\mathcal{G}s)$ for any $s \in C(\mathcal{E})$.

First, we will prove that $\mathcal{Q}$ transforms $C(\mathcal{E})$ into itself. To do this, it is sufficient to show that $\mathcal{F}s, \mathcal{G}s \in C(\mathcal{E})$ for each $s \in C(\mathcal{E})$. As we know that the product of continuous functions is continuous, by $(\mathcal{S}_1)$, it follows that if $s \in C(\mathcal{E})$ for $s \in C(\mathcal{E})$. In order to prove that $\mathcal{G}s \in C(\mathcal{E})$ for $s \in C(\mathcal{E})$, let $\ell \in \mathcal{E}$ be fixed, take $s \in C(\mathcal{E})$, and let (ℓ_n) be a sequence in $\mathcal{E}$ such that $\ell_n \to \ell$ as $n \to \infty$. We can consider that $\ell_n \geq \ell$ for n large enough without losing generality. For every n, we obtain

$$\begin{aligned}
|\mathcal{G}s(\ell_n) - \mathcal{G}s(\ell)| &\leq \frac{1}{\tau}\int_a^\ell \left[\left(\frac{\ell}{x}\right)^{-\tau} - \left(\frac{\ell_n}{x}\right)^{-\tau}\right] \Im_a^{w-1} y(x, s(x))\, \frac{dx}{x} \\
&+ \frac{1}{\tau}\int_\ell^{\ell_n} \left[\left(\frac{\ell_n}{x}\right)^{-\tau} - 1\right] \Im_a^{w-1} y(x, s(x))\, \frac{dx}{x} \\
&\leq \frac{\sigma(\|s\|)}{\tau}\left(\int_a^\ell \left[\left(\frac{\ell}{x}\right)^{-\tau} - \left(\frac{\ell_n}{x}\right)^{-\tau}\right] \Im_a^{w-1}\theta(x)\, \frac{dx}{x}\right. \\
&\left.+ \int_\ell^{\ell_n} \left[\left(\frac{\ell_n}{x}\right)^{-\tau} - 1\right] \Im_a^{w-1}\theta(x)\, \frac{dx}{x}\right) \\
&\leq \frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\left(\int_a^\ell \left[\left(\frac{\ell}{x}\right)^{-\tau} - \left(\frac{\ell_n}{x}\right)^{-\tau}\right] \frac{dx}{x}\right. \\
&\left.+ \int_\ell^{\ell_n} \left[\left(\frac{\ell_n}{x}\right)^{-\tau} - 1\right] \frac{dx}{x}\right) \\
&\leq \frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\left(\frac{a^\tau}{\tau}\left(\frac{\ell^\tau - \ell_n^\tau}{\ell_n^\tau \ell}\right) + \frac{1}{\tau}\left(\frac{\ell_n^\tau - \ell^\tau}{\ell_n^\tau}\right)\right. \\
&\left.+ \frac{\ell^{-\tau}}{\tau}(\ell_n^\tau - \ell^\tau) - (\ln \ell_n - \ln \ell)\right).
\end{aligned}$$

As $\ell_n \to \ell$, the right-hand side of the above inequality converges to zero. Thus, we conclude that $\mathcal{G}s(\ell_n) \to \mathcal{G}s(\ell)$. Therefore, $\mathcal{G}s \in C(\mathcal{E})$. This indicates that if s is in $C(\mathcal{E})$, then $\mathcal{Q}s$ is also in $C(\mathcal{E})$. Utilizing assumptions $(\mathcal{S}_2)$ and $(\mathcal{S}_3)$, for $s \in C(\mathcal{E})$ and $\ell \in \mathcal{E}$, we obtain

$$\begin{aligned}
|(\mathcal{Q}s)(\ell)| &= \left| f(\ell, s(\ell))\{-\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{x^{-\tau}} - 1\right) \Im_a^{w-1} y(x, s(x))\, \frac{dx}{x}\}\right| \\
&\leq (|f(\ell, s(\ell) - f(\ell,0)| + |f(\ell,0)|)\{\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{x^{-\tau}} - 1\right) \Im_a^{w-1} y(x, s(x))\, \frac{dx}{x}\} \\
&\leq [\varrho(|s(\ell)|) + H]\frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\{\int_a^\ell \left[\left(\frac{\ell}{x}\right)^{-\tau} - 1\right]\frac{dx}{x}\} \\
&\leq [\varrho(|s(\ell)|) + H]\frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\{\frac{1}{\tau}\left[1 - \left(\frac{a}{\ell}\right)^\tau\right] - \ln \ell + \ln a\} \\
&\leq [\varrho(|s(\ell)|) + H]\frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\|A\|,
\end{aligned}$$

where $A(\ell) = \frac{1}{\tau}\left[1 - \left(\frac{a}{\ell}\right)^\tau\right] - \ln \ell + \ln a$. Consequently,

$$\|\mathcal{Q}s\| \leq [\varrho(|s(\ell)|) + H]\frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\|A\|. \tag{14}$$

According to assumption $(\mathcal{S}_4)$, we observe that the operator $\mathcal{Q}$ maps the ball $B_{r^*} \subset C(\mathcal{E})$ into itself. Additionally, based on the most recent estimate, we deduce the following conclusion:

$$\begin{aligned}
\|\mathcal{F}B_{r^*}\| &\leq \varrho(r^*) + H, \\
\|\mathcal{G}B_{r^*}\| &\leq \frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\|A\|.
\end{aligned} \tag{15}$$

This result indicates that the operator $\mathcal{Q}$ transforms the set B_{r^*} into itself. Next, we will demonstrate how the operators $\mathcal{F}$ and $\mathcal{G}$ are continuous on the ball B_{r^*}. To begin, we establish the continuity of the operator $\mathcal{F}$ on the ball B_{r^*}. For this purpose, we consider a sequence $\{s_n\} \subset B_{r^*}$ and $s \in B_{r^*}$ such that $\|s_n - s\| \to 0$ as $n \to \infty$, and we aim to show that $\|\mathcal{F}s_n - \mathcal{F}s\| \to 0$ as $n \to \infty$. Indeed, for all $\ell \in \mathcal{E}$, utilizing assumption $(\mathcal{S}_2)$, we have

$$\begin{aligned}
|(\mathcal{F}s_n)(\ell) - (\mathcal{F}s)(\ell)| = |f(\ell, s_n(\ell) - f(\ell, s(\ell))| &\leq \varrho(|s_n(\ell) - s(\ell)|) \\
&\leq \varrho(\|s_n - s\|) \leq \|s_n - s\|.
\end{aligned}$$

Thus, we obtain

$$\|\mathcal{F}s_n - \mathcal{F}s\| \leq \|s_n - s\|.$$

Consequently, from the above inequality, we conclude that $\lim_{n\to\infty} \|\mathcal{F}s_n - \mathcal{F}s\| = 0$. Therefore, the operator $\mathcal{F}$ is continuous on the ball B_{r^*}. To prove the continuity of $\mathcal{G}$ on the ball B_{r^*}, we fix $\varepsilon > 0$ and consider arbitrary $z, s \in B_{r^*}$ such that $\|s - z\| \leq \varepsilon$. Then, for $\ell \in \mathcal{E}$, we obtain

$$\begin{aligned}
(\mathcal{G}s)(\ell) - (\mathcal{G}z)(\ell) &= -\frac{1}{\tau}\int_a^\ell \left(\frac{\ell^{-\tau}}{x^{-\tau}} - 1\right) \Im_a^{w-1}[y(x, s(x)) - y(x, z(x))]\, \frac{dx}{x} \\
|(\mathcal{G}s)(\ell) - (\mathcal{G}z)(\ell)| &\leq \frac{\omega_y(x, \varepsilon)}{\tau\Gamma(w)}\{\int_a^\ell \left(\frac{\ell^{-\tau}}{x^{-\tau}} - 1\right)\frac{dx}{x}\} \\
&\leq \frac{\omega_y(x, \varepsilon)}{\tau\Gamma(w)}\{\frac{1}{\tau}\left(1 - \left(\frac{a}{\ell}\right)^\tau\right) - \ln \ell + \ln a\} \\
&\leq \frac{\omega_y(x, \varepsilon)}{\tau\Gamma(w)}\|A\|,
\end{aligned}$$

where $A(\ell) = \frac{1}{\tau}\left(1 - \left(\frac{a}{\ell}\right)^{\tau}\right) + \ln(\frac{a}{\ell})$ and $\omega_y(x,\varepsilon) = \sup\{|y(\ell,u) - y(\ell,v)| \colon \ell \in \mathcal{E}, u, v \in [-x,x], |u-v| \le \varepsilon\}$. Thus, $\|\mathcal{G}s - \mathcal{G}z\| \le \frac{\omega_y(x,\varepsilon)}{\tau\Gamma(w)}\|A\|$. Given that $y(\ell,s)$ is uniformly continuous on the compact $\mathcal{E} \times [-x,x]$, it follows that as $\varepsilon \to 0$, $\omega_y(x,\varepsilon)$ tends to zero. Thus, the inequality above implies that $\lim_{\varepsilon\to 0}\|\mathcal{G}s - \mathcal{G}z\| = 0$. Consequently, the operator $\mathcal{G}$ exhibits continuity within B_{r^*}. Therefore, we establish that $\mathcal{Q}$ is a continuous operator on B_{r^*}. Additionally, we demonstrate that the operator $\mathcal{Q}$ satisfies (7) with respect to the measure of noncompactness ω_0 as defined in (8). Let $\varepsilon > 0$ be fixed, and $s \in S$ and $\ell_1, \ell_2 \in \mathcal{E}$ with $|\ell_1 - \ell_2| \le \varepsilon$ for any nonempty subset S of B_{r^*}. By utilizing assumption $(\mathcal{S}_2)$, we obtain

$$\begin{aligned}
|(\mathcal{F}s)(\ell_1) - (\mathcal{F}s)(\ell_2)| &= |f(\ell_1, s(\ell_1)) - f(\ell_2, s(\ell_2))| \\
&\le |f(\ell_1, s(\ell_1)) - f(\ell_1, s(\ell_2))| + |f(\ell_1, s(\ell_2)) - f(\ell_2, s(\ell_2))| \\
&\le \varrho(|s(\ell_1) - s(\ell_2)|) + \omega(f,\varepsilon) \\
&\le \varrho(\omega(s,\varepsilon)) + \omega(f,\varepsilon),
\end{aligned}$$

where

$$\omega(f,\omega) = \sup\{|f(\ell_1,s) - f(\ell_2,s)| \colon \ell_1,\ell_2 \in \mathcal{E}, |\ell_1 - \ell_2| \le \varepsilon, s \in [-x,x]\}.$$

Hence,

$$\omega(\mathcal{F}S,\varepsilon) \le \varrho(\omega(S,\varepsilon)) + \omega(f,\varepsilon).$$

Because $f(\ell,s)$ is uniformly continuous on the set $\mathcal{E} \times [-x,x]$, we conclude that $\omega(f,\varepsilon) \to 0$ as $\varepsilon \to 0$. Consequently, from the above inequality, we deduce

$$\omega_0(\mathcal{F}S) \le \varrho(\omega_0(S)). \tag{16}$$

Now, we estimate $\omega_0(\mathcal{G}S)$. Fix $\varepsilon > 0$, and because $\ln\ell$ and $A(\ell)$ are uniformly continuous on $\mathcal{E}$, there exists $\delta > 0$ (which can be taken with $\delta < \varepsilon$) such that for each $\ell_1, \ell_2 \in \mathcal{E}$ with $|\ell_2 - \ell_1| \le \delta < \varepsilon$

$$\left|\frac{\ell_2^\tau - \ell_1^\tau}{\ell_2^\tau}\right| \le \tau\varepsilon, \quad \left|\frac{\ell_2^\tau - \ell_1^\tau}{\ell_1^\tau \ell_2^\tau}\right| \le \frac{\tau\varepsilon}{a^\tau}, \quad |\ln\ell_2 - \ln\ell_1| \le \varepsilon.$$

Thus, we have

$$\begin{aligned}
|\mathcal{G}s(\ell_2) - \mathcal{G}s(\ell_1)| &\le \frac{1}{\tau}\int_a^{\ell_1}\left[\left(\frac{\ell_1}{x}\right)^{-\tau} - \left(\frac{\ell_2}{x}\right)^{-\tau}\right]\Im_a^{w-1}|y(x,s(x))|\,\frac{dx}{x} \\
&+ \frac{1}{\tau}\int_{\ell_1}^{\ell_2}\left[\left(\frac{\ell_2}{x}\right)^{-\tau} - 1\right]\Im_a^{w-1}|y(x,s(x))|\,\frac{dx}{x}. \\
&\le \frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\{\int_a^{\ell_1}\left[\left(\frac{\ell_1}{x}\right)^{-\tau} - \left(\frac{\ell_2}{x}\right)^{-\tau}\right]\frac{dx}{x} \\
&+ \frac{1}{\tau}\int_{\ell_1}^{\ell_2}\left[\left(\frac{\ell_2}{x}\right)^{-\tau} - 1\right]\frac{dx}{x}\} \\
&\le \frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\{\frac{1}{\tau}\left(\frac{\ell_2^\tau - \ell_1^\tau}{\ell_2^\tau}\right) + \frac{a^\tau}{\tau}\left(\frac{\ell_1^\tau - \ell_2^\tau}{\ell_2^\tau \ell_1^\tau}\right) \\
&+ \frac{1}{\tau}\left(\frac{\ell_2^\tau - \ell_1^\tau}{\ell_2^\tau}\right) - (\ln\ell_2 - \ln\ell_1)\} \\
&\le \frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\{2\varepsilon\}.
\end{aligned}$$

Therefore,

$$\omega(\mathcal{G}s,\varepsilon) \leq \frac{\|\theta\|_{\mathcal{L}^1}\sigma(\|s\|)}{\tau\Gamma(w)}\{2\varepsilon\}.$$

From this, it follows that

$$\omega_0(\mathcal{G}S) = 0. \tag{17}$$

Next, by Definition 2 and the estimates (15), (16), and (17), we have

$$\begin{aligned}\omega_0(\mathcal{Q}S) &= \omega_0(\mathcal{F}S\cdot\mathcal{G}S)\\ &\leq \|\mathcal{F}S\|\omega_0(\mathcal{G}S) + \|\mathcal{G}S\|\omega_0(\mathcal{F}S)\\ &\leq \|\mathcal{F}B_{r*_0}\|\omega_0(\mathcal{G}S) + \|\mathcal{G}B_{r*_0}\|\omega_0(\mathcal{F}S)\\ &\leq \frac{\|\theta\|_{\mathcal{L}^1}}{\tau\Gamma(w)}\|A\|\sigma(r*)\varrho(\omega_0(S)).\end{aligned}$$

Thus, because $\|\theta\|_{\mathcal{L}^1}\|A\|\sigma(r*) < \tau\Gamma(w)$ from assumption $(\mathcal{S}_4)$, we obtain that operator $\mathcal{Q}$ is a contraction on ball B_r* with respect to the measure of noncompactness ω_0. Therefore, Theorem 2 gives that the operator $\mathcal{Q}$ has at least one fixed point in B_r*. Consequently, the problem (1)–(2) has a unique solution in B_r*. The proof is now finished. □

An Example

Assume a hybrid fractional problem

$$\left[{}^{CH}\mathfrak{D}_a^{3/2} + {}^{CH}\mathfrak{D}_a^{1/2}\right]\left[\frac{s(\ell)}{1/3\ell + \ln(1/3+|s(\ell)|)}\right] = 0.2\ell^3\sin|s(\ell)|,\ \ell\in[1,e], \tag{18}$$

$$\begin{cases}\left(\frac{s(\ell)}{1/3\ell+\ln(1/3+|s(\ell)|)}\right)\Big|_{\ell=1} &= 0,\\ \left(\frac{s(\ell)}{1/3\ell+\ln(1/3+|s(\ell)|)}\right)^{(1)}\Big|_{\ell=1} &= 0,\\ \left(\frac{s(\ell)}{1/3\ell+\ln(1/3+|s(\ell)|)}\right)^{(2)}\Big|_{\ell=1} &= 0.\end{cases} \tag{19}$$

Corresponding to the problem (1)–(2), we have $w = 1/2, \tau = 1$,

$$f(\ell,s(\ell)) = 1/3\ell + \ln(1/3+|s(\ell)|),\ y(\ell,s(\ell)) = 0.2\ell^3\sin s(\ell).$$

Further, $H = 1$, and by a simple calculation, we obtain $\|A\| = 0.3704$. It is clear that the functions f and y satisfy $(\mathcal{S}_1)$ of Theorem 3. Furthermore, for any $\ell\in\mathcal{E}$, and $s_1(\ell), s_2(\ell)\in\Re$. We can assume that $|s_1(\ell)| < |s_2(\ell)|$. Then,

$$\begin{aligned}|f(\ell,s_2(\ell)) - f(\ell,s_1(\ell))| &= |\ln(1/3+|s_2(\ell)|) - \ln(1/3+|s_1(\ell)|)|\\ &\leq \ln(\frac{1/3+|s_2(\ell)|}{1/3+|s_1(\ell)|}) = \ln(1+\frac{|s_2(\ell)|-|s_1(\ell)|}{1/3+|s_1(\ell)|})\\ &\leq \ln(1+(|s_2(\ell)|-|s_1(\ell)|) \leq \ln(1+|s_2(\ell)-s_1(\ell)|).\end{aligned}$$

Therefore, assumption $(\mathcal{S}_2)$ of Theorem 3 is satisfied, with $\varrho(\ell) = \ln(1+\ell)$. Moreover, for any $\ell\in\mathcal{E}$ and $s\in\Re$, we obtain $|y(\ell,s(\ell))| = 0.2\ell^3|\sin s(\ell)| \leq 0.2\ell^3|s(\ell)|$. We can see that the condition $(\mathcal{S}_3)$ of Theorem 3 holds, that is, $\sigma(s) = s$ and $\theta(\ell) = 0.2\ell^3$. The inequality appearing in $(\mathcal{S}_4)$ of Theorem 3 has the expression

$$\begin{aligned} r^* &\le \exp\left(\frac{\Gamma(w)}{\|\theta\|_{\mathcal{L}^1}\|A\|} - 1/3\right) - 1/3 \\ &= \exp\left(\frac{1.7725}{3.9366 \times 0.3704} - 1/3\right) - 1/3 \\ &= 2.0832 \end{aligned}$$

and

$$r^* \le \frac{\Gamma(w)}{\|\theta\|_{\mathcal{L}^1}\|A\|} = 1.2156.$$

Thus, assumption $(\mathcal{S}_4)$ of Theorem 3 is satisfied for all $0 < r^* \le 2.0832$.

Hence, all the assumptions of Theorem 3 are fulfilled, and the problem (18)–(19) has at least one solution.

4. Concluding Remarks

In conclusion, this study delves into the sophisticated domain of hybrid Caputo–Hadamard fractional sequential differential equations (HCHFSDEs) and establishes vital results regarding the existence and uniqueness via the application of Darbo's fixed-point theorem. By merging the advantages of Caputo and Hadamard fractional derivatives, the hybrid approach offers a more encompassing framework for modeling sequential processes across diverse scientific and engineering domains. Through the utilization of Darbo's theorem, we have successfully addressed the fundamental question of the existence and uniqueness of solutions of such equations, thereby enhancing our comprehension of HCHFSDE dynamics. Moving forward, these findings are poised to catalyze further developments in modeling and analyzing dynamics systems, fostering innovation and progress in interdisciplinary research endeavors.

Author Contributions: Conceptualization, S.M. and M.Y.; methodology, A.H. and R.G.; software, M.Y. and S.M.; validation, H.A.N. and M.Y.; formal analysis, M.Y. and A.H.; investigation, R.G., M.Y. and S.M.; resources, R.G.; writing—original draft preparation, M.Y. and S.M.; writing—review and editing, A.H., R.G. and H.A.N.; supervision, M.Y.; project administration, A.H., M.Y. and S.M.; funding acquisition, R.G. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through project number PSAU/2024/01/78917.

Data Availability Statement: All data has been included in the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Meral, F.; Royston, T.; Magin, R. Fractional calculus in viscoelasticity: An experimental study. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 939–945. [CrossRef]
2. Oldham, K. Fractional differential equations in electrochemistry. *Adv. Eng. Softw.* **2010**, *41*, 9–12. [CrossRef]
3. Balachandran, K.; Matar, M.; Trujillo, J.J. Note on controllability of linear fractional dynamical systems. *J. Control Decis.* **2016**, *3*, 267–279. [CrossRef]
4. Hadamard, J. Essai sur l'etude des fonctions donnees par leur developpment de Taylor. *J. Math. Pures Appl.* **1892**, *8*, 101–186.
5. Ahmad, B.; Nieto, J.J. Sequential fractional diferential equations with three point boundary conditions. *Comput. Math. Appl.* **2012**, *64*, 3046–3052. [CrossRef]
6. Ahmad, B.; Nieto, J.J. Boundary value problems for a class of sequential in tegro diferential equations of fractional order. *J. Funct. Spaces Appl.* **2013**, *2013*, 149659. [CrossRef]
7. Ahmad, B.; Ntouyas, S.K. A higher-order nonlocal three-point boundary value problem of sequential fractional diferential equations. *Miskolc Math. Notes* **2014**, *15*, 265–278. [CrossRef]
8. Aqlan, M.H.; Alsaedi, A.; Ahmad, B.; Nieto, J.J. Existence theory for sequential fractional diferential equations with anti-periodic type boundary conditions. *Open Math.* **2016**, *14*, 723–735. [CrossRef]
9. Klimek, M. Sequential fractional diferential equations with Hadamard derivative. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 4689–4697. [CrossRef]

10. Mahmudov, N.I.; Awadalla, M.; Abuassba, K. Nonlinear sequential fractional dif ferential equations with nonlocal boundary conditions. *Adv. Difer. Equ.* **2017**, *2017*, 319. [CrossRef]
11. Ye, H.; Huang, R. On the nonlinear fractional diferential equations with Caputo sequential fractional derivative. *Adv. Math. Phys.* **2015**, *2015*, 174156. [CrossRef]
12. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Diferential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
13. Mohammadi, H.; Rezapour, S.; Etemad, S. On a hybrid fractional Caputo—Hadamard boundary value problem with hybrid Hadamard integral boundary value conditions. *Adv. Differ. Equ.* **2020**, *2020*, 455. [CrossRef]
14. Jarad, F.; Abdeljawad, T.; Baleanu, D. Caputo-type modification of the Hadamard fractional derivatives. *Adv. Differ. Equ.* **2012**, *2012*, 142. [CrossRef]
15. Baitiche, Z.; Guerbati, K.; Benchohra, M.; Henderson, J.; Avery, R. Boundary value problems for hybrid Caputo sequential fractional differential equations. *Mathematics* **2020**, *7*, 282. [CrossRef]
16. Benchohra, M.; Henderson, J.; Seba, D. Measure of noncompactness and fractional differential equations in Banach Spaces. *Commun. Appl. Anal.* **2008**, *12*, 419.
17. Darbo, G. Punti uniti in transformazioni a codominio noncompatto. *Rend. Semin. Mat. Univ. Padova* **1955**, *24*, 84–92.
18. Banas, J.; Olszowy, L. On a class of measures of noncompactness in Banach algebras and their application to nonlinear integral equations. *Z. Anal. Anwend.* **2009**, *28*, 475–498. [CrossRef]
19. Aghajani, A.; Banas, J.; Sebzali, N. Some generalizations of Darbo fixed point theorem and applications. *Bull. Belg. Math. Soc.-Simon Stevin* **2013**, *20*, 345–358. [CrossRef]
20. Shunan, L.; Bingyang, C. Beyond phonon hydrodynamics: Nonlocal phonon heat transport from spatial fractional-order Boltzmann transport equation. *AIP Adv.* **2020**, *10*, 6.
21. Marawan, A.-R. Applying fractional quantum mechanics to systems with electrical screening effects. *Chaos Solitons Fractals* **2021**, *150*, 111201.
22. Dubey, S.; Chakraverty, S. Hybrid techniques for approximate analytical solution of space- and time-fractional telegraph equations. *Pramana-J. Phys.* **2023**, *97*, 13. [CrossRef]
23. Aljoudi, S.; Ahmad, B.; Nieto, J.J.; Alsaedi, A. A coupled system of Hadamard type sequential fractional diferential equations with coupled strip conditions. *Chaos Solitons Fractals* **2016**, *91*, 39–46. [CrossRef]