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Abundant Closed-Form Soliton Solutions to the Fractional Stochastic Kraenkel–Manna–Merle System with Bifurcation, Chaotic, Sensitivity, and Modulation Instability Analysis

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Abstract: An essential mathematical structure that demonstrates the nonlinear short-wave movement across the ferromagnetic materials having zero conductivity in an exterior region is known as the fractional stochastic Kraenkel–Manna–Merle system. In this article, we extract abundant wave structure closed-form soliton solutions to the fractional stochastic Kraenkel–Manna–Merle system with some important analyses, such as bifurcation analysis, chaotic behaviors, sensitivity, and modulation instability. This fractional system renders a substantial impact on signal transmission, information systems, control theory, condensed matter physics, dynamics of chemical reactions, optical fiber communication, electromagnetism, image analysis, species coexistence, speech recognition, financial market behavior, etc. The Sardar sub-equation approach was implemented to generate several genuine innovative closed-form soliton solutions. Additionally, phase portraiture of bifurcation analysis, chaotic behaviors, sensitivity, and modulation instability were employed to monitor the qualitative characteristics of the dynamical system. A certain number of the accumulated outcomes were graphed, including singular shape, kink-shaped, soliton-shaped, and dark kink-shaped soliton in terms of 3D and contour plots to better understand the physical mechanisms of fractional system. The results show that the proposed methodology with analysis in comparison with the other methods is very structured, simple, and extremely successful in analyzing the behavior of nonlinear evolution equations in the field of fractional PDEs. Assessments from this study can be utilized to provide theoretical advice for improving the fidelity and efficiency of soliton dissemination.

Keywords: fractional derivative; bifurcation analysis; chaotic behaviors; sensitivity; modulation instability; Sardar sub-equation approach; fractional stochastic Kraenkel–Manna–Merle system; closed-form soliton solutions



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1. Introduction

Most of the procedures on the planet have many nonlinear phenomena and oscillating parameters. Nonlinear partial differential equations (NLPDEs) have been utilized to define real-world natural problems. Fractional calculus is a branch of mathematical analysis that generalizes the concepts of differentiation and integration to non-integer orders. The roots of fractional calculus can be traced back to the work of mathematicians like Leibniz, Euler, and Laplace. However, the modern development of fractional calculus gained momentum in the 19th century with the works of Liouville, Riemann, and others. There are various definitions of fractional derivatives and integrals, such as Riemann–Liouville, Grünwald–Letnikov, Caputo, conformable fractional derivative, and

others, each with its own properties and applications [1–3]. Fractional calculus has found applications in numerous fields, including physics, engineering, signal processing, control theory, signal transmission, chemical kinematics, finance, biology, viscoelastic materials and complex fluids, anomalous diffusion phenomena, electromagnetic, nano-technology, flat wave propagation, caustic wave in crystals, digital communication, optical fibers, and more. A subclass of fractional partial differential equations known as stochastic fractional partial differential equations (SFPDEs) is a strong mathematical foundation with a wide range of potential uses in scientific and technical disciplines. The outcomes of these SFPDEs provide additional understanding of the physical challenge by illuminating the evolution features of real-life complications. A new avenue for research into systems that seem wave-like in nature is opened up by the notion of soliton. For deeper consideration of nonlinear occurrence and realistic challenges, it is essential to discover closed-form soliton solutions of SFPDEs. According to the quick advancements in nonlinear sciences, a variety of simple and efficient approaches have been developed to obtain closed-form soliton solutions to NLPDEs, including the Hirota method [4,5], the Bernoulli sub-equation method [6,7], the F-expansion method [8], the (G'/G^2) -expansion method [9], the simple equation method [10], the modified auxiliary equation method [11,12], the two variable $(G'/G, 1/G)$ -expansion method [13–16], the Lie symmetric analysis [17], the polynomial complete discriminant system [18], the tanh-coth scheme [19], the Conservation laws method [20], the generalized exponential rational function approach [21,22], the binary bell polynomials method [23], the mapping method [24], the Shehu transform scheme [25], the sine–Gordon expansion [26], the Cole–Hopf transformation method [27,28], the Fan sub-equation technique [29], the unified method [30], the Khater method [31], the $r + m$ EDAM method [32], the spectral Tau method [33], the $\left(\frac{G'}{G'+G+A}\right)$ -expansion procedure [34–38], the sub-equation method [39], the collocation method [40], the finite element method [41], and the generalized G'/G -expansion method [42], and there may be enough others. In an earlier investigation, [13] pursued the two expansion algorithms to identify the rational, trigonometric, and hyperbolic solitary wave solutions for the fractional-order Hirota–Satsuma-coupled KdV system, which is highly beneficial for the field of analysis. Numerous features of this system have been studied in these investigations; in addition, multiple uses rely on it as well. A few years ago, Chen et al. [43] examined the periodic travelling waves of the modified KdV equation and rogue waves on the periodic background by implementing an algebraic method. Very recently, in [44], wave patterns were established for the discontinuous initial value problem to the generalized Gerdjikov–Ivanov equation by operating Whitham modulation. Lately, Iqbal et al. [38] applied the $\left(\frac{G'}{G'+G+A}\right)$ -expansion procedure most recently to analyze the fractional order $(4 + 1)$ -dimensional Fokas equation and the fractional order $(2 + 1)$ -dimensional breaking soliton equation to calculate the closed-form solutions that are better suited for nonlinear science. One of the most recently invented methods established in this area is the Sardar sub-equation procedure. The majority of fractional NLPDEs can be solved quickly and easily with this innovative approach, which is regarded as one of the most contemporary and comprehensive approaches in this field. A few years back, Rezazadeh et al. [45] produced some novel exact solutions to the $(3 + 1)$ -dimensional Wazwaz–Benjamin–Bona–Mahony equations by effectively applying the Sardar sub-equation procedure. Employing the proposed method, ref. [46] quite recently achieved the soliton solutions of the M-truncated paraxial wave equation. Scholars have proven the usefulness of the suggested method and expressed an attraction to understanding this nonlinear phenomenon scientifically. Very recently, Yasin et al. [47], discover the closed-form solutions of the $(3 + 1)$ -dimensional modified KdV–Zakharov–Kuznetsov equation with the aid of the suggested technique. The Kraenkel–Manna–Merle system (KMMS) refers to a specific nonlinear dynamical system proposed by Kraenkel, Manna, and Merle [48]. The fractional stochastic Kraenkel–Manna–Merle system (FSKMMS) is a mathematical model that combines elements of fractional calculus, stochastic processes, and nonlinear dynamics. The FSKMMS likely incorporates nonlinear dynamics to capture the intricate interactions between different components or variables within the system. It also

represents an interdisciplinary approach to understanding the dynamics of systems with memory effects, randomness, and nonlinear interactions. It is commonly recognized that the electromagnetic wave propagation problem in ordered magnetic materials, particularly in a ferromagnetic medium, becomes extremely nonlinear. The FSKMMS demonstrates exactly how nonlinear short waves spread in an external domain across ferromagnetic materials that have zero conductivity. Over the past few decades, there have been a lot of fascinating studies on ferromagnetic materials because of the enormous advancements in information technology to address the requirement for massive data as well as high-density storage. Gaining more insight into the unique properties of micro- and super-microstructures in nanoscale ferrous metals is indispensable. Typically, it is exceptionally challenging for FSKMMS to retrieve closed-form soliton solutions. Our recommended strategy can be used to solve most of the fractional NLPDEs, including FSKMMS, quickly and effectively without abandoning its generality. The FSKMMS has not yet been inspected using the aforementioned approach, to the best of our collective understanding. Considering the foregoing arguments, the intention of the present study was to implement the Sardar sub-equation strategy to determine the closed-form soliton solutions of the FSKMMS including the damping effect $\lambda = 0$. Utilizing the above-mentioned approach, certain innovative solutions to trigonometric, exponential, and rational functions have already been established, which are novel in the research. Andrew Kay invented nonlinear systems in 1952. Analysis of phase profiles of bifurcation, sensitivity, stability, and modulation instability is being used in nonlinear physical models more and more in recent decades [49–53]. In dynamical systems theory, bifurcation analysis, which includes phase portraiture, is a method for examining how a system's behavior varies with changes in parameters. This framework provides a useful mathematical basis for examining physical events in relation to classical mechanics. The relative significance of various inputs or factors in causing the variability of model results is revealed by sensitivity analysis. Using the Runge–Kutta method, we performed a thorough sensitivity analysis of the dynamical system. A basic idea in many domains of signal processing, control, and systems engineering is stability analysis. It is frequently associated with identifying the behavior of dynamic systems in the natural world, which can be mechanical, electrical, or chemical. Here, we analyzed the dynamical system's stability. This also addresses the investigated system's modulation instability analysis, which had not been reported before for that system. Scientists will find enormous utility in the solutions along with the several analysis provided, as they provide vital insights into pertinent physical phenomena. Meanwhile, we display the acquired figures to demonstrate the numerous diverse wave natures, which are highly beneficial for learning deeper about fractional NLPDEs.

This research's succeeding segments are organized as follows: Section 2 incorporates an extensive description and analysis of the preferred procedure alongside amenities. The inflections of the elected technique are explored in Section 3. Diverse analyses are examined in Section 4. Section 5 conveys a visual depiction together with a rationalization of the findings. The conclusions are laid out in Section 6.

2. Standard Process of the Methodology and Amenities

2.1. Brownian Motion

The Brownian motion $\{\mathbb{B}(t)\}_{t \geq 0}$ has a stochastic process and accomplishes the subsequent specification:

1. $\mathbb{B}(0) = 0$;
2. $\mathbb{B}(t)$ is continuous for $t \geq 0$;
3. $\mathbb{B}(t_2) - \mathbb{B}(t_1)$ is independent for $t_2 > t_1$;
4. A normal distribution is exhibited by $\mathbb{B}(t_2) - \mathbb{B}(t_1)$ with mean 0 as well as variance $t_2 - t_1$.

2.2. Fractional Derivative and Its Feature

A fractional derivative is a generalization of the concept of a derivative to non-integer orders. While the traditional derivative represents the rate of change of a function with

respect to a variable, fractional derivatives extend this notion to allow for non-integer orders of differentiation. Researchers have been exploring multiple versions of fractional derivatives over the years. Contributions by Riesz, Marchaud, Kober, Riemann–Liouville, Erdelyi, Hadamard, Grunwald–Letnikov, and Caputo have become the most recognized. A new fractional derivative categorized as the conformable derivative was just recently discovered by Mohammed et al. [24]. The concept of the conformable derivative seems natural and covers most of the requirements of the classical integral derivative, such as the vanishing derivatives for constant functions, the mean value theorem, the Rolle’s theorem, the chain rule, the power rule, linearity, and the quotient and product rules. It has advantages for modeling a wide range of real-world phenomena, as differential equations with conformable derivatives are easier to solve numerically than those with Riemann–Liouville or Caputo fractional derivatives.

Now, for $\delta \in (0, 1]$, the conformable derivative of $v : \mathbb{R}^+ \rightarrow \mathbb{R}$ is expressed as follows:

$$\mathcal{D}_x^\delta v(x) = \lim_{\epsilon \rightarrow 0} \frac{v(x + \epsilon x^{1-\delta}) - v(x)}{\epsilon}. \quad (1)$$

If we take into consideration that $v, w : \mathbb{R}^+ \rightarrow \mathbb{R}$ are differentiable and δ -differentiable functions, then the conformable derivative preserves each of the following features:

1. $\mathcal{D}_x^\delta \{\alpha_1 v(x) + \alpha_2 w(x)\} = \alpha_1 \mathcal{D}_x^\delta v(x) + \alpha_2 \mathcal{D}_x^\delta w(x);$
2. $\mathcal{D}_x^\delta(\alpha_1) = 0;$
3. $\mathcal{D}_x^\delta(v(w(x))) = x^{1-\delta} w'(x) v(w(x));$
4. $\mathcal{D}_x^\delta(x^n) = nx^{n-\delta};$
5. $\mathcal{D}_x^\delta v(x) = x^{1-\delta} \frac{dv}{dx}.$

Here, α_1 and α_2 stand for constants.

2.3. Methodology

This segment includes a description of the suggested method’s step-by-step workflow, which was established to produce innovative traveling wave solutions for noteworthy NLPDEs in fractional form. Say an address of a nonspecific fractional NLPDEs involving self-governing variables x with t is accomplished:

$$S(\mathcal{D}_x^\delta v, v \mathcal{D}_x^\delta v, \mathcal{D}_t^\delta v, v \mathcal{D}_t^\delta v, \mathcal{D}_x^\delta \mathcal{D}_x^\delta v, \mathcal{D}_t^\delta \mathcal{D}_t^\delta v, v, \dots) = 0 \quad (2)$$

wherever $v = v(x, t)$ is an unexplained function, a polynomial in v and its fractional derivative are represented by S . The succeeding phases in the foregoing advance:

Phase 1: Think about a fresh variable ψ that is the combination of all the self-regulating variables x as well as t :

$$v(x, t) = v(\psi), \quad \psi = \frac{\psi_1 x^\delta}{\delta} + \psi_2 t, \quad (3)$$

in which ψ_1, ψ_2 stands for nonzero constants. Operating Equation (3), we may make Equation (2) into an ordinary differential equation (ODE):

$$S_1(v, vv', v'', vv'', v''', \dots) = 0 \quad (4)$$

At this point, S_1 symbolizes a polynomial of v together with its several ordinary derivatives with the aid of the self-regulating variable ψ .

Phase 2: The arrangement shown below is able to build the solution of Equation (4):

$$v(\psi) = \sum_{l=0}^R b_l Y^l(\psi) \quad (5)$$

The homogeneous balancing principle may have been used for determining R . However, the coefficients b_l ($l = 0, 1, 2, \dots, R$) are easily calculated by solving a set of algebraic equations resulting from suggested approach. Moreover, $Y'(\psi)$ maintains the subsequent ODE:

$$Y'(\psi) = \sqrt{\eta + \zeta Y^2(\psi) + Y^4(\psi)}, \quad (6)$$

where η, ζ remain real constants. After resolving Equation (6), we discover the following subsequent possible outcomes:

Scenario 1: When $\zeta > 0$ and $\eta = 0$,

$$\begin{aligned} Y_1^\pm(\psi) &= \pm \sqrt{-mn\zeta} \operatorname{sech}_{mn}(\sqrt{\zeta}\psi) \\ Y_2^\pm(\psi) &= \pm \sqrt{mn\zeta} \operatorname{csch}_{mn}(\sqrt{\zeta}\psi); \end{aligned}$$

herein, $\operatorname{sech}_{mn}(\psi) = \frac{2}{me^\psi + ne^{-\psi}}, \operatorname{csch}_{mn}(\psi) = \frac{2}{me^\psi - ne^{-\psi}}$.

Scenario 2: When $\zeta < 0$ and $\eta = 0$,

$$\begin{aligned} Y_3^\pm(\psi) &= \pm \sqrt{-mn\zeta} \sec_{mn}(\sqrt{-\zeta}\psi) \\ Y_4^\pm(\psi) &= \pm \sqrt{-mn\zeta} \csc_{mn}(\sqrt{-\zeta}\psi); \end{aligned}$$

here, $\sec_{mn}(\psi) = \frac{2}{me^{i\psi} + ne^{-i\psi}}, \csc_{mn}(\psi) = \frac{2}{me^{i\psi} - ne^{-i\psi}}$.

Scenario 3: When $\zeta < 0$ and $\eta = \frac{\zeta^2}{4}$,

$$\begin{aligned} Y_5^\pm(\psi) &= \pm \sqrt{\frac{-\zeta}{2}} \tanh_{mn}\left(\sqrt{\frac{-\zeta}{2}}\psi\right) \\ Y_6^\pm(\psi) &= \pm \sqrt{\frac{-\zeta}{2}} \coth_{mn}\left(\sqrt{\frac{-\zeta}{2}}\psi\right) \\ Y_7^\pm(\psi) &= \pm \sqrt{\frac{-\zeta}{2}} (\tanh_{mn}(\sqrt{-2\zeta}\psi) \pm i\sqrt{mn} \operatorname{sech}_{mn}(\sqrt{-2\zeta}\psi)) \\ Y_8^\pm(\psi) &= \pm \sqrt{\frac{-\zeta}{2}} (\coth_{mn}(\sqrt{-2\zeta}\psi) \pm \sqrt{mn} \operatorname{csch}_{mn}(\sqrt{-2\zeta}\psi)) \\ Y_9^\pm(\psi) &= \pm \sqrt{\frac{-\zeta}{8}} \left(\tanh_{mn}\left(\sqrt{\frac{-\zeta}{8}}\psi\right) + \coth_{mn}\left(\sqrt{\frac{-\zeta}{8}}\psi\right) \right), \end{aligned}$$

in which $\tanh_{mn}(\psi) = \frac{me^\psi - ne^{-\psi}}{me^\psi + ne^{-\psi}}, \coth_{mn}(\psi) = \frac{me^\psi + ne^{-\psi}}{me^\psi - ne^{-\psi}}$.

Scenario 4: When $\zeta > 0$ and $\eta = \frac{\zeta^2}{4}$,

$$\begin{aligned} Y_{10}^\pm(\psi) &= \pm \sqrt{\frac{\zeta}{2}} \tan_{mn}\left(\sqrt{\frac{\zeta}{2}}\psi\right) \\ Y_{11}^\pm(\psi) &= \pm \sqrt{\frac{\zeta}{2}} \cot_{mn}\left(\sqrt{\frac{\zeta}{2}}\psi\right) \\ Y_{12}^\pm(\psi) &= \pm \sqrt{\frac{\zeta}{2}} (\tan_{mn}(\sqrt{2\zeta}\psi) \pm \sqrt{mn} \sec_{mn}(\sqrt{2\zeta}\psi)) \\ Y_{13}^\pm(\psi) &= \pm \sqrt{\frac{\zeta}{2}} (\cot_{mn}(\sqrt{2\zeta}\psi) \pm \sqrt{mn} \csc_{mn}(\sqrt{2\zeta}\psi)) \\ Y_{14}^\pm(\psi) &= \pm \sqrt{\frac{\zeta}{8}} \left(\tan_{mn}\left(\sqrt{\frac{\zeta}{8}}\psi\right) + \cot_{mn}\left(\sqrt{\frac{\zeta}{8}}\psi\right) \right) \end{aligned}$$

in which $\tan_{mn}(\psi) = -i \frac{me^{i\psi} - ne^{-i\psi}}{me^{i\psi} + ne^{-i\psi}}, \cot_{mn}(\psi) = i \frac{me^{i\psi} + ne^{-i\psi}}{me^{i\psi} - ne^{-i\psi}}$.

Phase 3: The polynomial of $Y^l(\psi)$ is now easily obtained by applying Equation (5) in Equation (4) together with Equation (6), and the algebraic equations for b_l ($l = 0, 1, 2, \dots, R$), ψ_1, ψ_2, ψ , ζ , η might be generated by adjusting the coefficient to zero. Upon completion all of these algebraic calculations, we quickly discover the necessary solutions to the provided fractional NLPDEs by exchanging the values of b_l ($l = 0, 1, 2, \dots, R$), ψ_1, ψ_2, ψ , ζ , η .

3. Application of the Method

This component focuses on how the Sardar sub-equation approach has been implemented to generate the newer, broadly applicable closed-form traveling wave solutions for fractional NLPDEs. The fractional stochastic Kraenkel–Manna–Merle system refers to an exclusive nonlinear dynamical system suggested by Kraenkel, Manna, and Merle. Soon after, this structure underwent extensive investigation in relation to information systems, signal processing, control systems, condensed matter physics, climate system modeling, optical fiber communication, chaotic systems, electromagnetism, signal processing, fluid dynamics, porous media, coastal engineering, epidemiology, microelectronics, biological systems, financial markets, chemical reactions, and ecological systems. It is an adaptable tool for modeling and analyzing a wide range of systems in science, engineering, finance, and other fields because of its multifaceted character and capacity to accurately represents challenging phenomena. So, the fractional stochastic Kraenkel–Manna–Merle system is represented here [24,32]:

$$\begin{aligned} \mathcal{D}_x^\delta v_t - v \mathcal{D}_x^\delta f + \lambda \mathcal{D}_x^\delta f &= \mu \mathcal{D}_x^\delta v \mathbb{B}_t, \\ \mathcal{D}_x^\delta f_t - v \mathcal{D}_x^\delta v &= \mu \mathcal{D}_x^\delta f \mathbb{B}_t \end{aligned} \quad (7)$$

where $v = v(x, t)$ endures the magnetization, and the external magnetic fields symbolized by $f = f(x, t)$ are associated to ferrite; $\mathcal{D}_x^\delta(\cdot)$ characterizes the conformal derivative operator $\delta \in (0, 1]$; λ corresponds to the damping coefficient, the noise intensity denoted by μ ; and $\mathbb{B}$ symbolizes Brownian motion. After exploiting the value for $\mu = 0$ together with $\delta = 1$ in the aforementioned equation, we discover the following equation:

$$\begin{aligned} v_{xt} - v f_x + \lambda f_x &= 0 \\ f_{xt} - v v_x &= 0 \end{aligned} \quad (8)$$

The above description incorporates short-wave nonlinear propagation in zero conductivity saturated in ferromagnetic material. Equation (8) exhibits Lax pairings and becomes integrable when damping is abolished ($\lambda = 0$).

To determine the wave equation of the fractional stochastic Kraenkel–Manna–Merle system, we assume the damping effect $\lambda = 0$ along with the provided following wave transformation:

$$v(x, t) = v(\psi) e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)}, \quad f(x, t) = f(\psi) e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)}, \quad \psi = \frac{\psi_1 x^\delta}{\delta} + \psi_2 t, \quad (9)$$

where $v(\psi)$, $f(\psi)$ are real functions, and ψ_1 and ψ_2 are nonzero constants. The following is noted:

$$\begin{aligned} \mathcal{D}_x^\delta v &= \psi_1 v' e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)}, \quad \mathcal{D}_x^\delta v_t = [\psi_1 \psi_2 v'' + \mu \psi_1 v' \mathbb{B}_t] e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)} \\ \mathcal{D}_x^\delta f &= \psi_1 f' e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)}, \quad \mathcal{D}_x^\delta f_t = [\psi_1 \psi_2 f'' + \mu \psi_1 f' \mathbb{B}_t] e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)} \end{aligned} \quad (10)$$

where the Ito correction term is represented by $\frac{1}{2} \mu^2 t$. Applying Equation (9) in Equation (7) together with Equation (10), we determine the following:

$$\begin{aligned} \psi_1 \psi_2 v'' - \psi_1 v f' e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)} &= 0 \\ \psi_1 \psi_2 f'' - \psi_1 v v' e^{(\mu \mathbb{B}(t) - \frac{1}{2} \mu^2 t)} &= 0 \end{aligned} \quad (11)$$

Evaluating expectations for each side of the above Equation (11), we develop the following:

$$\begin{aligned} \psi_2 v'' - v f' e^{-\frac{1}{2} \mu^2 t} \mathfrak{E} e^{[\mu \mathbb{B}(t)]} &= 0 \\ \psi_2 f'' - v v' e^{-\frac{1}{2} \mu^2 t} \mathfrak{E} e^{[\mu \mathbb{B}(t)]} &= 0 \end{aligned} \quad (12)$$

Here, $\mathbb{B}(t)$ is situated for normal standard distribution along side $\mathfrak{E}\left(e^{[\mu\mathbb{B}(t)]}\right) = e^{\frac{1}{2}\mu^2 t}$, with the aid of these concepts Equation (12) altered as given below:

$$\begin{aligned}\psi_2 v'' - v f' &= 0 \\ \psi_2 f'' - v v' &= 0.\end{aligned}\quad (13)$$

If we integrate once to the second member of the Equation (13), we attain the following:

$$f' = \frac{1}{2\psi_2} v^2 + \frac{a_0}{\psi_2}, \quad (14)$$

where a_0 means integrating constant. Utilizing Equation (14) to the first member of Equation (13), we obtain the following:

$$v'' - \frac{1}{2\psi_2^2} v^3 - \frac{a_0}{\psi_2^2} v = 0. \quad (15)$$

Here, the homogeneous balance process supplies us $R = 1$. So, the Equation (5) can be written as given below:

$$v(\psi) = \sum_{l=0}^1 b_l Y^l(\psi) = b_0 + b_1 Y(\psi). \quad (16)$$

After a successful calculation, we obtain the value of constants a_0 , b_0 , b_1 :

$$b_0 = 0, b_1 = \pm 2\psi_2, a_0 = \zeta\psi_2^2.$$

In concern to the constants directly above, the soliton solutions are as follows:

Scenario-1: When $\zeta > 0$ and $\eta = 0$,

$$v_1^\pm(\psi) = \pm 2\psi_2 \sqrt{-mn\zeta} \operatorname{sech}_{mn}(\sqrt{\zeta}\psi). \quad (17)$$

$$f_1^\pm(\psi) = -2mn\sqrt{\zeta}\psi_2 \tanh_{mn}(\sqrt{\zeta}\psi) + \zeta\psi\psi_2. \quad (18)$$

$$v_2^\pm(\psi) = \pm 2\psi_2 \sqrt{mn\zeta} \operatorname{csch}_{mn}(\sqrt{\zeta}\psi). \quad (19)$$

$$f_2^\pm(\psi) = -2mn\sqrt{\zeta}\psi_2 \coth_{mn}(\sqrt{\zeta}\psi) + \zeta\psi\psi_2. \quad (20)$$

At the moment of employing the modification in Equation (9) to Equation (17) on or after (20), linking v and f operates the solutions of Equation (7).

$$v_1^\pm(x, t) = \pm 2e^{(\mu\mathbb{B}(t) - \frac{1}{2}\mu^2 t)} \psi_2 \sqrt{-mn\zeta} \operatorname{sech}_{mn}\left(\sqrt{\zeta}\left(\frac{\psi_1 x^\delta}{\delta} + \psi_2 t\right)\right), \quad (21)$$

$$f_1^\pm(x, t) = e^{(\mu\mathbb{B}(t) - \frac{1}{2}\mu^2 t)} \left\{ -2mn\sqrt{\zeta}\psi_2 \tanh_{mn}\left(\sqrt{\zeta}\left(\frac{\psi_1 x^\delta}{\delta} + \psi_2 t\right)\right) + \zeta\psi_2\left(\frac{\psi_1 x^\delta}{\delta} + \psi_2 t\right) \right\}, \quad (22)$$

$$v_2^\pm(x, t) = \pm 2e^{(\mu\mathbb{B}(t) - \frac{1}{2}\mu^2 t)} \psi_2 \sqrt{mn\zeta} \operatorname{csch}_{mn}\left(\sqrt{\zeta}\left(\frac{\psi_1 x^\delta}{\delta} + \psi_2 t\right)\right), \quad (23)$$

$$f_2^\pm(x, t) = e^{(\mu\mathbb{B}(t) - \frac{1}{2}\mu^2 t)} \left\{ -2mn\sqrt{\zeta}\psi_2 \coth_{mn}\left(\sqrt{\zeta}\left(\frac{\psi_1 x^\delta}{\delta} + \psi_2 t\right)\right) + \zeta\psi_2\left(\frac{\psi_1 x^\delta}{\delta} + \psi_2 t\right) \right\}, \quad (24)$$

where μ represents the noise intensity, and ψ_1 and ψ_2 are nonzero constants.

Scenario-2: When $\zeta < 0$ and $\eta = 0$,

$$v_3^\pm(\psi) = \pm 2\psi_2 \sqrt{-mn\zeta} \operatorname{sec}_{mn}(\sqrt{-\zeta}\psi). \quad (25)$$

$$f_3^\pm(\psi) = \frac{-2mn\zeta\psi_2}{\sqrt{-\zeta}} \tan_{mn}\left(\sqrt{-\zeta}\psi\right) + \zeta\psi\psi_2. \quad (26)$$

$$v_4^\pm(\psi) = \pm 2\psi_2 \sqrt{-mn\zeta} \csc_{mn}\left(\sqrt{-\zeta}\psi\right). \quad (27)$$

$$f_4^\pm(\psi) = \frac{2mn\zeta\psi_2}{\sqrt{-\zeta}} \cot_{mn}\left(\sqrt{-\zeta}\psi\right) + \zeta\psi\psi_2. \quad (28)$$

In a similar fashion, using the change in Equation (9) to Equation (25) on or after Equation (28), relating v and f , we can carry out the solutions of Equation (7).

Scenario 3: When $\zeta < 0$ and $\eta = \frac{\zeta^2}{4}$,

$$v_5^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{-\zeta}{2}} \tanh_{mn}\left(\sqrt{\frac{-\zeta}{2}}\psi\right). \quad (29)$$

$$f_5^\pm(\psi) = \frac{-\sqrt{2}\zeta\psi_2 \left\{ \sqrt{\frac{-\zeta}{2}}\psi - \tanh_{mn}\left(\sqrt{\frac{-\zeta}{2}}\psi\right) \right\}}{\sqrt{-\zeta}} + \zeta\psi\psi_2. \quad (30)$$

$$v_6^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{-\zeta}{2}} \coth_{mn}\left(\sqrt{\frac{-\zeta}{2}}\psi\right). \quad (31)$$

$$f_6^\pm(\psi) = \frac{-\sqrt{2}\zeta\psi_2 \left\{ \sqrt{\frac{-\zeta}{2}}\psi - \coth_{mn}\left(\sqrt{\frac{-\zeta}{2}}\psi\right) \right\}}{\sqrt{-\zeta}} + \zeta\psi\psi_2. \quad (32)$$

$$v_7^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{-\zeta}{2}} \left(\tanh_{mn}\left(\sqrt{-2\zeta}\psi\right) \pm i\sqrt{mn} \operatorname{sech}_{mn}\left(\sqrt{-2\zeta}\psi\right) \right) \quad (33)$$

$$v_8^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{-\zeta}{2}} \left(\coth_{mn}\left(\sqrt{-2\zeta}\psi\right) \pm \sqrt{mn} \operatorname{csech}_{mn}\left(\sqrt{-2\zeta}\psi\right) \right). \quad (34)$$

$$v_9^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{-\zeta}{8}} \left(\tanh_{mn}\left(\sqrt{\frac{-\zeta}{8}}\psi\right) + \coth_{mn}\left(\sqrt{\frac{-\zeta}{8}}\psi\right) \right). \quad (35)$$

In a similar fashion, we can determine f from the Equation (33) to Equation (35) along with Equation (14), which was omitted for avoiding complication of calculation. Now, applying the conversion in Equation (9) to all the v and f , we can drive the solutions of Equation (7).

Scenario 4: When $\zeta > 0$ and $\eta = \frac{\zeta^2}{4}$,

$$v_{10}^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{\zeta}{2}} \tan_{mn}\left(\sqrt{\frac{\zeta}{2}}\psi\right). \quad (36)$$

$$f_{10}^\pm(\psi) = \psi_2 \sqrt{2\zeta} \left\{ \tan_{mn}\left(\sqrt{\frac{\zeta}{2}}\psi\right) - \sqrt{\frac{\zeta}{2}}\psi \right\} + \zeta\psi\psi_2. \quad (37)$$

$$v_{11}^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{\zeta}{2}} \cot_{mn}\left(\sqrt{\frac{\zeta}{2}}\psi\right). \quad (38)$$

$$f_{11}^\pm(\psi) = -\psi_2 \sqrt{2\zeta} \left\{ \cot_{mn}\left(\sqrt{\frac{\zeta}{2}}\psi\right) + \sqrt{\frac{\zeta}{2}}\psi \right\} + \zeta\psi\psi_2. \quad (39)$$

$$v_{12}^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{\zeta}{2}} \left(\tan_{mn}\left(\sqrt{2\zeta}\psi\right) \pm \sqrt{mn} \sec_{mn}\left(\sqrt{2\zeta}\psi\right) \right). \quad (40)$$

$$v_{13}^\pm(\psi) = \pm 2\psi_2 \sqrt{\frac{\zeta}{2}} \left(\cot_{mn}\left(\sqrt{2\zeta}\psi\right) \pm \sqrt{mn} \operatorname{csec}_{mn}\left(\sqrt{2\zeta}\psi\right) \right). \quad (41)$$

$$v_{14}^{\pm}(\psi) = \pm 2\psi_2 \sqrt{\frac{\zeta}{8}} \left(\tan_{mn} \left(\sqrt{\frac{\zeta}{8}} \psi \right) + \cot_{mn} \left(\sqrt{\frac{\zeta}{8}} \psi \right) \right). \quad (42)$$

Likewise, f can be found from the Equation (40) through Equation (42) together with Equation (14), which was left out in order to simplify the computation. Here, we can use the modification in Equation (9) to all the v and f to construct the solutions of Equation (7).

4. Diverse Analyses Concerning to the Fractional Equation

Bifurcation analysis, chaotic behaviors, sensitivity, and modulation instability analysis for the Equation (15) are explored in detail in the following paragraphs.

4.1. Bifurcation Analysis

Bifurcation analysis, comprising phase portraits, is a technique used in dynamical systems theory to study how the behavior of a system changes as parameters are varied. In essence, it examines the points at which a system undergoes qualitative changes in its behavior. It has applications across various fields, including physics, biology, chemistry, economics, and engineering. It also helps understand phenomena such as phase transitions, population dynamics, chemical reactions, and electronic circuits. Firstly, we generate a dynamical system from Equation (15) by means of the Galilean transformation. This dynamical system offers an effective mathematical foundation for analyzing physical phenomena within the context of classical mechanics.

$$\begin{cases} \frac{dv(\psi)}{d\psi} = S \\ \frac{dS(\psi)}{d\psi} = M_1 v^3(\psi) + M_2 v(\psi) \end{cases} \quad (43)$$

where $M_1 = \frac{1}{2\psi_2^2}$, $M_2 = \frac{a_0}{\psi_2^2}$. Now, the Hamiltonian function for the Equation (43) might be represented by

$$\mathbb{H}(v, S) = \frac{S^2}{2} - \frac{M_1 v^4}{4} - \frac{M_2 v^2}{2} = \mathbb{h},$$

here, $\mathbb{h}$ is situated for Hamiltonian constant. We have the following dynamical system:

$$\begin{cases} S = 0 \\ M_1 v^3 + M_2 v = 0 \end{cases}$$

to calculate the equilibrium points of Equation (43). The equilibrium points of the above dynamical system are $(0, 0)$, $(i\sqrt{\frac{M_2}{M_1}}, 0)$, $(-i\sqrt{\frac{M_2}{M_1}}, 0)$. The Jacobian of the system in Equation (43) is as follows:

$$D(v, S) = \begin{vmatrix} 0 & 1 \\ 3M_1 v^2 + M_2 & 0 \end{vmatrix} = -3M_1 v^2 - M_2.$$

Planar dynamical systems theory reveals the following to us:

1. The equilibrium position (v, S) represents a saddle point, when $D(v, S) < 0$;
2. The equilibrium position (v, S) denotes a center point, when $D(v, S) > 0$;
3. The equilibrium position (v, S) signifies a cuspid point, when $D(v, S) = 0$.

The content that follows is a list of possible outcomes that can be obtained through modifying the relevant parameters.

Instance 1: $M_1 > 0$ and $M_2 > 0$.

Considering the particular values for the parameters $a_0 = 1$, $\psi_2 = 2$, we observed that the equilibrium position $(0, 0)$ represents a saddle point, which is exhibited in Figure 1a.

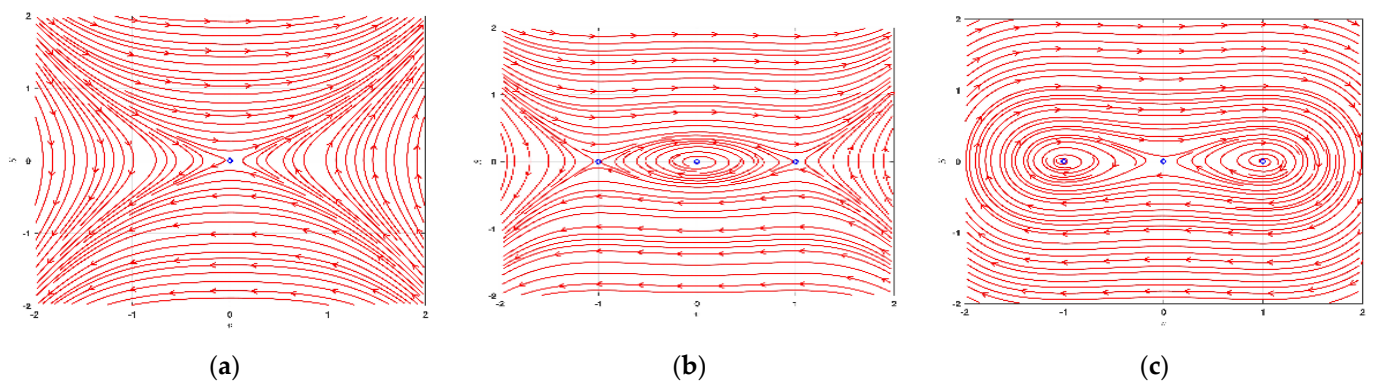


Figure 1. Bifurcation analysis of the proposed system with different state for M_1 and M_2 along with several values of the parameters: for (a) $a_0 = 1$, $\psi_2 = 2$, (b), $a_0 = -1$, $\psi_2 = 2$, and (c) $a_0 = -1$, $\psi_2 = -2$.

Instance 2: $M_1 > 0$ and $M_2 < 0$.

By selecting the fixed value of the parameters $a_0 = -1$, $\psi_2 = 2$, we detect that the equilibrium position $(0, 0)$ stands for a center point, and equilibrium position $(-1, 0)$ together with $(1, 0)$ stand for saddle points, which is displayed in Figure 1b.

Instance 3: $M_1 > 0$ and $M_2 < 0$.

After choosing the definite value of the parameters $a_0 = -1$, $\psi_2 = -2$, we notice that the equilibrium position $(0, 0)$ represents a saddle point, and equilibrium position $(-1, 0)$ together with $(1, 0)$ represent center points, which is shown in Figure 1c.

4.2. Chaotic Nature of the Dynamic System

In chaotic systems, small variations in input can lead to drastically different outcomes in which nonlinear dynamics clarifies ostensibly arbitrary behavior in these systems. To explore the chaotic nature, we augment a perturbed term in the planar dynamical systems that yields us the subsequent dynamical system:

$$\begin{cases} \frac{dv(t)}{dt} = S \\ \frac{dS(t)}{dt} = M_1 v^3(t) + M_2 v(t) + \Theta_1 \cos(\Theta_2 t) \end{cases} \quad (44)$$

where Θ_1 and Θ_2 stand for the amplitude and frequency of the dynamical system in Equation (44). The impact of the perturbed term $\Theta_1 \cos(\Theta_2 t)$ on the dynamic structure stated above is analyzed in the subsequent figures. In this instance, we identify the chaotic nature of the recommended dynamical system and illustrate the phase portraits of this system for several values of the parameters: in Figure 2a, $a_0 = -3$, $\psi_2 = 1$, $\Theta_1 = 5$, $\Theta_2 = \pi/2$; Figure 2b, $a_0 = -1$, $\psi_2 = -1$, $\Theta_1 = 5$, $\Theta_2 = \pi/2$; Figure 2c, $a_0 = 1$, $\psi_2 = -2$, $\Theta_1 = 5$, $\Theta_2 = \pi/2$; Figure 3a, $a_0 = 1$, $\psi_2 = -1$, $\Theta_1 = 2$, $\Theta_2 = \pi$; Figure 3b, $a_0 = -1$, $\psi_2 = -2$, $\Theta_1 = 5$, $\Theta_2 = \pi$; Figure 3c, $a_0 = -4$, $\psi_2 = 1$, $\Theta_1 = 1$, $\Theta_2 = \pi$; Figure 4a, $a_0 = 1$, $\psi_2 = -5$, $\Theta_1 = 1$, $\Theta_2 = 2\pi$; Figure 4b, $a_0 = -2$, $\psi_2 = 3$, $\Theta_1 = 5$, $\Theta_2 = 2\pi$; and Figure 4c, $a_0 = -1$, $\psi_2 = -2$, $\Theta_1 = 5$, $\Theta_2 = 2\pi$. Upon examining the provided figures, we observe highly complex and attractive behaviors. We detect the ringlet dynamics in Figure 2a,b and Figure 4a,c, besides those in Figures 2c and 4b; we also see the complex dynamics. In a similar way, one can observe the periodic nature in Figure 3a as well as Figure 3b together with Figure 3c, which show the surprising periodic nature. This improved conception of the system's sensitivity to fluctuations in parameters aid us in accomplishing the complex interactions among the frequency Θ_2 , the perturbation term, as well the dynamics of the system. The inference from this analysis allows us to launch a strong and progressive mathematical structure.

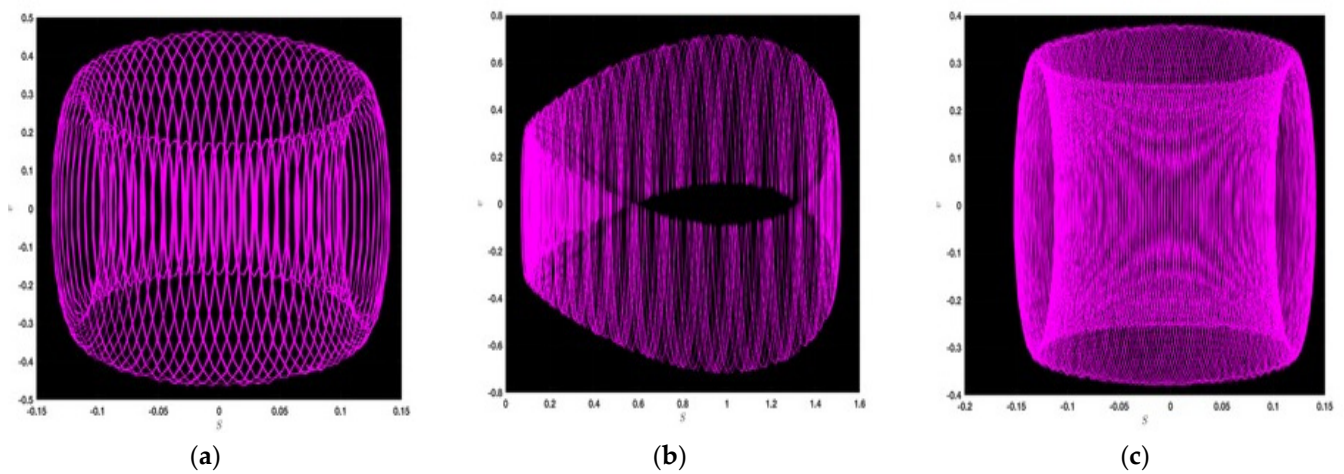


Figure 2. Chaotic nature of the dynamical system for the values (a) $a_0 = -3, \psi_2 = 1, \Theta_1 = 5, \Theta_2 = \pi/2$, (b) $a_0 = -1, \psi_2 = -1, \Theta_1 = 5, \Theta_2 = \pi/2$, and (c) $a_0 = 1, \psi_2 = -2, \Theta_1 = 5, \Theta_2 = \pi/2$.

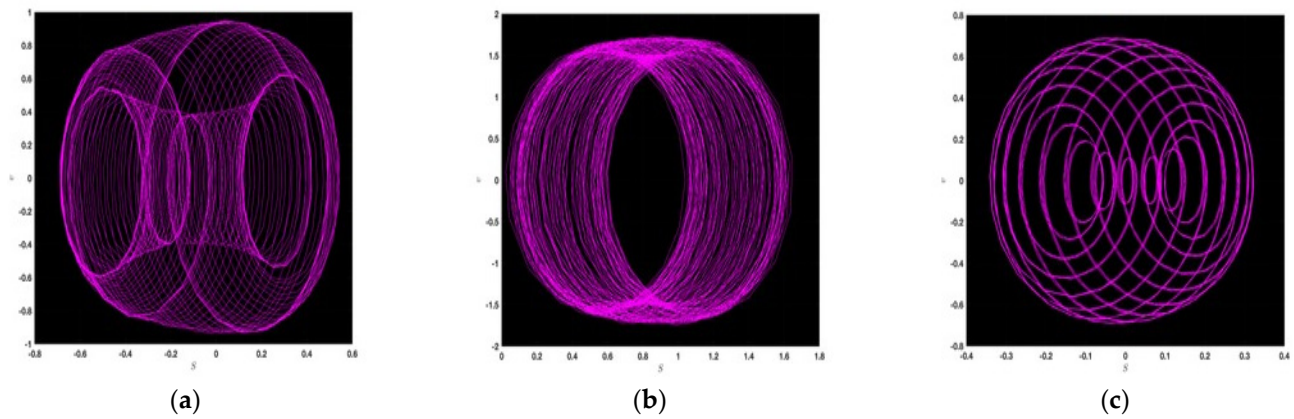


Figure 3. Chaotic nature of the dynamical system for arbitrary values (a) $a_0 = 1, \psi_2 = -1, \Theta_1 = 2, \Theta_2 = \pi$, (b) $a_0 = -1, \psi_2 = -2, \Theta_1 = 5, \Theta_2 = \pi$, and (c) $a_0 = -4, \psi_2 = 1, \Theta_1 = 1, \Theta_2 = \pi$.

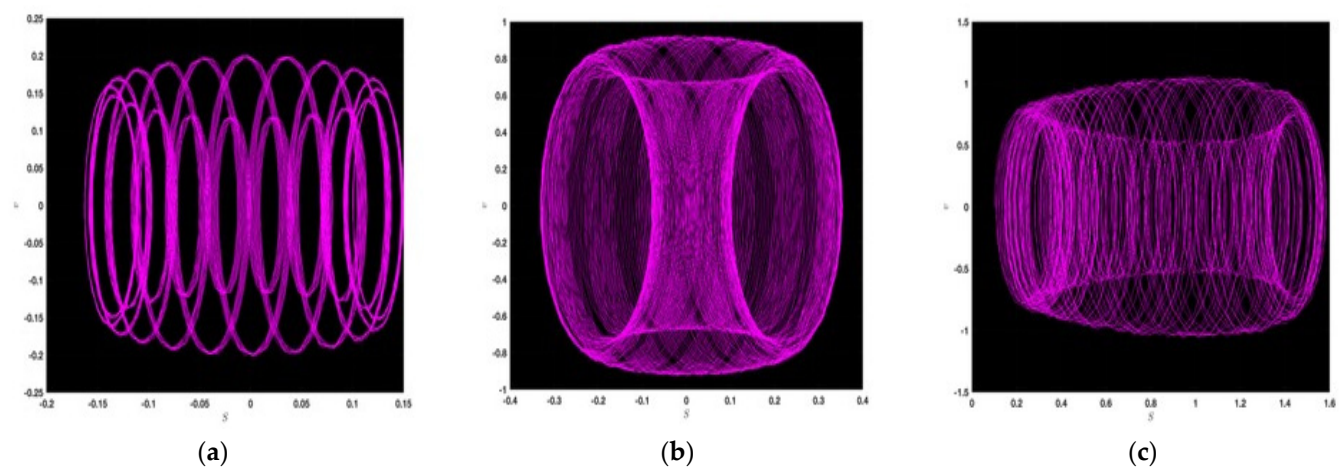


Figure 4. Chaotic nature of the dynamical system for the values, (a) $a_0 = 1, \psi_2 = -5, \Theta_1 = 1, \Theta_2 = 2\pi$, (b) $a_0 = -2, \psi_2 = 3, \Theta_1 = 5, \Theta_2 = 2\pi$, and (c) $a_0 = -1, \psi_2 = -2, \Theta_1 = 5, \Theta_2 = 2\pi$.

4.3. Sensitivity Analysis of the Dynamic System

Sensitivity analysis offers insights into the comparative importance of different inputs or parameters in driving the changeability of model aftereffects. It is a precious tool for

understanding the behavior of complex systems, advancing model accuracy, and making informed decisions in the face of ambiguity. Sensitivity analysis is extensively used in several fields, together with engineering, finance, environmental science, epidemiology, and policy analysis. To examine the sensitivity of the dynamical system described by Equation (43), we utilize the widely recognized and effective Runge–Kutta technique. Now, the Runge–Kutta technique is utilized to solve the following dynamical system to observe its sensitivity:

$$\begin{cases} \frac{dv(t)}{dt} = S \\ \frac{dS(t)}{dt} = M_1 v^3(t) + M_2 v(t) \end{cases} \quad (45)$$

Here, we pick the value of the parameters like $a_0 = -2$, $\psi_2 = -1$. The system's beginning conditions are provided by the following:

- (a) $v(0) = 0.1$ and $S = 0$; (b) $v(0) = 0$ and $S = 0.1$
 (c) $v(0) = 0.2$ and $S = 0$; (d) $v(0) = 0$ and $S = 0.2$

Figure 5 illustrates the outcomes that this competent plan manufactures. The dynamics of class v are demonstrated by blue curves, and dynamics of class S are displayed with orange curves. A close examination of the figures reveals that even minor modifications to the initial setting result in substantial modifications in the system's behavior.

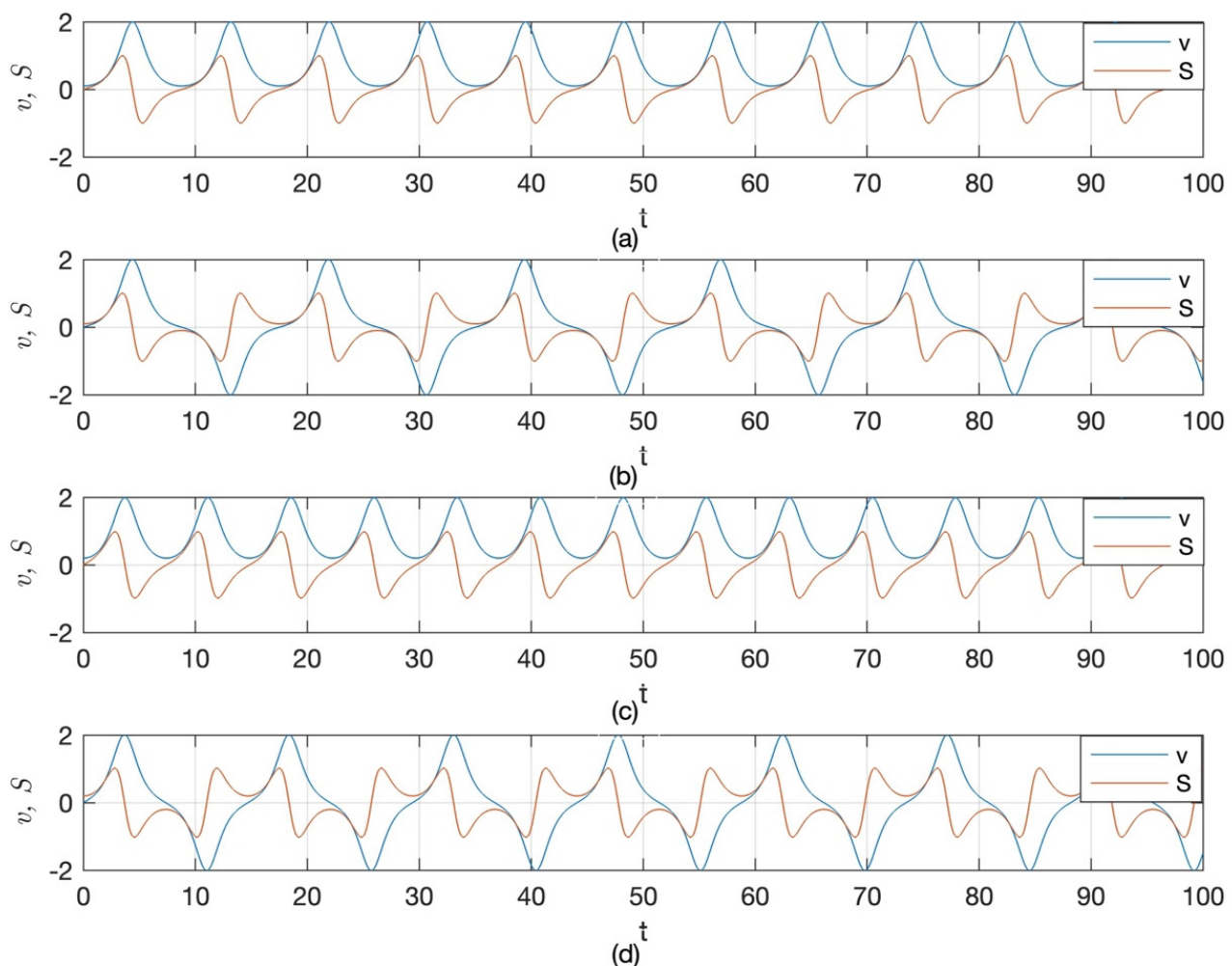


Figure 5. Sensitivity analysis of the proposed system with several initial conditions along with the parameters values $a_0 = -2$, $\psi_2 = -1$: (a) $v(0) = 0.1$ and $S = 0$, (b) $v(0) = 0$ and $S = 0.1$, (c) $v(0) = 0.2$ and $S = 0$, (d) $v(0) = 0$ and $S = 0.2$.

4.4. Modulation Instability

Modulation instability is an elementary phenomenon detected in nonlinear systems, remarkably in optics and fluid dynamics. It illustrates the unconstrained growth of perturbations or fluctuations in a system, running to the creation of patterns or structures. In optics, modulation instability arises when a beam of light interacts with a nonlinear medium, such as assured optical fibers or photonic devices. In fluid dynamics, modulation instability can demonstrate as the spontaneous generation of patterns or waves in fluid flows, such as the development of ripples on the surface of water or the occurrence of turbulence in a flow. Modulation instability has various applications, including the generation of optical frequency combs, which are used in telecommunications, spectroscopy, optical coherence tomography, nonlinear imaging, optical signal processing, wavelength conversion, and pulse shaping. It remains to be a lively area of research in modern physics and contemporary engineering.

In this part, we identify the traveling wave modulation instability of Equation (8). Consider the disturbed solution specified by the following:

$$\begin{aligned} v(x, t) &= \varrho \xi(x, t) + k_1 \\ f(x, t) &= \varpi \Omega(x, t) + k_2 \end{aligned} \quad (46)$$

in which the constants k_1 and k_2 aim for a steady flow attitude for Equation (8). Operating Equation (46) to Equation (8) and at the same time linearizing in ϱ and ϖ yields the following:

$$\begin{aligned} \varrho \xi_{xt} - (k_1 - \lambda) \varpi \Omega_x &= 0 \\ \varpi \Omega_{xt} - k_1 \varrho \xi_x &= 0 \end{aligned} \quad (47)$$

Differentiating the second equation of Equation (47), one can obtain $\varrho \xi_{xt} = \frac{\varpi}{k_1} \Omega_{xtt}$. Now, using this value in the first equation of Equation (47):

$$\varpi \Omega_{xtt} - k_1(k_1 - \lambda) \varpi \Omega_x = 0. \quad (48)$$

Assume the solution of the system in Equation (48):

$$\Omega = e^{i(\alpha_1 x + \beta_1 t)}, \quad (49)$$

where α_1 stands for the waveform quantity, and β_1 represents the frequency. Perform Equation (49) to Equation (48):

$$-\varpi \alpha_1 \beta_1^2 - k_1(k_1 - \lambda) \varpi \alpha_1 = 0. \quad (50)$$

Equation (50):

$$\beta_1^2 = -k_1(k_1 - \lambda). \quad (51)$$

From Equation (51), the following is obtained:

$$\beta_1 = -\sqrt{k_1(\lambda - k_1)}. \quad (52)$$

For the value of k_1 and λ , the right-hand side of Equation (52) gives us the negative value, which shows that any combination of the soliton solutions demonstrates a decaying pattern. Finally, Equation (52) confirms that any arrangement of the outcomes appears to decay. This reveals a stable dispersion of the model.

5. Graphical Explanation of Closed-Form Solutions

Graphs afford an adaptable framework for modeling complex relationships along with resolving a wide range of difficulties in modern science and beyond. They are exercised to represent complex information in a clear, concise, and easily understandable manner. Also, they play a decisive role in data analysis, decision making based on data-

driven insights, and communication across various domains, including business, science, education, and public policy. Numerous researchers have studied the FSKMMS with the aid of several techniques. However, the mechanical features of the FSKMMS in saturated ferromagnetic materials, a specific kind of the fractional NLPDEs, are presented here in this unique analysis by applying a novel procedure, the Sardar sub-equation approach. For the FSKMMS, we reveal the traveling wave along with soliton profile in 3D and contour forms with appropriate constant values. The mechanisms associated with the complex nonlinear physical phenomenon are competently described by the offered solutions. The most intriguing features of FSKMMS, which accounts for the many wonders of fractional nonlinear science, are better understood by researchers. This segment contains the visual depiction of the revealed closed-form soliton solutions to the stated previous equations. At first, we place the solution of Equation (21) in two formats, namely 3D with $x \in [0, 5]$, $t \in [0, 10]$ and contour with $x \in [0, 5]$, $t \in [0, 10]$; we obtain a singular shape soliton, which arise exposed in Figure 6 for the parameters $\psi_1 = 2$, $\psi_2 = 0.01$, $\delta = 0.5$, $m = -1$, $n = 1$, and $\zeta = 1$. Afterward, we depict the solution of Equation (22) as well in two different compositions: 3D using $x \in [-5, 5]$, $t \in [0, 5]$ and contour using $x \in [-5, 5]$, $t \in [0, 5]$, and we obtain the soliton shape disclosed in Figure 7 alongside the parameters $\psi_1 = 2$, $\psi_2 = 0.5$, $\delta = 1$, $m = -3$, $n = 2$, and $\zeta = 4$. The kink-shaped soliton solution presented in Equation (29) is given in Figure 8, in which the formats include 3D within $x, t \in [-10, 10]$ and contour within $x, t \in [-10, 10]$; we count the parameters $\psi_1 = 2$, $\psi_2 = 0.05$, $\delta = 1$, $m = -2$, $n = -1$, and $\zeta = -2$. Furthermore, Figure 9 describes the singular soliton solution of Equation (30), wherein we employ 3D with $x \in [0, 10]$, $t \in [-10, 10]$ and contour with $x \in [0, 10]$, $t \in [-10, 10]$, and the parameters bear $\psi_1 = 2$, $\psi_2 = 1$, $\delta = 0.7$, $m = -2$, $n = 1$, and $\zeta = -2$. Subsequently, we show the solution of Equation (31) as well in two different compositions, namely 3D with $x, t \in [-10, 10]$ and contour with $x, t \in [-10, 10]$; we thus obtain the kink-shaped soliton that is revealed in Figure 10 along with the parameters $\psi_1 = 2$, $\psi_2 = 2$, $\delta = 1$, $m = 2$, $n = -1$, and $\zeta = -2$. The dark kink-shaped soliton solution stated in Equation (32) is given in Figure 11, which uses 3D within $x \in [0, 10]$, $t \in [-10, 10]$ and contour within $x \in [0, 10]$, $t \in [-10, 10]$, incorporating the parameters $\psi_1 = -2$, $\psi_2 = -2$, $\delta = 0.7$, $m = -2$, $n = 1$, and $\zeta = -2$.

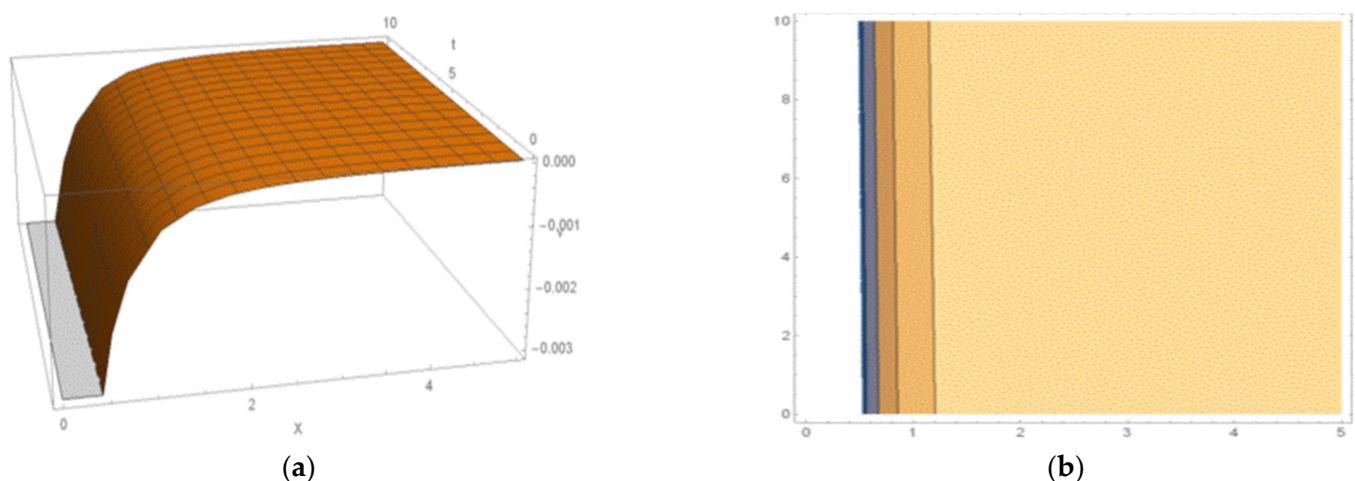


Figure 6. Singular shape soliton offered by Equation (21): (a) 3D structure and (b) the contour shape.

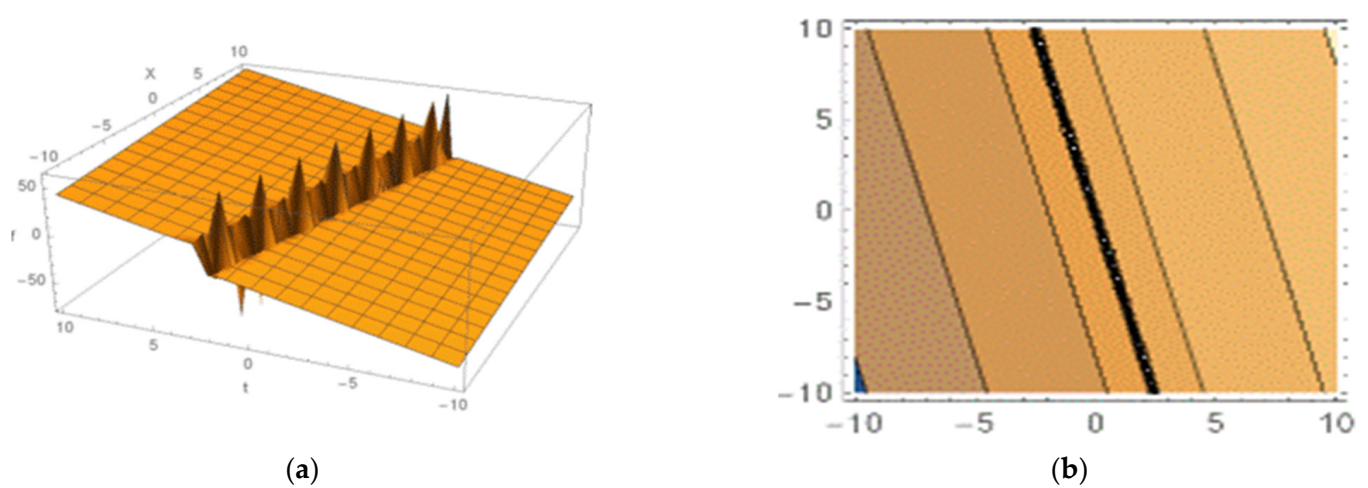


Figure 7. Soliton shape yielded by Equation (22): (a) 3D structure and (b) the contour shape.

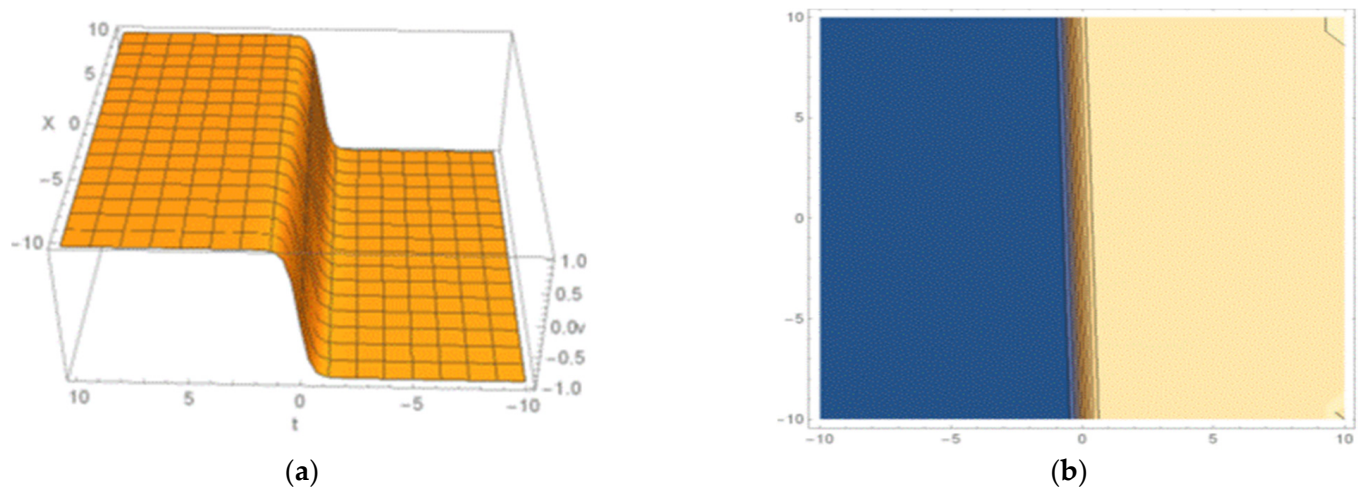


Figure 8. Kink-shaped soliton of Equation (29): (a) 3D structure and (b) the contour shape.

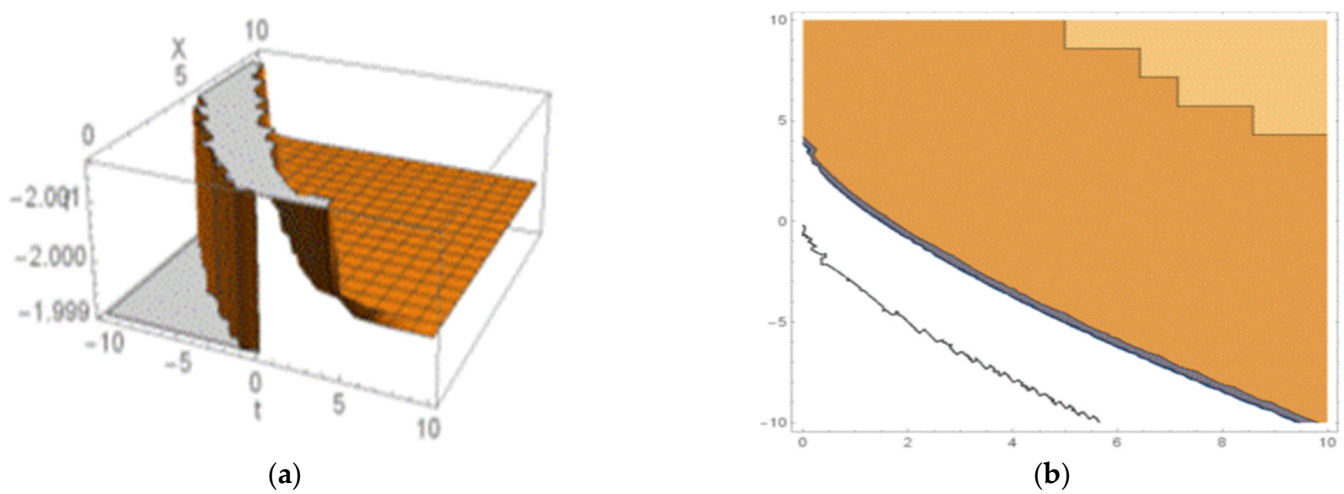


Figure 9. Singular soliton obtained from Equation (30): (a) 3D structure and (b) the contour shape.

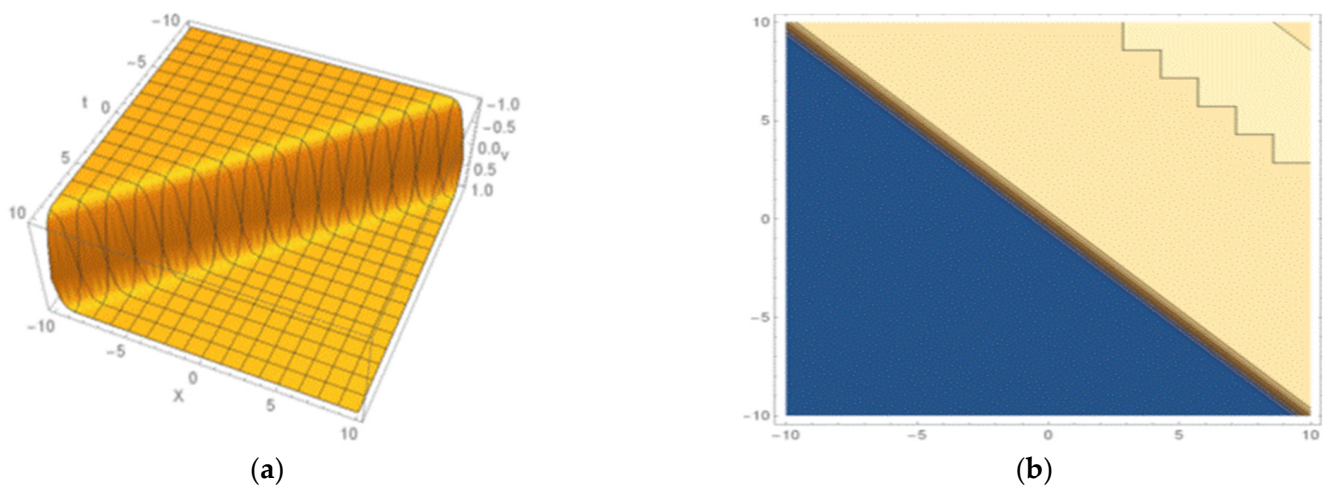


Figure 10. Kink-shaped soliton found by Equation (31): (a) 3D structure and (b) the contour shape.

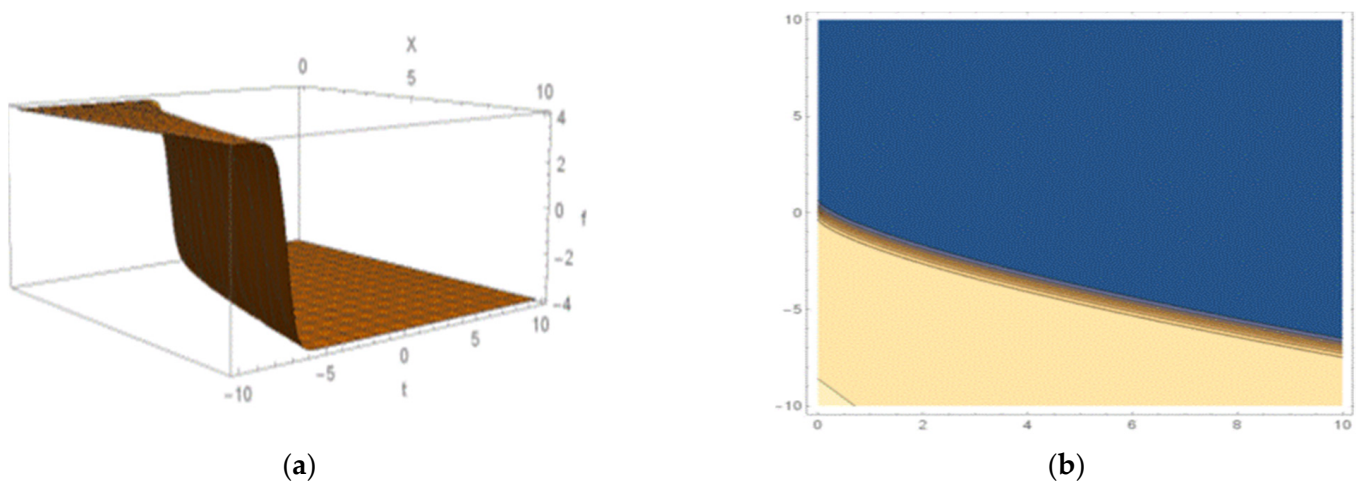


Figure 11. Dark kink-shaped soliton obtained through Equation (32): (a) 3D structure and (b) the contour shape.

The links between various types of solitons and their propagation trends, interactions, and the consequence of noise on them are illustrated in the 3D and contour shape of the concluded solutions containing various values for the parameters. The effectiveness of the Sardar sub-equation method in organizing complex nonlinear systems was proven by this visual study, which also highlights the significance of our findings. This graphic representation further emphasizes the important contributions made by the Sardar sub-equation technique for analyzing complex nonlinear occurrences and advancing our understanding of soliton behavior in the context of FSKMMS, which has not been recognized in the past literature.

6. Conclusions

The recommended model in this research is expected to be extremely beneficial in comprehending significant physical occurrences in signal transmission, biophysics, mechanical and electrical engineering, materials sciences, plasma physics, chaotic systems, ocean waves, fluid dynamics, porous media, optical fiber communication, electromagnetism, magneto-acoustic waves in plasma, epidemiology, photonics, and other fields. Electromagnetic wave propagation in ordered magnetic materials, particularly in a ferromagnetic medium, is essential to achieve faster and greater density storage fields. The present investigation revives analytical solutions for the FSKMMS in ferromagnetic materials. A

potent and innovative methodology, the Sardar sub-equation technique, was performed to produce closed-form soliton solutions of the suggested model in different forms of functions. In contemporary science, engineering, finance, and other fields, the innovative closed-form solutions, which might be applied to define magnetic fields in ferromagnets with zero conductivity, offered in this research work could be highly useful. To explore the nature of what was discovered, we completed phase portraiture of bifurcation analysis, chaotic behaviors, sensitivity, and modulation instability analysis. By using these analytical processes, we were able to verify that the system under investigation has a saddle point as well as a center point in its equilibrium position, sensitivity, and modulation instability and maintains a stable dispersion. The precise impact of each parameter on soliton interactions was thoroughly examined. An incredible wave shape associated with singular shape, soliton-shaped, kink-shaped, and dark kink-shaped soliton was generated by the fantastic physical features of the gathered solutions. We performed extensive research about the closed-form soliton solutions as well as their visualization in contour and 3D diagrams. The framework has marvelous potential for further development. Additionally, it is readily apparent that the advised strategy is simple to implement for gaining newer soliton solutions that have not been accomplished before. As a result, the suggested approach is genuine and adaptive for overcoming an extensive range of diverse fractional NLPDEs that have risen in the fields of mathematical physics as well as contemporary engineering technology.

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